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Set-membership errors-in-variables identification of MIMO linear systems

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Abstract

In this paper, we consider the problem of set-membership identification of multiple-input multiple-output (MIMO) linear models when both input and output measurements are affected by bounded additive noise. Firstly, we propose a general formulation that allows the user to take into account possible a-priori information on the structure of the MIMO model to be identified. Then, we formulate the problem in terms of a suitable polynomial optimization problem that is solved by means of a convex relaxation approach. To show the effectiveness of the proposed approach, we test the original MIMO identification algorithm on a simulation example, as well as on a set of input output experimental data, collected on a multiple-input multiple-output electronic process simulator.

Key words: set-memberhip identification , MIMO, bounded error

1 Introduction

Considerable research efforts have been devoted to the identification of multiple-input multiple-output (MIMO) systems in the last decades, and numerous algorithms have been proposed, including the subspace approach (see, e.g., Larimore (1983); Verhaegen (1994); Viberg (1995); Van Overschee and De Moor (1996); Chiuso (2007)), the approaches based on the maximum likelihood (ML) principle (see, e.g., Gibson and Ninness (2005); Wills and Ninness (2008)) and the ones exploiting the instrumental variables method to estimate the parameter of a multivariable transfer function (see, e.g., Stoica and Jansson (2000)). In particular in Verhaegen (1994), the author considers the problem of identifying a state-space multivariable model, when the system is excited by a known deterministic input and an unknown process noise, and measurements are affected by unknown noise. The work reports solutions based on Multivariable Output-Error State Space (MOESP) type algorithms. The paper by Viberg (1995) provides an interesting comparison among different classes of algorithms (4SID, IV-4SID, PEM, MOESP), for the identification of MIMO models within the framework of subspace-based methods. Interested readers can find additional details on subspace-based approaches in several books (see, e.g., Van Overschee and De Moor (1996)). The contributions by Gibson and Ninness (2005) and by Wills and Ninness (2008) consider a ML-based approach, where the authors focus on two important aspects of the algorithms exploited for the computation of the model estimate: the choice of the parameterization and the numerical robustness. An expectation-maximization approach is considered in Gibson and Ninness (2005), while a gradient-based search method is proposed in Wills and Ninness (2008). A discussion of the comparison between state-space-based and transfer function-based algorithms is presented in the paper by Stoica and Jansson (2000), where the authors also propose an instrumental variable algorithm based on a transfer function description of the MIMO model.

Interesting results have been reported throughout the literature related to the challenging problem of errors-in-variables (EIV) identification of MIMO systems. Early contributions can be traced back to the paper by Green and Anderson (1986), where identifiability conditions for such a class of systems are studied. One of the first algorithms for estimating EIV MIMO models has been proposed by Castaldi et al. (1999), where the simultaneous estimate of the model parameters and the noise covariance matrices are obtained through of a suitable prediction error method. The reader is referred to the survey paper by Söderström (2007) and the references the-

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rein for a thorough review of EIV identification of both SISO and MIMO systems proposed in the literature before 2007. Identification of EIV MIMO models is still a hot topic, as witnessed by the new algorithms proposed during recent years. Among the others, we mention the approach proposed by Diversi and Guidorzi (2012), based on the extension of the Frisch scheme to the multivariable case, and the generalized instrumental variable estimation (GIVE) approach proposed by Söderström (2012). An interesting recursive identification approach for the estimation of MIMO EIV linear systems with input static nonlinearity has been proposed by Mu and Chen (2015).

The classical approach to system identification is based on a statistical description of the experimental data uncertainty. An alternative to the stochastic description, inspired by the seminal work of Schweppe (1968), is the bounded-error or set-membership characterization, where measurement errors are assumed to be *unknown but bounded* (UBB), i.e., the measurement uncertainties are assumed to belong to a given bounded set. Such a description can be chosen in those cases where either a priori statistical information is not available, or the errors are better characterized in a deterministic way (e.g., systematic and class errors in measurement equipments, rounding and truncation errors in digital devices). Based on the UBB uncertainty description, a new paradigm called bounded-error or set-membership identification has progressively emerged in the last three decades. Interested readers are referred to the book Milanese et al. (1996), the special issues Norton (1994, 1995), the survey papers Milanese and Vicino (1991); Walter and Piet-Lahanier (1990) and the references therein for a thorough review of the fundamental principles of the theory.

The set-membership approach has been successfully applied to solve different classes of identification problems: estimation of single-input single-output (SISO) linear models with equation error (Milanese and Belforte (1982); Fogel and Huang (1982)) and error-in-variables model structures (Cerone (1993a,b); Cerone et al. (2011b,a, 2012b)), recursive identification (Chisci et al. (1998)), H_∞ identification (Milanese and Taragna (2005)), block-oriented (Cerone and Regruto (2006); Cerone et al. (2012a, 2013a)) and nonparametric nonlinear identification (Milanese and Novara (2004)), linear parameter varying model (Cerone and Regruto (2008); Cerone et al. (2013b)), identification from quantized data records (Casini et al. (2012); Cerone et al. (2013c)) conditional and robust identification (Garulli (1999); Garulli et al. (2000); Cerone et al. (2014)) just to cite a few.

However, most of the works available in the literature deal with single-input single-output (SISO) linear models, while only few papers address the problem of identification of MIMO linear models in the presence of bounded errors. In particular, identification of MIMO systems affected by bounded equation error is addressed in the paper by Wang et al. (2013) by means of an interval analysis-based approach. Since an equation error model structure is assumed, the problem of estimating the MIMO model parameters leads to a linear regression problem where the regressor is not affected by uncertainty. Therefore, in this case, interval analysis tools can be profitably applied to compute tight parameter bounds. The proposed algorithm is shown to outperform in terms of accuracy the optimal ellipsoidal algorithm proposed in Fogel and Huang (1982). However, such an approach does not explicitly cover more complex error structure (output-error, errors-in-variables) where the regressor is affected by uncertainty.

An output-error model structure is considered, instead, in the paper by Pouliquen et al. (2011), under the assumption that a bound on the energy of the output measurement error is known. They derive an optimal bounding ellipsoid algorithm for MIMO models, by extending previous results on ellipsoid algorithms for SISO systems. Stability and convergence results are presented. Zaiser and co-workers focus on the problem of computing parameter bounds for MIMO state-space model (Zaiser et al. (2014b)) and for MIMO ARX models (Zaiser et al. (2014a)), by assuming that both the input and the output sequences are corrupted by additive noise (errors-in-variables) bounded in the ℓ_∞ norm, a problem only apparently close to the one considered in this work. In fact, the work in Zaiser et al. (2014a) mainly focuses on the problem of estimating the order of the multivariable system to be identified; once the order has been estimated, standard interval analysis tools available in the literature are used to estimate the parameters. However, as usually done in the interval analysis-based algorithms, the correlation among different occurrences of the same uncertainty variable in the regressor are neglected, since each uncertainty variable is replaced by an independent interval.

In this work, we assume that the order of the system is a-priori known, and we focus on the derivation of an algorithm for computing tight parameter uncertainty intervals (PUI), by taking explicitly into account the correlation between the uncertainty variables affecting the regressor. We address the problem of computing the PUIs for MIMO linear models, with both input and output measurements corrupted by bounded noise. We consider a general description, in transfer function form, that allows the user to consider possible a-priori knowledge on the structure of each entry of the matrix transfer function. The evaluation of the parameter uncertainty intervals is formulated in terms of a suitable polynomial optimization problem, solved by a computationally efficient convex relaxation method.

The novelty of the contribution can be summarized as follows. The results presented in this paper rely on a deterministic assumption on the input-output measurement noise (i.e., the noise is assumed to be unknown but bounded); therefore, although the problem of errors-in-variables identification for both SISO and MIMO models has been widely studied (see, e.g., the survey paper Söderström (2007)), most of the algorithms available in the literature and briefly reviewed in this introduction (see, e.g., Green and Anderson (1986); Castaldi et al. (1999); Diversi and Guidorzi (2012); Söderström (2012); Mu and Chen (2015)) are based on the assumption that the noise affecting the data is a stochastic process. In this work, a different framework is considered, since the noise affecting the input and the output sequences is assumed to be bounded, while it is not required, in general, to be a random variable. Only few works can be found in the literature addressing the identification of MIMO systems in the presence of bounded noise (Wang et al. (2013); Pouliquen et al. (2011); Zaiser et al. (2014a)); all these works exploit interval analysis-based algorithms, where the (possible) correlation among different occurrences of the same uncertainty variable in the regressor are neglected; the approach presented in this paper overcomes this limitation. The proposed algorithm is the first attempt to extend the convex-relaxation based approach to set-membership identification, previously proposed by the authors for different classes of linear and nonlinear SISO systems, to the case of MIMO linear systems.

The paper is organized as follows. The problem to be solved is formulated in Section 2, while a polynomial optimization-based solution is proposed in Section 3. Section 4 provides a motivating example that shows the main ideas behind the proposed approach. Section 5 describes a convex relaxation technique to solve the polynomial optimization problem. The effectiveness of the proposed method is shown in Section 6 through a simulation example, while a further test on the identification of a MIMO electronic process simulator, from experimental data, is provided in Section 7. Concluding remarks end the paper.

2 Problem formulation

Let us consider the multiple-input multiple-output (MIMO) linear-time-invariant (LTI) system depicted in Fig. 1, where $\boldsymbol{x}(t)$ is the n_x dimensional input and $\boldsymbol{w}(t)$ is the n_w dimensional output. The MIMO LTI system to be identified is modeled by a discrete time system, that transforms $\boldsymbol{x}(t)$ into the noise-free output $\boldsymbol{w}(t)$, according to the following input-output mapping

$$\boldsymbol{w}(t) = \boldsymbol{G}(q^{-1})\boldsymbol{x}(t), \tag{1}$$

where, $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)\ \dots\ x_{n_x}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_x}$ and $\boldsymbol{w}(t) = [w_1(t)\ w_2(t)\ \dots\ w_{n_w}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_w}$ are the samples of the multivariable input and output respectively,

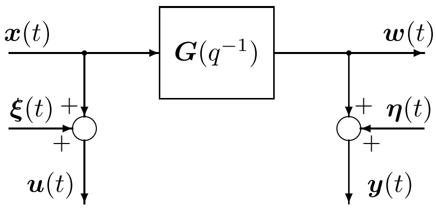

Fig. 1. Errors-in-variables basic setup for a MIMO linear dynamic system.

at time instant $t = 1, \dots, N$; N is the number of measurements and $\boldsymbol{G}(q^{-1})$ is the system matrix transfer function. The entry of $\boldsymbol{G}(q^{-1})$ relating the j-th input to the i-th output, is described by

$$G_{ij}(q^{-1}) = \frac{\sum_{k=0}^{m_{ij}} b_k^{(ij)} q^{-k}}{1 + \sum_{h=1}^{n_{ij}} a_h^{(ij)} q^{-h}}, \tag{2}$$

where $a_h^{(ij)} \in \mathbb{R}$, $(h = 1, \dots, n_{ij})$ and $b_k^{(ij)} \in \mathbb{R}$, $(k = 0, \dots, m_{ij})$ are the unknown parameters to be estimated. The i-th output of the system can be described by

$$w_i(t) = z_{i1}(t) + z_{i2}(t) \dots + z_{in_x}(t), \tag{3}$$

where z_{ij} is the contribution given by the j-th input to the i-th output, i.e.,

$$z_{ij}(t) = G_{ij}(q^{-1})x_j(t) \tag{4}$$

Let us call z_{ij} the ij-th *partial output*. On the basis of equation (4), we can relate $z_{ij}(t)$ and $x_j(t)$ through the following difference equation

$$\sum_{h=0}^{n_{ij}} a_h^{(ij)} z_{ij}(t-h) = \sum_{k=0}^{m_{ij}} b_k^{(ij)} x_j(t-k) \tag{5}$$

Both input and output data sequences are corrupted by additive noise $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ respectively

$$\boldsymbol{u}(t) = \boldsymbol{x}(t) + \boldsymbol{\xi}(t), \tag{6}$$
$$\boldsymbol{y}(t) = \boldsymbol{w}(t) + \boldsymbol{\eta}(t), \tag{7}$$

where the scalar noise variables $\xi_j(t)$ and $\eta_i(t)$, acting on the generic input $x_j(t)$ and the generic output $w_i(t)$ respectively, are assumed to range within given bounds $\Delta\xi_j$ and $\Delta\eta_i$, that is

$$\mid \xi_j(t) \mid \leq \Delta\xi, \forall t = 1, \dots, N \tag{8}$$
$$\mid \eta_i(t) \mid \leq \Delta\eta, \forall t = 1, \dots, N. \tag{9}$$

The unknown parameter vector $\boldsymbol{\theta} \in \mathbb{R}^p$ to be identified is

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_{11} \quad \dots \quad \boldsymbol{\theta}_{1n_x} \quad \boldsymbol{\theta}_{n_w1} \quad \dots \quad \boldsymbol{\theta}_{n_wn_x}]^{\mathrm{T}}, \tag{10}$$

where

$$\boldsymbol{\theta}_{ij} = \begin{bmatrix} a_0^{(ij)} & \dots & a_{n_{ij}}^{(ij)} & b_0^{(ij)} & \dots & b_{m_{ij}}^{(ij)} \end{bmatrix}, \tag{11}$$

and $p = \sum_{i=1}^{n_w} \sum_{j=1}^{n_x} (m_{ij} + n_{ij} + 1)$. The feasible parameter set (FPS) $\mathcal{D}_\theta$ is

$$\begin{aligned}\mathcal{D}_\theta =\{ & \boldsymbol{\theta} \in \mathbb{R}^p : \sum_{h=0}^{n_{ij}} a_h^{(ij)} z_{ij}(\tau_{ij} - h) = \\ & \sum_{k=0}^{m_{ij}} b_k^{(ij)} (u_j(\tau_{ij} - k) - \xi_j(\tau_{ij} - k)), \\ & z_{i1}(t) + z_{i2}(t) \ldots + z_{in_x}(t) = (y_i(t) - \eta_i(t)), \\ & \tau_{ij} = n_{ij} + 1, \ldots, N \\ & i = 1, \ldots, n_w,\ j = 1, \ldots, n_x \\ & \mid \xi_j(t) \mid \leq \Delta\xi, \ \mid \eta_i(t) \mid \leq \Delta\eta,\ t = 1, \ldots, N \}. \end{aligned} \tag{12}$$

Equation (12) provides an implicit exact description of the set of all possible values of the unknown parameter $\boldsymbol{\theta}$ that are consistent with measured data, error bounds and assumed model structure.

In this work, we address the problem of evaluating the parameter uncertainty intervals PUI_r, defined as

$$PUI_r = \left[\underline{\theta}^{(r)},\ \overline{\theta}^{(r)} \right] \quad \text{for } r = 1, \ldots, p \tag{13}$$

where $\theta^{(r)}$ is the r-th element of the vector $\boldsymbol{\theta}$, while

$$\underline{\theta}^{(r)} = \min_{\theta \in \mathcal{D}_\theta} \theta^{(r)}, \tag{14}$$

$$\overline{\theta}^{(r)} = \max_{\theta \in \mathcal{D}_\theta} \theta^{(r)}. \tag{15}$$

Thus, the computation of the PUI_r requires the solution to constrained optimization problems (14) and (15).

Remark 1 *It is worth noting that the formulation proposed in this section is quite general since it allows the user to take into account possible a-priori information on the structure of the MIMO system to be identified, i.e., the order of the numerator (m_{ij}) and denominator (n_{ij}) of each single transfer function G_{ij}.*
In the case such structural information is not available, according to the approach proposed in Stoica and Jansson (2000), all the scalar transfer functions G_{ij} are assumed to share the same denominator, and the order m_{ij} of the numerators of the transfer functions G_{ij} are assumed to satisfy $m_{ij} = n$ $\forall i = 1, \ldots, n_w$ and $\forall j = 1, \ldots, n_x$ where n is the order of the multivariable system to be identified. However, this approach may lead to overestimation of the degree of the denominators of some transfer functions G_{ij}. On the contrary, by exploiting the approach described in Zaiser et al. (2014a), the actual order of each single transfer function can be estimated.

3 Parameter bounds computation

In this section, we introduce an algorithm for the solution of problems (14) and (15).

The key idea of the proposed methodology is that the system parameters and the partial unmeasurable output signals z_{ij} can be simultaneously estimated through the solution of the following optimization problem

$$\begin{cases} \min\limits_{\theta, z, \eta, \xi} J(\boldsymbol{\theta}) \\ \text{s.t.} \\ \sum\limits_{h=0}^{n_{ij}} a_h^{(ij)} z_{ij}(\tau_{ij} - h) = \\ \quad = \sum\limits_{k=0}^{m_{ij}} b_k^{(ij)} (u_j(\tau_{ij} - k) - \xi_j(\tau_{ij} - k)), \\ \tau_{ij} = n_{ij} + 1, \ldots, N, \\ z_{i1}(t) + z_{i2}(t) \ldots + z_{in_x}(t) = (y_i(t) - \eta_i(t)), \\ i = 1, \ldots, n_w,\ j = 1, \ldots, n_x, \\ \mid \xi_j(t) \mid \leq \Delta\xi, \ \mid \eta_i(t) \mid \leq \Delta\eta,\ t = 1, \ldots, N \end{cases} \tag{16}$$

where also the samples of the unmeasurable partial output signals z_{ij} appear as decision variables of problem (16), together with the system parameters θ to be estimated. The functional $J(\boldsymbol{\theta})$ to be minimized is set to $J(\boldsymbol{\theta}) = \theta^{(r)}$ for the computation of $\underline{\theta}^{(r)}$, and to $J(\boldsymbol{\theta}) = -\theta^{(r)}$ when the computation of $\overline{\theta}^{(r)}$ is of interest.

Although problem (16) is a hard nonconvex optimization problem, it is worth noting that it falls into the class of the constrained semialgebraic optimization problems, for which some effective convex relaxations have been proposed in recent years. More specifically, it has been shown that, at least in principle, the global optimum of a constrained semialgebraic program can be approximated arbitrarily well by exploiting either the sum-of-squares-based decomposition approach proposed in Chesi et al. (2003) and Parrilo (2003), or the moment-based-approach in Lasserre (2001). The results presented in (Lasserre (2001); Chesi et al. (2003); Parrilo (2003)), allow the user to set up a hierarchy of convex linear matrix inequality (LMI) problems, guaranteed to converge to the global optimum of the original nonconvex polynomial problem as the order of relaxation goes to infinity (see the book Lasserre (2010) and the references therein for details). Further, in view of the recent results by Marshall (2009) and Nie (2014), it can be shown the convergence is finite, provided the problem satisfies a set of mild conditions (see Nie (2014) and the references in it for details).

However, direct application of such methods to large-scale identification problems (large number of parameters to be estimated and/or large set of experimental input-output data) might lead to intractable LMI problems, due to the requirements of memory storage and/or computational time. In order to overcome this limitation, ad-hoc approaches have been proposed in (Cerone et al. (2011b, 2012b,a, 2013a)), to reduce the computational complexity by exploiting some peculiar features of the polynomial optimization problems arising from the context of system identification. In the next section, we show that problem (16) enjoys the same sparsity structure of the problems considered in (Cerone et al. (2011b, 2012b,a, 2013a)) and, therefore, computationally effective implementation can be applied to solve MIMO identification problems with several hundreds of input-output data. However, we remark that set-membership approaches are particularly motivated for the case of small data set, i.e., when classical stochastic approaches cannot provide reliable results.

Remark 2 *It is worth noting that the problem of computing the PUIs could be directly formulated as a polynomial optimization problem, along the lines of our previous works for SISO systems (see, e.g., Cerone et al. (2012b)), by deriving a set of equality constraints directly from* (1)*. However, such a formulation would lead to an optimization problem with polynomial constraints of order* $n_x + 1$ *and, therefore, it would require higher computational efforts. On the contrary, the original formulation proposed in the paper, thanks to the inclusion of the partial output samples among the decision variables, leads to the polynomial optimization problem* (16) *where all the equality constraints are of order 2 (bilinear). In order to better clarify such an important fact, a simple motivating example is presented in the next section.*

4 A motivating example

To explain the strength of the approach proposed here for the computation of parameter bounds for MIMO linear systems, we consider the following simple four inputs, single output LTI model, where the output measurements are corrupted by bounded noise

$$\begin{aligned} w(t) &= \sum_{i=1}^{4} G_i(q^{-1})x_i(t) \\ y(t) &= w(t) + \eta(t),\ |\eta(t)| \leq \Delta\eta \end{aligned} \tag{17}$$

where

$$\begin{aligned} G_1(q^{-1}) &= \frac{b_0^{(1)} + b_1^{(1)}q^{-1}}{1 + a_1^{(1)}q^{-1} + a_2^{(1)}q^{-2}}, \\ G_2(q^{-1}) &= \frac{b_0^{(2)} + b_1^{(2)}q^{-1}}{1 + a_1^{(2)}q^{-1} + a_2^{(2)}q^{-2} + a_3^{(2)}q^{-3}}, \\ G_3(q^{-1}) &= \frac{b_2^{(3)}q^{-2}}{1 + a_1^{(3)}q^{-1} + a_2^{(3)}q^{-2} + a_3^{(3)}q^{-3}}, \\ G_4(q^{-1}) &= \frac{b_0^{(4)}}{1 + a_2^{(4)}q^{-2}}x_4(t). \end{aligned} \tag{18}$$

At least in principle, parameters bounds could be computed by straightforward generalization of the results for SISO systems proposed in our previous papers (see, e.g., Cerone et al. (2012b)). Such an approach, would lead to a set of polynomial optimization problems with a linear functional and polynomial constraints of order $n_x + 1 = 5$. As an example, we consider the computation of the lower bound on $b_0^{(1)}$

$$\begin{cases} \min\limits_{\theta,\eta} b_0^{(1)} \\ \text{s.t.} \\ (1 + a_1^{(1)}q^{-1} + a_2^{(1)}q^{-2}) \\ (1 + a_1^{(2)}q^{-1} + a_2^{(2)}q^{-2} + a_3^{(2)}q^{-3}) \\ (1 + a_1^{(3)}q^{-1} + a_2^{(3)}q^{-2} + a_3^{(3)}q^{-3}) \\ (1 + a_2^{(4)}q^{-2})(y(t) - \eta(t)) = \\ (b_0^{(1)} + b_1^{(1)}q^{-1})x_1(t) + \\ + (b_0^{(2)} + b_1^{(2)}q^{-1})x_2(t) + \\ + (b_2^{(3)}q^{-2})x_3(t) + b_0^{(4)}x_4(t), \\ \mid \eta_i(t) \mid \leq \Delta\eta,\ t = 1, \ldots, N \end{cases} \tag{19}$$

The equality constraints in problem (19) are polynomial of degree 5. Indeed, the higher-order terms are obtained through the multiplication of four parameters and the variables $\boldsymbol{\eta}$. In general, the constraints have degree $n_x + 1$, which depend on the number of inputs.

On the contrary, the original approach proposed in this paper leads to the following alternative formulation

$$\begin{cases} \min\limits_{\theta,z,\eta} b_0^{(1)} \\ \text{s.t.} \\ (1+a_1^{(1)}q^{-1}+a_2^{(1)}q^{-2})z_1(t) = (b_0^{(1)}+b_1^{(1)}q^{-1})x_1(t), \\ (1+a_1^{(2)}q^{-1}+a_2^{(2)}q^{-2}+a_3^{(2)}q^{-3})z_2(t) = \\ (b_0^{(2)}+b_1^{(2)}q^{-1})x_2(t), \\ (1+a_1^{(3)}q^{-1}+a_2^{(3)}q^{-2}+a_3^{(3)}q^{-3})z_3(t) = b_2^{(3)}q^{-2}x_3(t), \\ (1+a_2^{(4)}q^{-2})z_4(t) = b_0^{(4)}x_4(t), \\ z_1(t)+z_2(t)+z_3(t)+z_4(t) = (y(t)-\eta(t)), \\ \mid \eta_i(t) \mid \leq \Delta\eta,\ t=1,\ldots,N \end{cases} \tag{20}$$

that is a polynomial optimization problem with linear functional and bilinear constraints, independently from the actual number of inputs n_x.

Remark 3 *For the sake of simplicity, in this paper we focus on the case where the input and output data are corrupted by noise whose magnitude is bounded according to equations* (8) *and* (9)*. This kind of a-priori information is a natural choice since it is in agreement with the case, quite common in practice, where the errors affecting the experimental measurements are known to be bounded, while are not biased. However, the proposed approach can be straightforwardly generalized to consider the case of noise sequences that belong to any bounded set described by semialgebraic inequalities. In fact, such a case can be straightforwardly addressed by simply replacing the constraints* $\mid \eta_i(t) \mid \leq \Delta\eta,\ t=1,\ldots,N$ *in problem* (20) *with the constraints describing the semialgebraic set, since such a replacement does not modify the mathematical form of the optimization problem to be solved, which still remains semialgebraic.*

Remark 4 *By applying the available convex relaxation techniques that solve polynomial/semialgebraic optimization problems (see next section and the references therein for details), problem* (19) *would require a minimum relaxation order (maximum degree of the constraints divided by 2 and rounded to the next integer)* $\delta_{min}=3$*, while problem* (20) *would require* $\delta_{min}=1$*. More generally, application to the MIMO case of the approach presented in our previous works leads to* $\delta_{min}=(n_x+1)/2$*, which depends on the number of inputs* n_x*.*

Remark 5 *The computational complexity of the convex relaxation techniques exploited for solving both problems* (19) *and* (20) *depends exponentially on the order of relaxation* δ*. Therefore, since for problem* (19) *we have* $\delta_{min}=(n_x+1)/2$*, the approach presented in our previous works, when applied to the MIMO case leads to relaxed problems whose computational complexity depends exponentially on the number of inputs* n_x*. On the contrary, this is not true for the original approach presented in this paper. In fact, the minimum order of relaxation for problem* (20) *is* $\delta_{min}=1$*, i.e.* δ_{min} *does not depend on* n_x*.*

Remark 6 *In the case the same order of relaxation is used for solving problems* (19) *and* (20) *(e.g.,* $\delta=3$*), the obtained relaxed solution is expected to be significantly less conservative for* (20) *than for* (19)*, since in the first case the selected order of relaxation is significantly larger than the minimum value* $\delta_{min}=1$*, while it is just the minimum* $\delta=\delta_{min}=3$ *for* (19)*.*

The comparison between problems (19) and (20) shows that the approach proposed in the paper reduces both the computational complexity and the degree of conservativeness when the solution is obtained by applying convex relaxation techniques (as described in details in the next Section).

5 A convex relaxation approach

Since (16) is a semialgebraic optimization problem, at least in principle an approximation of its global optimal solution can be computed by directly applying the *dense* semidefinite (SDP) relaxation techniques proposed in Lasserre (2001); Chesi et al. (2003); Parrilo (2003). Such techniques are based on the solutions of a hierarchy of convex SDP problems, whose solution is guaranteed to monotonically converge to the exact parameters bounds defined in (16). However, it is worth noting that, for a given relaxation order δ, the application of the dense SDP-relaxation to problem (16) leads to convex optimization problems where the number of variables is $O(N^{2\delta})$, and the size of the largest linear matrix inequality (LMI) defining the feasible region of the relaxed problem is $O(N^{\delta})$. Thus, in practice, the application of the proposed approach is limited to the cases with a small number N of measurements (less than 10). In order to handle a larger number of measurements, the particular structure of the identification problem (16) has been analyzed to apply the *sparse* SDP-relaxation approach presented in the works Kojima et al. (2005); Lasserre (2006); Waki et al. (2008). In the following, we analyse the problem

$$
\begin{cases}
\min\limits_{\theta,z,\eta,\xi} J(\boldsymbol{\theta}) \\
\text{s.t.} \\
h_{t+(i-1)N}(\boldsymbol{\theta}, z_{ij}, \eta, \xi) = \sum\limits_{k=1}^{2} z_{ik}(t) - (y_i(t) - \eta_i(t)) \geq 0, \\
h_{t+(1+i)N}(\boldsymbol{\theta}, z_{ij}, \eta, \xi) = (y_i(t) - \eta_i(t)) - \sum\limits_{k=1}^{2} z_{ik}(t) \geq 0, \\
h_{t+(3+i)N}(\boldsymbol{\theta}, z_{ij}, \eta, \xi) = \Delta\eta - \eta_i(t) \geq 0, \\
h_{t+(5+i)N}(\boldsymbol{\theta}, z_{ij}, \eta, \xi) = \Delta\eta + \eta_i(t) \geq 0, \\
h_{t+(7+j)N}(\boldsymbol{\theta}, z_{ij}, \eta, \xi) = \Delta\xi - \xi_j(t) \geq 0, \\
h_{t+(9+j)N}(\boldsymbol{\theta}, z_{ij}, \eta, \xi) = \Delta\xi + \xi_j(t) \geq 0, \\
h_{\tau_{ij}+[11+2(i-1)+j]N}(\boldsymbol{\theta}, z_{ij}, \eta, \xi) = \\
\sum\limits_{k=1}^{n_{ij}} a_k^{(ij)} z_{ij}(\tau_{ij} - k) - \sum\limits_{l=0}^{m_{ij}} b_l^{(ij)} u_j(\tau_{ij} - l) + \\
+ \sum\limits_{l=0}^{m_{ij}} b_l^{(ij)} \xi_j(\tau_{ij} - l) + z_{ij}(\tau_{ij}) \geq 0, \\
h_{\tau_{ij}+[15+2(i-1)+j]N}(\boldsymbol{\theta}, z_{ij}, \eta, \xi) = \\
- \sum\limits_{k=1}^{n_{ij}} a_k^{(ij)} z_{ij}(\tau_{ij} - k) - \sum\limits_{l=0}^{m_{ij}} b_l^{(ij)} \xi_j(\tau_{ij} - l) + \\
+ \sum\limits_{l=0}^{m_{ij}} b_l^{(ij)} u_j(\tau_{ij} - l) - z_{ij}(\tau_{ij}) \geq 0, \\
\tau_{ij} = n_{ij} + 1, \ldots, N, \\
i = 1, 2, \; j = 1, 2, \; t = 1, \ldots, N
\end{cases}
\tag{21}
$$

that is equivalent to (16), where, for the sake of simplicity and without loss of generality, $n_w = n_x = 2$ is considered. More specifically, the result presented in Property 1 reported below proves that problem (21) enjoys the peculiar structured sparsity considered in the Kojima et al. (2005); Lasserre (2006); Waki et al. (2008).

Property 1 *Problem* (21) *enjoys the following features:*

P 1.1 *The functional involves only the variable* $\theta^{(r)}$.

P 1.2 *For all* $r = 1, \ldots, N$, $i = 1, 2$, *the linear constraints* $h_{r+(i-1)N} \geq 0$ *and* $h_{r+(i+1)N} \geq 0$ *depend only on the variables* $z_{ik}(r)$ *and the noise sample* $\eta_i(r)$.

P 1.3 *For all* $r = 1, \ldots, N$, $i = 1, 2$, *constraints* $h_{r+(3+i)N} \geq 0$ *and* $h_{r+(5+i)N} \geq 0$ *depend only on the noise sample* $\eta_i(r)$.

P 1.4 *For all* $r = 1, \ldots, N$, $i = 1, 2$, *constraints* $h_{r+(7+i)N} \geq 0$ *and* $h_{r+(9+i)N} \geq 0$ *depend only on the noise sample* $\xi_i(r)$.

P 1.5 *For all* $i = 1, 2$, $j = 1, 2$, $\tau_{ij} = n_{ij}, \ldots, N$, *constraints* $h_{\tau_{ij}+[11+2(i-1)+j]N} \geq 0$ *and* $h_{\tau_{ij}+[15+2(i-1)+j]N} \geq 0$ *depend only on the system parameters* $\boldsymbol{\theta}_{ij}$, *the variables* $z_{ij}(\tau_{ij} - k)$ *and the noise samples* $\xi_i(\tau_{ij} - l)$.

Thanks to Property 1, whose statement can be proved by direct inspection of equations (21), a peculiar sparsity pattern has been detected in problem (21). Therefore, by exploiting the results presented in Kojima et al. (2005); Lasserre (2006); Waki et al. (2008), we can formulate a sparse SDP-relaxed problem for (21) as described in the following.
Let $\Xi \in \mathbb{R}^{p+8N}$ be the collection of the optimization variables for the identification problem (21), i.e. $\Xi = \left[\Xi^{(11)}\ \Xi^{(12)}\ \Xi^{(21)}\ \Xi^{(22)}\ \eta_1\ \eta_2\ \xi_1\ \xi_2\right]^T$, where the entries of $\Xi^{(ij)} \in \mathbb{R}^{n_{ij}+m_{ij}+1+N}$ are given by the parameters θ_{ij} and the samples of the partial outputs z_{ij} of the transfer function G_{ij}, i.e. $\Xi^{(ij)} = [\theta_{ij}\ z_{ij}]$. In such a way, the first $p + 4N$ components of Ξ are the parameters θ_{ij} and the samples of the partial estimated output z_{ij} $\forall i, j$, while the components from position $p + 4N + 1$ to $p + 6N$ are the output noise variables η, and the last $2N$ components from position $p + 6N + 1$ to $p + 8N$ are the input noise variables ξ. Let us define the index sets $\mathcal{I}_r \subset \{1, 2, \ldots, p + 8N\}$ and $\mathcal{S}_r \subset \{1, \ldots, 20N\}$ as

$$
\begin{aligned}
\mathcal{I}_r = \{ & n_{11} + m_{11} + r, n_{11} + m_{11} + n_{12} + m_{12} + N + r, \\
& n_{11} + m_{11} + n_{12} + m_{12} + n_{21} + m_{21} + 2N + r, \\
& p + 3N + r, p + 4N + r, p + 5N + r\} \\
& \text{for } r = 1, \ldots, N
\end{aligned}
\tag{22}
$$

$$
\mathcal{I}_{r+N} = \{p + 6N + r, p + 7N + r\}, \quad \text{for } r = 1, \ldots, N \tag{23}
$$

$$
\begin{aligned}
\mathcal{I}_{r+2N} = \{ & 1, \ldots, n_{11} + m_{11}, \\
& n_{11} + m_{11} + r, \ldots, n_{11} + m_{11} + r + n_{11} \\
& p + 6N + r, \ldots, p + 6N + r + m_{11} + 1\} \\
& \text{for } r = 1, \ldots, N - n_{11}
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\mathcal{I}_{r+3N} = \{ & s + 1, \ldots, s + n_{12} + m_{12}, \\
& s + n_{12} + m_{12} + r, \ldots, s + n_{12} + m_{12} + r + n_{12} \\
& p + 7N + r, \ldots, p + 7N + r + m_{12} + 1\} \\
& \text{for } r = 1, \ldots, N - n_{12}, \\
& s = n_{11} + m_{11} + N
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\mathcal{I}_{r+4N} = \{ & s + 1, \ldots, s + n_{21} + m_{21}, \\
& s + n_{21} + m_{21} + r, \ldots, s + n_{21} + m_{21} + r + n_{21} \\
& p + 6N + r, \ldots, p + 6N + r + m_{21} + 1\} \\
& \text{for } r = 1, \ldots, N - n_{21}, \\
& s = n_{11} + m_{11} + n_{12} + m_{12} + 2N
\end{aligned}
\tag{26}
$$

$$\mathcal{I}_{r+5N} = \{s+1,\ldots,s+n_{22}+m_{22},\\ s+n_{22}+m_{22}+r,\ldots,s+n_{22}+m_{22}+r+n_{22}\\ p+7N+r,\ldots,p+7N+r+m_{22}+1\}\\ \text{for } r=1,\ldots,N-n_{22},\\ s=n_{11}+m_{11}+n_{12}+m_{12}+n_{21}+m_{21}+3N \tag{27}$$

$$\mathcal{S}_r = \{r, N+r, 2N+r, 3N+r, 4N+r, 5N+r, 6N+r,\\ 7N+r\}\\ \text{for } r=1,\ldots,N \tag{28}$$

$$\mathcal{S}_{r+N} = \{8N+r, 9N+r, 10N+r, 11N+r\}\\ \text{for } r=1,\ldots,N \tag{29}$$

$$\mathcal{S}_{r+2N} = \{12N+r, 16N+r\}\,, \quad \text{for } r=1,\ldots,N \tag{30}$$

$$\mathcal{S}_{r+3N} = \{13N+r, 17N+r\}\,, \quad \text{for } r=1,\ldots,N \tag{31}$$

$$\mathcal{S}_{r+4N} = \{14N+r, 18N+r\}\,, \quad \text{for } r=1,\ldots,N \tag{32}$$

$$\mathcal{S}_{r+5N} = \{15N+r, 19N+r\}\,, \quad \text{for } r=1,\ldots,N \tag{33}$$

By inspecting equations (22)-(33), it is possible to check that index sets $\mathcal{I}_r$ and $\mathcal{S}_r$ satisfy the following property, known in the literature as *running intersection property* (see, e.g., Lasserre (2006) for details).

Property 2 *For all $r=1,\ldots,6N$, the index sets $\mathcal{I}_r$ and $\mathcal{S}_r$ are such that:*

P 2.1 *The set of the variables indexes*

$$\mathcal{I}_0 = \{1,2,\ldots,p+8N\}$$

is the union of the sets $\mathcal{I}_r$, that is $\mathcal{I}_0 = \bigcup_{r=1}^{6N}\mathcal{I}_r$.

P 2.2 *The set of the constraints indexes $\mathcal{S}_0 = \{1,\ldots,6N\}$ defining $\mathcal{D}_{\theta z\eta\xi}$ is the union of the sets $\mathcal{S}_r$, that is $\mathcal{S}_0 = \bigcup_{r=1}^{6N}\mathcal{S}_r$.*

P 2.3 *The sets $\mathcal{S}_r$ are mutually disjoint.*

P 2.4 *For every $s \in \mathcal{S}_r$, the polynomial constraint $h_s(\boldsymbol{\theta}, z_{ij}, \eta, \xi) \geq 0$ defining $\mathcal{D}_{\theta z\eta\xi}$ depends only on the variables $\Xi(\mathcal{I}_r) = \{\Xi_i : i \in \mathcal{I}_r\}$.*

P 2.5 *The functional of identification problem* (16) *depends only on the variables $\Xi(\mathcal{I}_r) = \{\Xi_i : i \in \mathcal{I}_r\}$.*

P 2.6 *For every $r=1,\ldots,6N-1$,*

$$\mathcal{I}_{r+1} \cap \bigcup_{j=1}^{r}\mathcal{I}_j \subseteq \mathcal{I}_r.$$

■

For a given relaxation order $\delta \geq 1$, let us consider the SDP problems

$$\underline{\theta}_j^\delta = \min_{p\in\mathcal{D}^\delta_{\theta z\eta\xi}} \sum_{\alpha\in\mathcal{A}_{2\delta}} \Theta_{j\alpha}p_\alpha, \quad \overline{\theta}_j^\delta = \max_{p\in\mathcal{D}^\delta_{\theta z\eta\xi}} \sum_{\alpha\in\mathcal{A}_{2\delta}} \Theta_{j\alpha}p_\alpha, \tag{34}$$

where $\Theta_j = \{\Theta_{j\alpha}\}_{\alpha\in\mathcal{A}_{2\delta}}$ is the coefficient vector of the function $\theta^{(j)}$ in the basis $h = \{\Xi^\alpha\}_{\alpha\in A_{2\delta}}$, which is the canonical basis of the real-valued polynomials of degree 2δ in the variables vector Ξ. The feasible region $\mathcal{D}^\delta_{\theta z\eta\xi}$ is a convex set defined as

$$\mathcal{D}^\delta_{\theta z\eta\xi} = \{\, p : M_\delta(p,\mathcal{I}_r) \succeq 0,\ r=1,\ldots,6N\\ M_{\delta-1}(g_s,p,\mathcal{I}_r) \succeq 0,\ s\in\mathcal{S}_r, r=1,\ldots,6N\,\}\,, \tag{35}$$

where $M_\delta(p,\mathcal{I}_r)$ is the moment matrix of order δ associated to the variables $\Xi(\mathcal{I}_r)$, and $M_{\delta-1}(h_s,p,\mathcal{I}_r)$ is the localizing matrix (associated to the variables $\Xi(\mathcal{I}_r)$) obtained by taking into account the constraint $h_s \geq 0$, that defines the original semialgebraic feasible region $D_{\theta z\eta\xi}$. The δ-relaxed uncertainty intervals, defined as $PUI^\delta_{\theta_j} = \left[\underline{\theta}_j^\delta;\ \overline{\theta}_j^\delta\right]$, enjoy the following properties.

Property 3 *For all $k=1,\ldots,p$ and relaxation order $\delta \geq 1$, the δ-relaxed uncertainty interval $PUI^\delta_{\theta_j}$ satisfies the following properties.*

P 3.1 *The interval $PUI^\delta_{\theta_j}$ is guaranteed to contain the true parameter θ_j to be estimated, i.e. $\theta_j \in PUI^\delta_{\theta_j}$.*

P 3.2 *The interval $PUI^\delta_{\theta_j}$ becomes tighter as the relaxation order δ increases, that is $PUI^{\delta+1}_{\theta_j} \subseteq PUI^\delta_{\theta_j}$. Besides, $PUI^\delta_{\theta_j}$ converges to the tight interval PUI_{θ_j} as the LMI relaxation order goes to infinity, that is*

$$\lim_{\delta\to\infty}\underline{\theta}_j^\delta = \underline{\theta}_j, \quad \lim_{\delta\to\infty}\overline{\theta}_j^\delta = \overline{\theta}_j. \tag{36}$$

■

The proof of Property 3 follows from the structure of the index sets I_r and S_r highlighted in Property 2, the direct application of the results presented in Lasserre (2006) to problems (14)-(15) and the corresponding SDP-relaxed problems (34). Similar results to Property 3 hold for the relaxed intervals $PUI^\delta_{\theta_j}$.

Property 4 ***Computational complexity of the SDP-problems*** (34)

(i) *The number of free optimization variables q is*

$$2(6N)\begin{pmatrix} p+8N+2\delta \\ 2\delta \end{pmatrix}+$$
$$-2(6N-1)\begin{pmatrix} p+8N-1+2\delta \\ 2\delta \end{pmatrix}.$$

(ii) *The feasible region $\mathcal{D}^{\delta}_{\theta z \eta \xi}$ is described by:*

- $2(6N)$ *moment matrixes, each one of size* $\begin{pmatrix} p+8N+\delta \\ \delta \end{pmatrix}$,
- $20N$ *localizing matrixes, each one of size* $\begin{pmatrix} p+8N-1+\delta \\ \delta-1 \end{pmatrix}$. ■

For technical details on the computation of the number of optimization variables q and dimension of the LMIs, which describe the SDP relaxation of a sparse semialgebraic optimization problem, the reader is referred, e.g., to the book Lasserre (2010) and the references therein.

6 A simulation example

In order to show the effectiveness of the proposed algorithm, a numerical example is given.
We consider the following proper, fully observable and controllable state-space system, described by

$$\begin{aligned} \boldsymbol{\psi}(t+1) =& A\boldsymbol{\psi}(t)+B\boldsymbol{x}(t) \\ \boldsymbol{w}(t) =& C\boldsymbol{\psi}(t) \end{aligned} \quad (37)$$

where $\boldsymbol{\psi}(t)$ is the system state, $\boldsymbol{x}(t)$ is the input, $\boldsymbol{w}(t)$ the output and

$$A=\begin{bmatrix} 0.7 & 0.1 & 0.55 & 0 \\ -0.6 & 0.9 & 0.6 & -0.8 \\ 0 & 0 & 0.5 & 1 \\ 0.1 & 0 & 0 & -0.9 \end{bmatrix}$$
$$B=\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ C=\begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix}. \quad (38)$$

The MIMO transfer function is

$$G(q^{-1})=C(q^{-1}I-A)^{-1}B, \quad (39)$$

Table 1
PUI's for the parameters of the simulation example for $SNR = 15$ dB.

Parameter	PUI's	True Value	$\Delta_{\%}PUI_r$
a_1	[-1.247 -1.145]	-1.2	4.3
a_2	[-0.4434 -0.3573]	-0.4	11
a_3	[0.9082 0.9778]	0.949	3.7
a_4	[-0.3014 -0.234]	-0.271	13
$b_1^{(1,1)}$	[1.896 2.109]	2	5.3
$b_2^{(1,1)}$	[-1.4 -1.022]	-1.2	16
$b_3^{(1,1)}$	[-1.257 -1.019]	-1.14	10
$b_4^{(1,1)}$	[0.3995 0.6703]	0.54	25
$b_1^{(1,2)}$	[-1.088 -0.9335]	-1	7.6
$b_2^{(1,2)}$	[2.912 3.291]	3.1	6.1
$b_3^{(1,2)}$	[-2.883 -2.378]	-2.62	9.6
$b_4^{(1,2)}$	[0.5209 0.7677]	0.64	19
$b_1^{(2,1)}$	[1.893 2.088]	2	4.9
$b_2^{(2,1)}$	[-0.967 -0.6332]	-0.8	21
$b_3^{(2,1)}$	[-2.017 -1.783]	-1.9	6.1
$b_4^{(2,1)}$	[0.7722 1.039]	0.9	15
$b_1^{(2,2)}$	[0.8659 1.038]	1	9.1
$b_2^{(2,2)}$	[-2.183 -1.907]	-2.1	6.8
$b_3^{(2,2)}$	[1.81 2.152]	1.96	8.6
$b_4^{(2,2)}$	[-0.8826 -0.6651]	-0.74	14
$b_1^{(3,1)}$	[3.836 4.135]	4	3.7
$b_2^{(3,1)}$	[-3.256 -2.664]	-3	10
$b_3^{(3,1)}$	[-4.379 -3.944]	-4.16	5.2
$b_4^{(3,1)}$	[2.154 2.682]	2.45	11
$b_1^{(3,2)}$	[0.8476 1.083]	1	12
$b_2^{(3,2)}$	[-3.077 -2.685]	-2.9	6.8
$b_3^{(3,2)}$	[3.793 4.205]	3.99	5.2
$b_4^{(3,2)}$	[-2.293 -2.05]	-2.165	5.6

where I is the unit matrix of order 4. Since $G(q^{-1})$ is obtained from a state-space representation, all the transfer functions share the same denominator. Thus the transfer function from the j-th input to the i-th output is

$$G_{ij}(q^{-1})=\frac{\sum_{k=1}^{4} b_k^{(ij)} q^{-k}}{1+\sum_{h=1}^{4} a_h q^{-h}}, \quad (40)$$

Table 2
PUI's for the parameters of the simulation example for $SNR = 20$ dB.

Parameter	PUI's	True Value	$\Delta_{\%}PUI_r$
a_1	[-1.219 -1.18]	-1.2	1.6
a_2	[-0.4167 -0.3832]	-0.4	4.2
a_3	[0.9344 0.9618]	0.949	1.4
a_4	[-0.2834 -0.2572]	-0.271	4.9
$b_1^{(1,1)}$	[1.953 2.05]	2	2.4
$b_2^{(1,1)}$	[-1.278 -1.119]	-1.2	6.6
$b_3^{(1,1)}$	[-1.198 -1.083]	-1.14	5.1
$b_4^{(1,1)}$	[0.4784 0.5972]	0.54	11
$b_1^{(1,2)}$	[-1.035 -0.9682]	-1	3.3
$b_2^{(1,2)}$	[3.03 3.17]	3.1	2.3
$b_3^{(1,2)}$	[-2.72 -2.531]	-2.62	3.6
$b_4^{(1,2)}$	[0.5922 0.6947]	0.64	8
$b_1^{(2,1)}$	[1.954 2.048]	2	2.4
$b_2^{(2,1)}$	[-0.8778 -0.7205]	-0.8	9.8
$b_3^{(2,1)}$	[-1.96 -1.848]	-1.9	2.9
$b_4^{(2,1)}$	[0.8476 0.9608]	0.9	6.3
$b_1^{(2,2)}$	[0.9486 1.023]	1	3.8
$b_2^{(2,2)}$	[-2.145 -2.028]	-2.1	2.8
$b_3^{(2,2)}$	[1.892 2.035]	1.96	3.6
$b_4^{(2,2)}$	[-0.7903 -0.7047]	-0.74	5.7
$b_1^{(3,1)}$	[3.944 4.053]	4	1.4
$b_2^{(3,1)}$	[-3.097 -2.88]	-3	3.6
$b_3^{(3,1)}$	[-4.241 -4.077]	-4.16	2
$b_4^{(3,1)}$	[2.338 2.544]	2.45	4.2
$b_1^{(3,2)}$	[0.9448 1.043]	1	5
$b_2^{(3,2)}$	[-2.972 -2.819]	-2.9	2.6
$b_3^{(3,2)}$	[3.916 4.065]	3.99	1.9
$b_4^{(3,2)}$	[-2.211 -2.121]	-2.165	2.1

where the terms a_h do not depend on i and j. The system parameter $\theta = [\theta_{11}\ \ \theta_{12}\ \ \theta_{21}\ \ \theta_{22}\ \ \theta_{31}\ \ \theta_{32}]^{\mathrm{T}}$, where

$$
\begin{aligned}
\theta_{11} &= \left[a_1\ \ a_2\ \ a_3\ \ a_4\ \ b_1^{(1,1)}\ \ b_2^{(1,1)}\ \ b_3^{(1,1)}\ \ b_4^{(1,1)}\right]^T\\
\theta_{12} &= \left[a_1\ \ a_2\ \ a_3\ \ a_4\ \ b_1^{(1,2)}\ \ b_2^{(1,2)}\ \ b_3^{(1,2)}\ \ b_4^{(1,2)}\right]^T\\
\theta_{21} &= \left[a_1\ \ a_2\ \ a_3\ \ a_4\ \ b_1^{(2,1)}\ \ b_2^{(2,1)}\ \ b_3^{(2,1)}\ \ b_4^{(2,1)}\right]^T\\
\theta_{22} &= \left[a_1\ \ a_2\ \ a_3\ \ a_4\ \ b_1^{(2,2)}\ \ b_2^{(2,2)}\ \ b_3^{(2,2)}\ \ b_4^{(2,2)}\right]^T\\
\theta_{31} &= \left[a_1\ \ a_2\ \ a_3\ \ a_4\ \ b_1^{(3,1)}\ \ b_2^{(3,1)}\ \ b_3^{(3,1)}\ \ b_4^{(3,1)}\right]^T\\
\theta_{32} &= \left[a_1\ \ a_2\ \ a_3\ \ a_4\ \ b_1^{(3,2)}\ \ b_2^{(3,2)}\ \ b_3^{(3,2)}\ \ b_4^{(3,2)}\right]^T
\end{aligned}
\tag{41}
$$

Table 3
PUI's for the parameters of the simulation example for $SNR = 30$ dB.

Parameter	PUI's	True Value	$\Delta_{\%}PUI_r$
a_1	[-1.211 -1.187]	-1.2	0.98
a_2	[-0.4086 -0.3919]	-0.4	2.1
a_3	[0.9389 0.9578]	0.949	1
a_4	[-0.2779 -0.2637]	-0.271	2.6
$b_1^{(1,1)}$	[1.984 2.013]	2	0.74
$b_2^{(1,1)}$	[-1.233 -1.164]	-1.2	2.9
$b_3^{(1,1)}$	[-1.159 -1.126]	-1.14	1.4
$b_4^{(1,1)}$	[0.5213 0.5687]	0.54	4.3
$b_1^{(1,2)}$	[-1.009 -0.9833]	-1	1.3
$b_2^{(1,2)}$	[3.064 3.122]	3.1	0.95
$b_3^{(1,2)}$	[-2.663 -2.574]	-2.62	1.7
$b_4^{(1,2)}$	[0.6107 0.6662]	0.64	4.3
$b_1^{(2,1)}$	[1.977 2.015]	2	0.96
$b_2^{(2,1)}$	[-0.8199 -0.7603]	-0.8	3.8
$b_3^{(2,1)}$	[-1.923 -1.882]	-1.9	1.1
$b_4^{(2,1)}$	[0.8685 0.9237]	0.9	3.1
$b_1^{(2,2)}$	[0.9812 1.018]	1	1.9
$b_2^{(2,2)}$	[-2.132 -2.063]	-2.1	1.6
$b_3^{(2,2)}$	[1.931 1.995]	1.96	1.6
$b_4^{(2,2)}$	[-0.7593 -0.7254]	-0.74	2.3
$b_1^{(3,1)}$	[3.946 4.052]	4	1.3
$b_2^{(3,1)}$	[-3.059 -2.915]	-3	2.4
$b_3^{(3,1)}$	[-4.221 -4.101]	-4.16	1.4
$b_4^{(3,1)}$	[2.36 2.51]	2.45	3.1
$b_1^{(3,2)}$	[0.9509 1.041]	1	4.5
$b_2^{(3,2)}$	[-2.966 -2.84]	-2.9	2.2
$b_3^{(3,2)}$	[3.912 4.052]	3.99	1.8
$b_4^{(3,2)}$	[-2.22 -2.109]	-2.165	2.5

are the following

$$
\begin{aligned}
\theta_{11} &= [-1.2,\ -0.4,\ 0.949,\ -0.271,\ 2,\ -1.2,\ -1.14,\ 0.54]\\
\theta_{12} &= [-1.2,\ -0.4,\ 0.949,\ -0.271,\ -1,\ 3.1,\ -2.62,\ 0.64]\\
\theta_{21} &= [-1.2,\ -0.4,\ 0.949,\ -0.271,\ 2,\ -0.8,\ -1.9,\ 0.9]\\
\theta_{22} &= [-1.2,\ -0.4,\ 0.949,\ -0.271,\ 1,\ -2.1,\ 1.96,\ -0.74]\\
\theta_{31} &= [-1.2,\ -0.4,\ 0.949,\ -0.271,\ 4,\ -3,\ -4.16,\ 2.45]\\
\theta_{32} &= [-1.2,\ -0.4,\ 0.949,\ -0.271,\ 1,\ -2.9,\ 3.99,\ -2.165]
\end{aligned}
\tag{42}
$$

The system is excited by a random input sequence $\boldsymbol{x}(t)$ uniformly distributed in the interval $[-2,\ +2]$. The output measurements are corrupted by random addi-

tive noise $\boldsymbol{\eta}(t)$, uniformly distributed in the interval $[-\Delta\eta,\ +\Delta\eta]$. The error bounds $\Delta\eta$ are chosen in order to obtain three different values of the signal to noise ratio $SNR_w = 10\log\left\{\sum_{t=1}^{N} w_t^2 \Big/ \sum_{t=1}^{N} \eta_t^2\right\}$, namely 30 dB, 20 dB and 15 dB. The length of the data sequence is $N = 100$. The parameters are estimated by solving problem (16) according to the method presented in section 3. The software SparsePOP (Waki et al. (2008)) is used to convert the identification problem (16) into a corresponding SDP relaxed problem, solved numerically by the solver SeDuMi (Sturm (1999)).
Results on the evaluation of the system parameters bounds are reported in Tables 1-3, which show the parameter uncertainty intervals together with the true parameter values and the percentage relative error defined as

$$\Delta_{\%}PUI_r = \frac{\overline{\theta}^{(r)} - \underline{\theta}^{(r)}}{\overline{\theta}^{(r)} + \underline{\theta}^{(r)}}100. \tag{43}$$

It is worth noting that the true parameter value is always contained in the PUI, as expected. Furthermore, the percentage relative error is small (typically less than 15%) also for a significantly large amount of noise ($SNR = 15$dB).

7 Identification of a test bench MIMO electronic filter

The algorithm presented in Section 3 has been tested also on the experimental input-output data collected on a test bench MIMO electronic filter, with 2 inputs and 2 outputs, that is to be considered as an Electronic Process Simulator (EPS). This EPS is a purposely self-built electronic process simulator, with the aim of highlighting the main features of the MIMO identification procedure proposed in this paper, as described in Section 4. Indeed, although the channels of the MIMO system are characterized by completely different transfer functions (i.e., with different zeros and poles), through our approach we can formulate the identification problem in terms of linear and bilinear constraints only. The EPS, as such, is a real plant that can be easily connected to a laboratory data acquisition equipment to collect the measurements. Furthermore, this self-built EPS is an open system in the sense that the partial outputs are available for measurements, which can be used to perform an accurate validation of the identified model. Fig. 2 shows the experimental setup to collect the measurements. The system structure is reported in the block-diagram depicted in Fig. 3, where G_{11} is the transfer function of a second order low–pass filter with two complex-conjugate poles, characterized by a natural frequency of 95 Hz and a damping factor of 0.6. The transfer function has been practically built in the form of a Sallen–Key circuit. G_{12} is the transfer function of a high–pass filter with a pair of complex conjugated zeros with a natural frequency of 17 Hz and damping factor 0.2, and a pair of complex conjugated poles with a natural frequency of 83 Hz and damping factor 0.5. The transfer function was implemented by means of a Tow–Thomas circuit. G_{21} is a transfer function of a third order low–pass filter, with a couple of complex conjugated poles with a natural frequency of 120 Hz and damping factor 0.5, and a real pole at 160 Hz. The physical realization has been done by means of a Sallen–Key circuit, which implements the complex conjugated poles pair, and an RC circuit, for the additional real pole. G_{22} is the transfer function of a first order low–pass filter with a real pole at 80 Hz built in the form of a standard RC filter.

Here we assume that a-priori information on the structure of the single entries of the matrix transfer function are available. More precisely, the degrees of the numerators and denominators of all the transfer functions $G_{11}(s)$, $G_{12}(s)$, $G_{21}(s)$ and $G_{22}(s)$ are assumed to be known.
It is worth noting that, although the electronic circuit under study is a continuous-time system, the proposed identification procedure provides a discrete-time approx-

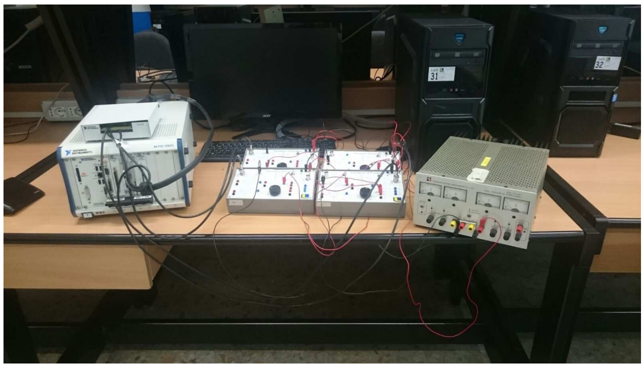

Fig. 2. The experimental MIMO system used as test bench.

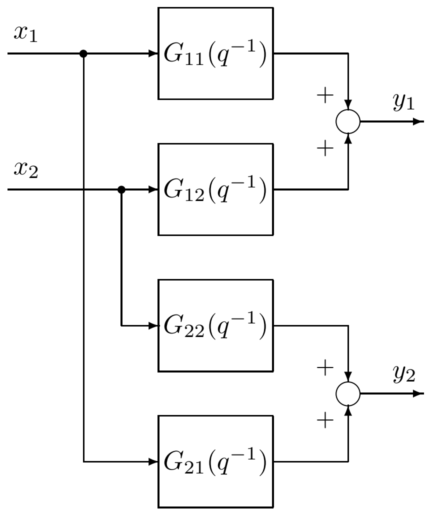

Fig. 3. Block-diagram description of the MIMO process simulator considered in the experimental test bench section.

imation of such a system.

On the basis of the a-priori information available on the transfer functions of the physical system, the following discrete-time model structure has been considered for the model to be identified

$$\begin{aligned} G_{11}(q^{-1}) &= \frac{\sum_{k=1}^{2} b_k^{(11)} q^{-k}}{1+\sum_{h=1}^{2} a_h^{(11)} q^{-h}} \\ G_{12}(q^{-1}) &= \frac{\sum_{k=0}^{2} b_k^{(12)} q^{-k}}{1+\sum_{h=1}^{2} a_h^{(12)} q^{-h}} \\ G_{21}(q^{-1}) &= \frac{\sum_{k=1}^{3} b_k^{(21)} q^{-k}}{1+\sum_{h=1}^{3} a_h^{(21)} q^{-h}} \\ G_{22}(q^{-1}) &= \frac{b_1^{(22)} q^{-1}}{1+a^{(22)} q^{-1}}. \end{aligned} \tag{44}$$

Therefore, the parameter $\theta = [\theta_{11} \;\; \theta_{12} \;\; \theta_{21} \;\; \theta_{22}]^T \in \mathcal{R}^{17}$, where

$$\begin{aligned} \theta_{11} &= \left[a_1^{(11)} \;\; a_2^{(11)} \;\; b_1^{(11)} \;\; b_2^{(11)}\right]^T \\ \theta_{12} &= \left[a_1^{(12)} \;\; a_2^{(12)} \;\; b_0^{(12)} \;\; b_1^{(12)} \;\; b_2^{(12)}\right]^T \\ \theta_{21} &= \left[a_1^{(21)} \;\; a_2^{(21)} \;\; a_3^{(21)} \;\; b_1^{(21)} \;\; b_2^{(21)} \;\; b_3^{(21)}\right]^T \\ \theta_{22} &= \left[a_1^{(22)} \;\; b_1^{(22)}\right]^T. \end{aligned} \tag{45}$$

The system has been excited by 2 uncorrelated random input sequences of 130 samples, uniformly distributed in the interval $[-1, +1]$. A National–Instruments PXI, equipped with a NI–6221 DAQ board, has been used to generate the input signal $\boldsymbol{x}(t)$ and to collect the signals $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, through a custom software developed in LabVIEW.

We have chosen a sampling frequency (f_s) of 4 kHz, which is suitably larger than the largest bandwidth of the frequency response of the transfer functions to be identified. More precisely, if we call f_b^i the bandwidth of the $i-th$ transfer function frequency response, then

$$f_b^{max} = \max_i f_b^i = 160 \text{ Hz}. \tag{46}$$

A practical choice of the sampling frequency (taken from van den Bosch and van der Klauw (1994)) is

$$10 f_b^{max} < f_s < 30 f_b^{max}. \tag{47}$$

In our case, $f_s = 4$ kHz satisfies the constraints 1600 Hz $< f_s <$ 4800 Hz. By choosing the sampling frequency too low leads to loss of information, thus the lower bound $10 f_b^{max} = 1600$ Hz is set in order to observe the main dynamics of the process. On the other hand, by choosing the sampling frequency too high leads to numerical problems, since the poles of the discrete-time system to identify cluster around the point $z = 1$ in the complex plane, which in turn makes it difficult to reliably determine the model. Furthermore, A system with a pole excess of two or more becomes nonminimum phase when sampled too fast (see Åström et al. (1984)).

From the precision of the measurement equipment, we have derived the upper bounds on the measurement errors, which are taken as $\Delta\xi = \Delta\eta = 0.003$ V. The estimation algorithm presented in Section 3 has been applied to the collected dataset. The software SparsePOP and SeDuMi have been used to solve the optimization problems. The parameter uncertainty intervals, the central estimate

$$\theta_c^{(r)} = \frac{\overline{\theta}^{(r)} + \underline{\theta}^{(r)}}{2} \tag{48}$$

and the percentage relative estimation errors are reported in Table 4, from which it can be seen that the estimation errors are relatively small (typically less than 1%) for almost all of the parameters. However, we draw the reader's attention on the fact that the optimization problem of this experimental example is ill-conditioned; indeed the parameters to be identified have significantly different magnitude. In fact, by looking at the central estimate values, the ratio between the largest parameter, $a_1^{(21)}$, and the smallest one, $b_3^{(21)}$, is close to 1000. It is known that, in general for ill-conditioned problems, the accuracy of the solution provided by numerical optimization algorithms may decrease and larger estimation error for small valued parameters may be obtained. This explains the occurrence of the larger estimation errors that can be noticed in Table 4 about parameters $b_1^{(11)}$, $b_2^{(11)}$, $b_1^{(21)}$, $b_2^{(21)}$ and $b_3^{(21)}$.

Since the proposed MIMO identification algorithm is applied to derive a discrete-time model of a real world continuous-time system, the quality of the obtained estimate is evaluated by comparing both the frequency domain and the time domain responses of the two systems, in accordance with the guidelines suggested in the literature (see, e.g., Schoukens et al. (2009) where the authors describe the 2009 SYSID benchmark problem).

The frequency response of the identified MIMO model, obtained by setting the value of the parameter to the central estimate $\theta_c^{(r)}$, has been compared with the frequency response of the electronic filter, obtained through a suitable frequency domain measurement procedure made available by the exploited National Instrument setup. The comparison presented in Fig. 4 shows that all the estimated frequency responses accurately match the measured ones.

The time-domain comparison between the measured

Table 4
PUI's and central estimates of the test bench transfer functions parameters.

Parameter	PUI's	$\theta_c^{(r)}$	$\Delta_{\%}PUI_r$
$a_1^{(11)}$	[-1.8211 -1.7941]	-1.8076	0.7461
$a_2^{(11)}$	[0.8153 0.8401]	0.8277	1.4931
$b_1^{(11)}$	[0.0239 0.0313]	0.0276	13.3985
$b_2^{(11)}$	[-0.0119 -0.0030]	-0.0074	59.6638
$a_1^{(12)}$	[-1.8567 -1.8448]	-1.8508	0.3210
$a_2^{(12)}$	[0.8628 0.8731]	0.8679	0.5902
$b_0^{(12)}$	[0.9285 0.9467]	0.9376	0.9723
$b_1^{(12)}$	[-1.8790 -1.8453]	-1.8621	0.9053
$b_2^{(12)}$	[0.9168 0.9336]	0.9252	0.9102
$a_1^{(21)}$	[-2.7230 -2.7123]	-2.7177	0.1973
$a_2^{(21)}$	[2.4778 2.4975]	2.4877	0.3968
$a_3^{(21)}$	[-0.7704 -0.7613]	-0.7658	0.5933
$b_1^{(21)}$	[0.0055 0.0069]	0.0062	10.9790
$b_2^{(21)}$	[0.0015; 0.0042]	0.0028	48.0829
$b_3^{(21)}$	[-0.0057 -0.0041]	-0.0049	15.7041
$a_1^{(22)}$	[-0.8676 -0.8660]	-0.8668	0.0918
$b_1^{(22)}$	[0.1324 0.1340]	0.1332	0.5973

($y_i(t)$) and the estimated ($\hat{w}_i(t)$) outputs has been performed on a validation set, which does not include the data exploited for performing the identification. The estimated signals $\hat{w}_i(t)$ have been computed as the output of the numerical model (44), whose parameters are the central estimate $\theta_c^{(r)}$, excited by the same inputs sequence $u_i(t)$ collected in the validation set. The obtained results, reported in Fig. 5 and 6, show that the estimated discrete-time system accurately reproduces the behaviour of the real-world continuous system.

8 Concluding remarks

Set-membership identification of MIMO systems from input-output measurements corrupted by bounded noise has been considered in the paper. We have proposed an algorithm for computing tight bounds on the system parameters, through the formulation of a suitable polynomial optimization problem, where the uncertainty affecting the data is properly handled. More precisely, we explicitly take into account the intrinsic correlation between successive occurrences of the same uncertain variables in the constraints that implicitly describe the feasible parameter set. The problem is then solved through a computationally efficient convex relaxation approach, by exploiting the peculiar sparsity structure of the problem. The effectiveness of the proposed approach is shown by means of a simulation example, where the percentage relative estimation error is quite small (typically less than 15%) also for a significantly large amount of noise ($SNR = 15$dB). We have applied also the presented algorithm to experimental data, obtained from a two input two output electronic circuit. The obtained discrete-time model, accurately reproduces both the frequency response and the time-domain behaviour of the real-world continuous-time electronic process simulator used to generate the data.

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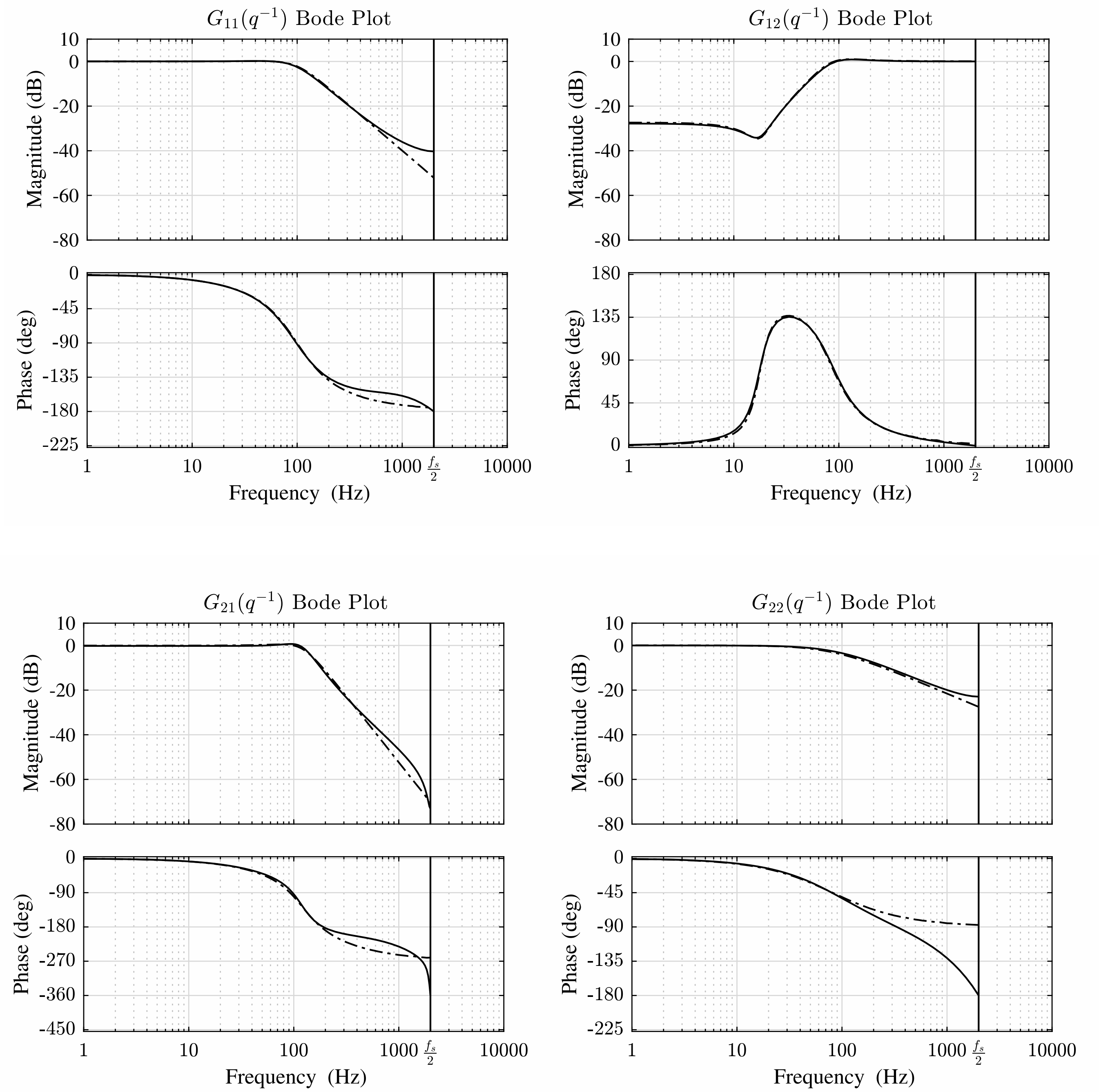

Fig. 4. Comparison between the bode plot of the frequency response of the identified transfer functions (solid) and the measured frequency response (dotted).

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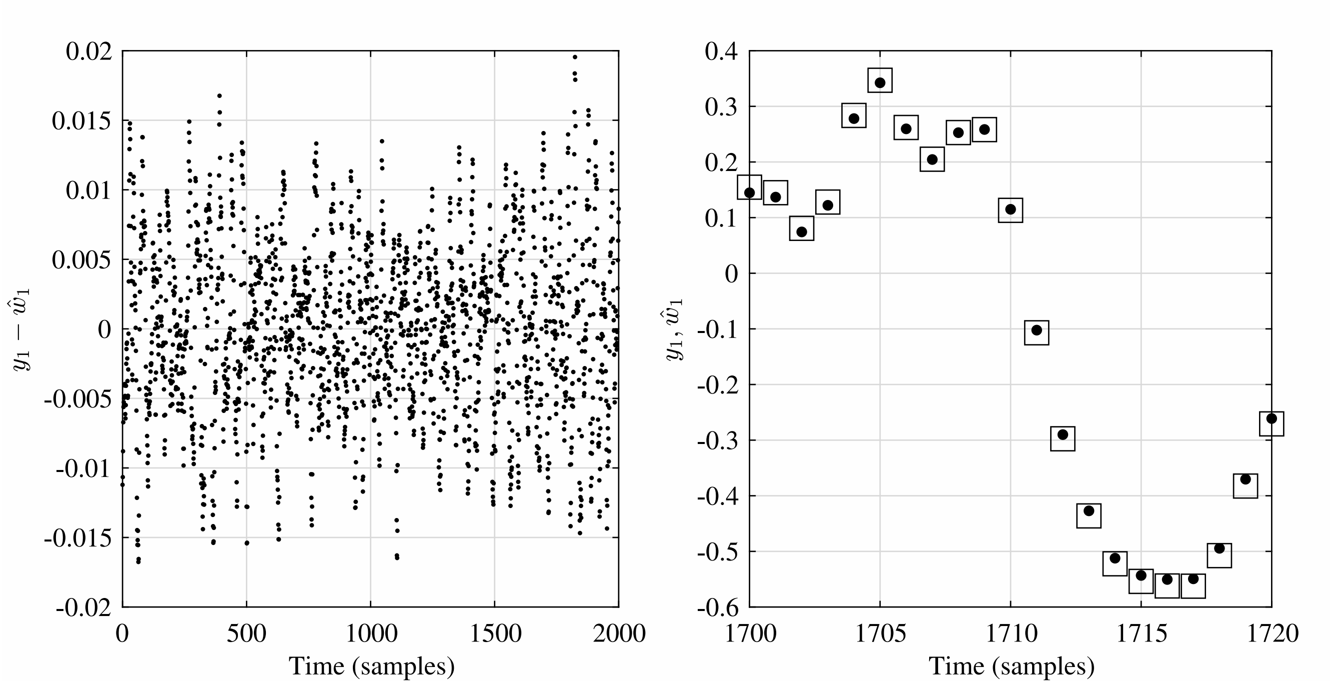
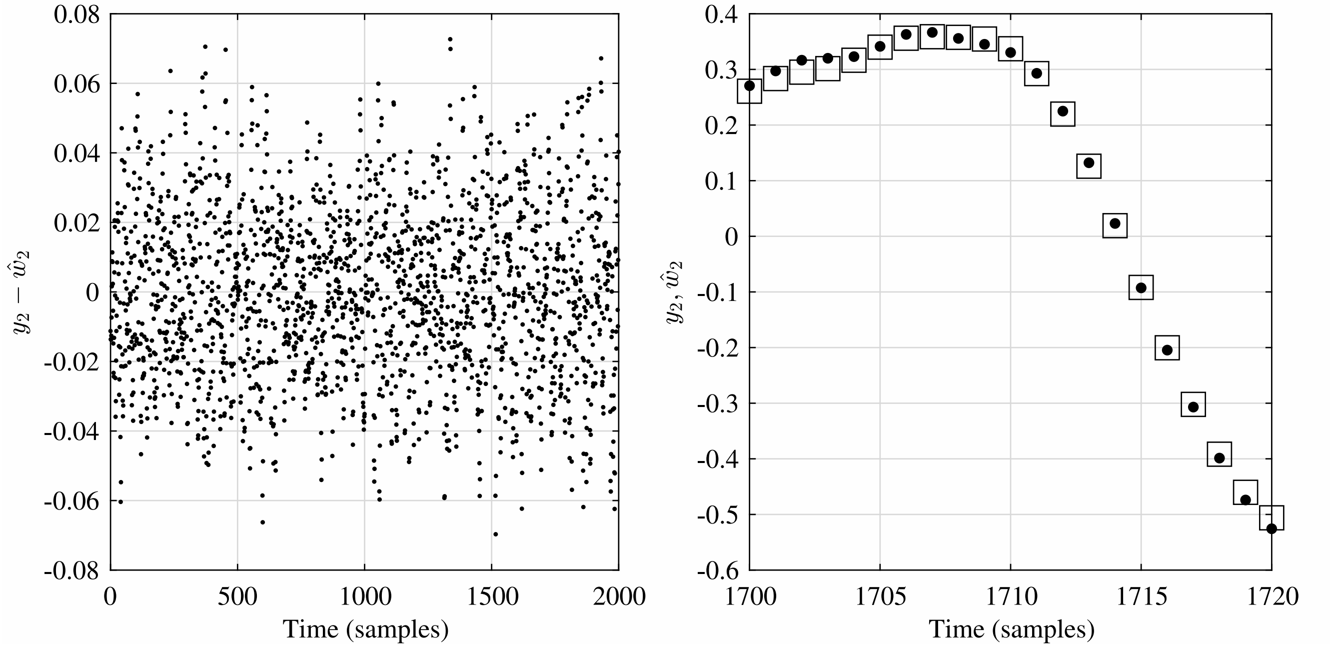


(a) Simulation error $y_1(t) - \hat{w}_1(t)$

(b) Comparison between $y_1(t)$ (squares) and $\hat{w}_1(t)$ (dots) on a small subset of the validation data

Fig. 5. Comparisons between the measured ($y_1(t)$) and the estimated ($\hat{w}_1(t)$) outputs has been performed on a validation set.



(a) Simulation error $y_2(t) - \hat{w}_2(t)$

(b) Comparison between $y_2(t)$ (squares) and $\hat{w}_2(t)$ (dots) on a small subset of the validation data

Fig. 6. Comparisons between the measured ($y_2(t)$) and the estimated ($\hat{w}_2(t)$) outputs has been performed on a validation set.

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