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27 April 2024

KÄHLER SUBMANIFOLDS AND THE UMEHARA ALGEBRA

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ABSTRACT. We show that the indefinite complex space form $\mathbb{C}^{r,s}$ is not a relative to the indefinite complex space form $\mathbb{CP}_l^N(b)$ or $\mathbb{CH}_m^N(b)$. We further study whether two Fubini-Study spaces are relatives or not.

1. INTRODUCTION

Problems about holomorphic and isometric embeddings are classical questions in complex and differential geometry. Starting with Bochner's paper [1] such questions have been studied extensively by many authors e.g. [2, 3, 6, 9, 10, 14, 15, 17, 19]. In his PhD. Thesis [2], E. Calabi obtained the existence, uniqueness and global extension of a local holomorphic isometry from a complex Kähler manifold into a complex space form (also called Fubini-Study spaces), among many other important results. Calabi's results show that the complex version of Nash's theorem is not true as it was recently asked in [12, mathoverflow]. In particular Calabi proved that any Fubini-Study space cannot be locally isometrically embedded into another Fubini-Study space with a different curvature sign with respect to the canonical Kähler metrics, and he further gave the sufficient and necessary condition for one Fubini-Study space to be embedded into another one.

The key object in Calabi's work is his *diastasis function*. Unlike the Kähler potential, Calabi's diastasis is unique. Actually, the diastasis is a clever choice of a potential around each point of an analytic Kähler manifold. Thanks to the diastasis Calabi was able to reduced metric tensor equations to functional identities. This idea turns out to be quite useful in problems with holomorphic and isometry immersions, quantization problems e.g. [5], [16, page 63] and even in related questions of number theory [3, 13, 14].

Umehara [18] later generalized Calabi's existence and uniqueness results for holomorphic isometries from a complex manifold with an indefinite Kähler metric into an indefinite complex space form. On the other hand, Umehara [17] studied an interesting question whether two complex space forms can share a common submanifold with the induced metrics. Following Calabi's idea, Umehara proved that two complex space forms with different curvature signs cannot share a common Kähler submanifold [17][18].

Two Kähler manifolds are called *relatives* when they share a common Kähler manifold i.e. a complex submanifold of one of them endowed with the induced metric is biholomorphically isometric to a complex submanifold of the other endowed with the induced

metric. It was shown in [6] that Hermitian symmetric spaces of different compact types are not relatives. In addition, the fact that Euclidean spaces and Hermitian symmetric spaces of compact types are not relatives follows from Umehara's result [17] and the classical Nakagawa-Takagi embedding of Hermitian symmetric spaces of compact type into complex projective spaces. Finally, it was shown in [11] that Euclidean spaces and Hermitian symmetric spaces of non-compact types are not relatives.

In this paper, we consider the relativity problems for two indefinite complex space forms as well as for two Fubini-Study spaces. We show that the flat indefinite complex space form $\mathbb{C}^{N,s}$ is not a relative with the non flat indefinite complex space forms $\mathbb{CP}_{s'}^{N'}(b)$, $\mathbb{CH}_{s'}^{N'}(b)$ (see Corollary 2.2 below). The relativity problem can be reformulated in terms of the so called Umehara's algebra. We use the techniques developed in [10] and [11] to obtain non-trivial improvements of Umehara's results [18] (see Theorem 2.1 below). In the last section we give necessary and sufficient conditions for two Fubini-Study space forms $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ to be relatives.

2. The Umehara algebra and the relatively problem

Umehara introduces in [18] the associate algebra $\Lambda(M)$ of a complex manifold M. Since the interest here is the local existence of a Kähler submanifold at some point p in M. We modify Umehara's definition as follows. Let $\mathcal{O}_p$ denote the local ring of germs of holomorphic functions at p. Define

$$\wedge_p := \left\{ f : f = \sum_{i=1}^n r_i |\chi_i|^2, \chi_i \in \mathcal{O}_p, r_i \in \mathbb{R} \right\},$$

and let K_p be the field of fractions of $\wedge_p$. Notice that the germs of real numbers, denoted by $\mathbb{R}_p$, belong to K_p. The main result is following local characterization of Umehara's algebra.

Theorem 2.1. *Let p be a fixed point on a complex manifold M and let $\chi_1, \cdots, \chi_s \in \mathcal{O}_p$ be non-constant germs of holomorphic functions at p such that $\chi_1(p) = \cdots = \chi_s(p) = 0$. For non-zero real numbers $r_1, \cdots, r_s$, the following statements hold:*

(i) $\log\left(1 + \sum_{i=1}^s r_i |\chi_i|^2\right) \notin K_p \setminus \mathbb{R}_p$;
(ii) $\exp\left(\sum_{i=1}^s r_i |\chi_i|^2\right) \notin K_p \setminus \mathbb{R}_p$;
(iii) If $\left(1 + \sum_{i=1}^s r_i |\chi_i|^2\right)^\alpha \in K_p \setminus \mathbb{R}_p$, then $\alpha \in \mathbb{Q}$.

The indefinite complex Euclidean space $\mathbb{C}^{N,s}(0 \leq s \leq N)$ is the complex linear space $\mathbb{C}^N$ with the indefinite Kähler metric

$$\omega_{\mathbb{C}^{N,s}} = \sqrt{-1}\left(\sum_{i=1}^{N-s} dz_i \wedge d\bar{z}_i - \sum_{j=1}^s dz_{N-s+j} \wedge d\bar{z}_{N-s+j}\right) = \sqrt{-1}\partial\bar{\partial}\left(\sum_{i=1}^{N-s} |z_i|^2 - \sum_{j=1}^s |z_{N-s+j}|^2\right).$$

The indefinite complex projective space $\mathbb{CP}_s^N(b)(0 \leq s \leq N)$ of constant holomorphic sectional curvature $b > 0$ is the open submanifold $\{(\xi_0, \cdots, \xi_N) \in \mathbb{C}^{N+1} : \sum_{i=0}^{N-s} |\xi_i|^2 - $

$\sum_{j=0}^{s-1} |\xi_{N-j}|^2\}/\mathbb{C}^*$ of $\mathbb{CP}^N$. The indefinite Kähler metric of $\mathbb{CP}^N_s(b)$ is given by

$$\omega_{\mathbb{CP}^N_s(b)} = \frac{\sqrt{-1}}{b} \partial\bar{\partial} \log \left(\sum_{i=0}^{N-s} |\xi_i|^2 - \sum_{j=0}^{s-1} |\xi_{N-j}|^2 \right).$$

The indefinite complex hyperbolic space $\mathbb{CH}^N_s(b)(0 \leq s \leq N)$ of constant holomorphic sectional curvature $b < 0$ is obtained from $\mathbb{CP}^N_{N-s}(-b)$ with indefinite Kähler metric

$$\omega_{\mathbb{CH}^N_s(b)} = -\frac{\sqrt{-1}}{(-b)} \partial\bar{\partial} \log \left(\sum_{i=0}^{s} |\xi_i|^2 - \sum_{j=0}^{N-s-1} |\xi_{N-j}|^2 \right).$$

Under inhomogeneous coordinates $(z_1, \cdots, z_N) = (\xi_1/\xi_0, \cdots, \xi_N/\xi_0)$, for instance, assuming $\xi_0 \neq 0$, the metrics are given by

$$\omega_{\mathbb{CP}^N_s(b)} = \frac{\sqrt{-1}}{b} \partial\bar{\partial} \log \left(1 + \sum_{i=1}^{N-s} |z_i|^2 - \sum_{j=0}^{s-1} |z_{N-j}|^2 \right)$$

and

$$\omega_{\mathbb{CH}^N_s(b)} = -\frac{\sqrt{-1}}{(-b)} \partial\bar{\partial} \log \left(1 + \sum_{i=1}^{s} |z_i|^2 - \sum_{j=0}^{N-s-1} |z_{N-j}|^2 \right).$$

In particular, when $s = 0$, $\mathbb{C}^{N,0}, \mathbb{CP}^N_s(b), \mathbb{CH}^N_s(b)$ are just the standard complex Euclidean, projective, hyperbolic space, respectively.

Furthermore, suppose that D is a complex manifold such that there exist holomorphic maps $F = (F_1, \cdots, F_N) : D \to \mathbb{C}^{N,s}$ and $L = (L_1, \cdots, L_{N'}) : D \to \mathbb{CP}^{N'}_{s'}(b)$ (or $\mathbb{CH}^{N'}_{s'}(b)$) with $F^*\omega_{\mathbb{C}^{N,s}} = L^*\omega_{\mathbb{CP}^{N'}_{s'}(b)}$ (or $F^*\omega_{\mathbb{C}^{N,s}} = L^*\omega_{\mathbb{CH}^{N'}_{s'}(b)}$) on D. Fixing $x \in D$, without loss of generality, we assume $F(x) = 0, L(x) = 0$ by composing the automorphisms on $\mathbb{C}^{N,s}$ and $\mathbb{CP}^{N'}_{s'}(b)$ (or $\mathbb{CH}^{N'}_{s'}(b)$). By the standard argument to get rid of $\partial\bar{\partial}$ as in the proof of Lemma 3.1, we have

$$\log \left(1 + \sum_{i=1}^{N'-s'} |L_i(z)|^2 - \sum_{j=0}^{s'-1} |L_{N'-j}(z)|^2 \right) = b \left(\sum_{i=1}^{N-s} |F_i(z)|^2 - \sum_{j=1}^{s} |F_{N-s+j}(z)|^2 \right)$$

or

$$\log \left(1 + \sum_{i=1}^{s'} |L_i(z)|^2 - \sum_{j=0}^{N'-s'-1} |L_{N'-j}(z)|^2 \right) = b \left(\sum_{i=1}^{N-s} |F_i(z)|^2 - \sum_{j=1}^{s} |F_{N-s+j}(z)|^2 \right).$$

This contradicts to Theorem 2.1 (i). Therefore, we proved the following corollary.

Corollary 2.2. *Let (D, ω_D) be a Kähler submanifold of $\mathbb{C}^{N,s}$. Then any open subset of D cannot be a Kähler submanifold of $\mathbb{CP}^{N'}_{s'}(b)$ or that of $\mathbb{CH}^{N'}_{s'}(b)$. In other words, $\mathbb{C}^{N,s}$ and $\mathbb{CP}^{N'}_{s'}(b)$ (or $\mathbb{CH}^{N'}_{s'}(b)$) cannot be relatives.*

It is a very interesting question to know whether the following statement is true. Let $\chi_1, \cdots, \chi_s \in \mathcal{O}_p$ be non-constant germs of holomorphic functions at p such that $\chi_1(p) = \cdots = \chi_s(p) = 0$. For non-zero real numbers $r_1, \cdots, r_s$, if $\alpha < 0$, then $(1 + \sum_{i=1}^{s} r_i|\chi_i|^2)^\alpha \notin K_p \setminus \mathbb{R}_p$. In particular, this statement implies that the indefinite complex space forms $\mathbb{CP}^N_l(b)$ and $\mathbb{CH}^N_m(b)$ are not relatives. Note that when $r_1, \cdots, r_s$ are all positive or all negative numbers, the statement is proved by Umehara [18].

3. Indefinite complex space forms

Lemma 3.1 (cf. Theorem 3.2 in [18])**.** *Let* h_σ, k_τ *be the germs of holomorphic functions at* $p \in U$ *with* $h_\sigma(p) = k_\tau(p) = 0$ *for* $1 \leq \sigma \leq r, 1 \leq \tau \leq s$. *Then there exist linearly independent germs of holomorphic functions* h'_σ, k'_τ *for* $1 \leq \sigma \leq r', 1 \leq \tau \leq s'$ *such that*

$$\sum_{\sigma=1}^{r} |h_\sigma|^2 - \sum_{\tau=1}^{s} |k_\tau|^2 = \sum_{\sigma=1}^{r'} |h'_\sigma|^2 - \sum_{\tau=1}^{s'} |k'_\tau|^2.$$

Proof. Choose a holomorphic coordinate $\{z\}$ at $p \in U$ such that $z(p) = 0$ and define $F(z) = \sum_{\sigma=1}^{r} |h_\sigma(z)|^2 - \sum_{\tau=1}^{s} |k_\tau(z)|^2$. It follows from the definition that $F(z)$ is a real analytic function of finite rank. By Theorem 3.2 in [18], there exit a pair of non-negative numbers (r', s'), a germ of holomorphic function ϕ_0 and germs of linearly independent holomorphic functions $\phi_1, \cdots, \phi_N$ at p, such that

$$(1) \qquad F(z) = \text{Re}(\phi_0(z)) + \sum_{\sigma=1}^{r'} |\phi_\sigma(z)|^2 - \sum_{\tau=1}^{s'} |\phi_{\tau+r'}(z)|^2,$$

with $\phi_1(0) = \cdots = \phi_{r'+s'}(0) = 0$. By comparing the Taylor expansion on the left and right sides of the equation (1), it follows that ϕ_0 is a constant function and thus $\phi_0 \equiv 0$. $\qquad\square$

From now on, we can assume, without loss of generality, that $\{f_1, \cdots, f_l, g_1, \cdots, g_m\}$ and $\{h_1, \cdots, h_r, k_1, \cdots, k_s\}$ are sets of linearly independent holomorphic functions.

Proof of Theorem 2.1 The idea of proof originates from [8] and [11]. We prove the Theorem 2.1 (i) (ii) by contradiction. Choose a holomorphic coordinate $\{z\}$ at $p \in U$ such that $z(p) = 0$. For Part (i), suppose, on the contrary, that $\log\left(1 + \sum_{i=1}^{n} r_i |\chi_i|^2\right) \in K_p \setminus \mathbb{R}_p$. By rewriting $1 + \sum_{i=1}^{n} r_i |\chi_i(z)|^2 = 1 + \sum_{\sigma=1}^{r} |h_\sigma(z)|^2 - \sum_{\tau=1}^{s} |k_\tau(z)|^2$ with $\{h_1, \cdots, h_r, k_1, \cdots, k_s\}$ linearly independent, we may assume

$$(2) \qquad \log\left(1 + \sum_{\sigma=1}^{r} |h_\sigma(z)|^2 - \sum_{\tau=1}^{s} |k_\tau(z)|^2\right) = \frac{\sum_{i=1}^{l} |f_i(z)|^2 - \sum_{j=1}^{m} |g_j(z)|^2}{\sum_{i=1}^{l'} |f'_i(z)|^2 - \sum_{j=1}^{m'} |g'_j(z)|^2}.$$

By intersecting p with a certain one dimensional complex plane, we may assume that (2) holds in an open set $U \subset \mathbb{C}$. By polarization, (2) is equivalent to

$$(3) \quad \log\left(1 + \sum_{\sigma=1}^{r} h_\sigma(z)\bar{h}_\sigma(w) - \sum_{\tau=1}^{s} k_\tau(z)\bar{k}_\tau(w)\right) = \frac{\sum_{i=1}^{l} f_i(z)\bar{f}_i(w) - \sum_{j=1}^{m} g_j(z)\bar{g}_j(w)}{\sum_{i=1}^{l'} f'_i(z)\bar{f}'_i(w) - \sum_{j=1}^{m'} g'_j(z)\bar{g}'_j(w)},$$

where $(z, w) \in U \times \text{conj}(U)$, $\text{conj}(U) = \{z \in \mathbb{C} | \bar{z} \in U\}$, and $\bar{\chi}_i(w) = \overline{\chi_i(\overline{w})}$.

Taking k-th derivative of the equation (3) in w for $k = 1, 2, \cdots$, and then evaluating at $w = 0$, we have the following matrix equation:

$$P = A \cdot X + \text{higher order terms in } X,$$

where

$$A = \begin{bmatrix} \cdots \frac{\partial \bar{h}_\sigma}{\partial w}(0) \cdots - \frac{\partial \bar{k}_\tau}{\partial w}(0) \cdots \\ \vdots \\ \cdots \frac{\partial^k \bar{h}_\sigma}{\partial w^k}(0) \cdots - \frac{\partial^k \bar{k}_\tau}{\partial w^k}(0) \cdots \\ \vdots \end{bmatrix}_{\infty \times (r+s)}, \quad X = \begin{bmatrix} \vdots \\ h_\sigma(z) \\ \vdots \\ k_\tau(z) \\ \vdots \end{bmatrix}_{(r+s) \times 1}, \quad \text{and } P^t = \begin{bmatrix} \vdots \\ p_k \\ \vdots \end{bmatrix}_{\infty \times 1},$$

with each p_k being rational function in $f_1(z), \cdots, f_l(z), g_1(z), \cdots, g_m(z)$ and $f_1'(z), \cdots, f_{l'}'(z), g_1'(z), \cdots, g_{m'}'(z)$.

We claim $\text{rank}(A) = r + s$, i.e. there exist $k = k_1, \cdots, k_{r+s}$ such that k_1-row to k_{r+s}-row in matrix A are linearly independent and all other rows can be written as linear combinations of k_1-row up to k_{r+s}-row. Reorganize the matrices A and P by deleting rows other than k_1-row to k_{r+s}-row, denoted the corresponding matrices by A_{r+s}, P_{r+s} respectively. We obtain the non-degenerate matrix equation

$$P_{r+s} = A_{r+s} \cdot X + \text{higher order terms in } X.$$

It follows by the implicit function theorem that each element in X is a Nash algebraic function in $f_1(z), \cdots, f_l(z), g_1(z), \cdots, g_m(z)$ and $f_1'(z), \cdots, f_{l'}'(z), g_1'(z), \cdots, g_{m'}'(z)$. Then one reach the contradiction by the similar argument in [11] and the reader may refer to [11] for the detailed proof. The idea is as follows. Suppose $f_1(z), \cdots, f_l(z), g_1(z), \cdots, g_m(z)$ and $f_1'(z), \cdots, f_{l'}'(z), g_1'(z), \cdots, g_{m'}'(z)$ are all Nash algebraic functions in z. So are all $h_1, \cdots, h_r, k_1, \cdots, k_s$ by the above argument. Then the left hand side of the equation (2) has logarithmic growth while the right hand side of the equation (2) has polynomial growth as z approaches the pole. If $f_1(z), \cdots, f_l(z), g_1(z), \cdots, g_m(z)$ and $f_1'(z), \cdots, f_{l'}'(z), g_1'(z), \cdots, g_{m'}'(z)$ are not all Nash algebraic functions. Then one can choose a maximal algebraic independent subset $S \subset \{f_1, \cdots, f_l, g_1, \cdots, g_m, f_1', \cdots, f_{l'}', g_1', \cdots, g_{m'}'\}$ such that any element in $\{f_1, \cdots, f_l, g_1, \cdots, g_m, f_1', \cdots, f_{l'}', g_1', \cdots, g_{m'}', h_1, \cdots, h_r, k_1, \cdots, k_s\}$ is Nash algebraic functions in z and elements in S. Denote elements in S by $\{X_1, \cdots, X_\kappa\}$ and we can write each $f_*(z), f_*'(z), g_*(z), g_*'(z), h_*(z), k_*(z)$ by Nash algebraic functions in z, X given by $\hat{f}_*(z, X), \hat{f}_*'(z, X), \hat{g}_*(z, X), \hat{g}_*'(z, X), \hat{h}_*(z, X), \hat{k}_*(z, X)$ respectively. By similar argument as in [11], we have

(4)
$$\log\left(1 + \sum_{\sigma=1}^{r} \hat{h}_\sigma(z, X)\bar{h}_\sigma(w) - \sum_{\tau=1}^{s} \hat{k}_\tau(z, X)\bar{k}_\tau(w)\right) = \frac{\sum_{i=1}^{l} \hat{f}_i(z, X)\bar{f}_i(w) - \sum_{j=1}^{m} \hat{g}_j(z, X)\bar{g}_j(w)}{\sum_{i=1}^{l'} \hat{f}_i'(z, X)\bar{f}_i'(w) - \sum_{j=1}^{m'} \hat{g}_j'(z, X)\bar{g}_j'(w)}$$

for independent variables z, w, X. Hence for fixed w the left hand side of the equation (4) has logarithmic growth while the right hand side of the equation (4) has polynomial growth as z approaches the pole. We again reach a contradiction.

Now we show $\text{rank}(A) = r+s$. Suppose $\text{rank}(A) = d < r+s$. Without loss of generality, we assume that the first d columns are linearly independent in the coefficient matrix A, writing $L_1, L_2, \cdots, L_d$. Then, for any n with $d < n \leq r+s$, the n-th column is linear combination of $L_1, L_2, \cdots, L_d$, i.e.

$$L_n = \sum_{i=1}^{d} C_i L_i.$$

In other words, the n-th element in $\{h_1, \cdots, h_r, k_1, \cdots, k_s\}$ can be written as linear combination of the first d elements by the Taylor expansion, meaning $\{h_1, \cdots, h_r, k_1, \cdots, k_s\}$ is not linear independent. This is a contradiction. Thus we complete the proof of Theorem 2.1 (i).

Part (ii) follows from the similar argument. The only difference is to take logarithmic differentiation in w.

For Part (iii), assume

(5) $$\left(1 + \sum_{\sigma=1}^{r} h_\sigma(z)\bar{h}_\sigma(w) - \sum_{\tau=1}^{s} k_\tau(z)\bar{k}_\tau(w)\right)^\alpha = \frac{\sum_{i=1}^{l} f_i(z)\bar{f}_i(w) - \sum_{j=1}^{m} g_j(z)\bar{g}_j(w)}{\sum_{i=1}^{l'} f_i'(z)\bar{f}_i'(w) - \sum_{j=1}^{m'} g_j'(z)\bar{g}_j'(w)}.$$

By taking logarithmic differentiation in w and applying the similar argument as for Part (i), we know that each element in $\{h_1(z), \cdots, h_r(z), k_1(z), \cdots, k_s(z)\}$ a Nash algebraic function in $f_1(z), \cdots, f_l(z), g_1(z), \cdots, g_m(z)$ and $f_1'(z), \cdots, f_{l'}'(z), g_1'(z), \cdots, g_{m'}'(z)$. Suppose $f_1(z), \cdots, f_l(z), g_1(z), \cdots, g_m(z)$ and $f_1'(z), \cdots, f_{l'}'(z), g_1'(z), \cdots, g_{m'}'(z)$ are all Nash algebraic functions in z. Then in (5) we have a Nash algebraic function to the power of α is equal to another Nash algebraic function. Thus $\alpha \in \mathbb{Q}$. Otherwise, one can choose a maximal algebraic independent subset $S \subset \{f_1, \cdots, f_l, g_1, \cdots, g_m, f_1', \cdots, f_{l'}', g_1', \cdots, g_{m'}'\}$ such that any element in $\{f_1, \cdots, f_l, g_1, \cdots, g_m, f_1', \cdots, f_{l'}', g_1', \cdots, g_{m'}', h_1, \cdots, h_r, k_1, \cdots, k_s\}$ is Nash algebraic functions in z and elements in S. By similar argument as above, we have

(6)
$$
\left(1 + \sum_{\sigma=1}^{r} \hat{h}_\sigma(z, X)\bar{h}_\sigma(w) - \sum_{\tau=1}^{s} \hat{k}_\tau(z, X)\bar{k}_\tau(w)\right)^\alpha = \frac{\sum_{i=1}^{l} \hat{f}_i(z, X)\bar{f}_i(w) - \sum_{j=1}^{m} \hat{g}_j(z, X)\bar{g}_j(w)}{\sum_{i=1}^{l'} \hat{f}_i'(z, X)\bar{f}_i'(w) - \sum_{j=1}^{m'} \hat{g}_j'(z, X)\bar{g}_j'(w)}
$$

for independent variables z, w, X. For fixed w, we have in (6) a Nash algebraic function to the power of α is equal to another Nash algebraic function. Thus again $\alpha \in \mathbb{Q}$. This completes the proof of Theorem 2.1 (iii). □

4. Fubini-Study spaces

Along this section we use Calabi's original notation $\mathbf{F}(n, b)$ for Fubini-Study spaces. Namely, as in [2, pages 16 and 17], we denote with $\mathbf{F}(n, b)$ a complex space form whose Kähler potential is locally given by

$$
\frac{1}{b} \log(1 + b|Z|^2)
$$

where $Z = (z_1, \cdots, z_n)$ and $|Z|^2 := \sum_{i=1}^{n} |z_i|^2$. So

$$
\mathbf{F}(n, b) = \begin{cases} \mathbb{CP}_0^n(b) & \text{if } b > 0, \\ \mathbb{CH}_0^n(b) & \text{if } b < 0. \end{cases}
$$

Theorem 4.1. *Let $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ be two complex space forms where $a, b \in \mathbb{R}^+$ are positive real numbers. Assume that:*

  (i) *there are positive integers s, r such that $sa = rb$,*
  (ii) $m + n + 1 > \max\left\{\binom{s+m}{s}, \binom{r+n}{r}\right\}$.

Then $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ are relatives.

For example $\mathbf{F}(3, a)$ and $\mathbf{F}(8, 2a)$ are relatives for any $a > 0$.

Proof. Let $\kappa = sa = rb > 0$ be the common value of the two numbers sa and rb. Then according to Calabi's [2, Theorem 13, page 21] both spaces $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ can be embedded into the bigger $\mathbf{F}(N, \kappa)$ where

$$
N := \max\left\{\binom{s+m}{s} - 1, \binom{r+n}{r} - 1\right\}.
$$

By Condition (ii) and the well-known intersection theorem [7, Theorem 7.2, page 48] every irreducible component Z of the intersection $\mathbf{F}(m, a) \bigcap \mathbf{F}(n, b)$ has positive dimension. Hence $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ are relatives since Z has an open subset of smooth points [7, Theorem 5.3, page 33]. □

Remark 4.2. Note that for positive integers s, m with $s \leq m$, $\mathbf{F}(m, s)$ and $\mathbf{F}(m, 1)$ are relatives since $(\cdots, f_j(z), \cdots) : U \subset \mathbb{C} \to \mathbb{C}^m \subset \mathbf{F}(m, s)$ and $(z, \cdots, z) : U \subset \mathbb{C} \to \mathbb{C}^m \subset \mathbf{F}(m, 1)$ with $f_j(z) = \sqrt{\frac{m^j}{s} \binom{s}{j}} z^j$ for $1 \leq j \leq s$ and $f_j(z) = 0$ for $s < j \leq m$ satisfy

$$\frac{1}{s} \log \left(1 + s \sum_j |f_j|^2 \right) = \log \left(1 + m|z|^2 \right).$$

Obviously, $2m + 1 \ll \binom{m+s}{s}$ for $s \gg 1$. For instance, $\mathbf{F}(2, 2)$ and $\mathbf{F}(2, 1)$ are relatives but $2 + 2 + 1 = 5 < \binom{2+2}{2} = 6$. This example shows that (ii) in the above Theorem is not a necessary condition for $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ to be relatives. Actually, this can be explained as follows: $\mathbf{F}(1, 1) \hookrightarrow \mathbf{F}(n, 1)$ by the totally geodesic embedding and $\mathbf{F}(1, 1) \hookrightarrow \mathbf{F}(n, b)$ for $b \in \mathbb{N}$, $n \geq b \geq 1$ by Calabi's result [2, Theorem 13, page 21]. Thus, $\mathbf{F}(n, b)$ and $\mathbf{F}(n, 1)$ are relatives but $2n + 1$ is not necessarily greater than $\binom{b+n}{b}$.

Theorem 4.3. *Let $a \neq 0, b \neq 0$. Suppose that $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ are relatives. Then $ab > 0$ and $a/b \in \mathbb{Q}$.*

Proof. Let $U \subset \mathbb{C}$ be a connected open set. Suppose that $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ are relatives. By composing with elements in holomorphic isometry groups, it is equivalent to the existence of holomorphic maps $H = (h_1, \cdots, h_m) : U \to \mathbf{F}(m, a)$, $K = (k_1, \cdots, k_n) : U \to \mathbf{F}(n, a)$ with $H(0) = 0, K(0) = 0$ such that

$$\frac{1}{a} \log \left(1 + a \sum_{i=1}^{m} |h_i(z)|^2 \right) = \frac{1}{b} \log \left(1 + b \sum_{i=1}^{n} |k_i(z)|^2 \right).$$

This is equivalent to

$$(7) \qquad 1 + a \sum_{i=1}^{m} |h_i(z)|^2 = \left(1 + b \sum_{i=1}^{n} |k_i(z)|^2 \right)^{a/b}.$$

If $ab < 0$, it follows from Umehara's argument [18] that the equation (7) cannot hold. Furthermore, $a/b \in \mathbb{Q}$ follows from Theorem 2.1(iii). $\qquad \square$

Corollary 4.4. *Let a, b be two positive real number such that $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ are relatives. Then there are $N \in \mathbb{N}$, $\kappa \in \mathbb{R}^+$ and holomorphic and isometric immersions $f : \mathbf{F}(n, b) \to \mathbf{F}(N, \kappa)$, $h : \mathbf{F}(m, a) \to \mathbf{F}(N, \kappa)$ such that*

$$\dim(f(\mathbf{F}(n, b)) \bigcap h(\mathbf{F}(m, a))) > 0.$$

Proof. Since $\frac{a}{b} \in \mathbb{Q}$ there are $r, s \in \mathbb{N}$ such that $ra = sb$. Set $\kappa := ra = sb$. Then by Calabi's Theorem [2, Theorem 13, page 21] there are holomorphic and isometric immersions $f : \mathbf{F}(n, b) \to \mathbf{F}(n', \kappa)$ and $j : \mathbf{F}(m, a) \to \mathbf{F}(m', \kappa)$. Taking $N := \max(m', n')$ we can assume that $N = m' = n'$. Since $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ are relatives there is a Kähler manifold Z, $\dim(Z) > 0$ and holomorphic and isometric immersions $G : Z \to \mathbf{F}(n, b)$ and $H : Z \to \mathbf{F}(m, a)$. Then $f \circ G$ and $j \circ H$ are holomorphic and isometric immersions of Z into $\mathbf{F}(N, \kappa)$. By Calabi's rigidity [2, Theorem 9, page 18] there is an isometry $u \in \mathrm{U}(N + 1)$ such that $f \circ G = u \circ j \circ H$. So by setting $h := u \circ j$ we get that $f(G(Z)) = h(H(Z)) \subset f(\mathbf{F}(n, b)) \bigcap h(\mathbf{F}(m, a))$ and $\dim(f(G(Z))) = \dim(Z) > 0$. $\qquad \square$

Corollary 4.5. *Let a, b be two positive real number such that $a/b \in \mathbb{Q}$. Suppose that $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ are relatives. Moreover, assume $sa = rb$ for positive integers s, r with*

$(s, r) = 1$. *Then*

(8) $$r + 1 \le \binom{s+m}{s} \text{ and } s + 1 \le \binom{r+n}{r}.$$

Proof. It follows from the proof of Theorem 4.3 that $\mathbf{F}(n, b)$ and $\mathbf{F}(m, a)$ are relatives if and only if there exist holomorphic maps $H = (h_1, \cdots, h_m) : U \subset \mathbb{C} \to F(m, a), K = (k_1, \cdots, k_n) : U \subset \mathbb{C} \to \mathbf{F}(n, a)$ with $H(0) = 0, K(0) = 0$ such that

$$\left(1 + a \sum_{i=1}^{m} |h_i(z)|^2\right)^s = \left(1 + b \sum_{i=1}^{n} |k_i(z)|^2\right)^r.$$

Note that the left hand side can be written as the sum of 1 and norm squares of P linearly independent holomorphic functions with $s \le P \le \binom{m+s}{s} - 1$ and the right hand side can be written as the sum of 1 and norm squares of Q linearly independent holomorphic functions with $r \le Q \le \binom{n+r}{r} - 1$. One knows $P = Q$ by Calabi's theorem (cf. also [4]). Thus we conclude (8). □

4.1. **Conditions (8) in Corollary 4.5 are not sufficient.** Here we show that $\mathbf{F}(2, 1)$ and $\mathbf{F}(2, \frac{3}{2})$ are not relatives. Observe that conditions (8) in Corollary 4.5 hold true. We also give a necessary condition on a, b for $\mathbf{F}(2, a)$ and $\mathbf{F}(2, b)$ to be relatives.

To show that that $\mathbf{F}(2, 1)$ and $\mathbf{F}(2, \frac{3}{2})$ are not relatives it is enough to show that there are no h_1, h_2, k_1, k_2 in the local ring $\mathcal{O}_{\mathbb{C}, 0}$ parameterizing local curves $\{h_1, h_2\}$ and $\{k_1, k_2\}$ through $(0, 0) \in \mathbb{C}^2$ such that

(9) $$(1 + |k_1(z)|^2 + |k_2(z)|^2)^2 = (1 + |h_1(z)|^2 + |h_2(z)|^2)^3$$

Expanding both sides we get

$$|\sqrt{2}k_1|^2 + |k_1^2|^2 + |\sqrt{2}k_2|^2 + |\sqrt{2}k_1 k_2|^2 + |k_2^2|^2 =$$

$$|\sqrt{3}h_1|^2 + |\sqrt{3}h_1^2|^2 + |h_1^3|^2 + |\sqrt{3}h_2|^2 + |\sqrt{6}h_1 h_2|^2 + |\sqrt{3}h_1^2 h_2|^2 + |\sqrt{3}h_2^2|^2 + |\sqrt{3}h_1 h_2^2|^2 + |h_2^3|^2$$

where we omitted the letter z from the argument of the functions.

According to [4, Proposition 3, page 102] (or by [2, Theorem 2, page 8]) there is a matrix $U \in \mathbf{U}(9)$ such that

(10) $$U \cdot H = K$$

where

$$H := [\sqrt{3}h_1, \sqrt{3}h_2, \sqrt{3}h_1^2, \sqrt{6}h_1 h_2, \sqrt{3}h_2^2, h_1^3, \sqrt{3}h_1^2 h_2, \sqrt{3}h_1 h_2^2, h_2^3]^t$$

and

$$K = [\sqrt{2}k_1, \sqrt{2}k_2, k_1^2, \sqrt{2}k_1 k_2, k_2^2, 0, 0, 0, 0]^t.$$

Now observe that the last four rows of U can be used to define 4 linearly independent affine curves through $(0, 0) \in \mathbb{C}^2$ of degree less or equal to 3. To see this, set $U = (u_{ij})$ and consider the polynomials $P_i \in \mathbb{C}[X, Y]$, $i = 6, 7, 8, 9$ defined by:

$$P_i := u_{i1}\sqrt{3}X + u_{i2}\sqrt{3}Y + u_{i3}\sqrt{3}X^2 + u_{i4}\sqrt{6}XY + u_{i5}\sqrt{3}Y^2 + u_{i6}X^3 + u_{i7}\sqrt{3}X^2 Y + u_{i8}\sqrt{3}XY^2 + u_{i9}Y^3$$

Then equation (10) implies that

$$P_i(h_1(z), h_2(z)) = 0$$

for $i = 6, 7, 8, 9$ and z near $0 \in \mathbb{C}$. Thus, $z \to (h_1(z), h_2(z))$ is a local parameterization of an irreducible connected component of each affine curve $P_i = 0$. Then the

four curves share a common component $\mathcal{P}$ through $(0,0)$. So we have three possibilities $\deg(\mathcal{P}) = 1, \deg(\mathcal{P}) = 2$ or $\deg(\mathcal{P}) = 3$. We will show that each of them yields a contradiction.

Case $\deg(\mathcal{P}) = 1$. This is the same as assuming that h_1, h_2 are linearly dependent. So there is a function $h(z)$ and complex numbers m_1, m_2 such that $h_1 = m_1 h$ and $h_2 = m_2 h$. Plugging this into equation (9) we get

$$(1 + |k_1(z)|^2 + |k_2(z)|^2)^2 = (1 + |mh(z)|^2)^3$$

where $m = \sqrt{|m_1|^2 + |m_2|^2}$. Changing the variable z with $z = h^{-1}(w)$ we get

$$(1 + |\widetilde{k}_1(w)|^2 + |\widetilde{k}_2(w)|^2)^2 = (1 + |mw|^2)^3$$

a further change of variables $w = \frac{z}{m}$ gives

$$(1 + |\widetilde{\widetilde{k}}_1(z)|^2 + |\widetilde{\widetilde{k}}_2(z)|^2)^2 = (1 + |z|^2)^3$$

but this is impossible. Namely, there are no holomorphic functions $\widetilde{\widetilde{k}}_1(z), \widetilde{\widetilde{k}}_2(z)$ as above since they should give a (local) isometric and holomorphic embedding $\mathbf{F}(1,1) \hookrightarrow \mathbf{F}(2, \frac{3}{2})$ which contradicts Calabi's results [2, Theorem 13, page 21].

Case $\deg(\mathcal{P}) = 2$. This implies that there exist $Q \in \mathbb{C}[X, Y]$, $\deg(Q) = 2$, such that Q divides the four polynomials P_6, P_7, P_8, P_9. That is to say,

$$P_i = Q \cdot (a_i X + b_i Y + c_i)$$

for $i = 6, 7, 8, 9$. But then P_6, P_7, P_8, P_9 are not linearly independent. Contradiction.

Case $\deg(\mathcal{P}) = 3$. In this case all polynomials are multiple of each other. Indeed, the irreducible curve parameterized by h_1, h_2 is an irreducible cubic hence up to multiple there is just one equation which define it. Contradiction.

So $\mathbf{F}(2,1)$ and $\mathbf{F}(2, \frac{3}{2})$ are not relatives hence conditions (8) in Corollary 4.5 are not sufficient conditions.

With the same idea we also get the following result.

Theorem 4.6. *Let $a, b \in \mathbb{N}$, $\gcd(a, b) = 1$, $1 < a < b$ and $a(a + 3) < 4b + 2$. Then $\mathbf{F}(2, 1)$ is not a relative neither of $\mathbf{F}(2, \frac{b}{a})$ nor of $\mathbf{F}(2, \frac{a}{b})$. Equivalently, under the above conditions $F(2, a)$ and $F(2, b)$ are not relatives.*

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