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Harmonic spectral components in time sequences of Markov correlated events

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The paper concerns the analysis of the conditions allowing time sequences of Markov correlated events give rise to a line power spectrum having a relevant physical interest. It is found that by specializing the Markov matrix in order to represent closed loop sequences of events with arbitrary distribution, generated in a steady physical condition, a large set of line spectra, covering all possible frequency values, is obtained. The amplitude of the spectral lines is given by a matrix equation based on a generalized Markov matrix involving the Fourier transform of the distribution functions representing the time intervals between successive events of the sequence. The paper is a complement of a previous work where a general expression for the continuous power spectrum was given. In that case the Markov matrix was left in a more general form, thus preventing the possibility of finding line spectra of physical interest. The present extension is also suggested by the interest of explaining the emergence of a broad set of waves found in the electro and magneto-encephalograms, whose frequency ranges from 0.5 to about 40Hz, in terms of the effects produced by chains of firing neurons within the complex neural network of the brain. An original model based on synchronized closed loop sequences of firing neurons is proposed, and a few numerical simulations are reported as an application of the above cited equation. [http://dx.doi.org/10.1063/1.4994039]

I. INTRODUCTION

In a previous paper[1] the power spectrum of time sequences of correlated events described by time functions $F_\alpha(t)$ and characterized by an arbitrary number N of states α has been considered. The correlation concerning the states of successive events is expressed by a $N \times N$ Markov matrix $||m||$ giving the conditional probability for the transition between two states of the sequence. The time intervals u_i between successive events i and $i+1$ are given by a set of N arbitrary distribution functions $q_{\alpha_i,\alpha_{i+1}}(u_i)$ depending on the states of these two events. A generalized Markov matrix $||M(u_i)||$ was then introduced expressing both the transition between the states α and the correlation between the time intervals u_i, The power spectrum was then expressed by a matrix equation in terms of the Fourier transform of this correlation matrix. Singular points of the matrix $(||I|| - ||M(\omega)||)$, where $||I||$ is the identity matrix, correspond to lines in the power spectrum. In Ref. 1 the search for singular points was performed by assuming that $||m||$ is an arbitrary Markov matrix, and the conclusion was that a line spectrum can be found under the condition that the functions $q_{\alpha_i,\alpha_{i+1}}(u_i)$ be delta functions of the $u_i s$. However, if $||m||$ is left in its general form, only a single line spectrum can be found, whose first line lowest frequency is higher or equal to the inverse of the smaller time interval in the sequence. The present paper shows that it is possible to generate a broad set of line power spectra covering all the possible frequency values and having a relevant physical interest by specializing the

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Markov matrix $||m||$. This particular Markov matrix determines closed loops sequences of events whose arbitrary distribution of the amplitude, form and time intervals is described by the correlation matrix $||M(\omega)||$, and is related to a steady physical condition of the system. The lowest frequency of the line spectrum is given by the inverse of the time needed to complete the loop and the whole power spectrum is generated by the irregularities concerning the time intervals between successive events in the sequence and/or the amplitude or the form of the functions representing the events. The paper gives a general matrix equation which allows to evaluate the full line spectrum in any situation represented by the correlation matrix $||M(\omega)||$ and functions $F_\alpha(t)$. The case of the generation of mixed power spectra created by random modulation of the time intervals between successive events due to external perturbations is also considered and expressions for both the line and the continuous power spectrum given.

An application concerning the interpretation of the brain electro and magneto encephalograms (EEG and MEG) in terms of closed loops of firing neurons is also given. The model allows to explain the range of frequencies observed in EEG and MEG, from $0.5Hz$ to about $4Hz$ (delta waves), up to about $40Hz$ (gamma waves), in terms of the number of connected neurons forming the closed loop. According to data regarding the propagation time of the emitted impulse between two near connected neurons, this number ranges from a few neurons (gamma waves) up to a few hundreds neurons (delta waves).

Numerical simulations are given to evaluate the amplitude of the signal generated by single loops of 30 neurons for different situations concerning the distribution of the amplitudes of the firing impulses, and of the time intervals u_i between successive firings. It was found that without taking into account two large, but not known, reduction factors, i.e. the transmission across the cerebral matter and the cranial bone of the electric impulses generated externally to the cell by the firing neurons, the amplitude of the sinusoidal wave relative to the first line of the spectrum was of the same order of magnitude of the observed one in alpha waves. As expected, taking into account the reduction factors cited above, a single closed loop sequence would give an EEG signal much smaller than the received ones. Thus a realistic model should consider the presence of a bundle of synchronized loops, which would give approximately the same line spectrum of a single loop, but with a stronger intensity. This possibility is briefly discussed in section VI, where it is assumed that a large group of synchronized loops may be generated by the fact that every neuron is directly connected with many nearby neurons and that the electric impulse which it emits during firing may produce a number of simultaneous firing within the connected ones.

II. SYNTHESIS OF THE MAIN RESULTS GIVEN IN THE PREVIOUS PAPER

In order to make the paper self-consistent, in this section the many assumptions and the general results given in Ref. 1, concerning the expression of the power spectrum of a time series of Markov correlated events, will be reported and briefly discussed without details about calculations. For that, the reader is referred to the original paper. In the next section these results will be used and extended to find the more general conditions about correlations giving rise to a line or a mixed spectrum: i.e. the case where a stochastic sequence of events present harmonic components together with pseudo-random ones.

This analysis will be made by studying the conditions under which singular points appear within the general expression of the power spectrum.

In papers[1,2] a physical system described by a time sequence of correlated events characterized by a number N of discrete states α_i, which can undergo transitions from state α_i to α_{i+1} along time intervals u_i, has been considered. Let $I(t)$ be a physical quantity relevant to the system, which can be expressed by a linear superposition of contributions $F_{\alpha_i,\alpha_{i+1}}(t, u_i)$ depending on the transition from state α_i to state α_{i+1}, from the time interval u_i, with arbitrary origin at t_i:

$$I(t) = \sum_{i=-\infty}^{\infty} F_{\alpha_i,\alpha_{i+1}}(t - t_i, u_i). \tag{2.1}$$

The following assumptions are made:

(a) the transition probability from the state α_i to state α_{i+1} is described by a $N \times N$ stochastic matrix $\|m\|$ whose elements $m_{\alpha_i,\alpha_{i+1}}$ represent the conditional probability that if the event i is in a state α event $i+1$ will be in the state α':

$$m_{\alpha_i,\alpha_{i+1}} = P(\alpha_{i+1} = \alpha' | \alpha_i = \alpha) \tag{2.2}$$

The sequence of states is thus represented by an homogeneous Markov chain.

(b) the sequence of the time intervals u_i is also dependent on the sequences of the states α and given by a set of arbitrary distribution functions $q_{\alpha_i,\alpha_{i+1}}(u_i)$. These functions allow to introduce a correlation between the intervals u_i characterizing the sequence, a possibility which was generally excluded in the calculation of the power spectrum of non Markovian series of events.[3,4] As shown in the next section, the correlation between the time intervals allows to generate new line components in the power spectrum of the sequence.

(c) A new $N \times N$ stochastic matrix $\|M(u)\|$, which contains also the state dependent distribution function $q_{\alpha_i,\alpha_{i+1}}(u_i)$, is now introduced. Its elements $M_{\alpha_i,\alpha_{i+1}}(u)$ are defined as:

$$M_{\alpha,\alpha'}(u) = m_{\alpha,\alpha'} q_{\alpha,\alpha'}(u), \tag{2.3}$$

where it is intended that α and α' are the states of a couple of successive events of the sequence and u is the time interval between this couple. Its transform:

$$M_{\alpha,\alpha'}(\omega) = m_{\alpha,\alpha'} Q_{\alpha,\alpha'}(\omega), \tag{2.4}$$

where:

$$Q_{\alpha,\alpha'}(\omega) = \int_0^{\infty} q_{\alpha,\alpha'}(u) \exp(i\omega u) du, \tag{2.5}$$

will be referred as the correlation matrix $\|M(\omega)\|$ of the Markov process. Detailed calculations for obtaining a compact expression of the power spectrum are given in Ref. 1 and are not reported here. Two cases were considered:

1. $F_{\alpha_i,\alpha_{i+1}}(t - t_i, u_i)$ does not depend on α_{i+1} and u, but only on the α. Thus:

$$I(t) = \sum_{i=-\infty}^{\infty} F_{\alpha_i}(t - t_i), \tag{2.6}$$

2. $I(t)$ is given by the more general Eq. (2.1), which can be useful to treat physical processes where the function describing an event i depends also on the state α_{i+1}, of the successive one and on the time interval u_i separating the two events.

Since the final expression of the power spectrum has in both cases the same singularities, for what concerns the line spectrum it is enough to consider the case 1). The general expression of the power spectrum $\Phi(\omega)$ in this simpler case is:

$$\Phi(\omega) = \nu \left\{ \overline{|S(\omega)|^2} + 2Re \left[\sum_{\alpha\alpha'} S^*_\alpha(\omega) S_{\alpha'}(\omega) p_\alpha \cdot \|K(\omega)\|_{\alpha\alpha'} \right] \right\}, \tag{2.7}$$

where:

$$\|K(\omega)\| = \|M(\omega)\| \cdot (\|I\| - \|M(\omega)\|)^{-1}. \tag{2.8}$$

In Eq. (2.7), ν is the average number of events per unit time, $\overline{|S(\omega)|^2}$ is the square modulus of the Fourier transform of $F_{\alpha_i}(t)$ averaged over all the events in the sequence, $S^*_\alpha(\omega)$ and $S_{\alpha'}(\omega)$ are the Fourier transforms of $F_\alpha(t)$ for a couple of states α and α', p_α is the fraction of states α present in the sequence, given for every state α by the components of the eigenvector of the transpose of the matrix $\|m\|$ corresponding to the eigenvalue 1, and Re means the real part of the expression within square brackets. The matrix $\|K(\omega)\|$ defined in Eq. (2.8), becomes singular when the determinant of:

$$(\|I\| - \|M(\omega)\|)^{-1}, \tag{2.9}$$

is zero. The values of ω corresponding to the singular points of $\|K(\omega)\|$ gives the line spectrum. In Eq. (2.8), $\|I\|$ is the identity matrix. An identical term occurs in the more complex expression of the power spectrum of a sequence of events described by Eq. (2.1) and given in paper.[1] This means that in both cases the occurrence of a line spectrum and the lines position along the ω axis are the same. For what concerns the line intensities, their expression is worked out in the next section, where the transforms $S_{\alpha,\alpha'}(\omega)$ will be calculated for the functions given by Eq. (2.1).

III. THE LINE SPECTRUM

In paper[1] the presence of line components was considered for an arbitrary expression of the stochastic matrix $\|m\|$. The results was quite similar to the one reported in previous papers[3,4] concerning a non Markovian sequence of events under more restrictive conditions. In this section we shall consider the case of the presence of line spectral components by considering appropriate forms of the matrix $\|m\|$ having a clear physical meaning. The analysis allows one to evidence any possible situation where Markov correlated sequences have harmonic solution together with stochastic ones. As stated in the previous section the presence of lines in the power spectrum of the sequence is related to singularities in the expression of the spectrum for given values of the angular frequency ω. These singular points occur when the determinant of the matrix appearing in $\|K(\omega)\|$, is zero:

$$DET\ (\|I\| - \|M(\omega)\|) = 0 \tag{3.1}$$

Let us first consider the general case for the matrix $\|m\|$, as it was assumed in paper.[1] In this case, in order to find solutions for Eq. (3.1), the necessary but not sufficient condition is:

$$q_{\alpha,\alpha'}(u) = \delta(u - u_{\alpha,\alpha'}) \tag{3.2}$$

where $u_{\alpha,\alpha'}$ is the time interval between two successive events characterized by the states α, α'. The Fourier transform of 3.2 can be written as:

$$Q_{\alpha,\alpha'}(\omega) = \exp(i\omega u_{\alpha,\alpha'}) \tag{3.3}$$

had to be assumed. The values of ω satisfying Eq. (3.1) were found as integer multiples of ω_o given by the relation:

$$\omega_n = n\omega_o = n\frac{2\pi}{u_o} \tag{3.4}$$

with n a positive or negative integer and u_o a time interval such that all the $u_{\alpha,\alpha'}$ are multiples of its value. A line at $\omega = 0$, whose amplitude is equal to the square of the average value of the signal, is always present, regardless of the correlation between the events in the sequence. Therefore, in the following the presence of this line will be tacitly assumed. It is easily seen that the condition expressed by Eq. (3.4) is very limitative, since u_o cannot be larger than the smaller interval $u_{\alpha,\alpha'}$ in the sequence and thus ω_o may become very large. When Eq. (3.4) holds, according to Eq. (4.24) of Ref. 1, the intensity of the line A_n at $\omega = n\omega_o$ is given by a simple relationship identical to the one found in papers:[3,4]

$$A_n = 2\pi\nu^2|\overline{S(\omega_n)}|^2, \tag{3.5}$$

where $|\overline{S(\omega_n)}|^2$ is the square modulus of the average of the Fourier transform of $F_\alpha(t)$ taken over the events of the sequence. Then, if ω_o is much larger than the inverse of the average time span of the functions $F_\alpha(t)$, all the A_n become very small and the line spectrum vanishes.

When, as in papers,[3,4] no correlation is assumed between time intervals separating subsequent events, no other possibility for a line spectrum exists.[5] However, in the present case of Markov correlated events, where a correlation between the time intervals is assumed, there are many possibilities to have other very interesting line components if the matrix $\|m\|$ takes a form suitable to describe physical processes where harmonic components are expected. This may be the case of sequences of events which form closed loops, as in the case of the application described in section V. In this case the expression of the line spectrum becomes more complex than the one given by Eq. (3.5), as shown in the following, since the limitative condition expressed by Eq. (3.4) is not further assumed, while Eq. (3.3), which makes $||K(\omega)||$ singular, is still valid. A closed loop involving N states α can be expressed by a $N \times N$ matrix $\|M(\omega)\|$ where all the rows are made up of zeroes except one term having the modulus equal 1 and, according to Eq. (3.3), the form:

$$M_{\alpha,\alpha'}(\omega) = m_{\alpha,\alpha'} \cdot \exp(i\omega u_{\alpha,\alpha'}) \tag{3.6}$$

with:

$$m_{\alpha,\alpha'} = 0 \tag{3.7}$$

except:

$$\begin{cases} m_{1,2} = & 1 \\ m_{2,3} = & 1 \\ & \vdots \\ m_{N-1,N} = & 1 \\ m_{N,1} = & 1 \end{cases} \tag{3.8}$$

In this case Eq. (3.1) becomes:

$$1 - \exp[i\omega \cdot (u_{1,2} + u_{2,3} + \ldots + u_{N,1})] = 0 \tag{3.9}$$

which has the simple solution:

$$\omega = \omega_n = n\omega_o \tag{3.10}$$

$$\omega_o = \frac{2\pi}{\sum_{i=1}^{N} u_{i,i+1}}, \tag{3.11}$$

where, for the closed loop, the index $N + 1$ corresponds to 1, as in Eq. (3.9), and n is an arbitrary positive or negative integer. It is obvious that in this case ω_o may become quite small depending on the value of N and on the average length of the time intervals $u_{i,i+1}$.

The matrix $\|M(\omega)\|$ given by Eqs. (3.6-3.8) generate a sequence strictly periodical and thus the continuous component of the spectrum for $\omega \neq \omega_n$ must vanish. As a stringent test of the correctness of Eq. (2.7), this result has been proved in a few practical cases with the use of a symbolic mathematical program like Macsyma. Furthermore the particular symmetries of the matrix $\|K(\omega)\|$ reported below, when Eqs. (3.6-3.8) hold, will also be used to evaluate the stochastic continuous component appearing in the power spectrum when the sequence ceases of being strictly periodical owing to external perturbations, as discussed in section IV. As anticipated, since the limitations generated by assuming Eq. (3.4) are now removed, also Eq. (3.5) becomes invalid and the expression of the line spectrum is given by the more general equation:

$$A_n = 2\pi\nu^2 Re\left\{\sum_{\alpha\alpha'} S^*_\alpha(\omega_n) S_{\alpha'}(\omega_n) p_\alpha \cdot C_{\alpha\alpha'}(\omega_n) \cdot \left[\sum_\alpha |C_{\alpha\alpha'}(\omega_n)|\right]^{-1}\right\} \tag{3.12}$$

where $C_{\alpha\alpha'}$ are the elements of the adjoint matrix:

$$\|C(\omega_n)\| = Adj\,(\|I\| - \|M(\omega_n)\|) \tag{3.13}$$

Details of the derivation of Eq. (3.12) are given in Ref. 1, where its simplification to Eq. (3.5), when Eq. (3.4) holds, is also given. In the following reference to A_n will be considered as relative to Eqs. (3.12-3.15) and not to Eq. (3.5).

When Eqs. (3.6-3.8) are assumed, all the elements of the matrix $\|C(\omega_n)\|$ are exponentials of the form $\exp[i\omega \cdot (u_i + u_{i+1} + \ldots)]$ with a variable number of time intervals u_α but such that the sum of the u_α relative to a couple of elements in symmetric position with respect to the principal diagonal of the matrix corresponds always to all the N time intervals within the loop. Since the modulus of these elements is equal to 1, the quantity within square brackets in Eq. (3.12), corresponding to the sum over a single state α, which appears N times within the set of the N^2 elements $C_{\alpha,\alpha'}(\omega_n)$ of $\|C(\omega_n)\|$, is simply $1/N$. Since also p_α is equal to $1/N$ for every state α, Eq. (3.12) simplifies to:

$$A_n = 2\pi\nu^2 \cdot 1/N^2 \cdot Re\left\{\sum_{\alpha\alpha'} S^*_\alpha(\omega_n) S_{\alpha'}(\omega_n) \cdot C_{\alpha\alpha'}(\omega_n)\right\}. \tag{3.14}$$

It should be noticed that Eqs. (3.12-3.14) are valid only for $\omega = \omega_n$, when Eq. (2.7) becomes invalid owing to the fact that the determinant given by Eq. (3.1) is zero. Since $\Phi(\omega) = 0$ for $\omega \neq \omega_n$ the real part of the expression within square brackets in Eq. (2.7) must be equal to $-0.5\overline{|S(\omega)|^2}$. Actually in the case where Eqs. (3.6-3.8) hold, the matrix $\|K(\omega)\|$ has the following special characteristics (yielding the above results):

(a) the elements on the principal diagonal are complex numbers all equal and with the real part equal to -0.5.
(b) the elements symmetric with respect to the principal diagonal are complex numbers having equal real part but opposite sign.
Thus the couple of products $S^*_\alpha S_{\alpha'}$ and $S_\alpha S^*_{\alpha'}$, obtained from the double series appearing in Eq. (2.7), when multiplied by $p_\alpha = 1/N$ and by the corresponding non diagonal elements of $\|K(\omega)\|$ and summed up, give a pure imaginary result and do not contribute to the real part Re. The contribution to the real part comes from the terms $S^*_\alpha S_\alpha$ and $S_{\alpha'} S^*_{\alpha'}$ etc corresponding to the diagonal elements of $\|K(\omega)\|$. When multiplied by p_α and by these elements and summed up they give a real part of $-0.5\overline{|S(\omega)|^2}$ making $\Phi(\omega) = 0$. It should be noticed that if external perturbations make the sequence not strictly periodical, a mixed power spectrum is instead generated, as shown in Section IV.

A similar test has been made to prove the correctness of Eqs. (3.12-3.14), giving the line spectrum. To this purpose a periodic sequence has been considered with N events in the period $T = Nu$ generated by a Markov matrix of the type described by Eqs. (3.6-3.8) with $N \times N$ elements. Furthermore, all the events were assumed to be represented by the same function $F_\alpha(t)$ and thus by the same transforms $S_\alpha(\omega_n) = S(\omega_n)$ separated by identical time intervals u. In this situation it is expected that, apart from the line at $\omega = 0$, the first line of the spectrum would be at an angular frequency $\omega_N = N\omega_o = 2\pi/u$, while the line at the lowest frequency $\omega_o = 2\pi/(Nu)$ and all its harmonics up to $n = N-1$ would vanish. This is what actually gives Eqs. (3.12-3.14). Taking into account that $S^*_\alpha(\omega_n) S_{\alpha'}(\omega_n) = |S(\omega_n)|^2$, this equation becomes:

$$A_n = 2\pi\nu^2 \cdot 1/N^2 \cdot |S(\omega_n)|^2 \cdot Re\left[\sum_{\alpha\alpha'} C_{\alpha\alpha'}(\omega_n)\right]. \tag{3.15}$$

After performing the calculations, it is found that for $n = \pm 1, \pm 2, \ldots. \pm (N-1)$:

$$Re\left[\sum_{\alpha\alpha'} C_{\alpha\alpha'}(\omega_n)\right] = 0, \tag{3.16}$$

and for $n = \pm N, \pm 2N, \ldots.$:

$$1/N^2 \cdot Re\left[\sum_{\alpha\alpha'} C_{\alpha\alpha'}(\omega_N)\right] = 1. \tag{3.17}$$

Then, since $\nu = 1/u$, one gets for $\omega = \omega_N$:

$$A_N = 2\pi \cdot \nu^2 |S(\omega_N)|^2 = 2\pi \cdot 1/u^2 \cdot |S(\omega_N)|^2, \tag{3.18}$$

which is the square of the first line of the Fourier transform of a periodic function whose period is u rather than $N \cdot u$, as it would be expected on the basis of the assumptions made. Harmonic spectral lines are also obtained for $n = \pm 2N, \pm 3N, \ldots.$, whose amplitude is again determined by the value of $|S(\omega_n)|^2$. Since in the application treated in Section V sequences of events produced by closed loops of tens or hundreds of interconnected firing neurons will be considered, some interesting aspects of these sequences may be discussed.

One important point concerns the fact that harmonic components of very low frequency, down to 0.5 Hertz, are generated from elementary events and time intervals of the order of few milliseconds. In this case, very low frequencies, as given by Eqs. (3.10-3.11), are generated when different states α along the sequence correspond, at least in part, to different values of the time intervals $u_{i,i+1}$ and/or to different amplitude or form of the functions $F_\alpha(t)$. Even the intensity of these low frequency harmonic components becomes larger when the difference in the above quantities corresponding to different states α increases, as it can be obviously expected. An extended discussion is given in Section V.

In all these cases the sequence is strictly periodic and no stochastic component is present. In the next section the case will be considered where the time occurrence of the events along the sequence is modulated around an average value by effect of external perturbations. In this case, which is rather common in real-world applications, a stochastic component is generated and a reduction of the intensity of the line components, particularly the high frequency harmonics, is obtained.

IV. RANDOM MODULATION OF TIME INTERVALS

In this section we shall consider again a sequence of events described by the matrix $\|M(\omega)\|$ given by Eqs. (3.6-3.8). The sequence is strictly periodic and its line spectrum is a set of lines with angular frequencies ω_n, given by Eqs. (3.10-3.11) and intensity given by Eq. (3.14). In applications to physical processes, as for instance to the case of a closed loop of a chain of interconnected firing neurons treated in Section V, it may happen that the sequence of the firing time intervals are not exactly repetitive over different cycles but, owing to external perturbations, randomly oscillate around an average value. In this case a line spectrum still exist with reduced line intensity, while a stochastic component, represented by a continuous spectral density component, appears. If the random oscillation is described by a Gaussian function centered on zero, as it will be assumed in the following, when the variance of the Gaussian is increased, the line spectrum gradually vanishes, starting from the highest harmonic frequencies, whereas the stochastic component increases.

The trick to evaluate the power spectrum in this case is related to the fact that the time origin of the functions representing the events may be defined arbitrarily, but obviously the sequence of events must remain unchanged. In order that the sequence of the time intervals u_i be also unchanged it is necessary to shift the origin of the event of a quantity τ_i when the time interval between two successive events is varied by $-\tau_i$ and viceversa:

$$F_{\alpha_i}(t-t_i) \rightarrow F_{\alpha_i}(t-t_i+\tau_i). \tag{4.1}$$

This corresponds of interpreting the time shift of the event as a change of its form and not a change of the time interval u_i. The values of the τ_i may be given by an arbitrary normalized distribution function $W(\tau_i)$ with the condition that:

$$\int_{-\infty}^{0} W(\tau_i)d\tau = \int_{0}^{-\infty} W(\tau_i)d\tau \tag{4.2}$$

as it is for instance the case of a Gaussian function centered on zero. The symmetry of $W(\tau)$ with respect to the origin is not strictly required but its absence gives rise to a more complicated expression for the reduction factor given below.

Obviously, when the set of the u_i remains unchanged, the change of the time origin expressed by Eq. (4.1) has an effect only on the Fourier transforms $S_\alpha(\omega)$ of the $F_\alpha(t-t_i)$ while the matrices $\|K(\omega)\|$ and $\|C(\omega)\|$, defined respectively by Eq. (2.8) and by Eq. (3.13), remain unchanged. The transform of the elementary event i becomes:

$$S'_{\alpha_i}(\omega) \rightarrow S_{\alpha_i}(\omega) \cdot \exp(i\omega\tau_i). \tag{4.3}$$

Taking into account that the shifts τ_i are not correlated with the states α_i, by averaging over the statistical ensemble, one obtains:

$$\overline{S'_{\alpha_i}(\omega)} \rightarrow S_{\alpha_i}(\omega) \cdot \overline{\exp(i\omega\tau_i)} = S_{\alpha_i}(\omega) \cdot \int_{-\infty}^{\infty} W(\tau)\exp(i\omega\tau)d\tau \tag{4.4}$$

where the bars indicate the average, and the integral, which represents the average of the exponential, coincides with the Fourier transform $W(\omega)$ of $W(\tau)$ multiplied by $\sqrt{2\pi}$. The line spectrum is still given by Eq. (3.14), however the transforms $S^*_\alpha(\omega)$ and $S_{\alpha'}(\omega)$ appearing in this equation must be averaged according to Eq. (4.4), since for uncorrelated events the average of their product coincides with the product of their averages. Thus, while the frequency of the lines remains unchanged, the lines intensity decreases depending on the amplitude of the modulations of the event position. According to Eq. (3.14) and assuming that $W(\tau)$ is symmetric about the origin and thus that its Fourier transform is real, the intensities of the lines A_n given by Eq. (3.14) now becomes:

$$A'_n = A_n \cdot [W(\omega_n)]^2. \tag{4.5}$$

Assuming that $W(\tau)$ is a Gaussian function of variance σ^2 centered at zero:

$$W(\tau) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\tau^2/2\sigma^2\right) \tag{4.6}$$

Eq. (4.5) becomes:

$$A'_n = A_n \cdot \exp\left(-\sigma^2{\omega_n}^2\right) \tag{4.7}$$

The factor $R_n = \exp\left(-\sigma^2{\omega_n}^2\right)$ acts as a reduction factor of the line intensity. Since $\overline{|\tau|}$ for the Gaussian distribution given by Eq. (4.6) is:

$$\overline{|\tau|} = \sigma/\sqrt{2\pi} \tag{4.8}$$

and:

$$\omega_n = n\frac{2\pi}{T} = n\frac{2\pi}{\overline{u_i}N} \tag{4.9}$$

where T is the period of the lowest frequency line A_1, $\overline{u_i}$ the average time interval between subsequent events in the sequence, N the number of the events in a period T. The reduction factor R_n can now be written:

$$R_n = \exp(-\sigma^2\omega_n^2) = \exp\left[-2\pi\overline{|\tau|}^2\omega_n^2\right] \tag{4.10}$$

By using Eq. (4.9), the expression of R_n can be set in an alternative form:

$$R_n = \exp\left[-8\pi^3\left(\overline{|\tau_i|}/\overline{u_i}\right)^2 \cdot (n/N)^2\right] \tag{4.11}$$

To have an hint of the reduction, with $\overline{|\tau_i|} \approx 0.5\overline{u_i}$, N = 50 and n = 1 ($\omega = \omega_o$) from Eq. (4.11) one gets:

$$R_1 = \exp(-0.025) = 0.975$$

and for n = 3 the 3^{rd} harmonic frequency:

$$R_3 = \exp(-0.225) = 0.798$$

The weak reduction of the line spectrum of the sequence described by Eqs. (3.6-3.8) shows that even in the presence of rather large fluctuations of the time intervals between the events, when N is sufficiently large, the low frequency lines of the spectrum remain almost unchanged, except for the highest harmonic components.

This issue will be further discussed in Section V where it is proposed to relate this process to the origin of the harmonic and the stochastic components of the brain activity observed in electro- and magneto-encephalograms. As shown in the previous section, when the sequence is unperturbed the stochastic component is totally absent. Conversely, in the present case the reduction of the intensity of the line spectrum is compensated by the presence of a stochastic component represented by a continuous spectral component. For the evaluation of this component it must be kept in mind that the random shifts of the events in the sequence were considered a dynamic change of the point assumed as the origin of the function $F_\alpha(t)$ defining the event α. This implies a change of the Fourier transform of the functions but leaves the set of the u_i and thus the matrix $||K(\omega)||$ unchanged. This fact makes the evaluation of the stochastic component very simple. In the previous section, the case of a strictly periodic sequence was considered and the fact that the stochastic component was, as expected, absent, was proved on the basis of the particular symmetries of $||K(\omega)||$ when Eqs. (3.6-3.8) hold. Also in the present case the non diagonal elements of $||K(\omega)||$ do not give any contribution to the real part of the expression within square brackets in Eq. (2.7), while all the diagonal elements have a real part equal to -0.5. According to Eq. (4.4), the quantity within square brackets in Eq. (2.7) becomes $-0.5 \cdot \overline{|S(\omega)|^2} \cdot \exp\left(-\sigma^2\omega^2\right)$. Since the term $\cdot\overline{|S(\omega)|^2}$ in the same equation remains unchanged, because refers to the square modulus of the transform of the same element and not of the product of the transforms of two elements in the same state α along the sequence, the power spectrum $\Phi_s(\omega)$, for $\omega \neq \omega_n$, becomes:

$$\Phi_s(\omega) = \nu\left(\overline{|S(\omega)|^2} \cdot \left(1 - \exp\left(-\sigma^2\omega^2\right)\right)\right). \tag{4.12}$$

and the total power spectrum becomes:

$$\Phi(\omega) = \Phi_s(\omega) + \sum_{n=-\infty}^{\infty} A_n R_n \cdot \delta(\omega - \omega_n), \tag{4.13}$$

with ω_n given by Eqs. (3.10-3.11).

It is interesting to notice that, when the modulation amplitude becomes very large and $R_n \to 0$, Eq. (4.12) reduces to:

$$\Phi_s(\omega) = \nu(\overline{|S(\omega)|^2}) \tag{4.14}$$

which is the power spectrum of a sequence of completely uncorrelated events, while the line spectrum $\Phi_l(\omega)$ vanishes. When, in the absence of modulation, $R_n = 1$, $\Phi_s(\omega)$ and $\Phi_l(\omega)$ reduce to:

$$\Phi_s(\omega) = 0$$

$$\Phi_l(\omega) = \sum_{n=-\infty}^{\infty} A_n \cdot \delta(\omega - \omega_n) \tag{4.15}$$

V. A NEURAL CORRELATION PROCESS

In this section, the power spectrum of a neural correlation process is described with the purpose of (i) giving an example of application of the theory developed above and (ii) proposing a simple model to explain the emergence of harmonic frequencies within the apparently chaotic electric impulses generated by firing neurons in the complex neural network of the brain. These waves are observed in electro and magneto encephalograms (EEG and MEG) and cover a range of frequencies from $0.5Hz$ up to about $40Hz$ for what concerns the fundamental component. Higher frequency harmonics, which may be observed as lines in the power spectrum of the detected signal, have the effect of giving a distortion of the sinusoidal shape of the signal. To this purpose the neurons, which are the basic elements of the complex neural network, may be schematically represented by an electrical device which has many inputs (neuron dendrites) connected through dissipative elements (synapsis) to the outputs (axons) of a set of presynaptic neurons, and many outputs, coming out from the ramification of its axon, connected with the dendrites of another set of postsynaptic neurons, again thorough synapses. The way of operating of the different types of neurons present in the brain is quite similar. In equilibrium condition, the electric potential internal to the cell (soma), in correspondence to the membrane, is approximately $-75mV$ with respect to the average electric potential of the brain, assumed to be zero.

When the neuron receives from its dendrites a positive charge beyond a given threshold, it reacts by firing through its axon a positive electric impulse (action potential) lasting about $1ms$, while the internal potential approaches zero from negative values. The process occurs by effect of the opening of ionic channels distributed at the surface of the cell membrane which allows a flow of positive Na+ ions to enter the cell. The positive charge received by its dendrites contributes only in part to this process. After firing, during a short refractory time interval, the neuron becomes negatively recharged, reaching its equilibrium state, owing in part to the loss of the positive charge transmitted through its axon to the several connected postsynaptic neurons which are negatively charged, and from an interplay of different ionic channels on its membrane.[6,7] It is important to notice that to the positive impulse internal to the cell during firing corresponds an electric current impulse generated externally to the cell, near to the cell membrane. Actually, in the equilibrium condition, an excess density of positive ions is present on the external side of the membrane to screen the electric field generated by the internal negative polarization, since in equilibrium, within a conducting medium, the electric field must be zero. By considering a virtual closed surface surrounding the neuron cell and this excess charge, in equilibrium the total charge internal to that surface must be zero (Gauss theorem). When the neuron fires an impulse of positive charge through its axon, during the transient a corresponding positive charge must enter the closed surface to compensate for the loss. This process corresponds to the emission of a negative impulse, having about the same characteristics of the firing one, but propagating radially, in all directions, towards the cranial bone. Assuming a spherical symmetry of the propagation, the signal received by an electrode on the cranial bone is reduced according to a factor equal to the ratio between the solid angle which the electrode surface is seen from the neuron and the total spherical angle, equal to $4\pi sr$. Two further reduction effects should be taken into account (i) the screening to the propagation of the electric impulse due to the ionic impedance of the cerebral matter and (ii) the electrical resistance of the cranial bone. Except for the line at zero frequency, the sign of the impulse has no influence on the power spectrum of the received signal.

Alternative models explaining the external currents associated with the firing of the neurons make reference to the chemical synapses, where the electric impulses coming from pre-synaptic neurons become a flow of neurotransmitters giving rise to external currents contributing to the EEG detected signals. This assumption corresponds to splitting the impulse transmitted by the neuron in thousands smaller impulses in correspondence to the synapses, creating an high density population of small impulses having a statistical distribution of time lags which allows to develop models showing the possibility of oscillatory solutions. In Ref. 8 a model is presented and used, based on a second order partial differential equation, which admits oscillatory solutions by introducing suitable parameters.

Once the mechanisms of emission of the electric impulses from the neurons have been identified, the question is what type of correlation between firing of different connected neurons may produce the sinusoidal signals with the broad range of frequencies observed in EEG and MEG. If in a single time sequence, the correlation concerns only couples of subsequent events and a linear superposition of the events is assumed, as observed in Section III, the only lines in the power spectrum are observed at zero frequency and at frequencies larger than the inverse of the smaller time interval separating each couple of events.[3,4] If the correlation is expressed by a generalized Markov matrix $\|M(\omega)\|$, given by Eq. (2.3), which extends the correlation also to successive time intervals, there is the possibility of specializing the matrix to describe situations giving rise to line power spectra and mixed power spectra with lines having a frequency as low as the observed ones in EEG and MEG. In this case, as shown in Section III, low frequency waves are generated by closed loops sequences of events, corresponding to the electric impulses externally emitted by a chain of interconnected firing neurons.

Alternatively, the possibility of oscillatory solutions for a population of interacting events in a complex system of time processes is possible under the assumption of long range interactions between the events associated to non linearity and criticality of the system.[9] For what concerns the brain, several suggestions have been made but, to our knowledge, not fully developed in a consistent model, where numerical evaluation of the amplitude of the EEG and MEG signal can be made.

In the present application it is proposed an original model which is based on the characteristics of closed loop sequences of firing neurons. In the following, as a starting point, the case of a single closed loop sequence will be considered, evaluating numerically in a few simulations the amplitude and frequency of the line power spectrum for different situations concerning the distribution of the amplitude and density of the firing impulses along the loop. The more realistic case of many synchronized loops, expected on the basis that the firing of a neuron may produce the simultaneous firing of other connected neurons, maintains most of the aspects observed in a single loop, apart the amplitude of the signal, and will be briefly discussed in Section VI.

By considering the case of a single sequence of correlated events, as stated above, the only possibility to have a line power spectrum characterized by low frequency components is that of receiving impulses from firing neurons correlated in a Markov chain giving rise to closed loops, as expressed by Eqs. (3.6-3.8). In this case the lowest angular frequency line ω_0 is given by Eq. (3.11), corresponding to a period equal to the sum of all the time intervals between the firing of the neurons forming the closed loop. By assuming that the correlated firing impulses in the sequence are relative to nearest neighbours neurons, such time intervals may be considered of the order of $5.0ms$. Thus closed loops between 60 to 400 neurons would give a lowest frequency line between respectively 3.3 and $0.5Hz$ (delta waves), while the highest observed frequencies of about $40Hz$ (gamma waves) would be produced by closed loops of only few neurons.

The amplitude of the fundamental line and its harmonics, appearing in the power spectrum, is given by Eq. (3.14). The low frequency lines cited above appear whenever the amplitude of the impulses emitted by the firing neurons of the sequence or the time intervals between subsequent firings are, even in a small part of the loop, different but repetitive for subsequent loops. For instance, if the sequence contains part of the neurons active in epileptic seizures, it is expected a rather large amplitude of the EEG signal during a seizure if these neurons behave differently than the normal ones. Even in the normal states, the fact that the lengths of the connections between successive neurons of the loop are slightly different will produce a low frequency EEG signal, as the first simulation reported below shows. The effects of these situations, which must be related to a physical condition persistent during the EEG observation, should be distinguished from the ones produced by random perturbations originated outside the loop. As discussed in Section IV, in this case the intensity of the lines appearing in the power spectrum, and particularly those concerning the harmonics of higher frequency, are attenuated, while a continuous component of the power spectrum is generated, giving rise to a mixed spectrum.

The results of the simulations concerning these interesting aspects of the proposed model are given below. Details of calculations are reported in the Appendix. They show how the amplitude of the low frequency oscillations, represented by the intensity of the lines in the power spectrum, are strongly dependent on these situations. In these simulations, single closed loops sequences of 30 neurons are considered giving a lowest frequency line of $6.66 Hz$ (alpha waves) if it assumed that the average time interval between subsequent firing neurons is exactly $5.0 ms$. According to what was discussed above, the electric impulse radially emitted outside of the firing neuron was assumed to be equal to the one internal to the neuron. The sign of the impulse has an influence only on the zero frequency line of the spectrum and corresponds to a d.c. component which is generally not detected in EEG measurements. The large correction relative to the reduction factor due to the ratio between the average solid angle relative to the area of the active electrode, as seen from the neurons of the loop, and the spherical solid angle of $4\pi sr$, has been considered by assuming a given spatial situation. Two further reduction factors, due to the effect of the impedance of the cerebral matter on the propagation of the impulse and to the effect of the resistivity of the cranial bone, could not be evaluated, but it was considered that their effect could produce a reduction of the received impulse even larger than the above correction. The possible spreading out of the impulse during the propagation would probably have no effect on the lowest frequency spectral line but will reduce the amplitude of the higher harmonics, making the signal more sinusoidal. The purpose of these calculations, in addition of presenting an application of the theory, was that of considering what could be the contribution of single closed loops of firing neurons to the EEG signal, and making a comparison with the average voltage values experimentally measured in different physical situation. It is found that the amplitude of the received signal corresponding to the first spectral line would be of the same order of magnitude than the measured one if the two attenuation factors cited above are not considered, but probably one or two orders of magnitude smaller if those factors are taken into consideration. Thus, as already observed, a realistic model should consider the contributions of many synchronized loops, a situation briefly discussed in the next section.

For what concerns the simulations, assuming that the electric signal received by an electrode is relative to a reference electrode being inactive and at zero potential, the following case studies have been considered:

(a) All the impulses of a closed loop of 30 neurons received by the active electrode are assumed to be equal in amplitude, while the time intervals between successive firing neurons are distributed according to a Gaussian distribution of mean value $5.0 ms$ and a variance of $1.5 ms$. This distribution could be due, for instance, to a different length of the connections between successive neurons forming the loop and remains unchanged for subsequent loops. Eq. (3.14), which give the intensity of the lines of the power spectrum, with the above assumption simplify to Eq. (3.15). The shape of the impulse emitted from each firing neuron of the loop was assumed to be approximated by a Gaussian function centered on zero, of amplitude $-75 mV$ and variance $1 ms$, values which are generally reported as results of direct measurements.[7] The angular frequency of the fundamental line ω_0 and of its harmonics, are given by Eq. (3.11) and depend on the sum of the firing time intervals of the neurons of the loop. Since that sum is depending on the seed used to initialize the random-gauss number generator, the reported values are not exactly repeatable over different loops generated with different sets of random numbers. As shown in the Appendix, in the present simulation it has been found that, without taking into account the two reduction factors cited above, an electrode seen with an area of $1cm^2$ from a distance of $2.0cm$ from the loop will receive, at the lowest frequency line $f_0 = 7.23Hz$, a sinusoidal wave of $174.306\mu V$ peak to peak amplitude. This value corresponds to one of the highest values observed in EEG,[10] but the neglected reduction factors cited above would make it much smaller. The values of the second and third harmonics, which are largely dependent on the set of the random-Gauss numbers, turn out to be about of the same order of magnitude, giving a rather large distortion to the sinusoid if it is not filtered.

(b) The situation cited above, that the low frequency lines vanish when also these time intervals become all equal, is simulated by assuming a very low value of the variance of the Gaussian distribution: $1 \cdot 10^{-3}ms$ instead of 1.5 ms. In this case it is found that the amplitude peak to peak of the fundamental wave comes out to be only $0.4\mu V$. When the time intervals between firing are assumed to be all equal, the sum of the real part of the 900 elements of the adjoint matrix $\|C(w_0)\|$ becomes exactly zero, a proof of the precision of the numerical calculations.

(c) A third and fourth simulation concern with the case where a few neurons of the loop behave differently from the others, as it may happen, for instance, in the case of epileptic seizures. In these simulations, we consider that in the loop of 30 neurons, a sequence of 10 neurons has a time interval between subsequent firing of $4.0ms$ in one case and $3.5ms$ in a second case, while the other neurons keep the interevent delay value of $5.0ms$. The purpose is to evidence the large increase of the amplitude of the spectral line at the fundamental angular frequency ω_0 with the increasing of the different behavior of part of the neurons forming the loop. For simplicity, also in these simulations we consider that the impulses emitted by the neurons are all equal and corresponding to those assumed above. Calculations are quite similar to the one reported above, the only difference being the generation of the matrix $\|M(\omega)\|$ appropriate to the present cases. As shown in the Appendix, in the case where 10 subsequent neurons have time firing intervals of $4.0ms$ it is found that the first spectral line corresponds to a sinusoid of $161.40\mu V$ peak to peak, while the second and third harmonics are of $113.07\mu V$ and of $53.78\mu V$, again peak to peak. In the case of $3.5ms$ intervals the corresponding values are $246.06\mu V$, $186.21\mu V$, $108.57\mu V$.

(d) A fifth simulation concerns the case where the loop of 30 neurons contains 10 subsequent neurons having a smaller (or higher) amplitude than the residual 20. In this simulation we consider that the amplitude of the ten neurons is smaller of 20% respect to the others, and that the time intervals between firing of subsequent neurons are all equal to $5.0ms$. This allows to separate the effects on the power spectrum of the variations of the pulses amplitude from those of the time intervals. This time it is necessary to use Eq. (3.14) instead of the simpler Eq. (3.15). The impulses are again represented by a Gaussian function centered on zero of amplitude $-75mV$ and variance of $1ms$, except the 10 neurons which have an amplitude reduced of 20%. Details of calculations are again given in the Appendix. The lowest frequency line f_0 occurs at $6.66Hz$ and the amplitude of the corresponding sinusoid is $154.28\mu V$ peak to peak, while for the second and the third harmonics the corresponding values are $9.94\mu V$ and $0.118\mu V$. In this case the signal turns out to be nearly a pure sinusoid. The line power spectrum remains unchanged if the amplitude of the 10 impulses is increased, instead of reduced, of 20%, except for what concerns the line at zero frequency.

VI. DISCUSSION AND CONCLUSIONS

As stated in the introduction, this work is a development of paper,[1] where general expressions have been worked out for the power spectrum of time sequences of correlated events, represented by the functions $F_{\alpha_i}(t)$, or, more generally, by the functions $F_{\alpha_i,\alpha_{i+1}}(t, u_i)$, making transitions between N different states α_i according to a correlation matrix $||M(\omega)||$ given by Eq. (2.4) and containing, as a factor, the Markov matrix $||m||$, given by Eq. (2.2). In the whole paper the matrix $||m||$ was left in its general form, giving to the expressions of the power spectrum a general validity. However, without specializing the Markov matrix, a specific expression for the line power spectrum was found only in the very limited case discussed in Section III.

In this paper it has been shown that it is possible to generate a broad set of line spectra involving arbitrary values for the lowest frequency, if the Markov matrix $||m||$ is given by Eqs. (3.6-3.8). These equations describe closed loops of events giving rise to a line power spectrum where the lowest frequency is the inverse of the duration of the loop. This line, together with its harmonics, is generated by irregularities in the time intervals separating successive events, and/or by differences in the functions representing the events in the sequence, as the simulations reported above clearly show. Eq. (3.12-3.14), worked out in the present paper using a procedure described in Ref. 1 and previously, in a simpler contest, in Refs. 3 and 4, give the line spectrum for every situation concerning the characteristics of the loop described by Eqs. (3.6-3.8). Obviously these equations are valid only when ω represents a singular point of the power spectrum, where a line exists. For others non singular points, the power spectrum is given by Eqs. (2.7-2.8). Mixed power spectra can be generated also for closed loops if external perturbations are present, as described in Sect. IV, otherwise Eqs. (3.6-3.8) define a strictly periodic function, and outside the singular points, where they are valid, Eqs. (2.7-2.8) give exactly zero, as shown in Sect. III. With this extension two general relationships are given which allow to evaluate the power spectrum, either continuous, line or mixed, for stochastic time sequences of events under the very general conditions described in Section II.

Another reason which suggested to extend the analysis concerning the line power spectra generated by time sequences of Markov correlated events was the tentative of explaining the existence of a large range of low frequency waves, down to 0.5 Hertz, observed in the spectra of the brain EEG and MEG. From this analysis, it turns out that for a time sequence of impulses whose correlation, concerning their form and time intervals, is described by the correlation matrix $||M(\omega)||$, and linearity for the superposition of the events is assumed, the only possibility to have line spectra involving frequencies lower than the repetition frequency of the impulses is the formation of closed loops, as discussed in Section III. The paper is thus exhaustive on this point of view: no other line spectra can be found under these conditions. Of course in the brain every neuron is connected with many neurons, and thus the assumption of a single chain of connected neurons out of the whole neuron population should be considered at least an oversimplification. Furthermore, as the simulations show, a single sequence would produce a signal too small to be detected. As it is generally assumed, the effect of these connections corresponds to a direct interaction of each neuron with many others, and thus a firing neuron may trigger the simultaneous firing of some of those connected neurons. Actually the impulse emitted by the firing neuron is split to thousands small impulses received through synapses by the dendrites of many nearby neurons, some of which may fire, since the triggering threshold of a neuron is much smaller than its firing impulse, and many of the neurons to which is connected may be receiving also part of the firing impulse of other nearby neurons whose firing is simultaneous to the considered one. Thus an increasingly growing number of simultaneous firings may occur along the chain, possibly giving rise to a bunch of synchronized single sequences having the characteristic of those discussed above. It may also be considered that, since the neuron behaves as a rectifying diode, producing a unidirectional current, a stable dynamical situation and electric charge conservation requires that the bunch forms a closed loop. In such a loop the synapses are the dissipative elements and the ionic currents recharging the neurons are the current generators. If this picture is accepted, the bunch would simply increase the amplitude of the line spectrum of a single loop, except for small corrections. For instance, since the connections between the neurons have different lengths, the synchronism is not perfect, but there are small time shifts between the impulses produced by neurons assumed to be synchronized. Since the sum of these impulses is received by the electrode as a single impulse smeared out, as discussed in Section V this would have a negligible influence on the low frequency lines of the spectrum. The main effect would be the reduction of the amplitude of the higher harmonics, making the received signal less distorted. Finally, together with the synchronized neurons there are firings of neurons not synchronized, giving a noise having a continuous power spectrum. The distribution of this continuous spectrum over a large range of frequencies will reduce its amplitude compared to the low frequency lines of the spectrum, where all the power of the synchronized loops is displayed.

As a conclusion it may be stated that all these considerations must be regarded as a proposal to develop a more definite model based on the properties of closed loops sequences of firing neurons to explain the origin of EEG and MEG. The mathematical part reported in the first four sections of this paper will be the mean for evaluating the numerical results of each proposed model, and compare them with the experimental results.

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APPENDIX

In the following, calculation details concerning the simulations a), b), c) and d) reported in Section V are given. The symbolic and numerical mathematical program Macsyma was used.

a) Since it has been assumed that all the impulses received by the electrode are identical, and only the time intervals between contiguous firing neurons in the sequence are randomly distributed according to a Gaussian function, the line power spectrum is given by Eq. (3.15). The matrix $||M(\omega)||$ given by Eqs. (3.6-3.8), has been generated by the Macsyma function *genmatrix* making use of the function *random-gauss(5, 1.5)* to obtain the set of the time intervals u_i of average value $5ms$, distributed according to a Gaussian function of variance $1.5ms$. Again Macsyma was used to get the adjoint matrix $||C(\omega)||$ by making the inverse of the matrix $(||I|| - ||M(\omega)||)$ and multiplying it by its determinant. Because for $\omega_n = n \cdot \omega_0$, as given by Eq. (3.11), this determinant is zero, during this operation ω must be left as a symbolic variable. Since every element of the inverse of the matrix $(||I|| - ||M(\omega)||)$ has at denominator the same determinant, the adjoint matrix $||C(\omega)||$ is not singular for $\omega = \omega_n$, and the term within square brackets in Eq. (3.15) can be easily evaluated for ω_0 and its harmonics. It should be noticed that this equation, and more generally Eqs. (3.12-3.13) are only valid for $\omega = \omega_n$. As shown in Section III, the power spectrum $\Phi(\omega)$, for other values of ω, is given by Eq. (2.7), which, in the present case of a closed loop without external perturbation, gives $\Phi(\omega) = 0$ for every value of ω and diverges for $\omega = \omega_n$. Since this term requires to make the real part of the 900 elements of $||C(\omega)||$ and summing them up, to use double precision during the calculations is mandatory. The same must be made for determining the value of ω_o, which is a singular point, as well as its harmonics. In the present simulations these terms are reported with few decimal digits. The term within square brackets in Eq. (3.15) comes out to be $2.890 \cdot 10^{-3}$ at $\omega_0 = 45.455[s^{-1}]$, corresponding to $f = 7.234Hz$.

According to Eq. (3.15) this term must be multiplied by the square modulus of the Fourier transform $S(\omega)$ of the function $v(t)$, representing the voltage pulse emitted by each neuron during firing, and by $2\pi\nu^2$, where ν represents the number of pulses emitted per unit time $\nu = 30 \cdot 7.234 = 217.02[s^{-1}]$ and $2\pi\nu^2 = 2.9592 \cdot 10^5[s^{-2}]$.

As stated above, $v(t)$ was assumed to be a Gaussian function of time centered in zero with amplitude of $v_0 = -75mV$ and variance $\sigma = 1ms$:

$$v(t) = v_0 \cdot \exp(-0.5 \cdot \sigma^{-2} \cdot t^2) = -0.075 \cdot \exp(-5 \cdot 10^5 \cdot t^2)\,[V], \tag{A1}$$

whose transform is:

$$S(\omega) = -0.075 \cdot 10^{-3} \cdot \exp(-5 \cdot 10^{-7} \cdot \omega^2),\ [V \cdot s],$$

and its square modulus is:

$$|S(\omega)|^2 = 56.25 \cdot 10^{-10} \cdot \exp(-10^{-6} \cdot \omega^2),\ [V^2 \cdot s^2].$$

Thus, since for $\omega = \omega_0 = 45.455$ the exponential is 0.9979, one gets: $2\pi\nu^2 \cdot |S(\omega_0)|^2 = 1.6645 \cdot 10^{-3} \cdot \exp(-10^{-6} \cdot \omega_0^2) = 1.6610 \cdot 10^{-3}\,[V^2]$. Finally we get for the power spectral line at ω_0 the value

$$A(\omega_0) = 1.6610 \cdot 10^{-3} \cdot 2.890 \cdot 10^{-3} = 4.8004 \cdot 10^{-6},\ [V^2].$$

The power spectrum $\Phi(f)$, referred to the frequency f instead of ω, and considered only for positive values of f, must be doubled in order to take into account that $\Phi(\omega)$ is calculated by assuming positive and negative values for ω.[11]

Thus at $f_0 = 7.234 Hz$ we get $A(f_0) = 9.601 \cdot 10^{-6}\ V^2$. In order to evaluate the amplitude of the sinusoid corresponding to this spectral line, it must be considered that it represents the power dissipated when that sinusoid is a voltage applied to a resistor of 1Ω. Thus the line is the square of the RMS value of such sinusoid. If a is the amplitude of the sinusoid then: $a = \sqrt{2} \cdot \sqrt{A(f_0)} = 4.381 \cdot 10^{-3}[V]$. This very large value of the signal corresponds to the emitted pulses by the neurons of the loop diffused internally to the whole cranial space. As stated above, the signal that would be received by an electrode having an area of $1cm^2$ internal to the cranium must be reduced by a factor corresponding to the ratio between the solid angle from which the electrode is seen by the neurons of the loop and the spherical solid angle 4π. If the electrode is distant $2cm$ from the loop and orthogonal, the reduction factor would be $1.989 \cdot 10^{-2}$ and the amplitude of the received sinusoid would be $a = 87.153 \cdot 10^{-6}$ corresponding to nearly $174.306[\mu V]$ peak to peak. This value is about the largest ones observed in EEG, but it does not take into account the two further strong reductions cited above, concerning the propagation of the electric pulse within the cerebral matter and the crossing of the cranial bone.

b) In order to make a control of the fact that when the distribution becomes more and more uniform the low frequency lines tend to zero, the same calculation has been made for a variance of $1 \cdot 10^{-3}$ instead of 1.5. The result is that the spectral line at ω_o is $A(\omega_o) = 2.4917 \cdot 10^{-10}[V^2]$, i.e. lower of about a factor $0.5 \cdot 10^{-4}$ than the previous result. If the time intervals are assumed to be all equals, the sum of the real parts of the elements of the adjoined matrix $||C(\omega_0||$ comes out to be exactly zero. This is also a test of the precision of the numerical calculations.

c) A third and fourth simulation concerns the case where a few neurons on the loop behave differently from the others, as it may happen, for instance, in the case of epileptic seizures. In these simulations, we consider that in the loop of 30 neurons, a sequence of ten neurons has a time interval between subsequent firing of $4.0ms$ in one case and $3.5ms$ in a second case, the other neurons conserving the value of $5ms$. The purpose is to evidence the large increase of the amplitude of the spectral line at the fundamental frequency ω_0 with the increasing of the different behavior of part of the neurons forming the loop. For simplicity, also in these simulations we consider that the impulses emitted by the neurons are all equal and corresponding to those assumed above. Calculations are quite similar to those reported above, the only difference being the generation of the matrix $M(\omega)$ appropriate to the present cases. We thus report only the results.

The case of $4.0ms$ gives for the amplitude of the spectral line at $f_0 = 7.099 Hz$ ($\omega_0 = 44.56159 s^{-1}$) and at the harmonics nf_0, with $n = 2, 3, ..$ the values $A(f_0) = 8.236 \cdot 10^{-6}[V^2]$, $A(2f_0) = 4.043 \cdot 10^{-6}[V^2]$, and $A(3f_0) = 0.9154 \cdot 10^{-6}[V^2]$ respectively. In the second case the corresponding results are $A(f_0) = 19.220 \cdot 10^{-6}[V^2]$, $A(2f_0) = 10.970 \cdot 10^{-6}[V^2]$, and $A(3f_0) = 3.732 \cdot 10^{-6}[V^2]$.

These values correspond to the square of the RMS of the sinusoids represented by these spectral lines. The amplitude a of these sinusoids is obtained from the expression given above. Their value is reported in Section V.

d) A fifth simulation concerns the case where the loop of 30 neurons contains 10 subsequent neurons having a smaller (or higher) amplitude than the residual 20. In this simulation we consider that the amplitude of the ten neurons is smaller of 20% respect to the others, and that the time intervals between firing of subsequent neurons are all equal to $5ms$. This allows to separate the effects on the power spectrum of the variations of the pulses amplitude from those of the time intervals. In this case, it is necessary to use Eq. (3.14) instead of the simpler Eq. (3.15). The pulses are again represented by a Gaussian function centered on zero of amplitude $-75mV$ and variance of $1ms$, except the 10 neurons which have an amplitude reduced of 20%. The Fourier transforms $S(\omega)$ are thus real, and the products $S_\alpha(\omega) \cdot S_{\alpha'}(\omega)$ reduce simply to the modulus $|S(\omega)|^2 A_\alpha A_{\alpha'}$, where A_α and $A_{\alpha'}$ are the amplitude of the transforms of $S_\alpha(\omega)$ and $S'_\alpha(\omega)$ for the states α and α'. It is thus possible to use again Macsyma *genmatrix* to generate the matrix of 30×30 elements representing all the possible products of the amplitudes A_α and $A_{\alpha'}$, which must be multiplied by $|S(\omega)|^2$ and term by term by the adjoint matrix $||C(\omega)||$, calculated as in the previous simulations. We report here only the results: first line at $f_0 = 6.666 Hz$, $A(f_0) = 7.5209 \cdot 10^{-6}[V^2]$, $A(2f_0) = 3.129 \cdot 10^{-8}[V^2]$, and $A(3f_0) = 4.439 \cdot 10^{-12}[V^2]$. Again the corresponding amplitudes of the sinusoids corresponding to these lines are reported in section V. In the present case it can be observed that the harmonic frequencies are practically absent and that the whole signal received by the electrodes is almost a pure sinusoid.

It can be further observed that if the amplitude of the ten pulses were incremented by 20% instead of reduced of the same quantity, the line power spectrum would have remained identical to the present one, except for the line at zero frequency.

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[11] NOTE: As in the previous paper[1] and in most mathematical texts, the power spectrum is given for positive and negative values of ω. Instead commercial spectrum analyzers give, in the case of a continuous spectrum, the quantity $\Phi(f) = 2 \cdot 2\pi \cdot \Phi(\omega)$, limited to only positive values of the frequency f. In the case of a line spectrum the conversion is simply $A(f) = 2 * A(\omega)$.