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# FINITE ELEMENT MODELS WITH NODE-DEPENDENT KINEMATICS ADOPTING LEGENDRE POLYNOMIAL EXPANSIONS FOR THE ANALYSIS OF LAMINATED PLATES

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## ABSTRACT

This work presents a new class of advanced plate FEM models with node-dependent kinematics. To capture the high local stress gradients in the analysis of composite structures, refined models are essential; while for the rest region of the structure, usually, low-order models can be sufficient. An innovative approach named as node-dependent kinematics is proposed to integrate models with different levels of refinement to reach an optimal balance between accuracy and solution costs in FEM analysis. Node-dependent kinematics is based on Carrera Unified Formulation (CUF), which introduces thickness functions defined on the thickness domain for the refinement of plate models. Both ESL (Equivalent Single Layer) and LW (Layer-wise) models adopting various approximation theories can be described and implemented in such a framework. CUF-type displacement functions allow the thickness functions to be related to specific FEM nodes before the interpolation of them over the in-plane domain of the element, leading to advanced FEM models with node-dependent kinematics. In this way, a kinematic variation can be conveniently obtained, which can bridge a global model to a locally refined model while keeping the displacement continuity. Hierarchical Legendre Expansions (HLE) are adopted to construct the shape functions in this work, which provides an approach to capture the localized effects without re-meshing on the structure. Governing equations for FEM models with node-dependent kinematics are derived from the Principle of Virtual Displacements (PVD). When used in the analysis of laminated plates with local effects to be accounted for, the proposed advanced plate models can reduce the computational costs greatly while guaranteeing accuracy without employing special global-local coupling methods.

## 1 INTRODUCTION

Composite materials have attracted considerable attention in recent several decades due to their outstanding properties in engineering applications, which has also boosted the demands for efficient structural analysis methods to capture their sophisticated mechanical responses.

Towards the optimal design of thin-walled laminated structures, various theories have been proposed. Based on Kirchoff-Love's hypothesis, Classical Plate Theory (CPT) [1] is the simplest 2D model. First-order Shear Deformation Theory (FSDT) [2] can account for transverse shear effects but give only constant transverse shear stresses through the thickness. A variety of Higher-Order Theories (HOT) have been suggested to improve the solution accuracy. Carrera [3] proposed Unified Formulation (CUF) as a new methodology to develop refined 1D and 2D models for the analysis of laminated structures. For 2D case, with the help of the introduced thickness functions, various theories can be integrated to formulate refined kinematics as either Equivalent Single Layer (ESL) or Layer-Wise (LW) models. The introduction of Fundamental Nucleus (FN) allows the governing equation to be derived in a unified and compact manner [4]. In the framework of CUF, various models based on either series expansions or interpolation polynomials have been put into practice as reported by Cinefra and Valvano [5] and Cinefra et al. [6].

In FEM analysis, to improve the solution accuracy especially in the area with high stress gradients, refinement is required. A  $h$ -version approach [7] increases the mesh refinement to capture local effects in detail, while a  $p$ -version approach [8] adopts higher-order polynomials as shape functions to approximate the structural deformation. The  $h$ - $p$ -version approach employs these two methods at the same time [9]. Note that in such mono-kinematic approaches, the element kinematics preserve over the whole model.

Some global-local methods can couple models with different kinematics. The so-called three field formulations [10] enforce the displacement compatibility with Lagrange multipliers at domain interfaces. Various methods have been developed based on such an idea, as presented by Aminpour et al. [11] adopting a spline method, and by Prager [12] who introduced an interface potential, as well as eXtended Variational Formulation (XVF) suggested by Blanco et al. [13, 14]. Application of multi-line method was reported by Carrera and Pagani [15, 16]. A superimposed zone can also help to construct global-local models. Arlequin method uses Lagrange multipliers to impose the compatibility in the overlapped area, as suggested by Dhia [17] and Dhia and Rateau [18]. Coupling of CUF-type models with different level of refinement with Arlequin method are reported on 1D modeling by Biscani et al. [19] and 2D modeling by Biscani et al. [20, 21]. On the contrast, multi-grid method superimposes additional elements with either refined mesh [22] or higher-order kinematics [23] on the global model directly, in which homogeneous boundary conditions were imposed on the overlapped zone. Note that in the aforementioned global-local methods, *ad hoc* coupling approaches are used, and usually two sets of FEM mesh grids are needed to refine the kinematics locally.

In the framework of CUF for 2D models, the kinematics are refined by increasing the number of expansion terms used in the thickness functions. By relating the definition of thickness functions to specific nodes, kinematics can be refined locally. In this sense, the FEM nodes will act as “anchor” of the 2D kinematics. With the help of the nodal shape functions, the nodal kinematics can be smeared over the in-plane domain of a 2D element. Such an approach allows the local refinement to be carried out simply by increasing the kinematic order defined on a specific set of nodes without changing the mesh. Thus, one set of versatile mesh grids can be used to build various global-local models without using any *ad hoc* coupling method. This method was proposed by Carrera et al. [24] in 1D refined models then extended to the analysis multi-layered composite plates by Zappino et al. [25]. It has been demonstrated that global-local models constructed with node-dependent kinematics can account for detailed local effects with fewer computational costs [25].

In the present work, node-dependent kinematics for 2D modeling is extended to  $p$ -version plate elements constructed with Hierarchical Legendre Expansions (HLE). The adopted shape functions were firstly suggested by Szabo and Babuška [26] for quadrilateral domains, then employed by Pagani et al. [27] and Carrera et al. [28] as cross-section functions in constructing refined beam models. Compared with Lagrange shape functions of the same polynomial order, HLE requires a fewer number of functions. Meanwhile, since some of the HLE-type shape functions do not possess a specific interpolation node but an artificial one, the nodal kinematics can be smeared over the element domain located in the transition zone in a more averaged way. In the proposed method, the polynomial degree  $p$  is taken as an input parameter. In the following sections, the theoretical basis is briefly explained, and the governing equations are derived from the principle of virtual displacements. The efficiency of the proposed modeling approach is illustrated through a numerical example on a composite plate.

## 2 PRELIMINARIES

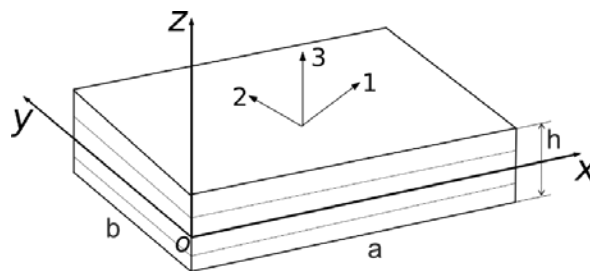


Figure 1: Geometry and reference system of a multi-layered plate model.

The geometry feature and reference system of a laminated plate are shown in Figure 1. The strains and stresses can be arranged as follows:

$$\boldsymbol{\varepsilon}_p^T = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}\}, \quad \boldsymbol{\varepsilon}_n^T = \{\varepsilon_{xz}, \varepsilon_{yz}, \varepsilon_{zz}\}. \quad (1)$$

$$\boldsymbol{\sigma}_p^T = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}, \quad \boldsymbol{\sigma}_n^T = \{\sigma_{xz}, \sigma_{yz}, \sigma_{zz}\}. \quad (2)$$

in which the subscript  $p$  represents the in-plane components, while  $n$  stands for the normal direction of the plate. The strains can be derived from the geometrical relations:

$$\boldsymbol{\varepsilon}_p = \mathbf{D}_p \mathbf{u}, \quad \boldsymbol{\varepsilon}_n = (\mathbf{D}_{np} + \mathbf{D}_{nz}) \mathbf{u}. \quad (3)$$

where the differential operator matrices are:

$$\mathbf{D}_p = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_y & \partial_x & 0 \end{bmatrix}, \quad \mathbf{D}_{np} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{nz} = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix}. \quad (4)$$

The strain and stress components can be related by the constitutive equations:

$$\boldsymbol{\sigma}_p = \tilde{\mathbf{C}}_{pp} \boldsymbol{\varepsilon}_p + \tilde{\mathbf{C}}_{pn} \boldsymbol{\varepsilon}_n, \quad \boldsymbol{\sigma}_n = \tilde{\mathbf{C}}_{np} \boldsymbol{\varepsilon}_p + \tilde{\mathbf{C}}_{nn} \boldsymbol{\varepsilon}_n. \quad (5)$$

$\tilde{\mathbf{C}}_{pp}$ ,  $\tilde{\mathbf{C}}_{pn}$ ,  $\tilde{\mathbf{C}}_{np}$  and  $\tilde{\mathbf{C}}_{nn}$  are the material coefficients rotated from the material system to the analysis system. For more details about the material coefficient matrices, the reader is referred to [25].

### 3 CARRERA UNIFIED FORMULATION FOR 2D MODELS

For refined 2D models, CUF describes the displacement field  $\mathbf{u} = \{u, v, w\}^T$  as a product of thickness functions  $F_\tau$  defined on the thickness domain and in-plane unknown vector  $\mathbf{u}_\tau(x, y)$ , which leads to:

$$\mathbf{u}(x, y, z) = F_\tau(z) \mathbf{u}_\tau(x, y) \quad \tau = 0, 1, \dots, N \quad (6)$$

where  $N$  is the number of expansion terms used in the thickness functions. When  $z$  is defined on the whole thickness domain and  $F_\tau$  employ series expansions, an Equivalent Single Layer (ESL) model can be constructed; when  $F_\tau$  are defined on a layer thickness domain and adopt interpolation polynomials, a Layer-Wise (LW) model can be obtained. Also, many traditional deformation theories can be described in such a manner as explained by Carrera et al. [4].

Particularly, a general case of Higher-order Deformation Theories (HOT) to the  $N$ th order can be expressed as follows:

$$\begin{cases} u(x, y, z) = u_0(x, y) + z \cdot u_1(x, y) + \dots + z^N \cdot u_N(x, y) \\ v(x, y, z) = v_0(x, y) + z \cdot v_1(x, y) + \dots + z^N \cdot v_N(x, y) \\ w(x, y, z) = w_0(x, y) + z \cdot w_1(x, y) + \dots + z^N \cdot w_N(x, y) \end{cases} \quad (7)$$

which can be written in the form of CUF by setting:

$$F_0 = z^0 = 1, \quad F_1 = z^1 = z, \quad \dots, \quad F_N = z^N \quad (8)$$

For a LW model based on Lagrange interpolation polynomials,  $F_\tau$  are defined in the thickness domain of layer  $k$ , which expressed in an isoparametric form is  $\zeta_k \in [-1, 1]$ . If  $N$  interpolation nodes are used in  $F_\tau^k$ , the thickness functions can be expressed as:

$$F_\tau^k(\zeta_k) = \prod_{i=0, i \neq \tau}^N \frac{\zeta_k - \zeta_{k_i}}{\zeta_{k_\tau} - \zeta_{k_i}} \quad (9)$$

In such a definition, the displacements on each interpolation node are physically meaningful. To keep the displacement continuity at layer interfaces, the displacements on the top surface of layer  $k$  should be set equal to those on the bottom surface of layer  $k+1$ . When a sufficient number of interpolation nodes are used to define the thickness functions, the interlaminar continuity of transverse shear stresses can be achieved, which has been demonstrated by Carrera et al. [6, 29].

#### 4 NODE-DEPENDENT KINEMATIC BEAM ELEMENTS

When applied to construct refined 2D FEM elements, the displacement functions read:

$$\mathbf{u}^{(k)}(x, y, z) = N_i(x, y)F_\tau^{(k)}(z) \mathbf{u}_{i\tau} \quad \tau = 1, \dots, N; \quad i = 1, \dots, M. \quad (10)$$

where  $F_\tau^{(k)}$  are the thickness functions, which in ESL models are  $F_\tau$  while in LW models will be  $F_\tau^k$ .  $N_i(x, y)$  are the nodal shape functions, which usually adopt Lagrange interpolation polynomials.  $\tau$  is the number of expansions used in the thickness functions, and  $M$  is the total number of nodes in an element. According to such a definition, the nodal kinematics preserve over the whole model, which means the level of kinematic refinement is uniform. For structures with higher stress gradients only within a limited area, such models might consume unnecessary computation resources.

CUF provides an approach to defining locally refined kinematics. By relating the thickness functions to specific nodes, a FEM element with node-dependent kinematics can be obtained, displacement functions are as follows:

$$\mathbf{u}^{(k)}(x, y, z) = N_i(x, y)F_\tau^{i(k)}(z) \mathbf{u}_{i\tau} \quad \tau = 1, \dots, N; \quad i = 1, \dots, M. \quad (11)$$

$F_\tau^{i(k)}$  are defined on node  $i$ , and become *nodal kinematics*. In a sense, node  $i$  acts as the ‘‘anchor’’ of  $F_\tau^{i(k)}$ . The nodal kinematics will be further interpolated over the in-plane domain of the element, as shown on the left-hand side of Figure 2, in which the kinematics are gradually refined from left to the right side of the element. Such elements with kinematic transition can act as a bridge between the local region with refined kinematics and a global model with lower-order kinematics, as illustrated on the right-hand side of Figure 2. Thus, the local kinematic refinement can be carried out without touching the mesh grids. Node-dependent kinematics has been successfully applied in building numerically efficient global-local models for 1D models by Carrera et al. [24] and 2D models by Zappino et al. [25].

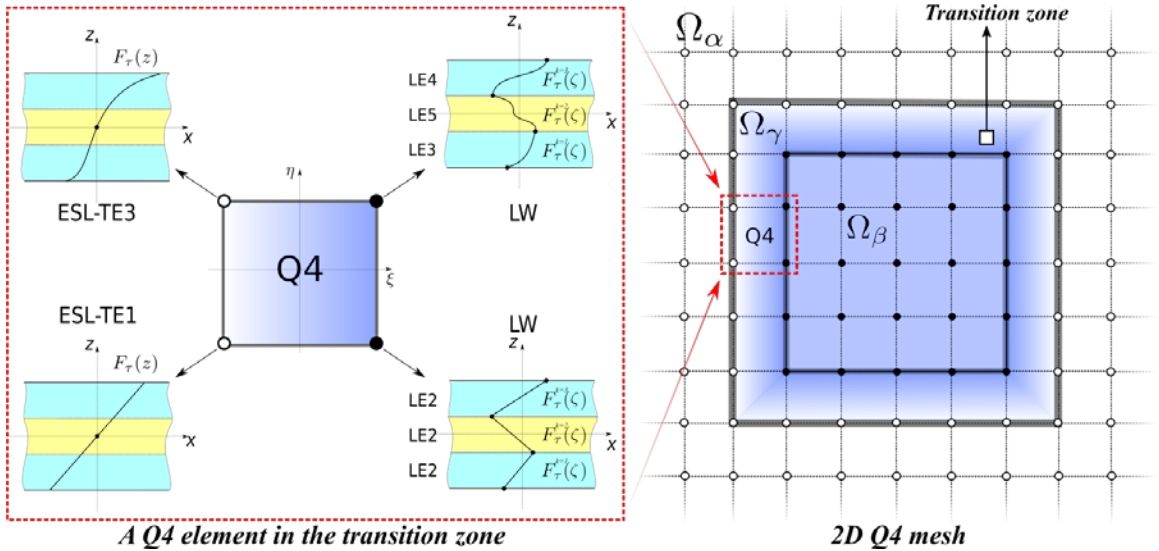


Figure 2: An example: a Q4 plate element with node-dependent kinematics.

## 5 HIERARCHICAL LEGENDRE EXPANSIONS

Szabó et al. [8] suggested a set of shape functions for a quadrilateral domain  $(r,s)$  defined on  $[-1,1]$ , which are based on Legendre polynomials. For a specific polynomial degree  $p-1$ , the set of functions are included in the set for polynomial degree  $p$ . Hierarchical Legendre Expansions (HLE) can also be employed as cross-section functions in refined 1D models as presented by Pagani et al. [27, 28]. In the present work, HLE is adopted as shape functions for 2D modeling and used in combination with node-dependent kinematics. In such a way, except for the kinematics refinement,  $p$ -version refinement is also considered.

Hierarchical Legendre shape functions consist of *nodal modes*, *edge modes*, and *internal modes*, as shown in Figure 2.

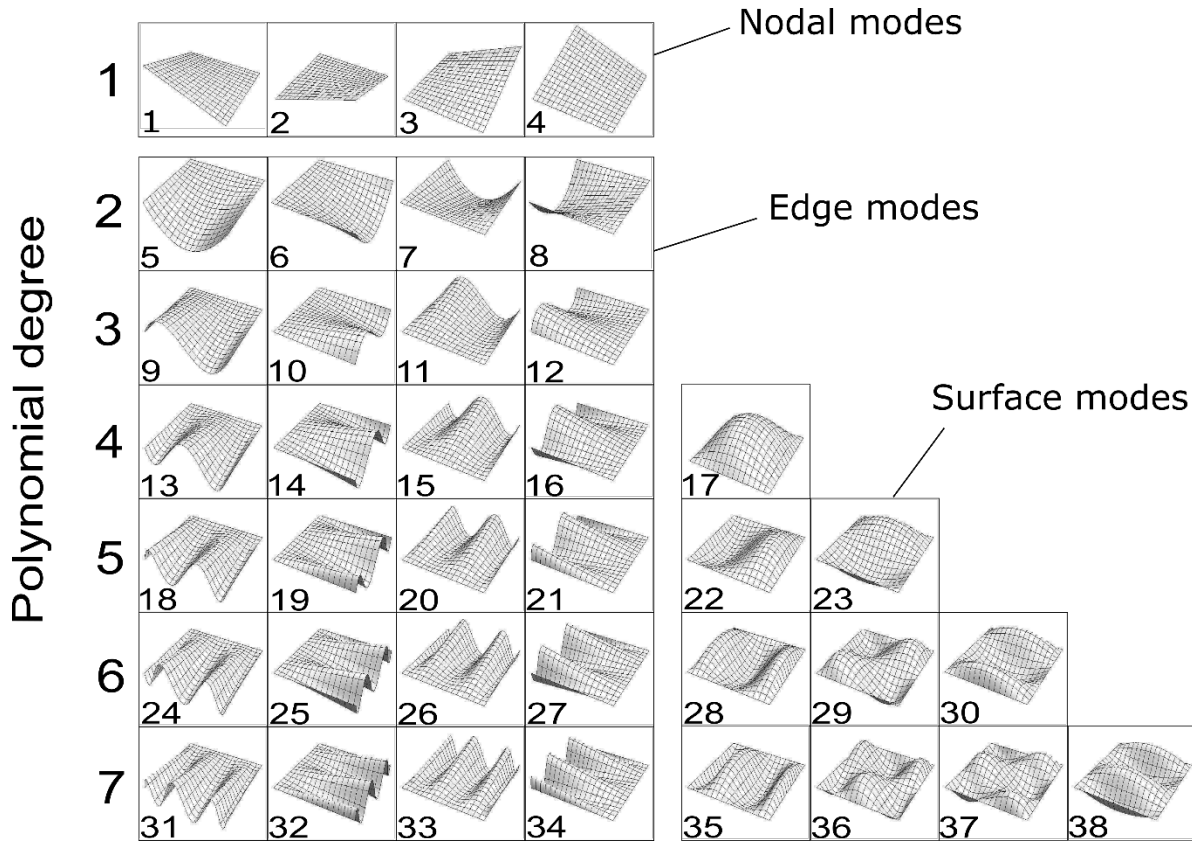


Figure 3: Hierarchical Legendre Expansions (HLE) as shape functions.

**Nodal modes:** are defined as Lagrange-type linear interpolation on the four vertex nodes of the quadrilateral, which are expressed as:

$$N_i = \frac{1}{4}(1-r_i r)(1-s_i s) \quad i=1,2,3,4 \quad (12)$$

in which  $(r_i, s_i)$  represent the local coordinates of the  $i$ th node of the four-node element in the isoparametric reference system.

**Edge modes:** are dominated by the deformation on the four edges and vary linearly along the corresponding perpendicular edges, which read:

$$\begin{aligned}
 N_i(r, s) &= \frac{1}{2}(1-s)\phi_p(r) & i = 5, 9, 13, 18, \dots \\
 N_i(r, s) &= \frac{1}{2}(1+r)\phi_p(s) & i = 6, 10, 14, 19, \dots \\
 N_i(r, s) &= \frac{1}{2}(1+s)\phi_p(r) & i = 7, 11, 15, 20, \dots \\
 N_i(r, s) &= \frac{1}{2}(1-r)\phi_p(s) & i = 8, 14, 16, 21, \dots
 \end{aligned} \tag{13}$$

where  $\phi_p$  is defined as:

$$\phi_p(s) = \sqrt{\frac{2p-1}{2}} \int_{-1}^s L_{p-1}(x) dx = \frac{L_p(s) - L_{p-2}(s)}{\sqrt{4p-2}}(L) \quad p = 2, 3, \dots \tag{14}$$

**Surface modes:** Describe the deformation shapes happen on the internal surface which will vanish on the edges and vertexes, which are:

$$N_i(r, s) = \phi_p(r)\phi_q(s) \quad p, q \geq 4 \tag{15}$$

Different from Lagrange polynomials, when the polynomial degree  $p$  increases to  $p+1$ , only the newly added shape functions need to be introduced in HLE. Meanwhile, most of the functions in HLE do not have a corresponding interpolation node, which implies in a higher-order 2D element, the nodal kinematics can be smeared more averagely. For those plate elements lie in the bridging zone, such a feature can help to improve the coupling performance. Moreover, it can be noticed that compared with Lagrange polynomials of the same polynomial order, HLE usually requires a fewer number of functions to construct a 2D element. Furthermore, with HLE the polynomial order  $p$  can be taken as an input parameter and  $p$ -version refinement can be conveniently realized.

## 6 FEM GOVERNING EQUATION

The governing equation for node-dependent kinematic 2D FEM elements is derived in this section. For a system in static equilibrium, the following equation holds:

$$\delta L_{int} = \delta L_{ext} \tag{16}$$

in which  $L_{int}$  stands for the stain energy and  $L_{ext}$  represents the work done by the external load.  $\delta$  indicates the virtual variation.  $\delta L_{int}$  can be written as follows:

$$\delta L_{int} = \int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = \int_{\Omega} \int_{A_k} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dA_k d\Omega \tag{17}$$

where  $A_k$  represents the thickness domain of the layer  $k$ , and  $\Omega$  indicates the in-plane domain of the element considered. The strains can be obtained as:

$$\begin{aligned}
 \boldsymbol{\varepsilon}_p^k &= F_{\tau}^{i(k)} \mathbf{D}_p (N_i \mathbf{I}) \mathbf{u}_{ir}^{(k)} \\
 \boldsymbol{\varepsilon}_n^k &= F_{\tau}^{i(k)} \mathbf{D}_{np} (N_i \mathbf{I}) u_{ir}^{(k)} + F_{\tau,z}^{i(k)} N_i \mathbf{I} \mathbf{u}_{ir}^{(k)}
 \end{aligned} \tag{18}$$

where  $\mathbf{I}$  is an  $3 \times 3$  identity matrix. By recalling the constitutive equations and the CUF-type displacement functions, the following expression can be obtained:

$$\delta L_{int} = \int_{\Omega} \int_{A_k} (\delta \boldsymbol{\varepsilon}_n^{kT} \boldsymbol{\sigma}_n^k + \delta \boldsymbol{\varepsilon}_p^{kT} \boldsymbol{\sigma}_p^k) dA_k d\Omega = \delta \mathbf{u}_{js}^{(k)T} \mathbf{K}_{j\tau s}^k \mathbf{u}_{ir}^{(k)} \tag{19}$$

$\mathbf{K}_{j\tau s}^k$  is a  $3 \times 3$  core unit of the stiffness matrix, which is also known as the *fundamental nucleus* (FN). The subscripts and Einstein summation convention allow the element stiffness matrix to be expressed and assembled conveniently, which has been elaborated by Carrera et al. [4].

Considering the external load:

$$\delta L_{ext} = \int_V \delta \mathbf{u}^T \mathbf{p} dV \quad (20)$$

By substituting the CUF-type displacement functions, one can have:

$$\delta L_{ext} = \delta \mathbf{u}_{js}^T \int_V N_j F_s^j \mathbf{p} dV = \delta \mathbf{u}_{js}^T \mathbf{P}_{js} \quad (21)$$

in which  $\mathbf{P}_{js}$  is the load vector. Thus, the governing equation for 2D FEM elements with node-dependent kinematics can be obtained as:

$$\delta \mathbf{u}_{js} : \mathbf{K}_{ijrs} \cdot \mathbf{u}_{ir} = \mathbf{P}_{js} \quad (22)$$

For a detailed description of the technique for the assembly of the stiffness matrix and load vector on a 2D FEM element with node-dependent kinematics, the reader is referred to [25].

## 7 NUMERICAL EXAMPLES

In this section, numerical results of a three-layered composite plate subjected to a concentrated load are reported. The square symmetrically laminated cross-ply plate considered is clamped on the four edges, and imposed to a point load  $P_z$  at the central point of the top surface, as illustrated in Figure 4. In such a case, a numerical singularity will be encountered, which is a great challenge for weak form solution approaches. The aim of this numerical example is to show how the proposed approach can construct numerically efficient models.

The plate has three layers with equal thickness  $h_1=h_2=h_3=h/3$ , and the stacking sequence is  $[0^\circ / 90^\circ / 0^\circ]$ . The length-to-thickness ratio is  $a/h=4$ . The FEM model is constructed based on CUF. The element shape functions employ Hierarchical Legendre Expansions (HLE). With node-dependent kinematics, in the local area with stress concentration, kinematic refinement is used. The material properties adopted are listed in Table 1, in which the subscript  $L$  indicates the fiber direction, while  $T$  represents the transverse direction.

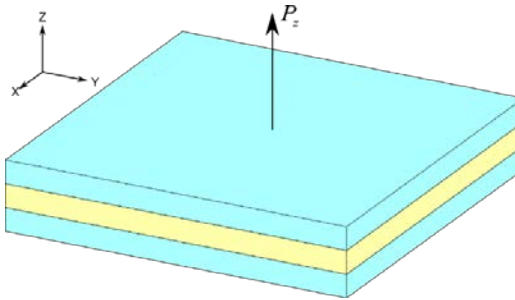


Figure 4: Three-layered composite plate with a concentrated load  $P_z$ .

$E_L / E_T$	$G_{LT} / E_T$	$G_{TT} / E_T$	$\nu_{LT}, \nu_{TT}$
25	0.5	0.2	0.25

Table 1: Material coefficients used for the three-layered composite plate.

The plate is discretized into 16 square elements adopting HLE to the same polynomial order  $p$  as shape functions (denoted as HLE- $p$ ), as illustrated in Figure 5. Meanwhile, in the central zone with strong local effects, LW models with Lagrange polynomials as thickness functions are adopted; the rest of the plate is modeled with ESL model based on a full linear model (TE1). In this work, shape function terms with the same feature position (on a node, edge or surface) are assigned to the same cross-section kinematic model. For comparison, models with uniformly kinematic refinement (LW-LE3) and HLE to



different polynomial order are also used. The transverse displacement  $\bar{w}$  is nondimensionalized according to the following equation:

$$\bar{w} = \frac{100E_r h^3}{P_z a^2} \cdot w \quad (23)$$

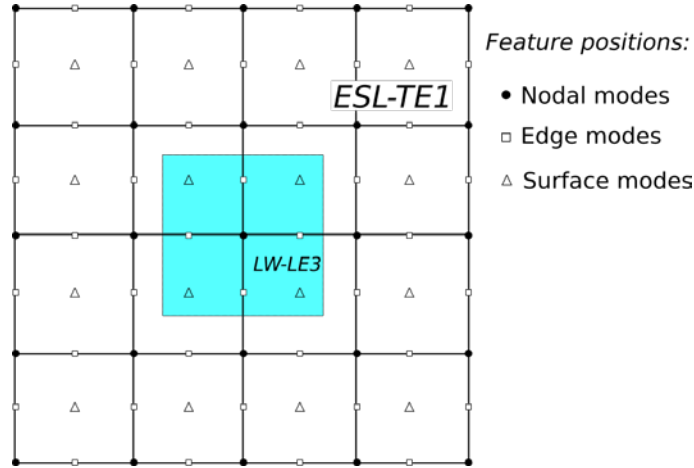


Figure 5: FEM discretization of the three-layered composite plate with HLE- $p$  as shape functions.

The obtained displacements  $\bar{w}$  are compared in Figure 6. As expected, with the increase of the polynomial order, the out-of-plane displacement  $\bar{w}$  at the loading point approaches infinity gradually. Still, more accurate modeling needs better refinement. Whereas, considering that singularity cannot be numerically reached, such effort is neither necessary nor desirable. Figure 6 also shows that, with the help of node-dependent kinematics, compared with HLE7-LE3, model HLE7-LE3/TE1 can reach almost the same accuracy but with a reduction of 67.8% in total degrees of freedom in this case.

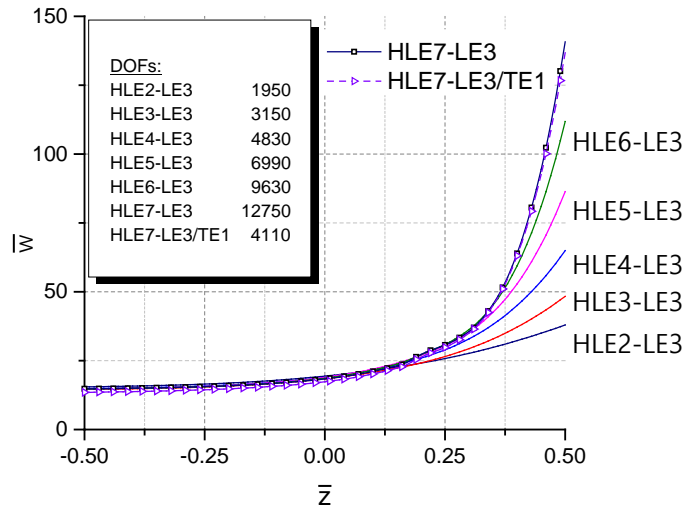


Figure 6: Through-the-thickness variation of transverse displacement  $\bar{w}$  at plate center  $(a/2, b/2)$ .

## 8 CONCLUSIONS

This work presents node-dependent kinematic 2D FEM models adopting Hierarchical Legendre Expansions (HLE) as shape functions. In the proposed approach, local kinematic refinement can be conveniently carried out, and numerically efficient global-local models can be constructed conveniently to account for localized effects.

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## REFERENCES

- [1] W. Koiter, "Koiter W. On the foundations of the linear theory of thin elastic shell. Proc Kon," *Proc Kon Nederl Akad Wetensch*, vol. 73, no. 3, pp. 169--195, 1970.
- [2] P. M. Naghdi, "The theory of shells and plates," in *Linear Theories of Elasticity and Thermoelasticity*, 1973, pp. 425--640.
- [3] E. Carrera, "Theories and finite elements for multilayered, anisotropic, composite plates and shells," *Archives of Computational Methods in Engineering*, vol. 9, no. 2, p. 87--140, 2002.
- [4] E. Carrera, M. Cinefra, M. Petrolo and E. Zappino, *Finite element analysis of structures through unified formulation*, John Wiley & Sons, 2014.
- [5] M. Cinefra and S. Valvano, "A variable kinematic doubly-curved MITC9 shell element for the analysis of laminated composites," *Mechanics of Advanced Materials and Structures*, vol. 23, no. 11, pp. 1312-1325, 2016.
- [6] E. Carrera, M. Cinefra, G. Li and G. M. Kulikov, "MITC9 shell finite elements with miscellaneous through-the-thickness functions for the analysis of laminated structures," *Composite Structures*, vol. 154, no. 15, pp. 360-373, 2016.
- [7] R. E. Bank, "The efficient implementation of local mesh refinement algorithms," *Adaptive Computational Methods for Partial Differential Equations*, pp. 74--81, 1983.
- [8] B. Szabó, A. Düster and E. Rank, "The p-Version of the Finite Element Method," in *Encyclopedia of computational mechanics*, 2004.
- [9] J. N. Reddy, *An introduction to the finite element method*, vol. 2, McGraw-Hill New York, 1993.
- [10] F. Brezzi and L. D. Marini, "The three-field formulation for elasticity problems," *GAMM-Mitteilungen*, vol. 28, no. 2, pp. 124--153, 2005.
- [11] M. A. Aminpour, J. B. Ransom and S. L. McCleary, "A coupled analysis method for structures with independently modelled finite element subdomains," *International Journal for Numerical Methods in Engineering*, vol. 38, no. 21, pp. 3695--3718, 1995.
- [12] W. Prager, "Variational principles of linear elastostatics for discontinuous displacements, strains, and stresses," *Recent Progress in Applied Mechanics*, pp. 463--474, 1967.
- [13] P. J. Blanco, R. A. Feijóo and S. A. Urquiza, "A variational approach for coupling kinematically incompatible structural models," *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 17, pp. 1577--1602, 2008.
- [14] P. J. Blanco, P. Gervasio and A. Quarteroni, "Extended variational formulation for heterogeneous partial differential equations," *Computational Methods in Applied Mathematics Comput. Methods Appl. Math.*, vol. 11, no. 2, pp. 141--172, 2011.
- [15] E. a. P. A. Carrera, "Analysis of reinforced and thin-walled structures by multi-line refined 1D/beam models," *International Journal of Mechanical Sciences*, vol. 75, pp. 278--287, 2013.
- [16] E. a. P. A. Carrera, "Multi-line enhanced beam model for the analysis of laminated composite structures," *Composites Part B: Engineering*, vol. 57, pp. 112--119, 2014.

- [17] H. B. Dhia, "Multiscale mechanical problems: the Arlequin method," *Comptes Rendus de l'Academie des Sciences Series IIB Mechanics Physics Astronomy*, vol. 12, no. 326, pp. 899--904, 1998.
- [18] H. B. Dhia and G. Rateau, "The Arlequin method as a flexible engineering design tool," *International journal for numerical methods in engineering*, vol. 62, no. 11, pp. 1442--1462, 2005.
- [19] F. Biscani, G. Giunta, S. Belouettar, E. Carrera and H. Hu, "Variable kinematic beam elements coupled via Arlequin method," *Composite Structures*, vol. 93, no. 2, pp. 697--708, 2011.
- [20] F. Biscani, G. Giunta, S. Belouettar, E. Carrera and H. Hu, "Variable kinematic plate elements coupled via Arlequin method," *International Journal for Numerical Methods in Engineering*, vol. 91, no. 12, pp. 1264--1290, 2012.
- [21] F. Biscani, P. Nali, S. Belouettar and E. Carrera, "Coupling of hierarchical piezoelectric plate finite elements via Arlequin method," *Journal of Intelligent Material Systems and Structures*, vol. 23, no. 7, pp. 749--764, 2012.
- [22] J. Fish, L. Pan, V. Belsky and S. Goma, "Unstructured multigrid method for shells," *International Journal for Numerical Methods in Engineering*, vol. 39, no. 7, pp. 1181--1197, 1996.
- [23] J. Fish, "The s-version of the finite element method," *Computers & Structures*, vol. 43, no. 3, pp. 539--547, 1992.
- [24] E. Carrera, M. Filippi, A. Pagani and E. Zappino, "Node-dependent kinematics, refined zig-zag and multi-line beam theories for the analysis of composite structures," in *58th AIAA/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, 2017.
- [25] E. Zappino, G. Li, A. Pagani and E. Carrera, "Global-local analysis of laminated plates by node-dependent kinematic finite elements with variable ESL/LW capabilities," *Composite Structures*, vol. 172, p. 1--14, 2017.
- [26] B. A. Szabo and I. Babuška, *Finite element analysis*, John Wiley & Sons, 1991.
- [27] A. Pagani, A. De Miguel, M. Petrolo and E. Carrera, "Analysis of laminated beams via Unified Formulation and Legendre polynomial expansions," *Composite Structures*, vol. 156, pp. 78--92, 2016.
- [28] E. Carrera, A. De Miguel and A. Pagani, "Hierarchical theories of structures based on Legendre polynomial expansions with finite element applications," *International Journal of Mechanical Sciences*, vol. 120, pp. 286--300, 2017.
- [29] E. Carrera, M. Cinefra and G. Li, "Refined finite element solutions for anisotropic laminated plates," *Composite Structures*, 2017.
- [30] E. Carrera, M. Cinefra, G. Li and G. Kulikov, "MITC9 shell finite elements with miscellaneous through-the-thickness functions for the analysis of laminated structures," *Composite Structures*, vol. 154, pp. 360 - 373, 2016.
- [31] M. Cinefra, G. Li, M. Petrolo and E. Carrera, "Refined 2D finite elements for composite plates and shells accounting for hygrothermal effects," in *European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS Congress 2016)*, Crete Island, Greece, June 5-10, 2016.