Finite beam elements based on Legendre polynomial expansions and node-dependent kinematics for the global-local analysis of composite structures

FINITE BEAM ELEMENTS BASED ON LEGENDRE POLYNOMIAL EXPANSIONS AND NODE-DEPENDENT KINEMATICS FOR THE GLOBAL-LOCAL ANALYSIS OF COMPOSITE STRUCTURES

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Keywords: Composite beams, Node-dependent kinematics, Hierarchical Legendre Expansions, Carrera Unified Formulation, Finite elements

ABSTRACT

This work introduces an innovative type of FEM beam models with node-dependent kinematics. A variety of global-local approaches have been proposed to reduce the consumption of computational resources in FEM analysis, in which mostly the main idea is to couple the elements in the locally refined region (with either refined mesh or higher-order theories) and those in the less refined area. As a new method to build FEM models in a global-local analysis, node-dependent kinematics makes it possible to construct elements with different kinematic theories on different nodes and implement a kinematic variation conveniently within an element to bridge a local model to a global one. With the help of the introduced cross-section functions, CUF allows the definition of different kinematics on each node and their interpolation over the axial domain of the beam element. Without using any *ad hoc* coupling method, beam elements with node-dependent kinematics have very compact and coherent formulations through the Fundamental Nucleus (FN). Corresponding FEM governing equation is derived from the Principle of Virtual Displacements (PVD), and the expressions of FNs of the stiffness matrix and load vector are given. Both ESL (Equivalent Single-layer) and LW (Layer-wise) models are addressed. In fact, in this work, Legendre polynomials are used to construct refined beam models, obtaining cross-section functions (nodal kinematics) with Hierarchical Legendre Expansions (HLE) and, eventually, LW accuracy. In the numerical examples, refined models with HLE are employed in the local area with a higher stress gradient, and in the less critical regions ESL models are adopted; meanwhile, in the kinematic transition zone, a beam element with node-dependent kinematics are used to connect these two domains. By comparing the numerical results with those in literature and from 3D FEM modeling, it is demonstrated that when used in the analysis of composite beams with local effects to be considered, node-dependent kinematic beam elements can reduce the computational costs significantly without losing numerical accuracy.

1 INTRODUCTION

Composite structures have been widely used in modern engineering, especially in aerospace applications. For multi-layered structures, the interlaminar continuity of the displacements and stresses should be properly captured for efficient design. Whereas, their heterogeneity has proposed a great number of challenges to numerical modeling.

On history, a great variety of 1D (bar and beam) for slender structures have been proposed. During the times when there was a lack of numerical analyzing approaches, some classical theories were developed, such as Euler-Bernoulli beam and Timoshenko beam [1]. Such traditional models have been widely applied to engineering applications. Such models are economic in FEM simulation, but fail to give a detailed approximation of the transverse stresses on the cross-sections of slender multi-layered structures. Many refined theories have been proposed to overcome such drawbacks. For reviews on such works, the readers are referred to Kapania and Raciti [2, 3] and Carrera [4].

Carrera proposed Unified Formulation (CUF), which can be adopted to derive refined 1D models accounting for detailed 3D stress field. The introduction of fundamental nuclei (FNs) allows the governing equations to be derived in a unified and compact manner. By increasing the polynomial order expanded on each cross-section, the exact solution can be approached in an axiomatic way. CUF can incorporate both series expansions and interpolation polynomials to construct refined 1D models. Equivalent Single Layer (ESL) model treats the stiffness on a cross-section in a smeared manner, which suits the theories based on series expansions, such as Taylor series, trigonometric series, hyperbolic series et al. Such models have been put into practice by Filippi et al. [5, 6]. On the contrast, Layer-Wise (LW) models can account for the physical boundaries and interfaces between layers on the cross-sections, which can naturally guarantee the interlaminar displacement continuity. Interpolation polynomials can be classified into two types: Lagrange-type polynomials which adopt interpolations points with specific coordinates; non-Lagrange type polynomials, such as Legendre and Chebyshev polynomials, which only use interpolation points on the domain boundaries. The latter ones are constructed with hierarchically defined polynomials. CUF-type models with Lagrange polynomials have been widely used in various situations, as summarized by Carrera et al. [7]. Application of Chebyshev polynomials in 1D models was reported by Filippi et al. [8].

Legendre polynomials used as shape functions to construct *p*-version FEM models were presented by Szabó, Düster and Rank [9]. The set of shape functions introduced by Szabó and Babuška [10] are classified into *vertex modes*, *side modes* and *internal modes*, and the lower-order functions are contained in the higher-order basis. Such type of Hierarchical Legendre Expansions (HLE) were adopted as cross-section functions to construct refined beam models by Carrera, De Miguel and Pagani [11] and Pagani et al. [12]. HLE makes the polynomial degree *p* an independent parameter, and the refinement of the kinematics on the cross-sections can be carried out by increasing *p*. Compared with Lagrange-type polynomials, a re-meshing on the cross-section can be avoided in such an approach.

The refinement of kinematics can improve the solution accuracy, but will also lead to increased degrees of freedom in FEM models. A compromise can be reached by global-local approaches. A variety of global-local methods have been proposed over the last several decades. Three-field formulations [13, 14, 15, 16, 17] are based on Lagrange multipliers, which enforce the displacement compatibility at domain interfaces. Arlequin method, proposed by Dhia [18] and Dhia and Rateau [19], combines two solution domains with an overlapping zone. Widely used techniques in commercial software, Rigid Beam Element (RBEi) and Multi-Point Constraints (MPCs) simply define the dependent degrees of freedom as linear functions of the independent degrees of freedom. Submodeling, a two-step technique approach provided in ABAQUS, uses the global displacement field to drive the deformation of the borders of a local model. A superelement, is formed by grouping a set of elements and condensing the so-called internal degrees of freedom, which requires special matrix operation techniques. These global-local approaches depend on extra constraints or *ad hoc* assumptions and need at least two sets separate meshes.

In the framework of CUF, cross-section functions can be defined on specific nodes, leading to 1D elements with node-dependent kinematics. Such elements can form a bridging zone, connecting a local area with refined kinematics to a global region approximated with lower-order cross-section functions. This approach can help to reduce the overall computational costs while guaranteeing the computational accuracy in the local zone with high stress gradients. This concept was firstly suggested by Carrera et al. [20] for 1D models, and then further developed for the analysis of laminated plates by Zappino et al. [21]. Comparatively, global-local models with node-dependent kinematics can avoid the using of *ad hoc* techniques, and keep the compactness of the FEM formulations.

In the present work, HLE is used as cross-sections functions in combination with node-dependent kinematics. Such an approach enables one to refine the kinematics locally ant any desirable node by simply increasing the polynomial degree *p*. Meanwhile, only one set of meshes is needed in constructing a series of global-local models. The related formulations are presented in the following sections. The application of the proposed method is illustrated though a numerical example on a two-layered cantilever composite beam.

2 PRELIMINARIES

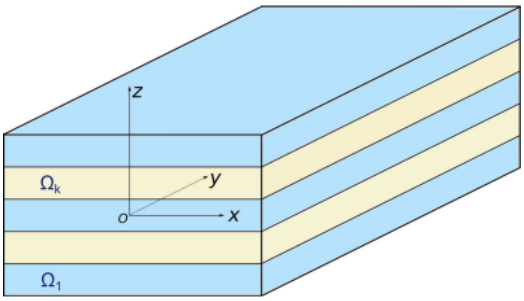

Figure 1: Reference system of a slender laminated structure.

For a slender laminated structure shown in Figure 1, when the longitudinal direction is set along the *y* direction, the cross-section domain will lie in the (*x*,*z*) plane. The strain and stress components are:

$$\boldsymbol{\varepsilon}^T = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xz}, \varepsilon_{yz}, \varepsilon_{xy}\} \tag{1}$$

$$\boldsymbol{\sigma}^T = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xz}, \sigma_{yz}, \sigma_{xy}\} \tag{2}$$

where the strain vector $\boldsymbol{\varepsilon}$ can be obtained by means of the differential operator matrix $\mathbf{D}$:

$$\boldsymbol{\varepsilon} = \mathbf{D}\,\mathbf{u} \tag{3}$$

For problems with infinitesimal strains, $\mathbf{D}$ in an explicit for is:

$$\mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \tag{4}$$

Meanwhile, the stresses and strains can be related through the constitutive equations:

$$\boldsymbol{\sigma} = \widetilde{\mathbf{C}}\,\boldsymbol{\varepsilon} \tag{5}$$

in which $\widetilde{\mathbf{C}}$ is the matrix of material coefficients rotated from the material system to the analysis coordinate system.

3 REFINED BEAM ELEMENTS BASED ON CUF

In the framework of CUF, beam models are refined through the cross-section functions $F_\tau(x,z)$, which lead to the following expression for the displacement field:

$$\mathbf{u} = \mathbf{u}_\tau(y) F_\tau(x,z), \qquad \tau = 1, \cdots, M \tag{6}$$

where $\mathbf{u}_\tau(y)$ is the axial displacement vector, *M* is the number of expansions used in the cross-section functions $F_\tau(x,z)$. To construct FEM models, the axial displacement vector can be approximated with Lagrangian shape functions and nodal unknowns as follows:

$$\mathbf{u}_\tau(y) = N_i(y)\mathbf{u}_{i\tau}, \qquad i = 1,\cdots,N_n \tag{7}$$

in which $N_i(y)$ are the shape functions, and N_n is the number of nodes within the same element, $\mathbf{u}_{i\tau}$ the nodal unknowns. Thus, the complete expression of displacement functions for FEM formulated according to CUF can be written as:

$$\mathbf{u} = N_i(y)F_\tau(x,z)\ \mathbf{u}_{i\tau}, \qquad \tau = 1,\cdots,M; \quad i = 1,\cdots,N_n \tag{8}$$

It can be found that, with the help of Einstein's summation convention, the displacement functions can be expressed in a compact way. The sub-indexes play an important role in describing various beam theories. For refined beam theories based on higher-order Taylor series expansion (TE), adopting the afore-described formulation, can be written as:

$$\begin{aligned} u_x &= u_{x_1} + xu_{x_2} + zu_{x_3} + x^2u_{x_4} + xzu_{x_5} + z^2u_{x_6} \\ u_y &= u_{y_1} + xu_{y_2} + zu_{y_3} + x^2u_{y_4} + xzu_{y_5} + z^2u_{y_6} \\ u_z &= u_{z_1} + xu_{z_2} + zu_{z_3} + x^2u_{z_4} + xzu_{z_5} + z^2u_{z_6} \end{aligned} \tag{9}$$

where:

$$F_1 = 1, \quad F_2 = x, \quad F_3 = z, \quad F_4 = x^2, \quad F_5 = xz, \quad F_6 = z^2 \tag{10}$$

For the widely-used Lagrange interpolation polynomial expansions (LE) defined on a quadrilateral domain (ξ,η), a model based on four interpolation nodes can expressed as:

$$F_1 = \frac{1}{4}(1-\xi)(1-\eta); \quad F_2 = \frac{1}{4}(1+\xi)(1-\eta); \quad F_3 = \frac{1}{4}(1+\xi)(1+\eta); \quad F_4 = \frac{1}{4}(1-\xi)(1+\eta). \tag{11}$$

in which $\xi,\eta \in [-1,1]$, and $F_1(-1,-1) = 1$, $F_2(1,-1) = 1$, $F_3(1,1) = 1$, $F_3(-1,1) = 1$. LE-type cross-section functions expanded on four and nine nodes can be defined accordingly.

4 HIERARCHICAL LEGENDRE EXPANSIONS

The cross-section functions can also adopt Hierarchical Legendre Expansions (HLE), which was employed firstly by Pagani et al. [12] for the refinement of beam models referring to the work of Szabó et al. [9]. Such type of cross-section functions treat the polynomial degree *p* as an independent variable. The functions for a quadrilateral domain (*r*,*s*) (*r* and *s* are defined on [-1,1]) presented by Szabó et al. [9] can be classified into vertex modes, side modes, and internal modes, as shown in Figure 2.

Vertex modes: These functions are defined as linear interpolation over the quadrilateral domain, which read:

$$F_\tau = \frac{1}{4}(1-r_ir)(1-s_is) \qquad \tau = 1,2,3,4 \tag{12}$$

where r_i and s_i stand for the local isoparametric coordinates of node *i* in the quadrilateral element with four nodes.

Side modes: Correspond to the edge-dominant shapes, which are defined as:

$$\begin{aligned} F_\tau(r,s) &= \frac{1}{2}(1-s)\phi_p(r) \qquad \tau = 5,9,13,18,... \\ F_\tau(r,s) &= \frac{1}{2}(1+r)\phi_p(s) \qquad \tau = 6,10,14,19,... \\ F_\tau(r,s) &= \frac{1}{2}(1+s)\phi_p(r) \qquad \tau = 7,11,15,20,... \\ F_\tau(r,s) &= \frac{1}{2}(1-r)\phi_p(s) \qquad \tau = 8,14,16,21,... \end{aligned} \tag{13}$$

in which ϕ_p is expressed as follows:

$$\phi_p(s)=\sqrt{\frac{2p-1}{2}}\int_{-1}^{s}L_{p-1}(x)\,\mathrm{d}x=\frac{L_p(s)-L_{p-2}(s)}{\sqrt{4p-2}}(L)\qquad p=2,3,\ldots \tag{14}$$

Internal modes: Describe the deformation shapes happen on the internal surface which will vanish on the edges and vertexes, which are:

$$F_\tau(r,s)=\phi_i(r)\phi_j(s)\quad \boxed{i,j\geq 4} \tag{15}$$

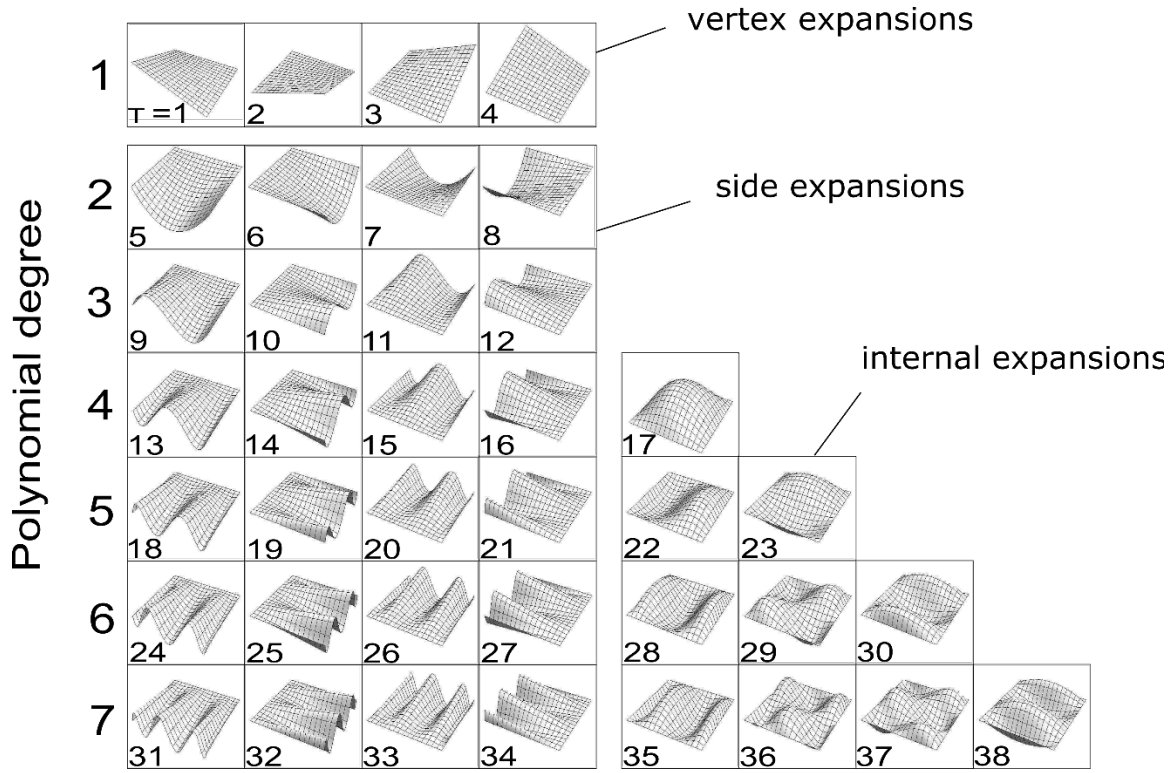

Figure 2: Hierarchical Legendre Expansions (HLE).

Since the set of functions for *p-1* are contained in those for *p*, they are described as *hierarchical*. In a sense, HLE combines the advantage of Taylor series and Lagrange interpolation. When used on a quadrilateral domain, compared with Lagrange polynomials with the same polynomial degree *p*, HLE requires a fewer number of expansions and can avoid the work on re-allocation of interpolation nodes and re-definition of the functions for a higher polynomial order.

5 NODE-DEPENDENT KINEMATIC BEAM ELEMENTS

In CUF-type displacement functions as in Equation (8), the cross-sections can be further related to its "anchoring" nodes *i*, leading to the following expression:

$$\mathbf{u}=N_i(y)F_\tau^i(x,z)\ \mathbf{u}_{i\tau},\qquad \tau=1,\cdots,M;\quad i=1,\cdots,N_n \tag{16}$$

Equation (15) describes a family of 1D FEM models with node-dependent kinematics. In such elements, each node can possess individually defined kinematics on the corresponding cross-section, then be smeared utilizing the nodal shape functions N_i to formulate an advanced element. A natural application of such an approach is to bridge a global model with less-refined kinematics to a sufficiently refined local model accounting for local effects. In the example shown in Figure 3, a beam with four nodes is formulated, and on each node, an individual set of cross-section functions are defined, and a kinematic transition is realized.

By avoiding using any *ad hoc* coupling technique, node-dependent kinematics can keep the compactness of the formulations, and refinement level of the local kinematics can be treated as an input

parameter. In such a way, a global-local model can be easily built on the basis of a global model, and only one set of meshes is needed for a series of analysis.

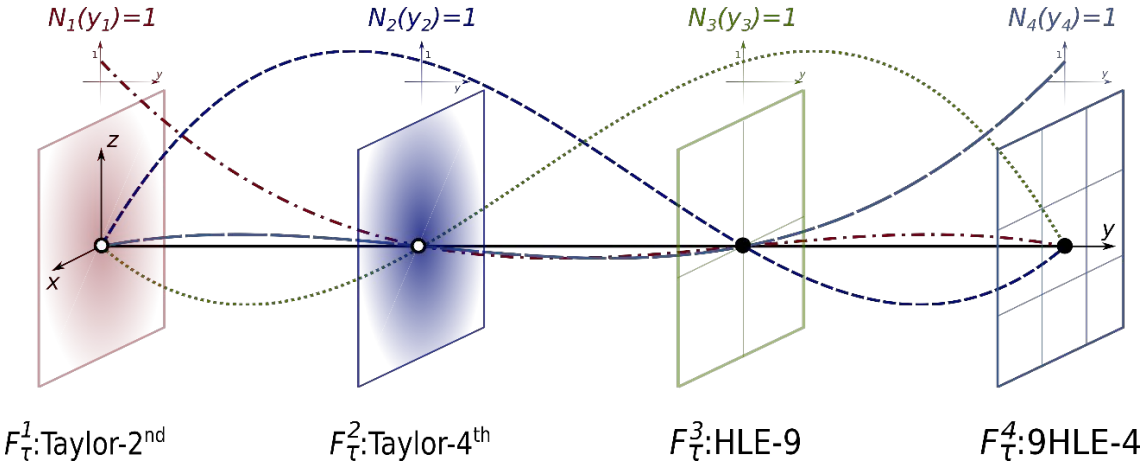

Figure 3: An example: a beam element (B4) with node-dependent kinematics.

6 FEM GOVERNING EQUATIONS

The governing equations are derived from the principle of virtual displacements (PVD). For elastic bodies in static equilibrium, one has:

$$\delta L_{int} = \delta L_{ext} \tag{17}$$

where δL_{int} stands for the variation of the strain energy caused by the deformations and δL_{ext} represents the work done on the virtual displacements by the external forces. δL_{int} can be expressed as:

$$\delta L_{int} = \int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV \tag{18}$$

By invoking CUF-type displacement functions, the geometric relations, and constitutive equations, the following equation can be obtained:

$$\delta L_{int} = \delta \mathbf{u}_{js}^T \cdot \int_V N_j \mathbf{I} F_s^j \mathbf{D}^T \mathbf{C} \mathbf{D} F_\tau^i \mathbf{I} N_i dV \cdot \mathbf{u}_{i\tau} = \delta \mathbf{u}_{js}^T \cdot \mathbf{K}_{ij\tau s} \cdot \mathbf{u}_{i\tau} \tag{19}$$

where $\mathbf{I}$ is an 3×3 identity matrix. $\mathbf{K}_{ij\tau s}$ is the *fundamental nucleus* (FN) for node-dependent kinematics, which is a basic unit of the stiffness matrix. The virtual work δL_{ext} done by the external load $\mathbf{p}$ reads:

$$\delta L_{ext} = \int_V \delta \mathbf{u}^T \mathbf{p} dV \tag{20}$$

The above equation can be further expanded in the form of CUF as:

$$\delta L_{ext} = \delta \mathbf{u}_{js}^T \int_V N_j \mathbf{I} F_s^j \mathbf{p} dV = \delta \mathbf{u}_{js}^T \mathbf{P}_{js} \tag{21}$$

where $\mathbf{P}_{j_s}$ represents the FN of the load vector. The governing equation for node-dependent kinematic 1D FEM models can be reached as follows:

$$\delta \mathbf{u}_{js}: \quad \mathbf{K}_{ij\tau s} \cdot \mathbf{u}_{i\tau} = \mathbf{P}_{js} \tag{22}$$

7 NUMERICAL EXAMPLES

In this section, a cantilever beam with two layers is analyzed with the proposed approach. The geometry features are as shown in Figure 4. The beam is clamped on one end and imposed to four point

loads on the vertexes of the loading end. The elastic properties of the materials used for the beam are as listed in Table 1. In the 1D FEM model, the structure is discretized into ten B4 elements as in Figure 4, and the transverse shear stress σ_{yz} on the cross-section at *y=2b/3* is considered. The whole structure is analyzed with models based on HLE, in which the polynomial order *p* is increased until a numerical convergence can be achieved. To reduce the computational costs, in the area far away from the region of interest on the clamped side, kinematics based on Taylor series are adopted. In the proposed approach, in the range of one element, a kinematic can be realized, and the less-refined model covers the range of three B4 elements.

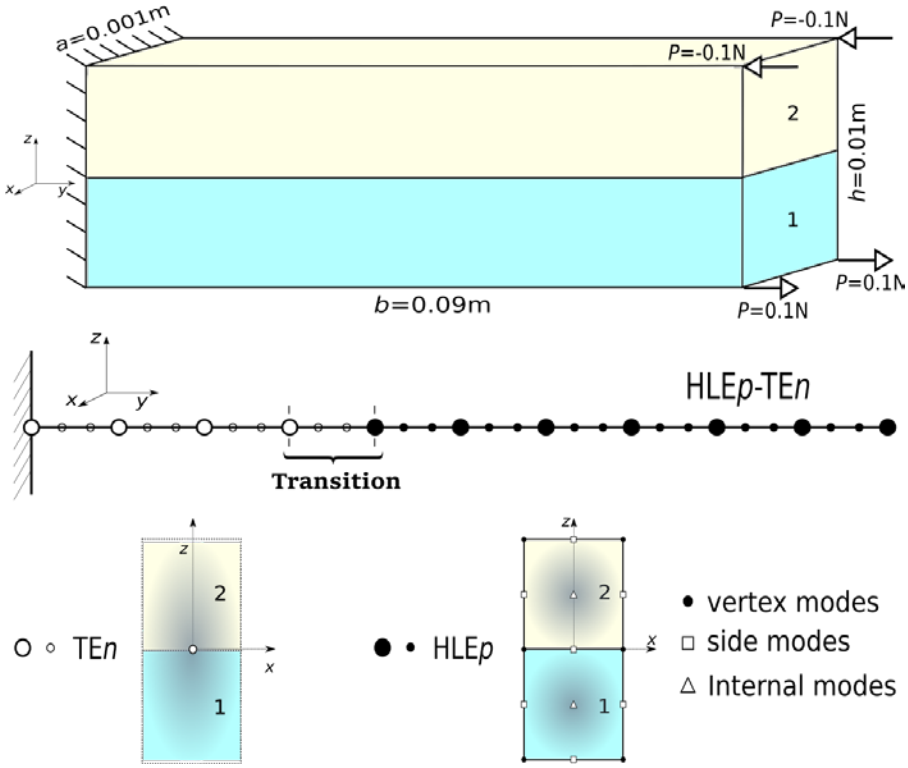

Figure 4: The two-layered composite cantilever beam.

	E_L[GPa]	E_T[GPa]	ν_{LT}	G_{LT}[GPa]
Material 1	30	1	0.25	0.5
Material 2	5	1	0.25	0.5

Table 1: Material coefficients used for the two-layered cantilever beam.

The results are summarized in Figure 5. By comparing the results obtained with different models, it can be found that, with uniformly adopted HLE kinematics, the transverse stress variation converges overall through-the-thickness domain when *p=6* is used. If the left-hand side of the structure adopts TE1 on the cross-section, the total degrees of freedom can be reduced by about 40% while keeping the solution accuracy, as shown in Figure 5.

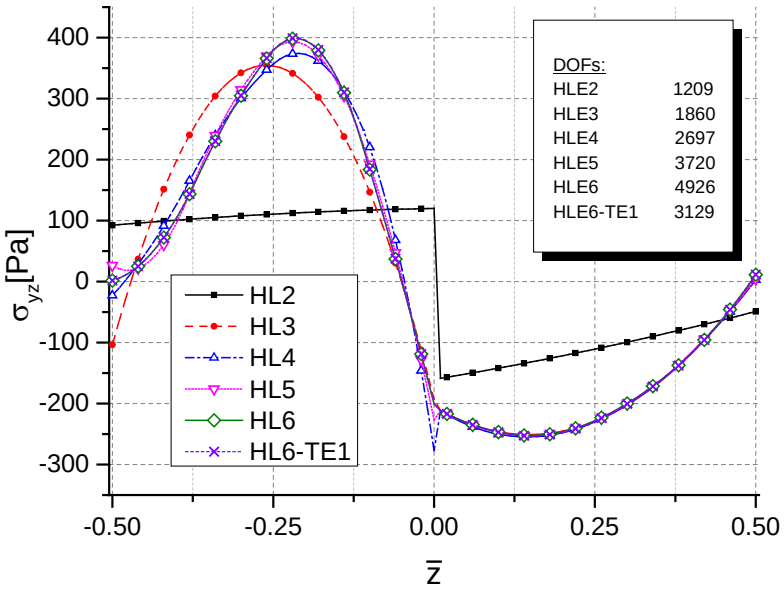

Figure 5: Through-the-thickness variation of σ_{yz} at $(0, 2b/3, \bar{z})$.

8 CONCLUSIONS

This work presents a class of advanced 1D FEM models for the analysis of laminated composite structures with node-dependent kinematics. This approach is applied to construct global-local models. Hierarchical Legendre polynomial Expansions (HLE) are adopted as cross-section functions for the refinement of local kinematics to account for local effects. By taking the polynomial degree *p* as an input parameter, and assigning refined kinematics at desirable nodes located in the area with high stress gradients, a series of FEM models can be built conveniently using only one set of meshes. Such an approach can help to improve the numerical efficiency in engineering simulations.

ACKNOWLEDGEMENTS

This research work has been carried out within the project FULLCOMP (FULLy analysis, design, manufacturing, and health monitoring of COMPosite structures), funded by the European Union Horizon 2020 Research and Innovation program under the Marie Skłodowska Curie grant agreement No. 642121. E. Carrera has been partially supported by the Russian Science Foundation (Grant No. 15-19-30002).

REFERENCES

[1] S. P. Timoshenko, "On the transverse vibrations of bars of uniform cross-section," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 43, no. 253, pp. 125-131, 1922.

[2] R. K. Kapania and S. Raciti, "Recent advances in analysis of laminated beams and plates. Part I-Sheareffects and buckling.," *AIAA Journal*, vol. 27, no. 7, pp. 923--935, 1989.

[3] R. K. Kapania and S. Raciti, "Recent advances in analysis of laminated beams and plates, part II: Vibrations and wave propagation," *AIAA Journal*, vol. 27, no. 7, pp. 935--946, 1989.

[4] E. Carrera, "Theories and finite elements for multilayered, anisotropic, composite plates and shells," *Archives of Computational Methods in Engineering*, vol. 9, no. 2, p. 87–140, 2002.

[5] M. Filippi, M. Petrolo, S. Valvano and E. Carrera, "Analysis of laminated composites and sandwich structures by trigonometric, exponential and miscellaneous polynomials and a MITC9 plate element," *Composite Structures*, vol. 150, pp. 103--114, 2016.

[6] E. Carrera, M. Filippi and E. Zappino, "Laminated beam analysis by polynomial, trigonometric, exponential and zig-zag theories," *European Journal of Mechanics-A/Solids,* vol. 41, pp. 58--69, 2013.

[7] E. Carrera, M. Cinefra, M. Petrolo and E. Zappino, Finite element analysis of structures through unified formulation, John Wiley & Sons, 2014.

[8] M. Filippi, A. Pagani, M. Petrolo, G. Colonna and E. Carrera, "Static and free vibration analysis of laminated beams by refined theory based on Chebyshev polynomials," *Composite Structures,* vol. 132, pp. 1248--1259, 2015.

[9] B. Szabó, A. Düster and E. Rank, "The p-Version of the Finite Element Method," in *Encyclopedia of Computational Mechanics,* 2004.

[10] B. A. Szabo and I. Babuška, Finite element analysis, John Wiley & Sons, 1991.

[11] E. Carrera, A. De Miguel and A. Pagani, "Hierarchical theories of structures based on Legendre polynomial expansions with finite element applications," *International Journal of Mechanical Sciences,* vol. 120, pp. 286--300, 2017.

[12] A. Pagani, A. De Miguel, M. Petrolo and E. Carrera, "Analysis of laminated beams via Unified Formulation and Legendre polynomial expansions," *Composite Structures,* vol. 156, pp. 78--92, 2016.

[13] W. Prager, "Variational principles of linear elastostatics for discontinuous displacements, strains, and stresses," *Recent Progress in Applied Mechanics,* pp. 463--474, 1967.

[14] M. A. Aminpour, J. B. Ransom and S. L. McCleary, "A coupled analysis method for structures with independently modelled finite element subdomains," *International Journal for Numerical Methods in Engineering,* vol. 38, no. 21, pp. 3695--3718, 1995.

[15] F. Brezzi and L. D. Marini, "The three-field formulation for elasticity problems," *GAMM-Mitteilungen,* vol. 28, no. 2, pp. 124--153, 2005.

[16] P. J. Blanco, R. A. Feijóo and S. A. Urquiza, "A variational approach for coupling kinematically incompatible structural models," *Computer Methods in Applied Mechanics and Engineering,* vol. 197, no. 17, pp. 1577--1602, 2008.

[17] E. Carrera, A. Pagani and M. Petrolo, "Use of Lagrange multipliers to combine 1D variable kinematic finite elements," *Computers & Structures,* vol. 129, pp. 194--206, 2013.

[18] H. B. Dhia, "Multiscale mechanical problems: the Arlequin method," *Comptes Rendus de l'Academie des Sciences Series IIB Mechanics Physics Astronomy,* vol. 12, no. 326, pp. 899--904, 1998.

[19] H. B. a. R. G. Dhia, "The Arlequin method as a flexible engineering design tool," *International Journal for Numerical Methods in Engineering,* vol. 62, no. 11, pp. 1442--1462, 2005.

[20] E. Carrera, M. Filippi, A. Pagani and E. Zappino, "Node-dependent kinematics, refined zig-zag and multi-line beam theories for the analysis of composite structures," in *58th AIAA/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference,* 2017.

[21] E. Zappino, G. Li, A. Pagani and E. Carrera, "Global-local analysis of laminated plates by node-dependent kinematic finite elements with variable ESL/LW capabilities," *Composite Structures,* vol. 172, p. 1–14, 2017.

[22] E. Carrera and E. Zappino, "Analysis of Complex Structures Coupling Variable Kinematics One-Dimensional Models," in *ASME 2014 International Mechanical Engineering Congress and Exposition,* 2014.