First-order displacement-based zigzag theories for composite laminates and sandwich structures: a review

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FIRST-ORDER DISPLACEMENT-BASED ZIGZAG THEORIES FOR COMPOSITE LAMINATES AND SANDWICH STRUCTURES: A REVIEW

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Abstract. *The paper gives a critical review and new accomplishments of the displacement-based zigzag theories for laminated composite and sandwich structures, with special emphasis to the underlying ideas, relative strengths and weaknesses. Some numerical results substantiate the conclusions.*

1. INTRODUCTION

Composite laminates and sandwich structures are characterized by high stiffness-to-weight and strength-to-weight ratios and by the possibility to be tailored according to the particular application. Sandwich structures exhibit further promising properties such as high-energy absorption, impact resistance, noise and vibration reduction, excellent damping behavior, [1, 2]**.** Due to these appealing properties, in the last thirty years an increasing number of high-performance and lightweight primary load-bearing structures in the aerospace, automotive, nuclear, naval and civil constructions, are comprised of relatively thick composite laminates and sandwich structures with one hundred or more layers. Such structures are inherently heterogeneous and anisotropic and may suffer unwanted failure due to transverse shearing and out-of-plane stretching.

As failure predictions call for an accurate evaluation of the stress field within each layer of laminated composite and sandwich structures, affordable computational costs for these structures are the main concern. In order for the computational costs to be affordable, accurate computational models for multilayered composite and sandwich structures need to be independent of number of layers and of their micro-structure.

Different approaches have been adopted to face these modeling challenges [3]. A first broad classification could be the following one: three-dimensional (3D) models and two-dimensional (2D) models. Example of 3D models can be found in [4,5,6,7,8,9]. The reduction of the 3D model to the 2D model rely generally on the use of the *axiomatic approach*, that is, on the a-priori assumed trough-the-thickness distribution of the primary variables. In the *displacement-based approach* [10,11], the primary variables are generalized displacements (displacement, rotation, etc.); in the *mixed approach* [12], both generalized displacements and stresses (generally, out-of-plane stresses) are adopted as primary variables.

Following [3], the 2D plate/shell models can be divided into Equivalent Single Layer (ESL) models (also known as Smeared Laminate models) and Layer-wise (LW) models according to the type of kinematics assumed.

Equivalent Single Layer (ESL) (also known as Smeared Laminate) models are generally based on a smooth expansion (generally, a power series expansion) over the whole laminate thickness of the displacement field in terms of the thickness-wise co-ordinate. This means that the kinematics along the thickness is assumed to be at least C^1-continuous and independent of the laminate lay-up. As a consequence, the multilayer structure is substituted with a plate/shell made by an equivalent single layer. While computationally not expensive, ESL models are generally not accurate, especially when through-the-thickness distribution of strains and stresses are the main concern and the laminate is highly heterogeneous.

Among the displacement-based ESL models, the most popular are the Classical Lamination Plate Theory (CLPT), based on the Kirchhoff's plate theory [13] and the First-order Shear Deformation Theory (FSDT), based on the Mindlin's and Reissner's plate theories [14,15] . Both CLPT and FSDT make use of a linear expansion of the in-plane displacement. In addition to the weaknesses of the ESL models, it is well-know that FSDT needs ad hoc shear correction factors to yield accurate results [14,15,16,17,18]. As well-known, CLPT and FSDT perform relatively well in predicting global quantities, such as, transverse displacement, fundamental natural frequency and buckling load for thin and moderately thick laminates that have a relatively low degree of transverse heterogeneity; however, their accuracy diminishes rapidly when they are used to predict the displacement and stress fields in highly heterogeneous and/or thick composite and sandwich laminates [11,19,16,20,21]. Improved predictions can be obtained using higher order through-the-thickness expansions of the displacements and/or stresses [22]. Computationally efficient analytical models for beams, plates and shells that account for transverse shear and thickness-stretch deformations have recently been advanced in [23,24,25]. Reviews on the ESL theories are given in [26,27,28].

Contrary to the ESL models, Layer-wise (LW) models [3,29,30,31,32,33,34,35,36] make a-priori assumptions on the distribution of the displacement and stress fields within each layer. Limiting the discussion to the displacement-based approach, LW models postulates a C^0-continuous kinematics, that is a distribution of displacements (first of all, in-plane displacements) continuous along the thickness with first derivative showing a jump at layer interfaces (the reason of this discontinuity will be explained hereinafter). According to [26], the LW models are further divided into (i) *layer dependent models*, wherein the number of kinematic variables increases with the number of layers; and (ii) *layer independent models*, wherein the number of variables remains constant and independent on the number of layers. Hereafter in this paper, the terminology Layer-wise (LW) models will be adopted only to indicate the layer dependent LW models, while the terminology Zigzag (ZZ) models will be adopted for the layer independent LW models.

Layer dependent LW models [29,30,31,34] are generally very accurate but computationally costly; in these models the number of unknowns increases with the number of layers, as in 3D models. As we said, in order for the computational costs to be affordable (this is especially relevant for nonlinear and/or progressive failure analysis of thick laminates made up of hundreds of layers), computational models for multilayered composite and sandwich structures need to be independent of number of layers and of their micro-structure, while preserving the ability to take into account the distortion of cross-section typical in composite laminate and sandwich structures.

From the above, we easily understand the strong interest for computational models allowing to take into account the layered nature of the composite laminates and sandwich structures, while preserving the affordable computational cost of the ESL models.

To date, ZZ models seem to be the best candidates to meet these requirements, [37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67]. In fact, they attempt to combine the advantages of ESL and LW models. In the ZZ models (as in ESL models), the number of kinematic unknowns does not depend on the number of layers. The in-plane displacements combine the smooth (generally, polynomial) functions defined across the entire laminate thickness (linear [37,38,39,40,41,52,55,56,57] or higher-order polynomial, [42,43,44,45,46,51,53,58,59] with the piecewise linear (i.e., zigzag) distributions. The zigzag contributions enable a more realistic modeling of the in-plane cross-sectional distortion in multilayered composites, giving rise to a computationally efficient theory for the modeling of relatively thick laminated composite and sandwich structures. The zigzag model predictions are often as accurate as those obtained by the computationally expensive LW and higher-order models. LW-based zigzag models have also been explored in [30,31].

Extensions of the kinematics of the Di Sciuva's First-order zigzag theory (linear zigzag model) include, among other: (i) extension to the linear [41] and nonlinear [48] multilayered shell theory; (ii) general lamination lay-up [42,47,49]; (iii) satisfaction of the shear stress-free boundary conditions on the top and bottom surfaces [44,46,50]; (iv) polynomial expansion of the global part to any degree [47,49]; (v) extension to the dynamics, buckling [16,47,49,48]; (vi) thermal effects [68],[69]; (vii) sublaminates approach [68,69,70,71,72,73,74]; (viii) inclusion of von Kármán geometrically nonlinear effects [16,48,49]; (ix) extension to the nonlinear theory of laminated composites with damaged interfaces [49,68,69,74,75,76]; (x) inclusion of the transverse normal strain and stress [74,77,78]; (xi) visco-elastic effects [79]; (xii) active control of beams, plates and shells [80].

Averill [55] pointed out two drawbacks of the classical zigzag models, (1) the transverse shear stresses derived from the constitutive equations vanish erroneously along clamped boundaries, and that (2) C^1-continuous finite element approximations are required for the transverse deflection variable leading to a type of approximation that is particularly undesirable for plate and shell finite elements.

In an attempt to resolve the aforementioned drawbacks of the classical zigzag models, Tessler, Di Sciuva and Gherlone developed the Refined Zigzag Theory (RZT), [61,62,63,64,65,66,67] that makes use of a set of novel zigzag functions and uses FSDT as a baseline. The transverse shear stresses are allowed to be discontinuous along the ply interfaces; this relaxation of stress continuity permits more accurate predictions of all response quantities including the transverse shear stresses that provide accurate average values of the ply-level stresses. RZT overcomes the key drawbacks of the original zigzag theories, i.e., (1) transverse shear stresses and forces do not vanish erroneously along clamped edges, and (2) since the strains are defined in term of first derivatives of the kinematic variables, computationally efficient C^0-continuous finite elements are readily formulated. The RZT has shown to be very accurate over a wide range of aspect ratios and material systems, including thick laminates with a high degree of transverse shear flexibility and heterogeneity [61,62,63,64,65,66,67].

Another first-order zigzag theory largely used in the literature [53,58,81,82,83,84,85] is the Murakami's zigzag theory [52], which shares with RZT the same kinematics, but different zigzag functions. As will be seen later, Murakami's zigzag functions are basically of geometrical nature, in the sense that the jump in their thickness-wise derivative depends only on the thickness of each layer, when the same jump in the derivative of the RZT zigzag functions depends on the lay-up and on the transverse shear elastic compliance of the layers.

In closing this introductory overview of the zigzag approaches for the analysis of laminated composite laminates and sandwich structures, it should be noted that they belong to the more general modern approaches, known either as Multi-scale or Global-local approaches [86,87,88,89,90,91,92], where the displacement field within each layeris represented as a superposition of a layer-wise field, defined over each layer, and a global field spanning the entire laminate. Thus, the governing equations exhibit explicit coupling between effects associated with different (global and local) length scales. This multi-scale representation allows the accuracy of layer-wise theories to be achieved at a reduced computational cost [88]. It also facilitates the use of highly refined kinematic descriptions in regions of critical interest with reduced-order descriptions elsewhere in the structure, and affords a straight-forward means of transitioning between such descriptions. The distribution of available computational resources in this manner can improve both accuracy and efficiency, and can be carried out adaptively to maximize its potential benefits.

Aim of this paper is to give a review of the first-order displacement-based zigzag theories for composite laminates and sandwich structures, with special emphasis to the underlying ideas, relative strengths and weaknesses. First, the basic ideas on which rests the first-order Di Sciuva's zigzag theory are presented. This will serve as a starting point for introducing the main concern of the paper: a rewiev and comparison of the first-order Refined Zigzag Theory (RZT) and the first-order Murakami's Zigzag theory. The paper ends with the presentation of some Numerical results and the Conclusions.

2. GEOMETRICAL PRELIMINARIES

We consider a multi-layered plate of constant thickness $2h$, built-up of a finite number N of linearly elastic anisotropic layers, each of them exhibiting different mechanical properties, see Figure 1. The layers are joined by N-1 interfaces and may be either perfectly bonded or have displacement jumps across the interface due to interfacial damage. The points of the plate are referred to a global orthogonal Cartesian co-ordinate system ($\mathbf{x}$,,z), with $\mathbf{x} \equiv (x_\alpha, \alpha = 1,2)$ the reference plane of the plate and z the thickness-wise co-ordinate. In the body of paper, Italic indices range from 1 to 3, Greek indices range from 1 to 2 and the summation convention on the repeated indices on a monomial expression is adopted. In general, superscript (k) placed in brackets on the right of any quantity identifies its affiliation to the layer k. The layer k has a constant thickness, $2h^{(k)}$. The coordinates of the bottom and top faces and of the mid-surface of the k*th* layer are denoted by $z_B^{(k)}$, $z_T^{(k)}$ and $z_M^{(k)}$, respectively. The interfacial coordinate between the k*th* and (k+1)*th* layers is denoted by $z^{(k)}$. $z_{(k)}$, so that $z_{(k)} = z_T^{(k)} = z_B^{(k+1)}$.

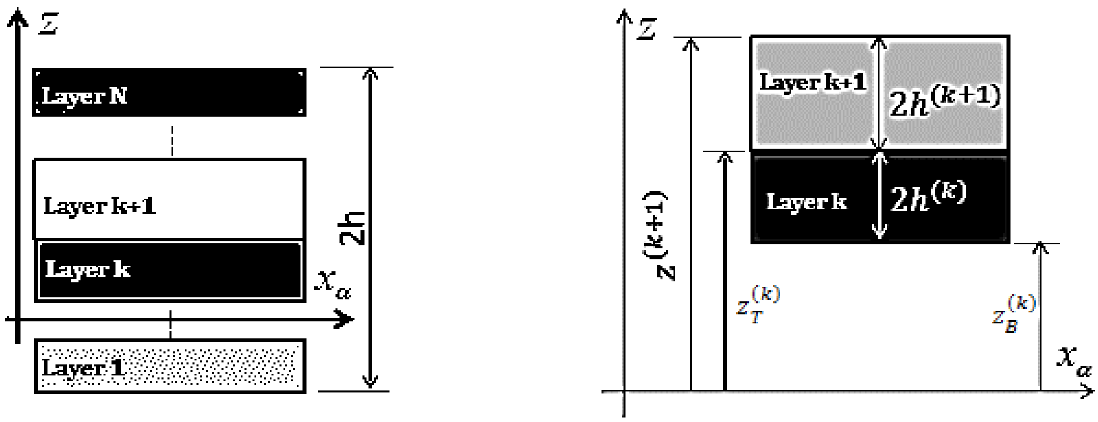


Fig. 1-Geometry and layer numbering of the multilayered plate.

For further convenience, we introduce non-dimensional local layer coordinates, $\zeta^{(k)}$, such that $\zeta^{(k)} \in [-1,+1]$, and k=1,...,N,

$$\zeta^{(k)} = \frac{z^{(k)} - z_M^{(k)}}{h^{(k)}} = \frac{z^{(k)} - z_B^{(k)}}{h^{(k)}} - 1 \text{ with } z^{(k)} \equiv z \in [z_B^{(k)}, z_T^{(k)}].$$

The origin of this local layer coordinate is located $z_M^{(k)}$abroad the coordinate system.

3. CONTACT AND BOUNDARY CONDITIONS

In order to understand the assumed kinematics of zigzag models for multilayered plates, we recall some important issues from 3D theory of elasticity. From 3D theory of elasticity it is well known that displacements and stresses must satisfy interlayer continuity conditions at the interface between two perfectly bonded layers and boundary conditions on the top and bottom surfaces.

3.1 Interlayer continuity or contact conditions

(i) continuity of the displacement field (geometric)

$$U_i^{(k)}\left(\boldsymbol{x}, z_T^{(k)}\right) = U_i^{(k+1)}\left(\boldsymbol{x}, z_B^{(k+1)}\right) \tag{1}$$

(ii) continuity of the transverse shear and normal stresses and of the gradient of the transverse normal stress (static)

$$\sigma_{iz}^{(k)}\left(\boldsymbol{x}, z_T^{(k)}\right) = \sigma_{iz}^{(k+1)}\left(\boldsymbol{x}, z_B^{(k)}\right) \tag{2}$$

$$\partial_z \sigma_{zz}^{(k)}\left(\boldsymbol{x}, z_T^{(k)}\right) = \partial_z \sigma_{zz}^{(k+1)}\left(\boldsymbol{x}, z_B^{(k)}\right) \tag{3}$$

where $U_i^{(k)}(\boldsymbol{x}, z)$ is the displacement component along the x_i $(i = 1,2,z)$ co-ordinate axis, and σ_{iz} are the components of the stress tensor.

3.2 Boundary conditions on the top (T) and bottom (B) surfaces

If the bottom and top surfaces of the laminate are loaded only by the pressure $\bar{p}_z^{(B)}(\boldsymbol{x}, -h)$ and $\bar{p}_z^{(T)}(\boldsymbol{x}, +h)$, then on the bottom (B) and top (T) surfaces of the laminate the following boundary conditions apply:

(i) On the transverse shear stresses (free-transverse shear stresses boundary conditions)

$$\sigma_{\alpha z}^{(1)}\left(\boldsymbol{x}, z_B^{(1)}\right) = \sigma_{\alpha z}^{(N)}\left(\boldsymbol{x}, z_T^{(N)}\right) = 0 \tag{4}$$

(ii) On the transverse normal stress and its gradient

$$\sigma_{zz}^{(1)}\left(\boldsymbol{x}, z_B^{(1)}\right) = \bar{p}_z^{(B)}(\boldsymbol{x}, -h); \qquad \sigma_{zz}^{(N)}\left(\boldsymbol{x}, z_T^{(N)}\right) = \bar{p}_z^{(T)}(\boldsymbol{x}, +h) \tag{5}$$

$$\partial_z \sigma_{zz}^{(1)}\left(\boldsymbol{x}, z_B^{(1)}\right) = \partial_z \sigma_{zz}^{(N)}\left(\boldsymbol{x}, z_T^{(N)}\right) = 0 \tag{6}$$

i.e., the transverse normal stresses must be equals to the applied pressure and the transverse shear stresses and the gradient of the transverse normal stress must vanishe.

4. EQUIVALENT SINGLE LAYER (ESL) THEORIES

As we said, Equivalent Single Layer (ESL) theories are generally based on a smooth expansion (generally, a power series expansion) over the whole laminate thickness of the displacement field in terms of the thickness-wise co-ordinate.

As a general expression for the expansion, we can assume the following one [41],

$$U_i\ (\mathbf{x},z) = \mathcal{Z}_{r(i)}(z)\widehat{U}_i^{(r(i))}(\mathbf{x}) = (\mathbf{x},z)\ \ (i=1,2,3),\ \ r(i) = 0,1,2,\dots,R_i \tag{7}$$

where $\mathcal{Z}_{r(i)}(z)$ stands for any set of a-priori chosen linearly independent functions. Generally, they are assumed to be a power series of z,

$$\mathcal{Z}_{r(i)}(z) = z^r\ ,\quad r = 0,1,2,..,R_i \tag{8}$$

or their orthogonal corresponding Legendre's polynomial of first kind. They are known as polynomial theories.

This means that the kinematics along the thickness is assumed to be at least C^1-continuous and independent of the laminate lay-up. As a consequence, the multilayer structure is substituted with a plate/shell made by an equivalent single layer.

Almost all the polynomials theories assume

$$R_1 = R_2 > R_3 \tag{9}$$

In addition, many theories make the further assumption, $R_1 = R_2 > R_3 = 0$. In order to emphasize the order of expansion adopted for the in-plane displacement (i=1,2) and the transverse displacement (i=3), the following notation is adopted, $\{R, R_3\} -$ order polynomial theory.

4.1 ESL First-order Shear Deformation Theory-FSDT

This theory assumes the following kinematics [3,14,15],

$$\begin{aligned} U_\alpha\ (\mathbf{x},z) &= \widehat{U}_\alpha^{(0)}(\mathbf{x}) + z\widehat{U}_\alpha^{(1)}(\mathbf{x}) \qquad (\alpha=1,2),\ \ R_\alpha = 1 \\ U_3\ (\mathbf{x},z) &= \widehat{U}_3^{(0)}(\mathbf{x}) = w^{(0)}(\mathbf{x}) \qquad\qquad\qquad R_3 = 0 \end{aligned} \tag{10}$$

Following the previous terminology, this ESL First-order shear deformation theory is a $\{1,0\} -$ order polynomial theory.

4.2 ESL Third-order Shear Deformation Theory-TSDT

Most of the third-order shear deformation theories start by assuming the following kinematics [10]

$$\begin{aligned} U_\alpha\ (\mathbf{x},z) &= \widehat{U}_\alpha^{(0)}(\mathbf{x}) + z\widehat{U}_\alpha^{(1)}(\mathbf{x}) + z^2\widehat{U}_\alpha^{(2)}(\mathbf{x}) + z^3\widehat{U}_\alpha^{(3)}(\mathbf{x}) \qquad (\alpha=1,2),\ \ R_\alpha = 3 \\ U_3\ (\mathbf{x},z) &= \widehat{U}_3^{(0)}(\mathbf{x}) = w^{(0)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R_3 = 0 \end{aligned} \tag{11}$$

Following the previous terminology, this ESL Third-order shear deformation theory is a $\{3,0\} -$ order polynomial theory.

Requiring the transverse shear stresses to vanish on the top and bottom surfaces of the plate ($z = \pm h$), the number of generalized in-plane displacements reduces from 4 to 2. For example, one well-known TSDT is Reddy's TSDT [3,22]. In this theory, for the in-plane displacements the following expression is assumed

$$U_\alpha(\mathbf{x}, z) = \widehat{U}_\alpha^{(0)}(\mathbf{x}) + z\widehat{U}_\alpha^{(1)}(\mathbf{x}) + z^3\widehat{U}_\alpha^{(3)}(\mathbf{x}) \tag{12}$$

where

$$\widehat{U}_\alpha^{(3)}(\mathbf{x}) = -\frac{1}{3h^2}\left(\partial_\alpha w^{(0)}(\mathbf{x}) + \widehat{U}_\alpha^{(1)}(\mathbf{x})\right) \tag{13}$$

In Eq.(13) and in the following, $\partial_i\ (.) = \frac{\partial(.)}{\partial x_i}$.

While computationally not expensive, ESL models are generally not accurate, especially when through-the-thickness distribution of strains and stresses are the main concern and the laminate is highly heterogeneous.

5. FIRST-ORDER DI SCIUVA's ZIGZAG THEORY-DSCZZT

Guided by these constraints, the majority of zigzag theories attempt to model accurately the transverse shear deformability and cross-section warping by satisfying the continuity constraints on the displacement field (Fig.2a) and on the inter-laminar shear stresses (Fig.2b). In other words, the displacement field is C^0 -continuous across the plate thickness, as well as the transverse shear stresses.

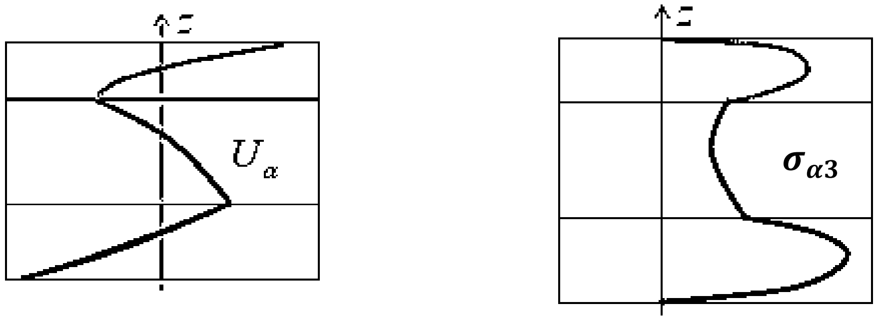


Fig. 2 - Thickness-wise distribution of the (a) in-plane displacements and (b) transverse shear stresses. Both the distributions are of the zigzag type, that is, their derivative along z is discontinuous at the interface (zigzag behavior).

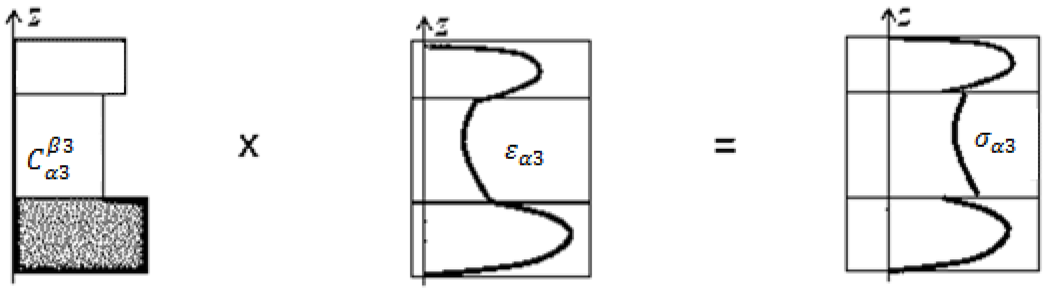
As most of the plate theories assume constant transverse displacement in the z-direction and zero transverse normal stress, the contact conditions on the transverse normal stress and its gradient (Eqs. 5-6) are a-priori satisfied, thus reducing the interface static contact conditions to the transverse shear stresses (Eq. 4). For material possessing a plane of elastic symmetry parallel to the reference plane of the plate, Hooke's law allows to write these contact conditions in terms of transverse shear elastic coefficients and transverse shear strains.

$$C_{\alpha 3}^{\beta 3(k)}\epsilon_{\beta 3}^{(k)}\left(\mathbf{x}, z_T^{(k)}\right) = C_{\alpha 3}^{\beta 3(k+1)}\epsilon_{\beta 3}^{(k+1)}\left(\mathbf{x}, z_B^{(k+1)}\right) \tag{14}$$

where $2\epsilon_{\beta 3} = \partial_z\ U_\alpha\ + \partial_\alpha\ U_z$ are the transverse shear components of the strain tensor and and $C_{\alpha 3}^{\beta 3(k)}$the transverse shear components of the elasticity tensor.

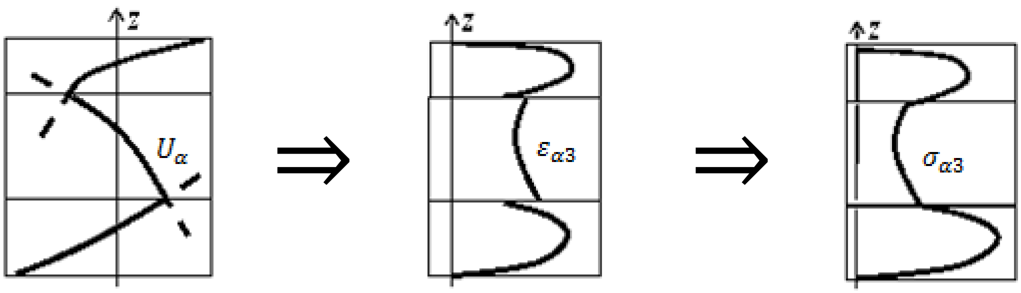
From Eq. (14) we easily deduce that in general the contact conditions on the transverse shear stresses cannot be met when the transverse shear elastic coefficients vary between two adjacent layers. Usually, the thickness-wise distribution of the elastic coefficients in laminated plates are piecewise constant functions of the thickness coordinate. Then, if the through-the-thickness distribution of the in-plane displacements are assumed to be smooth functions, i.e., at least C^1 -continuous functions of the *z*-coordinate, the transverse shear stress will have jumps across the interfaces proportional to the jumps in the transverse shear elastic coefficients. We conclude that in order to be the **transverse shear stresses continuous at the interfaces, the transverse shear strains must have a jump at the interfaces.**

By taking into account the expression of the transverse shear strain, we conclude that the **gradient of the in-plane displacements must be discontinuous at the interfaces,** that is the **in-plane displacements need to be $\boldsymbol{C^{(0)}}$ −continuous,** that is, $\partial_z\ U_\alpha^{(k)}(\mathbf{x}, z_T^{(k)}) \neq \partial_z\ U_\alpha^{(k+1)}\left(\mathbf{x}, z_B^{(k+1)}\right)$. Figure 4 shows graphically this.



(a) **(b)** **(c)**

Fig. 3 - Typical thickness-wise distributions of (a) transverse shear elastic stiffness coefficients, (b) transverse shear strains, and (c) transverse shear stresses, in a laminated structures if the in-plane displacements are assumed to be at least C^1 -continuous functions of the *z*-coordinate.



(a) **(b)** **(c)**

Fig. 4 - (a) C^0 -continuous in-plane displacements (zig-zag pattern); (b) Jumps in the transverse shear strains; (c) C^0 -continuous transverse shear stresses (zig-zag pattern).

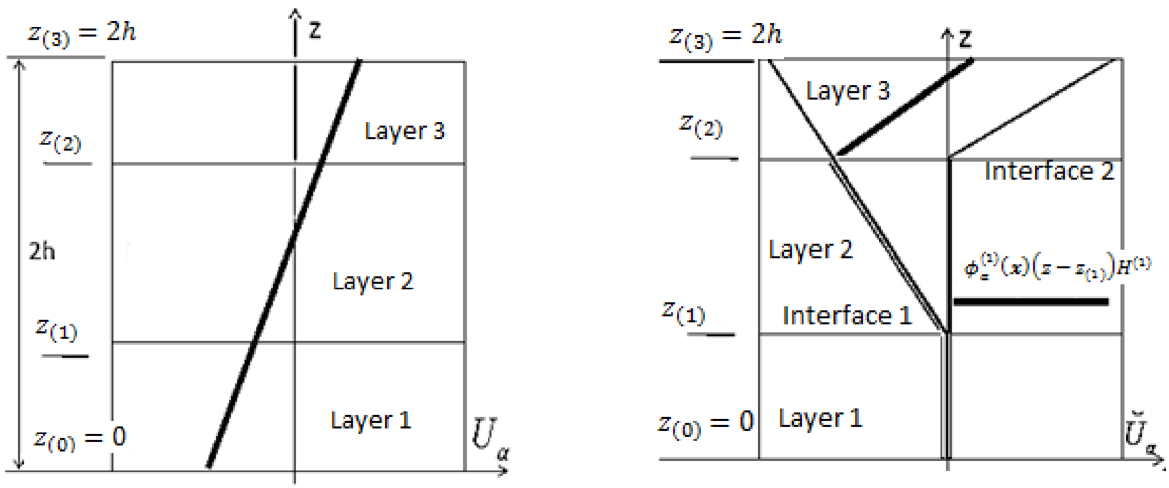
Zigzag theories seek to meet this requirements by adopting a kinematics having a number of generalized displacements as the corresponding ESL theory [16,37,38,40]. To do this, the in-plane displacement field within each layer is taken to be composed of the superposition of global and local contributions, while the transverse displacement is assumed to be constant over the thickness.

$$U_\alpha^{(k)}(\mathbf{x},z) = \widehat{U}_\alpha^{(k)}(\mathbf{x},z) + \breve{U}_\alpha^{(k)}(\mathbf{x},z) \;\; (\alpha = 1,2)$$
$$U_z\,(\mathbf{x},z) = w(\mathbf{x}) \tag{15}$$

In Eq.(15) $\widehat{U}_\alpha^{(k)}(\mathbf{x},z)$ gives the global kinematics (see, Eq. 7) and $\breve{U}_\alpha^{(k)}(\mathbf{x},z)$ the local enrichment.

In order to introduce the underlying ideas of the zigzag theories without excessive mathematical formalism, I will refer to the First-order zigzag plate theory.

In this zigzag theory, the Global kinematics is the same as that of the ESL First-Order Shear Deformation Theory (Eq. 10), that is, a linear expansion of the in-plane displacements (Figure 5(a)) [16,37,38,40].



(a) **(b)**

Fig. 5 - (a) Global ESL FSDT and (b) local enrichment in the Di Sciuva's FSDZZT.

The local enrichment, known as ***zigzag function***, is given by the expression

$$\breve{U}_\alpha^{(k)}(\mathbf{x},z) = \sum_{q=1}^{k-1} \emptyset_\alpha^{(q)}(\mathbf{x})(z - z_{(q)})H^{(q)} \tag{16}$$

where $\emptyset_\alpha^{(q)}$ are unknown generalized displacements and

$$H^{(q)} = H\big(z - z_{(q)}\big) = \begin{cases} 0 & z \le z_{(q)} \\ 1 & z > z_{(q)} \end{cases}$$

is the Heaviside unit function.

Eq. (16) is the expansion for the local enrichment used to model multilayered plates by enforcing the continuity of the transverse shear stresses at the interfaces (C^0 − contribution), as is evident from the analytical expression and from the graphs shown in figure 5(b) for the case of a three-layer laminate.

It is easy to convince that the zigzag local enrichment,

(a) is a continuous piece-wise linear function of the z coordinate (it satisfy a-priori the interface continuity conditions on the displacements);

(b) has piece-wise constant first derivative;

(c) adds N-1 unknown generalized displacements $\emptyset_{\alpha}$ for each coordinate directions;

(d) is zero on the first layer.

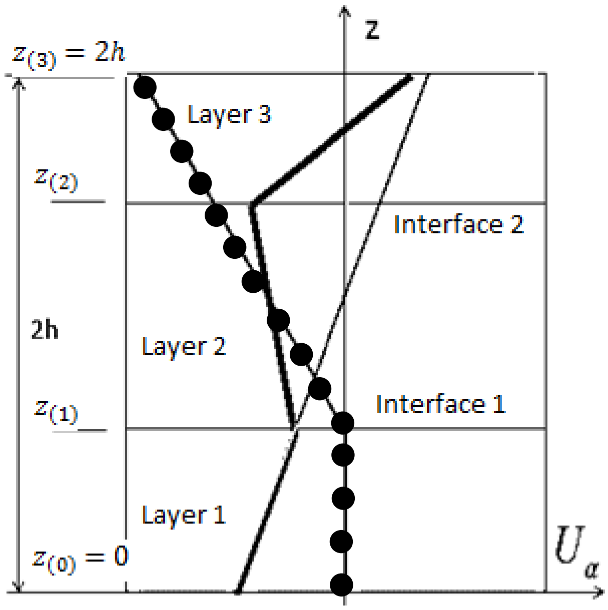


Fig. 6 - Global ESL FSDT (———), local zigzag enrichment (●—●—●) and resulting kinematics (━━━) of Di Sciuva's FSDZZT.

Figure 6 shows the global contribution (ESL FSDT), the zigzag function relative to the contrivìbution of the first interface (see, Fig. 5(b)) and the resulting kinematics adopted in First-order Di Sciuva's zigzag theory-FSDZZT, [16,37,38,40].

Since this zigzag function satisfy a-priori the interface continuity conditions on the displacements, it adds N-1 unknown generalized displacements for each co-ordinate directions. By taking into account that there are N-1 interface continuity conditions on the transverse shear stresses to be satisfied, it follows that, by satisfying these contact conditions, we can compute the N-1 unknowns generalized displacements $\emptyset_{\alpha}$ in terms of the global generalized displacements.

For example, for laminates with layers of orthotropic materials, the result is the following one,[16,37,38,40]

$$\widehat{U}_{\alpha}^{(k)}(\mathbf{x}, z) = \widehat{U}_{\alpha}^{(0)}(\mathbf{x}) + z\widehat{U}_{\alpha}^{(1)}(\mathbf{x}); \quad \breve{U}_{\alpha}^{(k)}(\mathbf{x}, z) = f_{DSC\alpha}^{(k)}(z)\gamma_{\alpha}(\mathbf{x}) \tag{17}$$

where

$$\gamma_{\alpha} = \widehat{U}_{\alpha}^{(1)}(\mathbf{x}) + \partial_{\alpha}w(\mathbf{x}) \tag{18}$$

is the shear strain at reference plane and the zigzag function has the following very simple expression

$$f_{DSC\alpha}^{(k)}(z) = \sum_{q=1}^{k-1} a_\alpha^{(q)} (z - z_{(q)}) H^{(q)} \tag{19}$$

where

$$a_\alpha^{(q)} = \left(\frac{C_{\alpha 3}^{\alpha 3(q)}}{C_{\alpha 3}^{\alpha 3(q-1)}} - 1\right)\left(1 + \sum_{j=1}^{q-1} a_\alpha^{(j)}\right) \tag{20}$$

are the ***interface continuity constants***. They are functions only of the transverse shear elastic coefficients.

Note that this first-order zigzag plate theory as five unknowns, as in ESL FSDT! This is the strength of the Di Sciuva's zigzag theory. On the other hand, because of the fulfilment of the continuity conditions the model estimates constant transverse shear stresses across the plate thickness. Thus, it is not able to fulfill the conditions of zero transverse shear stresses on the top and bottom plate surfaces. This is the weakness of this zigzag theory.

6. FIRST-ORDER REFINED ZIGZAG THEORY-RZT

As anticipated in the Introduction, in addition to the advantages highlighted above, and to the following ones (i) no shear correction factors are required, (ii) very accurate distribution of the transverse shear stresses are obtained from the integration of the local equilibrium equations, (iii) finite elements based on the First-order Shear Deformation Zigzag Theory are free from shear locking effects, the Di Sciuva's First-order Shear Deformation Zigzag Theory also suffers some deficiencies due to the fulfillment of the contact conditions of the transverse shear stresses. These weaknesses can be summarized as follows, (i) constant transverse shear stresses distribution along the plate thickness, (ii) classical clamped boundary conditions give zero transverse shear strains and, thus, zero transverse shear stresses at the clamped edges, when these last are obtained from the constitutive equations, (iii) due to the presence of $\partial_\alpha w$ (see, Eqs. (17) and (18)) in the in-plane displacement field, C^1 shape functions are required, in formulating beam, plate and shell finite elements. FSDT requires C^0 shape functions.

The first and third deficiency are specific of the First-order Shear Deformation Zigzag theory, i.e., higher-order zigzag theories does not suffer of these drawbacks. The second apply in the general case. The third deficiency, concerning the need for C^1 finite elements, has been addressed and solved in [93] in the framework of the Third-order Shear Deformation Zigzag theory.

To overcome the first two drawbacks, Tessler, Di Sciuva and Gherlone in a series of recent papers [61,62,63,64,65] advanced a new version of the first-order zigzag theory, named Refined Zigzag Theory-RZT.

Concerning the kinematics, the starting point is the same as in the Di Sciuva's first-order shear deformation zigzag plate theory, that is, Eq.(15), as the same is the global contribution, Eq. (17a) (that of the first-order shear deformation theory) and the local enrichment, that is, a piecewise linear zigzag function, whose starting mathematical expression is

$$\breve{U}_\alpha\left(\mathbf{x}, \zeta^{(k)}\right) = L_B\left(\zeta^{(k)}\right)\breve{U}_{\alpha B}^{(k)}(\mathbf{x}) + L_T\left(\zeta^{(k)}\right)\breve{U}_{\alpha T}^{(k)}(\mathbf{x}) \tag{20}$$

where $\breve{U}^{(k)}_{\alpha B}(\mathbf{x})$ and $\breve{U}^{(k)}_{\alpha T}(\mathbf{x})$ stand for the displacements on the bottom and top surfaces of the layer k, and

$$L_B\left(\zeta^{(k)}\right) = \frac{1}{2}\left(1 - \zeta^{(k)}\right); \quad L_T\left(\zeta^{(k)}\right) = \frac{1}{2}\left(1 + \zeta^{(k)}\right) \qquad -1 \leq \zeta^{(k)} \leq +1 \tag{21}$$

are linear Lagrange's interpolation polynomials.
So, the zigzag enrichment adds 2N unknowns generalized displacements for each co-ordinate directions, $\breve{U}^{(B,k)}_{\alpha}(\mathbf{x})$ and $\breve{U}^{(T,k)}_{\alpha}(\mathbf{x})$.

Let us summarize what has been achieved. For each co-ordinate directions, we have 2N unknown generalized displacements and 2(N-1) constraints (interface continuity conditions). This result in 2 free generalized displacements. In RZT this freedom has been exploited by adding the condition that the local contribution is zero on the top and bottom surfaces of the whole plate, as depicted in figure 7 for a three-layer laminate.

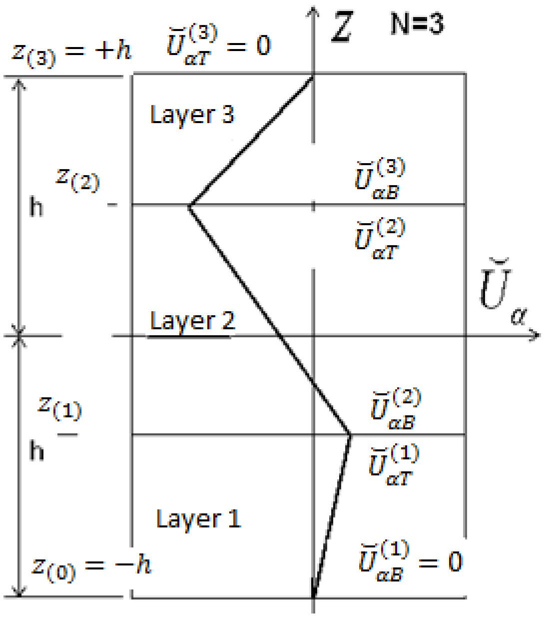


Fig. 7 – Refined Zigzag function for a three-layer laminate.

Where is the difference? The difference is in the expression of the refined zigzag function, $f_{RZT}(z)$.

In an attempt to overcome the drawbacks of Di Sciuva's first-order zigzag theory, i.e., transverse shear stresses are constant over the whole thickness of the plate, transverse shear stresses vanish at clamped end and C^1 shape functions are required in the finite element formulation, in the Refined zigzag theory the continuity conditions on the transverse shear stresses are partially relaxed.

Generalizing the previous results of the Di Sciuva's first-order zigzag theory, FSDZZT, Eq. (17b), RZT starts by writing the local kinematics in the following way

$$\breve{U}^{(k)}_{\alpha}(\mathbf{x}, z) = f^{(k)}_{RZT\alpha}(z)\psi_{\alpha}(\mathbf{x}) \tag{22}$$

where $\psi_\alpha(\mathbf{x})$ is the amplitude of the zigzag function. Note that it is independent of the other kinematic variables, then in RZT the number of independent generalized kinematic variables is greater than that of the ESL FSDT; specifically, they are 4 for the beam and 7 for the plate and shell.
As a consequence, transverse shear stress reads (orthotropic material)

$$\sigma_{\alpha 3}^{(k)} = C_{\alpha 3}^{\alpha 3(k)} \epsilon_{\alpha 3}^{(k)}\left(\mathbf{x}, z^{(k)}\right) = \frac{1}{2} C_{\alpha 3}^{\alpha 3(k)}\left(\gamma_\alpha(\mathbf{x}) + \psi_\alpha(\mathbf{x}) \partial_z f_{RZT\alpha}^{(k)}(z)\right) \tag{23}$$

with γ_α given by Eq. (18). Now we introduce the auxiliary transverse shear strain measure [61],

$$\eta_\alpha(\mathbf{x}) = \gamma_\alpha(\mathbf{x}) - \psi_\alpha(\mathbf{x}) \tag{24}$$

Note that when

$$\eta_\alpha(\mathbf{x}) = 0 \quad \Rightarrow \quad \psi_\alpha(\mathbf{x}) = \gamma_\alpha(\mathbf{x}) \tag{25}$$

and RZT reduces to Di Sciuva's FSDZZT.

Substituting Eq. (24) into Eq. (23), yields

$$\sigma_{\alpha 3}^{(k)} = \frac{1}{2} C_{\alpha 3}^{\alpha 3(k)} \eta_\alpha(\mathbf{x}) + \underline{\frac{1}{2} C_{\alpha 3}^{\alpha 3(k)}\left(1 + \partial_z f_{RZT\alpha}^{(k)}(z)\right)\psi_\alpha(\mathbf{x})} \tag{26}$$

As anticipated, instead to require the transverse shear stress to be continuous at the interfaces, in the Refined Zigzag Theory we require to be continuous only the underlined term in Eq. (26), that is

$$C_{\alpha 3}^{\alpha 3(k)}\left(1 + \partial_z f_{RZT\alpha}^{(k)}(z)\right) \tag{27}$$

Satisfying the interface static continuity condition, Eq. (2), yields

$$\partial_z f_{RZT\alpha}^{(k)}(z) = \frac{G_{\alpha 3}}{C_{\alpha 3}^{\alpha 3(k)}} - 1 \tag{28}$$

As before, there are two free additional conditions to uniquely determine the zigzag function.
As we said, in the RZT the following two additional conditions

$$\breve{U}_\alpha^{(1)}(\mathbf{x}, z_B) = \breve{U}_\alpha^{(N)}(\mathbf{x}, z_T) = 0 \tag{29}$$

are imposed (see, Fig. 7). As a consequence, we obtain

$$\frac{2h}{G_{\alpha 3}} = \int_{-h}^{h} \frac{1}{C_{\alpha 3}^{\alpha 3(k)}} dz = \sum_{k=1}^{N} \frac{2h^{(k)}}{C_{\alpha 3}^{\alpha 3(k)}} \tag{30}$$

Note that the slope of zigzag function depends on the lay-up and on the transverse shear elastic compliance of the layers.

7. FIRST-ORDER MURAKAMI'S ZIGZAG THEORY-MZT

A first-order zigag theory formally identical to the first-order ZZT has been advanced by Murakami [52].

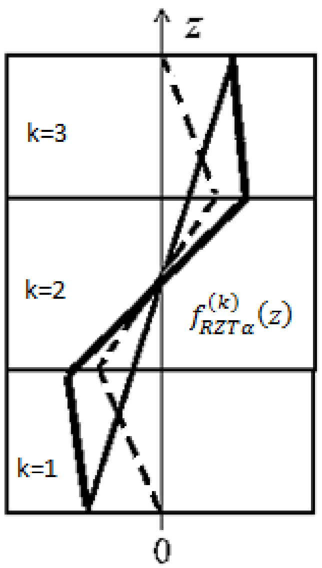
Murakami's zigzag theory (MZT) shares with the RZT the same global displacement field, Eq. (18a) and a formally identical local enrichment, Eq. (22), the difference being in the expression of the zigzag functions. Following Eq. (22), we write the local contribution as

$$\breve{U}_\alpha^{(k)}(\mathbf{x}, z) = f_{MZT\alpha}^{(k)}(z)\psi_\alpha(\mathbf{x}) \tag{31}$$

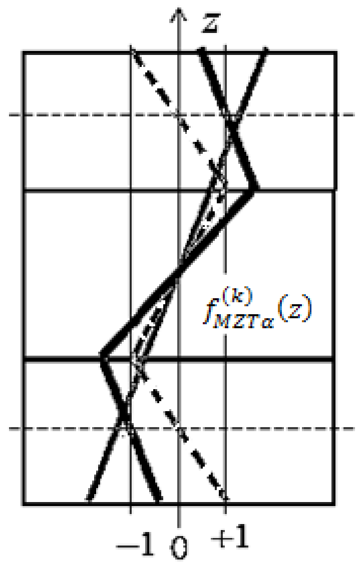
where

$$\partial_z f_{MZT\alpha}^{(k)}(z) = \frac{(-1)^{(k)}}{h^{(k)}} \quad \Rightarrow \quad f_{MZT\alpha}^{(k)}(z) = (-1)^{(k)} \frac{z - z_M^{(k)}}{h^{(k)}} \tag{32}$$

So, RZT and MZT have the same number of independent kinematic variables. From the above, it is evident that Murakami's zigzag function depends only on the thickness of each layer; it is independent on the mechanical properties of layers. Figure 8 gives plots of the global contribution (ESL FSDT), the zigzag function and the resulting kinematics, for RZT and MZT, for a three-layer symmetric laminate.



(a)



(b)

Fig. 8 – Kinematics of (a) RZT and (b) MZT. (———) global contribution (ESL FSDT), (– – –) local contribution (zigzag function), (▬▬) resulting kinematics, for a three-layer symmetric laminate.

8. NUMERICAL RESULTS

In order to substantiate the accuracy of the first-order zigzag theory and show the relative performances of RZT and MZT, in this Section some numerical results are quoted. Further comparisons can be found in [84,85,94,[95].

All the numerical results presented in this Section refer to the bending, under transverse bi-sinusoidal loading, and natural frequencies of symmetric (0/90/core/90/0) square sandwich plates of side a. The faces are comprised of two-layer regular cross-ply laminate (0/90) having thickness equal to 0.1 times the total thickness of the sandwich plate.

Table 1 gives the mechanical properties of unidirectional lamina with respect to the corresponding mechanical properties of core, the geometry and the boundary conditions of the sandwich plates under investigation.

	$E_1^{(f)}/E_1^{(c)}$	$E_2^{(f)}/E_2^{(c)}$	$a/2h$	$t^{(c)}/t^{(f)}$	Boundary conditions
Sandwich (1)	$5x10^5$	$1x10^5$	6	8	Simply supported
Sandwich (2)	$4.8x10^2$	$1.2x10$	10	8	Fully clamped

Table 1- Geometry, material data and boundary conditions of the sandwich plates.

8.1 Bending of sandwich plate

Figures 8 and 9 refer to sandwich plate (1) under transverse sinusoidal loading

$$\bar{p}_z^{(T)}(\mathbf{x}) = \bar{p}_0 \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a} \tag{33}$$

Numerical results as estimated by the 3D theory of elasticity [4], [5], the ESL FSDT, the RZT, and the MZT are compared.

Figure 9 shows estimates for the normalized transverse deflection at the center of the plate as a function of the span-to-thickness ratio a/2h and figure 10 shows the normalized in-plane displacements and transverse shear stresses. With the exception of 3D, the transverse shear stresses are evaluated by integration of the 3D local equilibrium equations. Concerning shear correction factors, as noted in the Introduction, no shear correction factors have been used in RZT and MZT, i.e., $k_x^2 = k_y^2 = 1$. In the ESL FSDT, in addition to the case, $k_x^2 = k_y^2 = 1$, ad hoc shear correction factors are used, $k_x^2 = 0.0242, k_y^2 = 0.0225$. They have been evaluated following the approach proposed by Ferreira [98]. It is of interest to note that these values are very different from those usually adopted in ESL FSDT, $k = \frac{5}{6};\ 1;\ \frac{\pi^2}{12}$. Moreover, in order to enhance some weakness of the MZT, the unidirectional orthotropic lamina is assumed to have equal transverse shear stiffness in the principal orthotropic directions 1 and 2, that is, $G_{13} = G_{23}$.

It is concluded that, (i) RZT is very accurate in predicting global (transverse displacement) and local (thickness-wise distribution) of displacement and normal in-plane stresses (not shown in figure). Even when considering the integrated shear stresses, clearly emerges the greater accuracy of the predictions made by RZT than MZT and ESL FSDT. Moreover, these results show that (i) ESL FSDT improves its accuracy if a suitable shear correction factor is adopted (fig. 9); (ii) MZT predicts the same deflections of the FSDT model with unitary shear correction factors (fig. 9).

A point of interest is the behavior of MZT for this specific sandiwch construction. As we said, the orthotropic unidirectional lamina constituting the faces has $G_{13} = G_{23}$. This means that the faces are free of zigzag effect (that is, the actual in-plane displacement doesn't change slope in the faces), while the zigzag effect will be present at the interface between core and face. This behavior is very well caught by RZT theory, while MZT completely fails (fig. 10, top). What happens is this. Having to vanish the zigzag effect in the case of two adjacent layers having the same transverse shear stiffness (this is the case for the faces in our sandwich), since the MZZ functions are independent of the transverse shear stiffness of the layers (see, Eq. (32)), in order to vanish the zigzag effect in the faces, the amplitude of the zigzag function, i.e., $\psi_{\alpha}(\mathbf{x})$ in Eq. (31), should be zero. This implies zero contribution of the zigzag effect on the whole thickness and, as a consequence, the same trend of the in-plane displacements provided by ESL FSDT (Fig. 10, top). This deficiency also affects the distribution prediction of transverse shear stresses (Fig. 10, bottom).

8.2 Natural frequencies of square sandwich plate

Aim of this investigation is to assess the reliability of the RZT changing the problem under investigation. Here we investigate the natural frequencies of sandwich plate (2), which is different from sandwich plate (1) for material characteristics of the faces and core, aspect ratio and boundary conditions (table 1). Note that in this sandwich the core is stiffer than sandwich (1) and the plate is fully clamped. As a consequence, the ad hoc shear correction factors are changed. They are $k_x^2 = 0.0825, k_y^2 = 0.1446$.

Figure 11 show the estimated free undamped frequencies for the lowest six modes (m is the number of half waves in the x_1 direction and n that in the x_2 direction). As for the bending problem of sandwich (1), RZT provides estimates very close to the 3D elasticity for all modes. FSDT with ad-hoc shear correction factors is also accurate, even if its accuracy decreases for higher modes (fifth and sixth), indicating the need to introduce other values of the correction factors as the modal number increases.

9. CONCLUDING REMARKS

Aim of this paper is to give a review of the first-order displacement-based zigzag theories for composite laminates and sandwich structures, with special emphasis to the underlying ideas, relative strengths and weaknesses. To do this, the Di Sciuva's first-order zigzag theory, the first-order Refined Zigzag Theory, and the Murakami's first-order zigzag theory have been reviewed and numerically compared. It is shown that RZT and Murakami's Zigzag theory share the same kinematics, but different zigzag functions. Specifically, unlike RZT zigzag functions, Murakami's zigzag functions are basically of geometrical type, in the sense that the jump in their thickness-wise derivative depends only on the thickness of each layer (see, Eq. (32)), when the same jump in the derivative of the RZT zigzag functions depends on the lay-up and on the transverse shear elastic compliance of the layers (see, Eq. (28)). This makes Murakami's zigzag functions approach generally unreliable, especially when the laminate does not present a repetitive or periodic lay-up, as pointed out by the same Murakami [52] and recently also by Gherlone [84], Iurlaro et al [85], [94] and Iurlaro [95]. In extreme cases, like that of sandwich taken used for the numerical assessment presented in this paper, the Murakami's zigzag approach completely fails, in the sense that Murakami's zigzag functions do not give any contribution and Murakami's approach provides the same incorrect results of FSDT with unitary shear correction factors. On the contrary, also for this sandwich plates, RZT provides a very accurate estimates.

From the present numerical investigation and others available in the open literature [61, 62,63,64,65,66,67], it seems reasonable to draw the conclusion that RZT is very accurate in predicting global (transverse displacement) and local (thickness-wise distribution) of displacement and normal in-plane stresses.

By taking into account the additional strength of RZT, mainly, the possibility to develop C^0 beam [97,98,99,100,101,102,103], plate and shell finite elements, [104,105,106,107,108], like ESL FSDT, in my opinion RZT represents, among the first-order zigzag theories, the best compromise between computational efficiency and numerical accuracy.

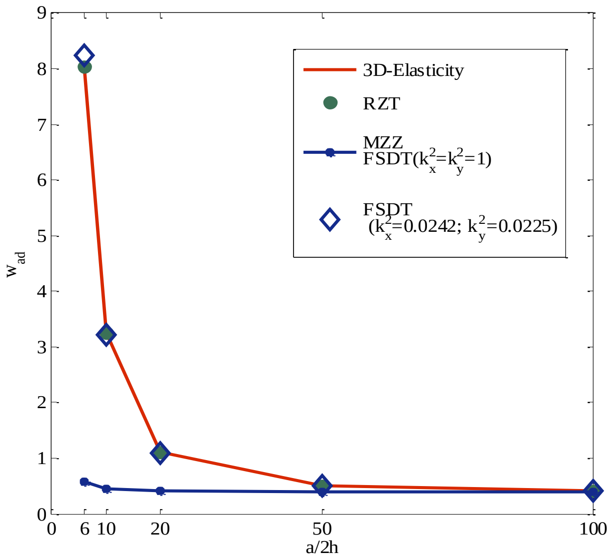


Fig. 9 - Normalized transverse deflection at the center of the plate as a function of the span-to-thickness ratio a/2h.

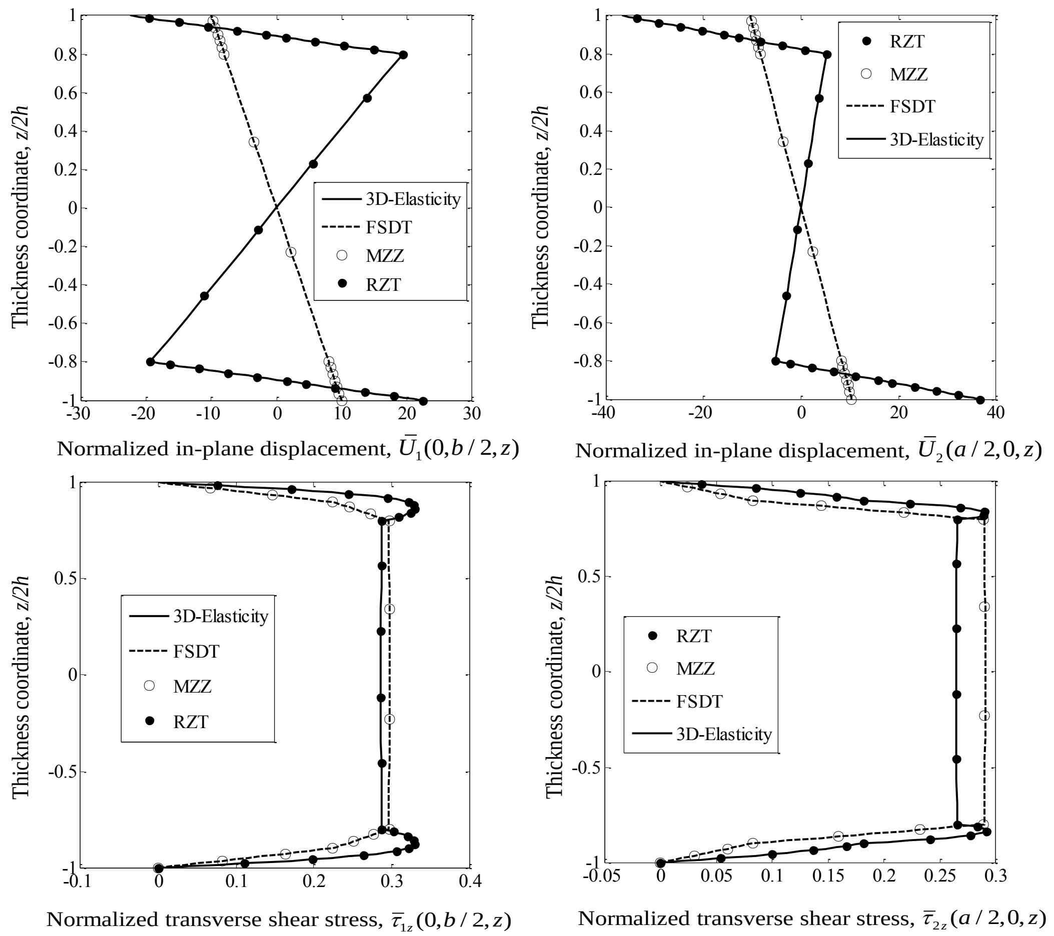


Fig. 10 – Normalized in-plane and transverse shear stresses. With the exception of 3D, the transverse shear stresses are evaluated by integration the 3D local equilibrium equations.

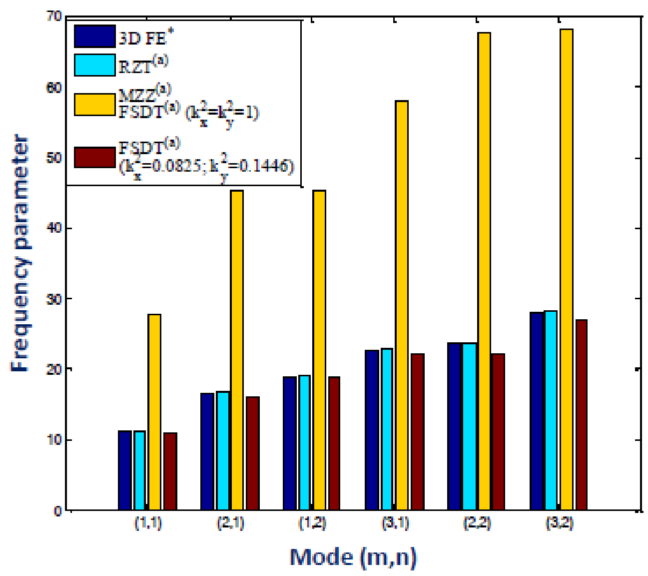


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Fig. 11 – Natural frequencies for sandwich plate (2).

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