Real rank of monomials

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Abstract

In this paper we give an upperbound for the real Waring rank of monomials and we determine the real rank of monomials which have at least one of their exponents equal to one. Moreover, we characterize ternary monomials for which the real rank equals the complex rank.

1 Introduction

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_0, \ldots, x_n]$ be the ring of polynomials with coefficients in $\mathbb{K}$. Let $F \in S$ be a form – i.e. a homogeneous polynomial – of degree d. The Waring problem for F asks for the least integer s such that there exist s linear forms $L_1, \ldots, L_s$ over $\mathbb{K}$ with

$$F = \sum_{i=1}^{s} c_i L_i^d, \text{ where } c_i \in \mathbb{K};$$

such an s is called the *Waring rank* over $\mathbb{K}$ of the form F, denoted by $\mathrm{rk}_{\mathbb{K}}(F)$.

The Waring problem is a very classical question whose study goes back to the work of Sylvester [17] and other geometers and algebraists of the XIX century; see [14] for historical details. Even if it has a long history, it was just recently that the Waring problem for general forms over the complex numbers was solved by Alexander and Hirschowitz [1]. The computation of the Waring rank for specific polynomials is more difficult and, in general, still unknown. The case of complex binary forms goes back to Sylvester, see [10]. More recently, some progress has been made: the complex Waring rank of monomials (and sums of pairwise coprime monomials) have been determined in [8] and the complex rank of other isolated families of polynomials have been computed in [7]. Some algorithm have been proposed, but they require technical restrictions to compute the rank, see [2, 5, 11, 16].

The Waring problem is generalized to the setting of tensors, where the question is to minimally decompose a tensor as a sum of rank 1-tensors. Tensors and their decompositions are very interesting for applications to signal processing, complexity of matrix multiplications, **P** vs. **NP** problem; see [15].

While the Waring problem over the complex numbers is witnessing some progress, the corresponding questions over the real numbers are very difficult, yet very interesting for the aforementioned applications. The real Waring rank of bivariate monomials was fully studied in [4]. Another direction of research has been focused on determining the typical real ranks of forms for instance in [12] and [3].

The aim of the present paper is to make progress on computing the real ranks of given forms; specifically our objective is to relate complex and real rank of monomials. In particular, we conjecture the following.

Conjecture 1. *Let $M = x_0^{a_0} \cdots x_n^{a_n}$ be a degree d monomial with $a_0 \leq \ldots \leq a_n$. Then,*

$$\mathrm{rk}_{\mathbb{R}}(M) = \mathrm{rk}_{\mathbb{C}}(M) \text{ if and only if } a_0 = 1.$$

First, we give an upper bound for the real rank of monomials (Theorem 3.4). This allows us to prove that having one exponent equal to one is a sufficient condition for a monomial to have real rank equal to the complex rank.

In Theorem 3.8, we prove Conjecture 1 in the case of three variables. Our method involve the use of Elimination Theory and therefore doesn't generalize easily to the case of more variables. However, we present a family of monomials in any number of variables, with exponents at least two, for which the real rank is strictly bigger than the complex rank.

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2 Background

In this section we recall the notation and results we need in the rest of the paper.

Let $T = \mathbb{K}[y_0, \ldots, y_n]$ be the ring of polynomial differential operators acting by differentiation on S, i.e.

$$y_i \circ x_j := \frac{\partial}{\partial x_i} x_j.$$

This extends linearly to all the elements $\partial \in T$; for simplicity we omit $\circ$.

Definition 2.1. Let $F \in S$ be a form of degree d. We define the *annihilator*, or *perp ideal*, of F as follows

$$F^{\perp} := \{\partial \in T \mid \partial F = 0\}.$$

The apolar ideals of a given form F gives information about the Waring problem:

Lemma 2.2 (**Apolarity Lemma**, [14, **Lemma** 1.15])**.** *Let $F \in S$ be a form of degree d. The Waring rank of F is s, i.e. there exist s linear forms $L_1, \ldots, L_s$ such that $F = \sum_{i=1}^{s} L_i^d$ if and only if there exists an ideal of s reduced points $I_{\mathbb{X}} \subset F^{\perp}$. Moreover these s points are dual to the linear forms L_i's.*

A set of reduced points $\mathbb{X}$ in $\mathbb{P}^n$ whose ideal $I_{\mathbb{X}}$ is contained in $F^{\perp}$ is called *apolar*.

The Waring rank over $\mathbb{C}$ of monomials was solved essentially using the Apolarity Lemma.

Theorem 2.3 ([8, **Corollary** 3.3])**.** *Let* $M = x_0^{a_0} x_1^{a_1} \dots x_n^{a_n}$ *with* $0 < a_0 \leq a_1 \leq \dots \leq a_n$. *Then*

$$\mathrm{rk}_{\mathbb{C}}(M) = \frac{1}{a_0+1} \prod_{i=0}^{n} (a_i + 1).$$

We recall the following result from [6] characterizing the ideals of apolar sets of points giving minimal Waring decomposition. We use it in the next section.

Proposition 2.4 ([6, **Proposition** 21])**.** *Let* $F \in S$ *be a form with* $\mathrm{rk}_{\mathbb{C}}(F) = r$. *Then, any ideal* $I_{\mathbb{X}}$ *of an apolar set of* r *points is a complete intersection.*

3 Results

In this section we prove our main results. As mentioned above, the real rank of binary monomials has been computed in [4].

Proposition 3.1 ([4], **Proposition** 4.4)**.** *Let* $M = x^a y^b$. *Then,* $\mathrm{rk}_{\mathbb{R}}(M) = a + b$.

Remark 3.2. Let $M = x^a y^b$ and assume that $a \leq b$. By [8], we know that $\mathrm{rk}_{\mathbb{C}}(x^a y^b) = b + 1$. Hence, from the theorem above, we get that Conjecture 1 holds for binary forms.

The main tool used in [4] is the following application of Descartes' rule of signs.

Lemma 3.3 ([4, **Lemma** 4.1-4.2])**.** *Let* F *be a real polynomial*

$$F = x^d + c_1 x^{d-1} + \dots + c_{d-1} x + c_d.$$

Then, for any $i < d$, *there exists a choice of the coefficients* c_j*'s such that* F *has* d *distinct real roots and such that* $c_i = 0$.

This allows us to give a first upper bound for the real rank of monomials.

Theorem 3.4. *Let* $M = x_0^{a_0} \dots x_n^{a_n}$ *with* $0 < a_0 \leq \dots \leq a_n$, *then*

$$\mathrm{rk}_{\mathbb{R}}(M) \leq \frac{1}{2a_0} \prod_{i=0}^{n} (a_i + a_0).$$

Proof. The perp ideal is $M^{\perp} = (x_0^{a_0+1}, \dots, x_n^{a_n+1})$. Then, we can consider the following forms

$$\begin{aligned}
G_1 &= x_0^{a_0+1} g_1(x_0, x_1) + x_1^{a_1+1} h_1(x_0, x_1) \\
G_2 &= x_0^{a_0+1} g_2(x_0, x_2) + x_2^{a_2+1} h_2(x_0, x_2) \\
&\vdots \\
G_n &= x_0^{a_0+1} g_n(x_0, x_n) + x_n^{a_n+1} h_n(x_0, x_n),
\end{aligned}$$

where the $\deg g_i = a_i - 1$ and $\deg h_i = a_0 - 1$, for any $i = 1, \dots, n$.

Each G_i is a binary form of degree $a_0 + a_i$ where the monomial $x_0^{a_0} x_i^{a_i}$ is missing. Thus, by Lemma 3.3, there exists a choice of g_i and h_i such that G_i has $a_0 + a_i$ distinct real roots, say $p_{i,j}$ with $j = 1, \dots, a_0 + a_i$. Therefore, the ideal $(G_1, \dots, G_n) \subset M^\perp$ is the ideal of the following set of distinct points:

$$\mathbb{X} = \left\{ [1 : p_{1,j_1} : \dots : p_{n,j_n}] \;\middle|\; 1 \le j_i \le a_0 + a_i, \quad \text{for } i = 1, \dots, n \right\}.$$

By Apolarity Lemma 2.2, the proof is complete. □

From this upper bound, it follows that the sufficient condition in Conjecture 1 holds.

Corollary 3.5. *Let $M = x_0 x_1^{a_1} \dots x_n^{a_n}$, then the real rank of M equals its complex rank.*

Proof. Since $a_0 = 1$, the upper bound given by Theorem 3.4 equals the complex rank of M. □

Remark 3.6. Moreover, for $M = x_0 x_1^{a_1} \dots x_n^{a_n}$, we can give a family of real decompositions of M. The perp ideal is $M^\perp = (x_0^2, x_1^{a_1+1}, \dots, x_n^{a_n+1})$. Consider a set of real numbers

$$\left\{ p_{i,j} \in \mathbb{R} \;\middle|\; i = 1, \dots, n \text{ and } j = 1, \dots, a_i + 1 \right\},$$

such that

$$\sum_{j=1}^{a_i+1} p_{i,j} = 0, \text{ for any } i,$$

and, for each i, the $p_{i,j}$'s are all distinct. Hence, we have a set $\mathbb{X}$ of $(a_1 + 1) \cdot \ldots \cdot (a_n + 1)$ distinct reduced points in $\mathbb{P}^n$ given by

$$\mathbb{X} = \left\{ [1 : p_{1,j_1} : \dots : p_{n,j_n}] \;\middle|\; 1 \le j_i \le a_i + 1, \; i = 1, \dots, n \right\}.$$

We define the forms

$$G_i = \prod_{j=1}^{a_i+1} (x_i - p_{i,j} x_0), \text{ for } i = 1, \dots, n.$$

Since $\sum_{j=1}^{a_i+1} p_{i,j} = 0$, the G_i's are in the perp ideal of M and they define an ideal of $(a_1 + 1) \cdot \ldots \cdot (a_n + 1)$ real distinct points in $M^\perp$ which gives a minimal real Waring decomposition of M.

Example 3.7. É necessario questo esempio? Non mi sembra che spieghi molto di piú che il Remark sopra.

We are now ready to prove Conjecture 1 in the case of three variables.

Theorem 3.8. *Let $M = x_0^{a_0} x_1^{a_1} x_2^{a_2}$ be a monomial with $a_0 \le a_1 \le a_2$. Then $\mathrm{rk}_\mathbb{R}(M) = \mathrm{rk}_\mathbb{C}(M)$ if and only if $a_0 = 1$.*

Proof. If $a_0 = 1$, then the real rank is equal to the complex rank by Corollary 3.5.

Conversely, let us assume $a_0 \geq 2$. Assume by contradiction that $\mathrm{rk}_{\mathbb{C}}(M) = \mathrm{rk}_{\mathbb{R}}(M)$. The perp ideal of M is $M^{\perp} = (x_0^{a_0+1}, x_1^{a_1+1}, x_2^{a_2+1})$. By Proposition 2.4, there exists a complete intersection inside $M^{\perp}$ which admits $(a_1+1)(a_2+1)$ distinct real solutions. Such a complete intersection is given by a system of two polynomial equations of the form

$$F = x_1^{a_1+1} + x_0^{a_0+1} \cdot P = 0$$
$$G = x_2^{a_2+1} + x_0^{a_0+1} \cdot Q = 0,$$

where P and Q are arbitrary homogeneous polynomials of degrees $a_1 - a_0$ and $a_2 - a_0$ in $\mathbb{C}[x_0, x_1, x_2]$ respectively.

We want to show that for any choice of P, Q there always exists a non real solution of the system, thus contradicting the assumption. From [9], we know that every point of a minimal apolar set of M does not lie on the hyperplane $y_0 = 0$, then we may dehomogenize the equations by x_0; for the sake of simplicity we continue to denote the dehomogenized polynomials by F and G. Now, the idea is to compute $\mathrm{Res}(F, G, x_1)$, namely the resultant of F and G with respect to x_1. The resultant is the determinant of the Sylvester matrix

$$\mathrm{Sylv}(F, G, x_1) = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots & \dots \\ \vdots & \vdots & 1 & \dots & \dots & \dots \\ \vdots & \vdots & \dots & x_2^{a_2+1} + (\text{lower terms}) & \vdots & \vdots \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ \vdots & \vdots & \dots & \dots & \dots & x_2^{a_2+1} + (\text{lower terms}) \end{pmatrix},$$

where the number of 1's is the degree in x_1 of Q; the $a_1 + 1$ polynomials in x_2 along the main diagonal are of the form $x_2^{a_2+1} + (\text{lower terms})$, where the lower degree part has degree at most $a_2 - a_0$ in x_2. Since every other term has degree at most $a_2 - a_0$ in x_2, the resultant is a polynomial in $\mathbb{C}[x_2]$ of the form $\mathrm{Res}(F, G, x_1) = x_2^{(a_1+1)(a_2+1)} + (\text{lower terms})$, where the lower degree part has degree at most $(a_1+1)(a_2+1) - a_0 - 1$. Since $a_0 \geq 2$, by Descartes' rule we have that $\mathrm{Res}(F, G, x_1)$ cannot admit all real solutions. By [[13], Proposition 3, pag 164] and [[13], Exercise 10, pag 167] all the solutions of $\mathrm{Res}(F, G, x_1)$ can be lifted to solutions of the system of polynomial equations $F = G = 0$. This is a contradiction and hence the proof is complete. □

In the following result, we show the upperbound given in Proposition 3.4 is not sharp.

Proposition 3.9. *Let* $M = x_0^2 \dots x_n^2$, *then* $\mathrm{rk}_{\mathbb{R}}(M) \leq (3^{n+1} - 1)/2$.

Proof. We explicitly give an apolar set of points for M as follows. For any $i = 0, \dots, n$, let us consider the set

$$\mathbb{X}_i = \Big\{ [p_0 : \dots : p_{i-1} : 1 : p_{i+1} : \dots : p_n] \in \mathbb{P}^n \mid p_i \in \{0, \pm 1\} \Big\}.$$

We can easily determine the cardinality of $\mathbb{X} = \bigcup_{i=0}^{n} \mathbb{X}_i$. From all n-tuples $(p_0, \dots, p_n)$ with $p_i = 0, \pm 1$, we need to discard $(0, \dots, 0)$, since it does not give any point in the projective space. We are double counting, since $(p_0, \dots, p_n)$ and $(-p_0, \dots, -p_n)$ define the same point in the projective space. Thus $|\mathbb{X}| = (3^{n+1} - 1)/2$.

Now, we can give a real decomposition of M using this set of points. For each point $P = [p_0 : \dots : p_n]$, we denote with L_P the corresponding linear form $p_0x_0 + \dots + p_nx_n$; moreover, for each point $P \in \mathbb{X}$, we denote with $n(P)$ the number of entries different from 0. Hence, for each $i = 1, \dots, n+1$, we set

$$R_i = \sum_{\substack{P \in \mathbb{X} \\ n(P)=i}} L_P^{2n+2}.$$

By direct computation, we obtain

$$\frac{(2n+2)!}{2} x_0^2 \dots x_n^2 = \sum_{i=1}^{n+1} (-2)^{n+1-i} R_i.$$

□

Example 3.10. For $n = 1$, we have the following real decomposition of $M = x_0^2x_1^2$:

$$12x_0^2x_1^2 = R_2 - 2R_1 = (x_0 + x_1)^4 + (x_0 - x_1)^4 - 2(x_0^4 + x_1^4).$$

For $n = 2$, we have that $\mathrm{rk}_{\mathbb{C}}(x_0^2x_1^2x_2^2) = 9$ and $10 \leq \mathrm{rk}_{\mathbb{R}}(x_0^2x_1^2x_2^2) \leq 13$ by the real decomposition:

$$\begin{gathered} 360x_0^2x_1^2x_2^2 = R_3 - 2R_2 + 4R_1 = \\ = (x_0+x_1+x_2)^6 + (x_0+x_1-x_2)^6 + (x_0-x_1+x_2)^6 + (x_0-x_1-x_2)^6 + \\ -2[(x_0+x_1)^6 + (x_0-x_1)^6 + (x_0+x_2)^6 + (x_0-x_2)^6 + (x_1+x_2)^6 + (x_1-x_2)^6] + \\ +4(x_0^6 + x_1^6 + x_2^6). \end{gathered}$$

On the other hand, we have that the monomials considered in the latter proposition have real rank strictly bigger than the complex rank.

Proposition 3.11. *Let $M = x_0^a \cdots x_n^a$ with $a \geq 2$. Then, $\mathrm{rk}_{\mathbb{R}}(M) > \mathrm{rk}_{\mathbb{C}}(M)$.*

Proof. The perp ideal of M is $M^\perp = (x_0^{a+1}, \dots, x_n^{a+1})$ and its complex rank is $\mathrm{rk}_{\mathbb{C}}M = (a+1)^n$. Similarly as before, by **??**, if we assume that the real rank is equal to the complex rank, we have to look at ideals of apolar sets of points given by complete intersections, namely sets of points that are intersection of n forms of degree $a+1$ in $M^\perp$. Hence we have

$$F_i = \sum_{j=1}^{n} \alpha_{i,j} x_j^{d+1} \text{ for } i = 1, \dots, n.$$

The system of polynomial equations $F_1 = \dots = F_n = 0$ reduces to one equation of the form

$$\beta x_1^{d+1} + \gamma x_2^{d+1} = 0,$$

whose roots are not all reals. Hence, we have a contradiction. □

Remark 3.12. The last result confirms our Conjecture 1. It is worth to remark that our conjecture is equivalent to a conjecture about *studying how many real roots a specific system of polynomial equations can have.*

Conjecture 2. *Let $a_1, \ldots, a_n$ be positive integers. Then, consider the polynomial system*

$$\begin{cases} x_1^{a_1} + F_1 & = 0 \\ & \vdots \\ x_n^{a_n} + F_n & = 0 \end{cases}$$

where $F_i \in \mathbb{C}[x_1, \ldots, x_n]$. If $\deg(F_i) \leq a_i - 3$, for any $i = 1, \ldots, n$, then the system cannot have $a_1 \cdots a_n$ distinct real solutions.

This conjecture can be theoretically interesting for real and complex analysts, even beside our application of it. We haven't found in the literature any reference going on that direction.

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