Waring loci and the Strassen conjecture

WARING LOCI: STRASSEN'S AND COMON'S CONJECTURES.

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Abstract. Given a homogeneous polynomial of degree d, we define a *Waring decomposition* as an additive decomposition as sum of dth powers of linear polynomials and we say that it is *minimal* if it has the shortest possible length. A homogeneous polynomial may have infinitely many minimal Waring decomposition and we define its *Waring locus* as the set of linear forms that can appear in a minimal Waring decomposition; we call *locus of forbidden points* the complement.

In this paper, we give a complete description of Waring locus (and forbidden points) for quadrics, monomials, binary forms and plane cubics.

1. Introduction

Let $S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{i\geq 0} S_i$ be the standard graded polynomial ring in $n + 1$ variables and complex coefficients where S_i denotes the $\mathbb{C}$-vector space of degree i homogeneous polynomials, or *forms*.

A *Waring decomposition of* $F \in S_d$ is an expression of the form

$$F = L_1^d + \ldots + L_r^d,$$

for linear forms $L_i \in S_1$. The *Waring rank*, or simply *rank*, of F is

$$\mathrm{rk}(F) := \min\{r : F = L_1^d + \ldots + L_r^d,\ L_i \in S_1 \text{ for } 1 \leq i \leq r\}.$$

In the last decades, there is been a lot of work trying to compute Waring ranks and (minimal) Waring decompositions of homoegeneous polynomials. The main result in this subject is due to J.Alexander and A.Hirschowitz who determined the rank of a generic form [AH95].

This attention is mostly due to the relations with the theory of symmetric tensors and their decompositions as sums of rank 1 tensors which have applications in Algebraic Statistics, Biology, Quantum Field Theory and more, e.g. see [Lan12].

In this paper, we investigate the possible minimal Waring decompositions of a given form F, namely the Waring decompositions where the length is equal to the rank of F. Of particular interest are the cases when the minimal Waring decomposition is *unique*, called in the literature the *identifiable* cases, e.g. see [CC06, BCO14, COV15].

Definition 1. Given a form F, we define the *Waring locus* of F as the set of linear forms that appear in a minimal Waring decomposition of F, namely

$$\mathcal{W}_F := \{[L] \in \mathbb{P}(S_1) : \exists L_2, \ldots, L_r \in S_1,\ F = L^d + L_2^d + \ldots + L_r^d,\ r = \mathrm{rk}(F)\};$$

and we define the *locus of forbidden points* as its complement, namely the set of linear forms that *cannot* appear in a minimal Waring decomposition of F,

$$\mathcal{F}_F := \mathbb{P}(S_1) \setminus \mathcal{W}_F.$$

Although their approach and their results have a different nature, in [BC13], the authors started to point out the importance of the study of the Waring loci of homogeneous polynomials.

From our definitions, it is clear that in the identifiable cases, or when a form has finitely many minimal Waring decompositions, we have Waring locus equal to a finite set of points; hence, our main contribution is for forms which admit infinitely many minimal Waring decomposition.

In these cases, it is more interesting to describe the structure of the forbidden points and we have been able to give a complete answer in the following cases:

(1) quadrics, i.e. degree 2 forms, see Proposition 3.1;
(2) monomials, see Theorem 3.2;
(3) binary forms, i.e. two variable, see Theorem 3.3;
(4) plane conics, i.e. degree 3 forms in three variables, see Section 3.4.

In Section 2, we introduce the basics facts and our main tool: the *Apolarity Lemma*, Lemma 2.1. Itprovides a very explicit receipt to find Waring decompositions of an homogeneous polynomials F; in particular, it states that Waring decompositions of F corresponds to ideals of reduced points contained in the *perp ideal* $F^\perp$, namely the ideal of polynomials annihilating F by acting as differentials. The reason why we have been able to succeed in the computation of forbidden points in the cases listed above is that those are the cases when we can give a very precise description of the perp ideals and then look for all the possible (minimal) set of reduced points contained in them.

In Section 3, we explain in details all our computations and results.

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2. Basics

We introduce first the basic notions on *Apolarity theory* that we used in our computations; for an extended explanation see also [IK99, Ger96].

We consider two polynomial rings $S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{i\geq 0} S_i$ and $T = \mathbb{C}[X_0, \ldots, X_n] = \bigoplus_{i\geq 0} T_i$ with standard gradation, where S has the structure of a T-module via differentiation; namely, we consider the *apolarity action* given by

$$g \circ F := g(\partial_{x_0}, \ldots, \partial_{x_n})F, \quad \text{for } g \in T,\ F \in S.$$

In particular, we define the *perp ideal of* $F \in S_d$ is

$$F^\perp = \{\partial \in T : \partial \circ F = 0\}.$$

We say that $F \in \mathbb{C}[y_0, \ldots, y_m]$ *essentially involves* $n+1$ variables if $\dim(F^\perp)_1 = m - n$. In other words, if F essentially involves $n+1$ variables, there exist linear forms $l_0, \ldots, l_n \in \mathbb{C}[y_0, \ldots, y_m]$ such that $F \in \mathbb{C}[l_0, \ldots, l_n] \simeq S$.

We are interest in finding the *minimal* Waring decompositions of a form $F \in S_d$ and, as already mentioned before, the main tool is the following.

Lemma 2.1 (Apolarity Lemma). *Let F be a degree d form. The following are equivalent*

(1) *there exists a set of reduced points $\mathbb{X} \subset \mathbb{P}^n$ such that $I_\mathbb{X} \subset F^\perp$ and $|\mathbb{X}| = s$;*
(2) *there exists a set of s linear forms $L_1, \ldots, L_s \in S_1$ such that $F = L_1^d + \ldots + L_s^d$.*

A set of reduced points $\mathbb{X} \subset \mathbb{P}^n$ such that the first condition of Apolarity Lemma holds is said to be *apolar to* F. Moreover, given a apolar set, say $\mathbb{X} = P_1 + \ldots + P_s$ where $P_i = [p_{i,0} : \ldots : p_{i,n}] \in \mathbb{P}^n$, we have that a Waring decomposition of F is given by the linear forms $L_i := p_{i,0}x_0 + \ldots + p_{i,n}x_n$.

Example 1. Consider the monomial $M = xyz \in \mathbb{C}[x, y, z]$. It is easy to check that $M^\perp = (X^2, Y^2, Z^2)$; hence, we can easily find the ideal $I = (X^2 - Y^2, X^2 - Z^2)$ corresponding to the four reduced points $[1 : \pm 1 : \pm 1]$; thus, we have the Waring decomposition of M as

$$M = \frac{1}{24}\left[(x + y + z)^3 - (x - y + z)^3 - (x + y - z)^3 + (x - y - z)^3\right].$$

From the Apolarity, we can describe the Waring locus of a form F in terms of the apolar points to F, namely

$$\mathcal{W}_F = \{P \in \mathbb{P}^n : P \in \mathbb{X},\ I_{\mathbb{X}} \subset F^\perp \text{ and } |\mathbb{X}| = \mathrm{rk}(F)\}.$$

The following result, also mentioned in [BL13] in the case of tensors, allows us to study a form F in the ring of polynomials with the essential number of variables. In particular, we want to show that, if $F \in \mathbb{C}[y_0, \ldots, y_m]$ essentially involves $n + 1$ variables and $\mathbb{X}$ is a minimal set of of points apolar to F, then $\mathbb{X} \subset \mathbb{P}^m$ is contained in a n-dimensional linear subspace of $\mathbb{P}^m$. Hence, $\mathcal{W}_F \subset \mathbb{P}^n$ contains all points belonging to any minimal set of points apolar to F.

Proposition 2.2. *Let $F \in k[x_0, \ldots, x_n, x_{n+1}, \ldots, x_m]$ be a degree d form such that $(F^\perp)_1 = (X_{n+1}, \ldots, X_m)$. If*

$$F = \sum_1^r L_i^d$$

where $r = \mathrm{rk}(F)$ and the L_i are linear forms in $k[x_0, \ldots, x_n, x_{n+1}, \ldots, x_m]$, then

$$L_i \in k[x_0, \ldots, x_n] \subset k[x_0, \ldots, x_n, x_{n+1}, \ldots, x_m]$$

for all $i, 1 \leq i \leq r$.

Proof. We proceed by contradiction. Assume that $L_1 = x_{n+1} + \sum_{i \neq n+1} a_i x_i$, that is to assume that L_1 actually involves the variable x_{n+1}. By assumption $\mathrm{rk}(F - L_1^d) < r = \mathrm{rk}(F)$. However, since L_1 is linearly independent with $x_1, \ldots, x_n$ we can apply the following fact (see [CCC15, Proposition 3.1]): if y is new variable, then

$$\mathrm{rk}(F + y^d) = \mathrm{rk}(F) + 1.$$

Hence, $\mathrm{rk}(F - L_1^d) = \mathrm{rk}(F) + 1$ and this is a contradiction. □

Remark 1. Using the previous result, performing a linear change of variables and restricting the ring, we may always assume that $F \in S_d$ essentially involves $n + 1$ variables; hence, we always interpret $\mathcal{W}_F, \mathcal{F}_F$ as subsets of $\mathbb{P}^n$.

3. Results

In this section, we explain in details our computations and results.

3.1. **Quadrics.** We study first the elements of S_2, i.e. quadrics in $\mathbb{P}^n$. We recall that to each quadric Q we can associate a symmetric $(n+1)\times(n+1)$ matrix A_Q and that $\mathrm{rk}(Q)$ equals the rank of A_Q.

Proposition 3.1. *If $Q \in S_2$ essentially involves $n+1$ variables, then $\mathcal{F}_Q = V(Q) \subset \mathbb{P}^n$.*

Proof. After a change of variables we may assume that $Q = x_0^2 + \ldots + x_n^2$. A point $P = [a_0 : \ldots : a_n]$ is a forbidden point for Q if and only if

$$\mathrm{rk}(Q - \lambda L_P^2) = n+1, \text{ for all } \lambda \in \mathbb{C}$$

where $L_P = \sum_0^n a_i x_i$. Thus, P is a forbidden point for Q if and only if the symmetric matrix corresponding to the quadratic form $Q - \lambda L_P^2$ has non-zero determinant for all $\lambda \in \mathbb{C}$ and therefore, P is a forbidden point if and only if the symmetric matrix A_{L^2} corresponding to L^2 only have zero eigenvalues. Since A_{L^2} is a rank one matrix, A_{L^2} has at most a non-zero eigenvalue. Note that

$$\begin{pmatrix} a_0 & \ldots & a_n \end{pmatrix} A_{L^2} = (a_0^2 + \ldots + a_n^2)\begin{pmatrix} a_0 & \ldots & a_n \end{pmatrix}.$$

Also note that, if $\sum_0^n a_i^2 = 0$, then $A_{L^2}^2 = 0$ and thus zero is the only eigenvalue. Thus $\sum_0^n a_i^2$ is the only possible non-zero eigenvalue of A_{L^2}. Hence, P is a forbidden point if and only if $\sum_0^n a_i^2 = 0$ and the conclusion follows. □

3.2. **Monomials.** In this section, we consider monomials $x_0^{d_0} \ldots x_n^{d_n} \in \mathbb{C}[x_0, \ldots, x_n]$ where we order the exponents as $0 < d_0 = \ldots = d_m < d_{m+1} \leq \ldots \leq d_n$. In [CCG12], the authors proved an explicit formula for the Waring rank of monomials, i.e.

$$\mathrm{rk}(x_0^{d_0} \ldots x_n^{d_n}) = \frac{1}{d_0+1}\prod_{i=0}^{n}(d_i+1).$$

We also know from [BBT13] that minimal sets of apolar points of monomials are complete intersections, namely they are given by the intersection of n hypersurfaces in $\mathbb{P}^n$ of degrees $d_1+1, \ldots, d_n+1$ intersecting properly.

Theorem 3.2. *If $M = x_0^{d_0} \cdots x_n^{d_n} \in S$ with $0 < d_0 = \ldots = d_m < d_{m+1} \leq \ldots \leq d_n$, then*

$$\mathcal{F}_M = V(X_0 \cdot \ldots \cdot X_m) \subset \mathbb{P}^n.$$

Proof. It is easy to check that $M^\perp = (X_0^{d_0+1}, \ldots, X_n^{d_n+1})$. Consider any point $P = [p_0 : \ldots : p_n] \notin V(x_0 \cdot \ldots \cdot x_m)$, we may assume $p_0 = 1$. We construct the following hypersurfaces in $\mathbb{P}^n$, for any $i = 1, \ldots, n$ given by union of $d_i + 1$ hyperplanes, respectively,

$$H_i = \begin{cases} X_i^{d_i+1} - p_i^{d_i+1} X_0^{d_i+1} & \text{if } p_i \neq 0; \\ \\ X_i^{d_i+1} - X_i X_0^{d_i} & \text{if } p_i = 0. \end{cases}$$

The ideal $I = (H_1, \ldots, H_n)$ is contained in $M^\perp$ and $V(I)$ is the set of reduced points $[1 : q_1 : \ldots : q_n]$ where

$$q_i \in \begin{cases} \{\xi_i^j p_i \mid j = 0, \ldots, d_i\}, & \text{if } p_i \neq 0, \text{ where } \xi_i^{d_i+1} = 1; \\ \\ \{\xi_i^j \mid j = 0, \ldots, d_i - 1\} \cup \{0\}, & \text{if } p_i = 0, \text{ where } \xi_i^{d_i} = 1. \end{cases}$$

Thus, we have a set of $\mathrm{rk}(M)$ distinct points apolar to M and containing the point P; hence, $P \in \mathcal{W}_M$. To complete the proof we consider $P \in V(X_0 \cdots X_m)$. In this case, it follows from a trivial generalization of [CCG12, Remark 3.3] that there is no set of points apolar to M and containing P. Hence, $P \in \mathcal{F}_M$. □

Remark 2. In the case $d_0 \geq 2$, the second part of the proof can be explained as a trivial consequence of the formula for the rank of monomials. Indeed, in the same notations as the theorem, for any $i = 1, \ldots, m$, from the formula we have that $\mathrm{rk}(M) = \mathrm{rk}(\partial_{x_i} \circ M)$. Therefore, given any minimal Waring decomposition of $M = \sum_{j=1}^{r} L_j^d$, by differentiating both sides, we need to have $\partial_{x_i} \circ L_j \neq 0$ for all $i = 1, \ldots, m$, which is equivalent to say that $[L_j] \notin V(X_0 \cdot \ldots \cdot X_m)$.

3.3. **Binary forms.** In this section we deal with the case $n = 1$, that is the case of forms in two variables. The knowledge on the Waring rank of binary forms goes back to J.J. Sylvester [Syl51]. In our terminology, we have that, if $F \in \mathbb{C}[x, y]_d$, then $F^\perp = (g_1, g_2)$ and $\deg(g_1) + \deg(g_2) = d + 2$; moreover, if we assume $d_1 = \deg(g_1) \leq d_2 = \deg(g_2)$, then $\mathrm{rk}(F) = d_1$ if g_1 is square free and $\mathrm{rk}(F) = d_2$ otherwise. See [CS11] for a deeper study about rank of binary forms.

Theorem 3.3. *Let F be a degree d binary form and let $g \in F^\perp$ be an element of minimal degree. Then,*

(1) *if* $\mathrm{rk}(F) < \lceil \frac{d+1}{2} \rceil$, *then* $\mathcal{W}_F = V(g)$;
(2) *if* $\mathrm{rk}(F) > \lceil \frac{d+1}{2} \rceil$, *then* $\mathcal{F}_F = V(g)$;
(3) *if* $\mathrm{rk}(F) = \lceil \frac{d+1}{2} \rceil$ *and d is even, then $\mathcal{F}_F$ is finite and not empty;*
if $\mathrm{rk}(F) = \lceil \frac{d+1}{2} \rceil$ *and d is odd, then* $\mathcal{W}_F = V(g)$.

Proof. (1) It is is enough to note that the decomposition of F is unique and the unique apolar set of points is $V(g)$.

(2) As mentioned above, in this case we have that $F^\perp = (g_1, g_2)$, where $d_1 = \deg(g_1) < \deg(g_2) = d_2$, $d_1 + d_2 = d + 2$, g_1 is not square free, and $\mathrm{rk}(F) = d_2$. In particular, g_1 is an element of minimal degree in the perp ideal. We first show that $\mathcal{F}_F \supseteq V(g_1)$. Let $P = V(l) \in V(g_1)$ for some linear form l, that is l divides g_1. We want to show that there is no apolar set of points to F containing P. Thus, it is enough to show that there is no square free element of $F^\perp$ divisible by l. Since g_1 and g_2 have no common factors, and l divides g_1, it follows that the only elements of $F^\perp$ divisible by l are multiple of g_1, thus they are not square free. Hence, $P \in \mathcal{F}_F$. We now prove that $\mathcal{F}_F \subseteq V(g_1)$ by showing that, if $P = V(l) \notin V(g_1)$, then $P \in \mathcal{W}_F$. Note that l does not divide g_1 and consider

$$F^\perp : (l) = (l \circ F)^\perp = (h_1, h_2)$$

where $c_1 = \deg(h_1), c_2 = \deg(h_2)$ and $c_1 + c_2 = d + 1$. Since h_1 is a minimal degree element in $F^\perp$ and l does not divide g_1, we have $h_1 = g_1$ and $c_2 = d_2 - 1$. Thus $\mathrm{rk}(F) = \mathrm{rk}(l \circ F) + 1$. Let h be a degree $d_2 - 1$ square free element in $(l \circ F)^\perp = F^\perp : (l)$. Hence, $P \in V(lh)$ and $V(lh)$ is a set of d_2 points apolar to F.

(3) Let $F^\perp = (g_1, g_2)$, $d_1 = \deg(g_1)$, and $d_2 = \deg(g_2)$. If d is odd, then $d_2 = d_1 + 1$ and $\mathrm{rk}(F) = d_1$; thus g_1 is a square free element of minimal degree and F has a unique apolar set of d_1 distinct points, namely $V(g_1)$. This proves the d odd case. If d is even, then $d_1 = d_2 = \mathrm{rk}(F)$ and F has infinitely apolar sets of $\mathrm{rk}(F)$ distinct points. However, for each $P \in \mathbb{P}^1$ there is a unique set of $\mathrm{rk}(F)$ points (maybe not distinct) apolar to F and containing P. That is, there is a unique element (up to scalar) $g \in (F^\perp)_{d_1}$ vanishing at P. Thus, $P \in \mathcal{F}_F$ if and only if g is not square free. There

are finitely many not square free elements in $(F^\perp)_{d_1}$ since they correspond to the intersection of the line given by $(F^\perp)_{d_1}$ in $\mathbb{P}(T_{d_1})$ with the hypersurface given by the discriminant; note that the line is not contained in the hypersurface since $(F^\perp)_{d_1}$ contains square free elements. □

Remark 3. We can provide a geometric interpretation of Theorem 3.3 for F a degree d binary form of rank d, the maximal possible. In this case, after a change of variables, we can assume $F = xy^{d-1}$. To see geometrically that $[0:1] \in \mathcal{F}_F$, we consider the point $[y^d]$ on the degree d rational normal curve of $\mathbb{P}^d$. Note that $[F]$ belongs to the tangent line to the curve in $[y^d]$. Thus, it is easy to see that there does not exist a hyperplane containing $[F]$ and $[y^d]$ and cutting the rational normal curve in d distinct points. To prove geometrically that $\mathcal{F}_F = \{[0:1]\}$ one can argue using Bertini's theorem. However, for forms of lower rank, we could not find a straightforward geometrical explanation.

We can improve part (3) of Theorem 3.3 for d even adding a genericity assumption.

Proposition 3.4. *Let $d = 2h$. If $F \in S_d$ is a generic form of rank $h+1$, then $\mathcal{F}_F$ is a set of $2h^2$ distinct points.*

Proof. Let $\Delta \subset \mathbb{P}^{h+1}$ be the variety of degree $h+1$ binary forms having at least a factor of multiplicity two. Note that forms having higher degree factors, or more than one repeated factor, form a variety of codimension at least one in Δ. In particular, a generic line L will meet Δ in $\deg \Delta$ distinct points each point corresponding to a form of the type $B_1^2 B_2 \dots B_h$ and B_i is not proportional to B_j if $i \neq j$.

Note that $F^\perp = (g_1, g_2)$ where $\deg(g_1) = \deg(g_2) = h+1$. Since the Grassmannian of lines in $\mathbb{P}^{h+1}$ has dimension $2h$, the form F determines a generic line and viceversa. The non square free elements of $(F^\perp)_{h+1}$ corresponds to $L \cap \Delta$ where L is the line given by $(F^\perp)_{h+1}$. By genericity, $L \cap \Delta$ consists of exactly $\deg(\Delta)$ points each corresponding to a degree $h+1$ form f_i having exactly one repeated factor of multiplicity two. Since $T/F^\perp$ is artinian, then $\gcd(f_i, f_j) = 1$ $(i \neq j)$. Hence,

$$\mathcal{F}_F = \bigcup_i V(f_i)$$

is a set of $h \deg(\Delta)$ distinct points and the result is now proved. □

We can also iterate the use of Theorem 3.3 to construct a Waring decomposition for a given binary form. Let $F \in S_d$ with rank r large enough, so that the Waring decomposition is not unique, we can think of constructing such a decomposition one addend at the time.

From our result, we know that in this case the forbidden locus is a closed subset $\mathcal{F}_F = V(g)$ where g is an element in $F^\perp$ of minimal degree; hence, we can pick any point $[L_1]$ in the open set $\mathbb{P}^1 \setminus V(g)$ to start our Waring decomposition of F. Consider now $F_1 = F - L_1^d$. If the rank of F_1, which is simply one less than the rank of F, is still large enough not to have a unique decomposition, we can proceed in the same way as before. We may observe that $\mathcal{F}_{F_1} = \mathcal{F}_F \cup [L_1]$. Indeed, by Theorem 3.3, $\mathcal{F}_{F_1} = V(g_1)$, where g_1 is an element of minimal degree of $F_1^\perp$. Since $\mathrm{rk}(F_1) = \mathrm{rk}(F) - 1$, we have that $\deg(g_1) = \deg(g) + 1$, in particular it has to be $g_1 = gL_1^\perp$, where $L_1^\perp$ is the linear differential operator annihilating L_1.

Hence, we can continue to construct our decomposition for F by taking any point $[L_2] \in \mathbb{P}^1 \setminus V(g_1)$ and then looking at $F_2 = F - L_1^d - L_2^d$. We can continue this procedure until we get a form F_i with a unique Waring decomposition; namely, until $i = r - \lceil \frac{d+1}{2} \rceil$, if d is odd, and $i = r - \lfloor \frac{d+1}{2} \rfloor$, if d is even. In other words, we have proven the following result.

Proposition 3.5. *Let F be a degree d binary form of rank $r \geq \lceil \frac{d+1}{2} \rceil$; for any choice of distinct $L_1, \ldots, L_s \notin \mathcal{F}_F$, where $s = r - \lceil \frac{d+1}{2} \rceil$, if d is odd, and $s = r - \lfloor \frac{d+1}{2} \rfloor$, if d is even, there exists a unique minimal Waring decomposition for F involving $L_1^d, \ldots, L_s^d$.*

3.4. **Plane cubics.** In this section we describe $\mathcal{W}_F$ (and $\mathcal{F}_F$) for $n = 2$ and $F \in S_3$, that is for plane cubics. For simplicity, we denote $S = \mathbb{C}[x, y, z]$ and $T = \mathbb{C}[X, Y, Z]$.

We use the following characterization of plane cubics adapted from the table given in [LT10].

Type	Description	Normal form	Waring rank	Result
(1)	triple line	x^3	1	Theorem 3.2
(2)	three concurrent lines	$xy(x+y)$	2	Theorem 3.3
(3)	double line + line	x^2y	3	Theorem 3.2
(4)	smooth	$x^3 + y^3 + z^3$	3	Theorem 4.4
(5)	three non-concurrent lines	xyz	4	Theorem 4.4
(6)	line + conic (meeting transversally)	$x(yz + x^2)$	4	Theorem 3.7
(7)	nodal	$xyz - (y+z)^3$	4	Theorem 3.8
(8)	cusp	$x^3 - y^2z$	4	Theorem 4.5
(9)	general smooth ($a^3 \neq -27, 0, 6^3$)	$x^3 + y^3 + z^3 + axyz$	4	Theorem 3.9
(10)	line + tangent conic	$x(xy + z^2)$	5	Theorem 3.10

Note. In case (9), $a^3 \neq 0, 6^3$ so that the rank is actually 4 and $a^3 \neq -27$ for smoothness of the Hessian canonical form [Dol12].

Remark 4. We have already analyzed several cases:
(1),(3),(5): they are monomials and it follows from Theorem 3.2;
(2): these forms can be seen as forms in two variables, hence it follows from Theorem 3.3(3);
(4): smooth plane cubics can be seen as sums of pairwise coprime monomials with high exponents which are analyzed separatly in the next section, see Theorem 4.4;
(8): plane cubic cusps can be seen as the kind of sums of pairwise coprime monomials that we have analyzed in Theorem 4.5.

We now study plane cubics of rank four. First, we need the following lemma.

Lemma 3.6. *Let F be a plane cubic and let $\mathbb{X}$ be a set of four distinct points apolar to F. If $\mathbb{X}$ has exactly three collinear points, then F is a cusp, that is F is of type (8).*

Proof. We can assume that the three collinear points lie on the line defined by X and the the point not on the line is $[1 : 0 : 0]$. Thus, $XY, XZ \in F^\perp$ and $F = x^3 + G(y, z)$. By [CCC15, Proposition 3.1] we have that $\mathrm{rk}(F) = 1 + \mathrm{rk}(G)$ and thus $\mathrm{rk}(G) = 3$. Since all degree three binary cubics of rank three are monomials we get that, after a change of variables, G can be written as LM^2, where $L, M \in \mathbb{C}[x, y, z]$ are linear forms. Hence, $F = x^3 + LM^2$ and this completes the proof. □

Among the rank 4 plane cubics, we have already analyzed the cusps. Now, we consider families (6),(7) and (9). Due to Lemma 3.6, we can actually study these families using the approach described int he following remark.

Remark 5. Let F be a rank four plane cubic which is not a cusp. Since F is not a binary form, $\mathcal{L} = (F^\perp)_2$ is a net of conics and we let $\mathcal{L} = \langle C_1, C_2, C_3 \rangle$. Since F is not a cusp, all set of four points apolar to F are the complete intersection of two conics. Thus, when we look for minimal Waring decomposition of F, we only need to look at pencil of conics contained in $\mathcal{L}$ with four distinct base points.

In particular, fixing a point $P \in \mathbb{P}^2$, we can consider the pencil $\mathcal{L}(-P)$ of plane conics in $\mathcal{L}$ passing through P. If $\mathcal{L}(-P)$ has four distinct base points, then $P \in \mathcal{W}_F$; otherwise, we have that the base locus of $\mathcal{L}(-P)$ is not reduced and $P \in \mathcal{F}_F$. In the plane $\mathbb{P}(\mathcal{L})$, we consider the degree three curve Δ of reducible conics in $\mathcal{L}$. We recall a pencil of conics $\mathcal{L}'$ has four distinct base points, no three of them collinear, if and only if the pencil contains exactly three reducible conics. In conclusion, given a point $P \in \mathbb{P}^2$, we consider the line $\mathbb{P}(\mathcal{L}(-P)) \subset \mathbb{P}(\mathcal{L})$: if the line is a proper secant line of Δ, that is it cuts Δ in three distinct points, we have that $P \in \mathcal{W}_F$; otherwise, $P \in \mathcal{F}_F$. Thus we have to study the dual curve $\check{\Delta} \subset \check{\mathbb{P}}(\mathcal{L})$ of lines not intersecting Δ in three distinct points.

An equation for $\check{\Delta}$ can be found with a careful use of elimination. To explicitly find $\mathcal{F}_F$ we the consider the map:

$$\phi : \mathbb{P}(S_1) \longrightarrow \check{\mathbb{P}}(\mathcal{L})$$

such that $\phi([a : b : c]) = [C_1(a, b, c) : C_2(a, b, c) : C_3(a, b, c)]$. Note that ϕ is defined everywhere and that it is generically $4 : 1$. In particular,

$$\mathcal{F}_F = \phi^{-1}(\check{\Delta}).$$

Theorem 3.7. *If $F = x(yz + x^2)$, then $\mathcal{F}_F = V(XYZ(X^2 - 12YZ))$.*

Proof. Let $\mathcal{L} = (F^\perp)_2$ and let $\mathcal{C}_1 : C_1 = X^2 - 6YZ = 0$, $\mathcal{C}_2 : C_2 = Y^2 = 0$, and $\mathcal{C}_3 : C_3 = Z^2 = 0$ be the conics generating $\mathcal{L}$. In the plane $\mathbb{P}(\mathcal{L})$ with coordinate α, β and γ, let Δ be the cubic of reducible conics in $\mathcal{L}$. By computing we get the following equation for Δ:

$$\det \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & -3\alpha \\ 0 & -3\alpha & \gamma \end{bmatrix} = 0,$$

that is,

$$\alpha\beta\gamma - 9\alpha^3 = 0.$$

In this case, Δ is the union of the conic $\mathcal{C} : 9\alpha^2 - \beta\gamma = 0$ and the secant line $r : \alpha = 0$. The line r corresponds to $\mathcal{L}(-[1 : 0 : 0])$ and then, by Remark 5, we have that $[1 : 0 : 0] \in \mathcal{F}_F$.

By Remark 5, in order to completely describe $\mathcal{F}_F$, we have to study two family of lines in $\mathbb{P}(\mathcal{L})$: the tangents to the conic $\mathcal{C}$ and all the lines passing through the intersection points between the line r and the conic $\mathcal{C}$, that is through the points $[0 : 0 : 1]$ and $[0 : 1 : 0]$. More precisely the point $P = [X : Y : Z]$ is in $\mathcal{F}_F$ if and only if the line L (of the plane $\mathbb{P}(\mathcal{L})$)

$$L : C_1(P)\alpha + C_2(P)\beta + C_3(P)\gamma = 0,$$

that is,

$$L : \alpha(X^2 - 6YZ) + \beta Y^2 + \gamma Z^2 = 0,$$

falls in one of the following cases:

(i) L is tangent to the conic $\mathcal{C} : \beta\gamma - 9\alpha^2 = 0$;
(ii) L passes through the point $[0 : 1 : 0]$;
(iii) L passes through the point $[0 : 0 : 1]$.

In case (ii) and (iii) we get that $Y^2 = 0$ and $Z^2 = 0$, respectively. So $V(YZ) \subset \mathcal{F}_F$.

Now, by assuming $P \notin \{YZ = 0\}$. By an easy computation we get that the line L is tangent to the conic $\mathcal{C}$ if $X^2(X^2 - 12YZ) = 0$.

It follows that $\mathcal{F}_F = V(XYZ(X^2 - 12YZ))$. See Figure 1.

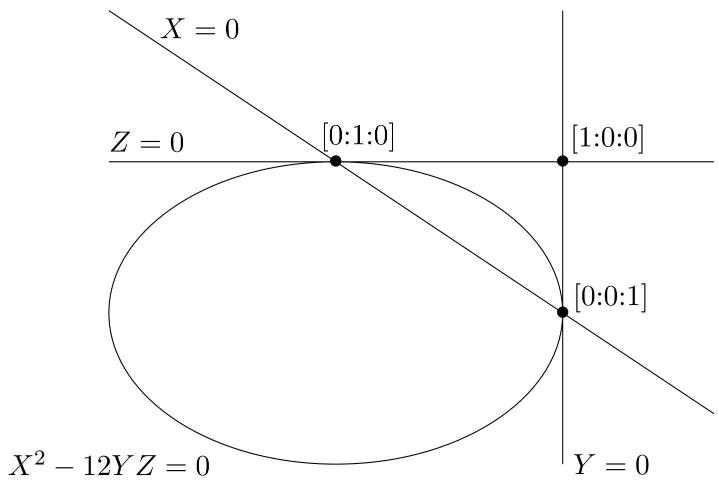

FIGURE 1. The forbidden points of $F = x(yz + x^2)$.

□

We now consider family (7), that is nodal cubics.

Theorem 3.8. *If* $F = y^2z - x^3 - xz^2$, *then*

$$\mathcal{F}_F = V(g_1g_2)$$

where $g_1 = X^3 - 6Y^2Z + 3XZ^2$ *and* $g_2 = 9X^4Y^2 - 4Y^6 - 24XY^4Z - 30X^2Y^2Z^2 + 4X^3Z^3 - 3Y^2Z^4 - 12XZ^5$.

Proof. Note that $[1 : 0 : 0] \in \mathcal{F}_F$. In fact $F + x^3 = z(y^2 - xz)$ represents a conic and a line tangent to it, namely it is in the family (10) and hence it has rank equal to five.

Let $\mathcal{L} = (F^\perp)_2$ and denote by $\mathcal{C}_1 : C_1 = XY = 0$, $\mathcal{C}_2 : C_2 = X^2 - 3Z^2 = 0$ and $\mathcal{C}_3 : C_3 = Y^2 + XZ = 0$ its generators.

In the plane $\mathbb{P}(\mathcal{L})$ with coordinates α, β, and γ let Δ be the cubic of reducible conics in $\mathcal{L}$. By computing we see Δ that is defined by

$$\det \begin{bmatrix} \beta & \frac{1}{2}\alpha & \frac{1}{2}\gamma \\ \frac{1}{2}\alpha & \gamma & 0 \\ \frac{1}{2}\gamma & 0 & -3\beta \end{bmatrix} = 0,$$

that is,

$$3\alpha^2\beta - 12\beta^2\gamma - \gamma^3 = 0.$$

In this case, we have that Δ is an irreducible smooth cubic. Hence, we have that

$$\mathcal{F}_F = \{P \in \mathbb{P}^2 : \mathbb{P}(\mathcal{L}(-P)) \text{ is a tangent line to } \Delta \subset \mathbb{P}(\mathcal{L})\}. \tag{1}$$

Thus, we are looking for points P such that the line

$$C_1(P)\alpha + C_2(P)\beta + C_3(P)\gamma = 0$$

is tangent to Δ. We consider two cases, namely $C_1(P) = 0$ and $C_1(P) \neq 0$.

If $C_1(P) \neq 0$, we compute α from the equation of the line and we substitute in the equation of Δ. Then it is enough to compute the discriminant D of the following form in β and γ

$$3(C_2\beta + C_3\gamma)^2\beta - 12C_1^2\beta^2\gamma - C_1^2\gamma^3$$

and we get $D = 27C_1^4 g_1^2 g_2$. Thus, if $C_1(P) \neq 0$, $P \in \mathcal{F}_F$ if and only if $P \in V(g_1g_2)$.

If $C_1(P) = 0$, by direct computation we check that $\mathcal{F}_F \cap V(C_1) = V(g_1g_2) \cap V(C_1)$. Hence the proof is completed.

□

Remark 6. In this paper we consider $\mathcal{F}_F$, and $\mathcal{W}_F$, as varieties and not as schemes. However, we found that in Theorem3.8 that the ideal of is $(g_1^2g_2)$.

Remark 7. The description of the forbidden locus for a plane cubic given in (1) reminds an old observation made by De Paolis. De Paolis gave an algorithm to construct a decomposition of a general plane cubic as sum of 4 cubes of linear forms whenever starting from a given linear form such that, the line defined by the linear form intersect the Hessian of the plane cubic in precisely three points. This algorithm is been recently recalled in [Ban14].

We now consider the case of cubics in family (9) and we use the map ϕ defined in Remark 5.

Theorem 3.9. *If $F = x^3 + y^3 + z^3 + axyz$ belongs to family (9), then*

(1) *if $\left(\frac{a^3-54}{9a}\right)^3 \neq 27$, then $\mathcal{F}_F = \phi^{-1}(\check{\Delta})$ where $\check{\Delta}$ is the dual curve of the smooth plane cubic*

$$\alpha^3 + \beta^3 + \gamma^3 - \frac{(a^3-54)}{9a}\alpha\beta\gamma = 0;$$

(2) *otherwise, $\mathcal{F}_F$ is the union of three lines pairwise intersecting in three distinct points.*

Proof. Let $\mathcal{L} = (F^\perp)_2$ and denote by $\mathcal{C}_1 : C_1 = aX^2 - 6YZ = 0$, $\mathcal{C}_2 : C_2 = aY^2 - 6XZ = 0$, and $\mathcal{C}_3 : C_3 = aZ^2 - 6XY = 0$ its generators. In the plane $\mathbb{P}(\mathcal{L})$ with coordinates α, β, and γ let Δ be the cubic curve of reducible conics. By computing we get an equation for Δ

$$\det \begin{bmatrix} a\alpha & -3\gamma & -3\beta \\ -3\gamma & a\beta & -3\alpha \\ -3\beta & -3\alpha & a\gamma \end{bmatrix} = (a^3-54)\alpha\beta\gamma - 9a\alpha^3 - 9a\beta^3 - 9a\gamma^3 = 0.$$

In the numerical case $\left(\frac{a^3-54}{9a}\right)^3 \neq 27$, we have that Δ is a smooth cubic curve. Thus, we have that

$$\mathcal{F}_F = \{P \in \mathbb{P}^2 : \mathbb{P}(\mathcal{L}(-P)) \text{ is a tangent line to } \Delta \subset \mathbb{P}(\mathcal{L})\}.$$

Hence we get $\mathcal{F}_F$ as described in Remark 5 using the map ϕ.

Otherwise, Δ is the union of three lines intersecting in three distinct points Q_1, Q_2 and Q_3. Hence,

$$\mathcal{F}_F = \{P \in \mathbb{P}^2 : Q_i \in \mathbb{P}(\mathcal{L}(-P)) \text{ for some } i\}$$

and the proof is now completed. □

Example 2. Consider $a = -6$, thus we are in (1) case of Theorem 3.9. We can compute $\mathcal{F}_F$ using `Macaulay2` [GS]. We get

$$\mathcal{F}_F = V(g_1, g_2),$$

where $g_1 = X^3 + Y^3 - 5XYZ + Z^3$ and $g_2 = 27X^6 - 58X^3Y^3 + 27Y^6 - 18X^4YZ - 18XY^4Z - 109X^2Y^2Z^2 - 58X^3Z^3 - 58Y^3Z^3 - 18XYZ^4 + 27Z^6$.

We conclude with the family (10), that is cubics of rank five.

Theorem 3.10. *If $F = x(xy + z^2)$, then $\mathcal{F}_F = \{[1 : 0 : 0]\}$.*

Proof. Let L be a linear form. The following are equivalent:

(1) $[L] \in \mathcal{F}_f$;
(2) $\mathrm{rk}(F - \lambda L^3) = 5$ for all $\lambda \in \mathbb{C}$;
(3) $F - \lambda L^3 = 0$ is the union of an irreducible conic and a tangent line, for all $\lambda \in \mathbb{C}$;
(4) F and L^3 must have the common factor L, that is, the line $L = 0$ is the line $x = 0$. .

It easy to show that (1) and (2) are equivalent.
For the equivalence between (2) and (3) see the table in Subsection 3.4.
If (3) holds, then all the elements in the linear system given by F and L^3 are reducible; note that the linear system is not composed with a pencil. Thus, by the second Bertini's Theorem, the linear system has the fixed component $x = 0$.
To see that (4) implies (3), note that for all $\lambda \in \mathbb{C}$, the cubic $x(xy + z^2 + \lambda x^2) = 0$ is the union of an irreducible conic and a tangent line.

$\square$

3.5. The forms $x_0^a(x_1^b + \ldots + x_n^b)$ and $x_0^a(x_0^b + x_1^b + \ldots + x_n^b)$.

Proposition 3.11. *Let $F = x_0^a(x_1^b + \ldots + x_n^b)$ and $G = x_0^a(x_0^b + x_1^b + \ldots + x_n^b)$, where $n \geq 2$, $a + 1 \geq b \geq 3$. Then*

$$\mathcal{W}_F = \mathcal{W}_G = V(X_1X_2, X_1X_3, \ldots, X_1X_n, X_2X_3, \ldots, X_2X_n, \ldots, X_{n-1}X_n) \setminus \{[1 : 0 : \ldots : 0]\},$$

that is, the coordinate lines through the point $[1 : 0 : \ldots : 0]$ *minus the point* $[1 : 0 : \ldots : 0]$.

Proof. By Propositions 4.4 and 4.9 in [**?**] we know that

$$\mathrm{rk}(F) = \mathrm{rk}(G) = (a + 1)n.$$

If $[1 : 0 : \ldots : 0] \in \mathcal{W}_F$, we have $\mathrm{rk}(F - \lambda x_0^{a+b}) < (a + 1)n$ for some $\lambda \in \mathbb{C}$. A contradiction, by Propositions 4.9 in [**?**]. Hence $[1 : 0 : \ldots : 0] \in \mathcal{F}_F$.
Analogously, using Propositions 4.4 in [**?**], we get that $[1 : 0 : \ldots : 0] \in \mathcal{F}_G$.
Now let $\partial = \alpha_1 X_1 + \ldots + \alpha_n X_n$, where the $\alpha_i \in \mathbb{C}$ are non-zero, for every i. By Propositions 4.4 and 4.9 in [**?**], we have

$$\mathrm{rk}(\partial \circ F) = \mathrm{rk}(\partial \circ G) = (a + 1)n.$$

Let $I_{\mathbb{X}} \subset F^{\perp}$ be the ideal of a set of points giving a Waring decomposition of F, i.e. the cardinality of $\mathbb{X}$ is equal to $\mathrm{rk}(F)$. Thus, $I_{\mathbb{X}'} = I_{\mathbb{X}} : (\partial)$ is the ideal of the points of $\mathbb{X}$ which are outside the linear space $\partial = 0$. Since

$$I_{\mathbb{X}'} = I_{\mathbb{X}} : (\partial) \subset F^{\perp} : (\partial) = (\partial \circ F)^{\perp},$$

we have that

$$(a + 1)n = \mathrm{rk}(F) = |\mathbb{X}| \geq |\mathbb{X}'| \geq \mathrm{rk}(\partial \circ F) = (a + 1)n.$$

It follows that $\mathbb{X}$ does not have points on the hyperplane $\partial = 0$. Thus

$$\mathcal{W}_F \subseteq V(X_1X_2, X_1X_3, \ldots, X_1X_n, X_2X_3, \ldots, X_2X_n, \ldots, X_{n-1}X_n) \setminus \{[1:0:\ldots:0].$$

The opposite inclusion follows from the proof of Proposition 4.4 in [**?**].
Similarly for G. □

4. Strassen's conjecture

Fix the following notation:

$$S = k[x_{1,0}, \ldots, x_{1,n_1}, \ldots\ldots, x_{m,0}, , \ldots, x_{m,n_m}],$$
$$T = k[X_{1,0}, \ldots, X_{1,n_1}, \ldots\ldots, X_{m,0}, \ldots, X_{m,n_m}].$$

For $i = 1, \ldots, m$, we let

$$S^{[i]} = k[x_{i,0}, \ldots, x_{i,n_i}],$$
$$T^{[i]} = k[X_{i,0}, \ldots, X_{i,n_i}],$$
$$F_i \in S^{[i]}_d,$$

and

$$F = F_1 + \cdots + F_m \in S_d.$$

If we consider $F_i \in S$, then we write

$$F_i^\perp = \{g \in T \mid g \circ F_i = 0\}.$$

On the other hand, if we consider $F_i \in S^{[i]}$, then we also write

$$F_i^\perp = \left\{g \in T^{[i]} \mid g \circ F_i = 0\right\}.$$

Conjecture 1 (Strassen's conjecture)**.** *If $F = \sum_{i=1}^s F_i \in S$ is a form such that $F_i \in S^{[i]}$ for all $i = 1, \ldots, s$, then*

$$\mathrm{rk}(F) = \mathrm{rk}(F_1) + \ldots + \mathrm{rk}(F_s).$$

Conjecture 2. *If $F = \sum_{i=1}^s F_i \in S$ is a degree $d \geq 3$ form such that $F_i \in S^{[i]}$ for all $i = 1, \ldots, s$, then any minimal Waring decomposition of F is a sum of minimal Waring decompositions of the forms F_i.*

In view of Conjecture 2 it is natural to formulate the following conjecture in term of Waring loci. As already explained in Remark 1, we look at $\mathcal{W}_{F_i} \subset \mathbb{P}^{n_i}_{X_{i,0},\ldots,X_{i,n_i}} \subset \mathbb{P}^N$.

Conjecture 3. *If $F = \sum_{i=1}^s F_i \in S$ is a degree $d \geq 3$ form such that $F_i \in S^{[i]}$ for all $i = 1, \ldots, s$, then*

$$\mathcal{W}_F = \bigcup_{i=1,\ldots,r} \mathcal{W}_{F_i} \subset \mathbb{P}^N, \quad \textit{where } N = n_1 + \ldots + n_s + s - 1.$$

Remark 8. Note that Conjectures 2 and 3 are false in degree two. For example, let $F = x^2 - 2yz$. The rank of F is three, but it is easy to find a Waring decomposition of F that is not the sum of x^2 plus a Waring decomposition of the monomial yz; i.e.

$$F = (x+y)^2 + (x+z)^2 - (x+y+z)^2.$$

Lemma 4.1. *Conjecture 2 and Conjecture 3 are equivalent and they imply Strassen's conjecture for $d \geq 3$.*

Proof. Clearly Conjecture 2 implies both Conjecture 1 both Conjecture 3. To complete the proof, we assume that Conjecture 3 holds. If $F = \sum_i L_i^d$ is a minimal decomposition of F, then each L_i appears in a minimal decomposition of $F_{j(i)}$, thus L_i only involves the variables of $S^{[j(i)]}$. Setting all the variables not in $S^{[j(i)]}$ equal zero in the expression $F = \sum_i L_i^d$, we get a decomposition of $F_{j(i)}$. Note that all the obtained decompositions of the F_j are minimal, otherwise $\mathrm{rk}(F) > \sum_i \mathrm{rk}(F_i)$. Hence Conjecture 2 is proved by assuming Conjecture 3. □

In order to study our conjectures we prove the following.

Proposition 4.2. *Let $F = \sum_{i=1}^s F_i \in S$ be a form such that $F_i \in S^{[i]}$ for all $i = 1, \ldots, s$. If the following conditions hold*

(1) *for each $1 \leq i \leq s$ there exists a linear derivation $\partial_i \in T^{[i]}$ such that*
$$\mathrm{rk}(\partial_i \circ F_i) = \mathrm{rk}(F_i),$$
(2) *Strassen's conjecture holds for $F_1 + \ldots + F_s$,*
(3) *Strassen's conjecture holds for $\partial_1 F_1 + \ldots + \partial_s F_s$,*

then F satisfies Conjecture 3.

Proof. Let's consider the linear form $t = \alpha_1\partial_1 + \ldots + \alpha_s\partial_s$, with $\alpha_i \neq 0$ for all $i = 1, \ldots, s$.

Let $I_{\mathbb{X}} \subset F^{\perp}$ be the ideal of a set of points giving a Waring decomposition of F, i.e. the cardinality of $\mathbb{X}$ is equal to $\mathrm{rk}(F)$. Thus, $I_{\mathbb{X}} : (t)$ is the ideal of the points of $\mathbb{X}$ which are outside the linear space $t = 0$. We can look at
$$I_{\mathbb{X}} : (t) \subset F^{\perp} : (t) = (t \circ F)^{\perp}.$$
By the assumptions we get that $\mathrm{rk}(F) = \mathrm{rk}(t \circ F)$, hence the set of points corresponding to $I_{\mathbb{X}} : (t)$ has cardinality equal to $\mathrm{rk}(F)$; it follows that $\mathbb{X}$ does not have points on the hyperplane $t = 0$.

Claim. If $P = [a_{1,0} : \ldots : a_{1,n_1} : \ldots : a_{s,0} : \ldots : a_{s,n_s}]$ belongs to $\mathcal{W}_F$ then in the set $\{a_{1,0}, \ldots, a_{s,0}\}$ there is *exactly* one non-zero coefficient.

The claim follows from the first part, since if we have either no or at least two non-zero coefficients in the set $\{a_{1,0}, \ldots, a_{s,0}\}$ it is easy to find a linear space $\{t = 0\}$ containing the point P and contradicting the assumption that it belongs to the Waring locus of F.

Let's consider $\mathbb{X}_i := \mathbb{X} \smallsetminus \{x_{i,0} = 0\}$, for all $i = 1, \ldots, s$. Similarly as above, by looking at
$$I_{\mathbb{X}_i} = I_{\mathbb{X}} : (\partial_i) \subset F^{\perp} : (\partial_i) = (\partial_i \circ F_i)^{\perp}$$
we can conclude that the cardinality of each $\mathbb{X}_i$ is at least $\mathrm{rk}(F_i)$. Moreover, by the claim, we have that the $\mathbb{X}_i$'s are all distinct. By additivity of the rank, we conclude that
$$\mathbb{X} = \bigcup_{i=1,\ldots,r} \mathbb{X}_i, \quad \text{with } \mathbb{X}_i \cap \mathbb{X}_j = \emptyset, \text{ for all } i \neq j, \quad \text{and} \quad |\mathbb{X}_i| = \mathrm{rk}(F_i), \text{ for all } i = 1, \ldots, s.$$

Hence, we have that the sets $\mathbb{X}_i$ give minimal Waring decompositions of the forms $\partial_i \circ F_i$'s and, by Proposition 1, they lie in $\mathbb{P}^{n_i}_{X_{i,0},\ldots,X_{i,n_i}}$, respectively. Since $\mathbb{X}$ gives a minimal Waring decomposition of F, specializing to zero the variables not in $S^{[i]}$ we see that $\mathbb{X}_i$ gives a minimal Waring decomposition of F_i. Hence, it follows $\mathcal{W}_F \subset \bigcup_{i=1,\ldots,s} \mathcal{W}_{F_i}$.

The other inclusion is trivial. □

There are several family of forms for which we can apply Proposition 4.2 as shown in the following lemma.

Lemma 4.3. *If F is one of the following degree d forms*

(1) *a monomial $x_0^{d_0} \cdot \ldots \cdot x_n^{d_n}$ with $d_i \geq 2$ for $0 \leq i \leq n$;*
(2) *a binary form $F \neq LM^{d-1}$;*
(3) *$x_0^a(x_1^b + \ldots + x_n^b)$ with $n \geq 2$ and $a+1 \geq b > 2$*
(4) *$x_0^a(x_0^b + x_1^b + \ldots + x_n^b)$ with $n \geq 2$ and $a+1 \geq b > 2$*
(5) *$x_0^a G(x_1, \ldots, x_n)$ such that $G^\perp = (g_1, \ldots, g_n)$, $a \geq 2$, and $\deg g_i \geq a+1$*

then there exists a linear derivation ∂ such that

$$\mathrm{rk}(\partial \circ F) = \mathrm{rk}(F).$$

Proof. (1) Let $F = x_0^{d_0} \cdots x_n^{d_n}$ with $d_0 \leq \ldots \leq d_n$, then we know by [CCG12] that $\mathrm{rk}(F) = (d_1+1)\cdots(d_n+1)$. If we let $\partial = X_0$, then $\mathrm{rk}(F) = \mathrm{rk}(\partial \circ F)$.

(2) We know that $F^\perp = (g_1, g_2)$ with $\deg(g_i) = d_i$, $d_1 \leq d_2$ and $d_1 + d_2 = d+2$. We have to consider different cases.

a) If $d_1 < d_2$ and g_1 is square-free, then $\mathrm{rk}(F) = \deg(g_1)$. Consider any linear form $\partial \in T_1$ which is not a factor of g_1. Then, $(\partial \circ F)^\perp = F^\perp : (\partial) = (h_1, h_2)$, with $\deg(h_1) + \deg(h_2) = d+1$. Since ∂ is not a factor of g_1, then we have that $g_1 = h_1$ and, since it is square-free, we have that $\mathrm{rk}(\partial \circ F) = \mathrm{rk}(F)$.

b) If $d_1 < d_2$ and g_1 is not square-free, say $g_1 = l_1^{m_1} \cdots l_s^{m_s}$, with $m_1 \leq \ldots \leq m_s$, then we have $\mathrm{rk}(F) = \deg(g_2)$. Fix $\partial = l_1 \in T_1$. Then, we have that $(\partial \circ F)^\perp = F^\perp : (l_1) = (h_1, h_2)$ with $\deg(h_1) + \deg(h_2) = d+1$. Since $l_1^{m_1-1} \cdots l_s^{m_s} \in (\partial \circ F)^\perp$, but not in $F^\perp$ it has to be $h_1 = l_1^{m_1-1} \cdots l_s^{m_s}$. In particular, $\mathrm{rk}(\partial \circ F) = \deg(h_2) = \deg(g_2) = \mathrm{rk}(F)$.

c) If $d_1 = d_2$, we can always consider a non square-free element $g \in (F^\perp)_{d_1}$. Indeed, if both g_1 and g_2 are square-free, then it is enough to consider one element lying on the intersection between the hypersurface in $\mathbb{P}(S_{d_1})$ defined by the vanishing of the discriminant of polynomials of degree d_1 and the line passing through $[g_1]$ and $[g_2]$.

Say $g = l_1^{m_1} \cdots l_s^{m_s}$, with $m_1 \leq \ldots \leq m_s$. Fix $\partial = l_1 \in T_1$. Hence, we conclude similarly as part b).

(3) If $F = x_0^a(x_1^b + \ldots + x_n^b)$ with $b, n \geq 2$ and $a+1 \geq b$, then we have that $\mathrm{rk}(F) = (a+1)n$, by [?]. If we set $\partial = X_1 + \ldots + X_n$, then $\partial \circ F = x_0^a(x_1^{b-1} + \ldots + x_n^{b-1})$ and the rank is preserved.

(4) If $F = x_0^a(x_0^b + \ldots + x_n^b)$ with $b, n \geq 2$ and $a+1 \geq b$, then we have that $\mathrm{rk}(F) = (a+1)n$, by [?]. If we set $\partial = X_1 + \ldots + X_n$, then $\partial F = x_0^a(x_1^{b-1} + \ldots + x_n^{b-1})$ and the rank is preserved.

(5) If $F = x_0^a G(x_1, \ldots, x_n)$ with $G^\perp = (g_1, \ldots, g_n)$, $a \geq 2$, and $\deg g_i \geq a+1$, we know that $\mathrm{rk}(F) = d_1 \cdots d_n$, by [?]. If we consider $\partial = X_0$, then we have that $\partial \circ F = x_0^{a-1} G(x_1, \ldots, x_n)$ and the rank is preserved. □

We can now prove the following.

Theorem 4.4. *Let $F = \sum_{i=1}^s F_i \in S$ be a form such that $F_i \in S^{[i]}$ for all $i = 1, \ldots, s$. If each F_i is one of the following,*

(1) *a monomial $x_0^{d_0} \cdot \ldots \cdot x_n^{d_n}$ with $d_i \geq 2$ for $0 \leq i \leq n$;*
(2) *a binary form $F \neq LM^{d-1}$;*

(3) $x_0^a(x_1^b + \ldots + x_n^b)$ *with* $b, n \geq 2$ *and* $a + 1 \geq b$
(4) $x_0^a(x_0^b + x_1^b + \ldots + x_n^b)$ *with* $b, n \geq 2$ *and* $a + 1 \geq b$
(5) $x_0^a G(x_1, \ldots, x_n)$ *such that* $G^\perp = (g_1, \ldots, g_n)$*,* $a \geq 2$*, and* $\deg g_i \geq a + 1$

then Conjecture 3 holds for F*.*

We can prove Conjecture 3 in a few cases without using Proposition 4.2. Note, for example, that Proposition 4.2 cannot be applied if one of the summand is a monomial with lowest exponent equal to one since Lemma 4.3 does not hold.

Theorem 4.5. *Conjecture 3 is true for a form* F *of degree* $d \geq 3$

$$F = x_0 x_1^{a_1} \ldots x_n^{a_n} + y_0^{b_0} y_1^{b_1} \ldots y_m^{b_m} \in k[x_0, x_1, \ldots x_n, y_0, y_1, \ldots y_m],$$

with $d = 1 + \sum_{i=0}^n a_i = \sum_{i=0}^m b_i$ *and* $b_0 \leq b_i$ $(i = 1, \ldots, m)$*.*

We need preliminary results in order to give the proof.

Lemma 4.6. *Let* $F_1, F_2 \in k[z_{1,1}, \ldots, z_{1,n_1}, z_{2,1}, \ldots, z_{2,n_2}]$ *be two monomials of the same degree, in different sets of variables, say*

$$F_1 = z_{1,1}^{d_{1,1}} \cdots z_{1,n_1}^{d_{1,n_1}}; \quad F_2 = z_{2,1}^{d_{2,1}} \cdots z_{2,n_2}^{d_{2,n_2}}.$$

Then
(i) $(F_1 + F_2)^\perp = (F_1)^\perp \cap (F_2)^\perp + (\Pi d_{2,i}!\ Z_{1,1}^{d_{1,1}} \cdots Z_{1,n_1}^{d_{1,n_1}} - \Pi d_{1,i}!\ Z_{2,1}^{d_{2,1}} \cdots Z_{2,n_2}^{d_{2,n_2}})$;
(ii) length $T/(F_1 + F_2)^\perp =$ length $T/(F_1)^\perp +$ length $T/(F_2)^\perp - 2$,
where $T = k[Z_{1,1}, \ldots, Z_{1,n_1}, Z_{2,1}, \ldots, Z_{2,n_2}]$

Proof. The results easily follow by observing that

$$(F_1 + F_2)^\perp = Z_{1,1}^{d_{1,1}+1}, \ldots, Z_{1,n_1}^{d_{1,n_1}+1}, Z_{2,1}^{d_{2,1}+1}, \ldots, Z_{2,n_2}^{d_{2,n_2}+1},$$

$$Z_{1,1}Z_{2,1}, \ldots, Z_{1,n_1}Z_{2,1}, \ldots, Z_{1,1}Z_{2,n_2}, \ldots, Z_{1,n_1}Z_{2,n_2}, \Pi d_{2,i}!\ Z_{1,1}^{d_{1,1}} \cdots Z_{1,n_1}^{d_{1,n_1}} - \Pi d_{1,i}!\ Z_{2,1}^{d_{2,1}} \cdots Z_{2,n_2}^{d_{2,n_2}};$$

$$(F_1)^\perp = Z_{1,1}^{d_{1,1}+1}, \ldots, Z_{1,n_1}^{d_{1,n_1}+1}, Z_{2,1}, \ldots, Z_{2,n_2}$$

$$(F_2)^\perp = Z_{1,1}, \ldots, Z_{1,n_1}, Z_{2,1}^{d_{2,1}+1}, \ldots, Z_{2,n_2}^{d_{2,n_2}+1}.$$

and by the exact sequence

$$0 \longrightarrow T/(I \cap J) \longrightarrow T/I \oplus T/J \longrightarrow T/(I + J) \longrightarrow 0 \qquad (\dagger)$$

where I, J are ideals in T.

□

Lemma 4.7. *Let* $F = x_0 x_1^{a_1} \ldots x_n^{a_n} + y_0^{b_0} y_1^{b_1} \ldots y_m^{b_m}$ *as in Theorem 4.5, then*

$$\text{length } T/(F^\perp : (X_0 + Y_0) + (X_0 + Y_0)) = \text{rk}F - 2,$$

where $T = k[X_0, X_1, \ldots X_n, Y_0, Y_1, \ldots Y_m]$*.*

Proof. We have

$$F^\perp : (X_0+Y_0) + (X_0+Y_0) = ((X_0+Y_0)\circ F)^\perp + (X_0+Y_0)$$

$$= \left(x_1^{a_1}\cdots x_n^{a_n} + y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m}\right)^\perp + (X_0+Y_0)$$

since $X_0 \in (x_1^{a_1}\cdots x_n^{a_n} + y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m})^\perp$ this is

$$= \left(x_1^{a_1}\cdots x_n^{a_n} + y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m}\right)^\perp + (X_0, Y_0).$$

Now, if $b_0 > 1$, by Lemma 4.6 we get

$$\left(x_1^{a_1}\cdots x_n^{a_n} + y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m}\right)^\perp + (Y_0)$$

$$= (x_1^{a_1}\cdots x_n^{a_n})^\perp \cap (y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m})^\perp + ((b_0-1)!\Pi_{i=1}^m b_i! X_1^{a_1}\cdots X_n^{a_n} - \Pi_{i=1}^n a_i! Y_0^{b_0-1}Y_1^{b_1}\cdots Y_m^{b_m}) + (Y_0)$$

$$= (x_1^{a_1}\cdots x_n^{a_n})^\perp \cap (y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m})^\perp + ((b_0-1)!\Pi_{i=1}^m b_i! X_1^{a_1}\cdots X_n^{a_n}) + (Y_0)$$

since $Y_0 \in (x_1^{a_1}\cdots x_n^{a_n})^\perp$ this is

$$= (x_1^{a_1}\cdots x_n^{a_n})^\perp \cap ((y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m})^\perp + (Y_0)) + (X_1^{a_1}\cdots X_n^{a_n})$$

and since $X_1^{a_1}\cdots X_n^{a_n} \in (y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m})^\perp$:

$$= ((x_1^{a_1}\cdots x_n^{a_n})^\perp + (X_1^{a_1}\cdots X_n^{a_n})) \cap ((y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m})^\perp + (Y_0)).$$

So by the exact sequence (1) we get

$$\text{length } T/(F^\perp : (X_0+Y_0) + (X_0+Y_0))$$

$$= \text{length } T/((x_1^{a_1}\cdots x_n^{a_n})^\perp + (X_1^{a_1}\cdots X_n^{a_n})) + \text{length } T/((y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m})^\perp + (Y_0))$$

$$-\text{length } T/((x_1^{a_1}\cdots x_n^{a_n})^\perp + (X_1^{a_1}\cdots X_n^{a_n}) + ((y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m})^\perp + (Y_0))$$

$$= \text{length } T/((x_1^{a_1}\cdots x_n^{a_n})^\perp + (X_1^{a_1}\cdots X_n^{a_n})) + \text{length } T/((y_0^{b_0-1}y_1^{b_1}\cdots y_m^{b_m})^\perp + (Y_0)) - 1$$

$$= \Pi_{i=1}^n (a_i+1) - 1 + \Pi_{i=1}^m (b_i+1) - 1 = \text{rk} F - 2.$$

In case $b_0 = 1$, since $F^\perp : (X_0+Y_0) + (X_0+Y_0) = \left(x_1^{a_1}\cdots x_n^{a_n} + y_1^{b_1}\cdots y_m^{b_m}\right)^\perp + (X_0, Y_0)$, by Lemma 4.6 we get

$$\text{length } T/(F^\perp : (X_0+Y_0) + (X_0+Y_0)) = \text{length } \widetilde{T}/\left(x_1^{a_1}\cdots x_n^{a_n} + y_1^{b_1}\cdots y_m^{b_m}\right)^\perp$$

$$= \text{length } \widetilde{T}/\,(x_1^{a_1}\cdots x_n^{a_n})^\perp + \widetilde{T}/(y_1^{b_1}\cdots y_m^{b_m})^\perp - 2 = \text{rk} F - 2,$$

where $\widetilde{T} = k[X_1, \dots X_n, Y_1, \dots Y_m]$.

$\square$

Lemma 4.8. *Notation as in Lemma 4.7. Let*

$$F = x_0x_1^{a_1}\dots x_n^{a_n} + y_0^{b_0}y_1^{b_1}\dots y_m^{b_m}$$

with $d = 1 + \sum_{i=0}^{n} a_i = \sum_{i=0}^{m} b_i \geq 3$ *and* $b_0 \leq b_i$ $(i = 1,\dots,m)$, *then*

$$\mathcal{W}_F \subset \{X_0Y_0 = 0\} \subset \mathbb{P}^{n+m+1}.$$

.

Proof. Let $I_{\mathbb{X}} \subset F^{\perp}$ be a minimal set of apolar points for F, thus

$$|\mathbb{X}| = \mathrm{rk}F.$$

It is enough to show that there are no points of $\mathbb{X}$ lying on the hyperplanes $\lambda X_0 + \mu Y_0 = 0$, for $\lambda\mu \neq 0$. After a change of coordinates, we may assume $\lambda = \mu = 1$.

We consider $I_{\mathbb{X}'} = I_{\mathbb{X}} : (X_0+Y_0)$ the ideal of the set of points in $\mathbb{X}$ which do not lie on $X_0+Y_0 = 0$.

The cardinality of $\mathbb{X}'$ is at least the length of the ring $T/(F^{\perp} :\ (X_0 + Y_0) + (X_0 + Y_0))$, that is, by Lemma 4.7,

$$|\mathbb{X}'| \geq \mathrm{rk}F - 2.$$

It follows that on the hyperplane $X_0 + Y_0 = 0$ we have at most two points of $\mathbb{X}$.

Claim: In degree 1, the ideal $I_{\mathbb{X}} : (X_0 + Y_0) + (X_0 + Y_0)$ differs from $F^{\perp} : (X_0 + Y_0) + (X_0 + Y_0)$.

Proof of Claim. As already computed in the proof of Lemma 4.7, we have that $F^{\perp}\ : (X_0 + Y_0) + (X_0 + Y_0)$ contains two linear forms, namely X_0 and Y_0.

Now assume that

$$L = \alpha_0X_0 + \dots + \alpha_nX_n + \beta_0Y_0 + \dots + \beta_mY_m \in I_{\mathbb{X}} : (X_0 + Y_0).$$

Thus, we have that $L(X_0 + Y_0) \in I_{\mathbb{X}} \subset F^{\perp}$.

In case $b_0 > 1$, since $X_0^2, X_0Y_0,\dots, X_0Y_m, X_1Y_0,\dots, X_nY_0 \in F^{\perp}$ we get

$$(\alpha_1X_0X_1 + \dots + \alpha_nX_0X_n + \beta_0Y_0^2 + \beta_1Y_0Y_1 + \dots + \beta_mY_0Y_m) \circ F = 0,$$

and from this easily follows that $\alpha_1 = \dots = \alpha_n = \beta_0 = \beta_1 = \dots = \beta_m = 0$. Hence, $L = \alpha_0X_0$ and so $\alpha_0X_0(X_0 + Y_0) \in I_{\mathbb{X}}$.

Now consider the hyperplane $Y_0 = 0$.

By Lemma 4.6 we get

$$\begin{aligned}
F^{\perp}+(Y_0) &= (x_0x_1^{a_1}\cdots x_n^{a_n})^{\perp}\cap(y_0^{b_0}y_1^{b_1}\dots y_m^{b_m})^{\perp}+(\Pi b_i!X_0X_1^{a_1}\cdots X_n^{a_n}-\Pi a_i!Y_0^{b_0}Y_1^{b_1}\dots Y_m^{b_m})+(Y_0) =\\
&= (x_0x_1^{a_1}\cdots x_n^{a_n})^{\perp}\cap(y_0^{b_0}y_1^{b_1}\dots y_m^{b_m})^{\perp}+(X_0X_1^{a_1}\cdots X_n^{a_n})+(Y_0) \subseteq\\
&\subseteq ((x_0x_1^{a_1}\cdots x_n^{a_n})^{\perp}+(X_0X_1^{a_1}\cdots X_n^{a_n}, Y_0))\cap((y_0^{b_0}y_1^{b_1}\dots y_m^{b_m})^{\perp}+(X_0X_1^{a_1}\cdots X_n^{a_n}, Y_0)) =\\
&= ((x_0x_1^{a_1}\cdots x_n^{a_n})^{\perp}+(X_0X_1^{a_1}\cdots X_n^{a_n}))\cap((y_0^{b_0}y_1^{b_1}\dots y_m^{b_m})^{\perp}+(Y_0))
\end{aligned}$$

Hence

$$\begin{aligned}
&\text{length } T/(F^{\perp}+(Y_0))\\
&\geq \text{length } T/((x_0x_1^{a_1}\cdots x_n^{a_n})^{\perp}+(X_0X_1^{a_1}\cdots X_n^{a_n})) + \text{length } T/((y_0^{b_0}y_1^{b_1}\dots y_m^{b_m})^{\perp}+(Y_0)) - 1\\
&= 2\Pi_{i\geq 1}(a_i+1)+\Pi_{i\geq 1}(b_i+1)-2 = \mathrm{rk}F + \Pi_{i\geq 1}(a_i+1)-2 > \mathrm{rk}F.
\end{aligned}$$

So

$$\text{length } T/(F^{\perp}+(Y_0)) > \mathrm{rk}F.$$

Hence Y_0 is not a zero divisor for $I_{\mathbb{X}}$ and there are points of $\mathbb{X}$ lying on the hyperplane $Y_0 = 0$. Since, by [CCC15], there are no points of $\mathbb{X}$ on the linear space defined by the ideal (X_0, Y_0), and since $\alpha_0 X_0(X_0 + Y_0) \in I_{\mathbb{X}}$, it follows that $\alpha_0 = 0$. So $I_{\mathbb{X}} : (X_0 + Y_0) + (X_0 + Y_0)$ contains only the linear form $X_0 + Y_0$, and thus in case $b_0 > 1$ the Claim is proved.

In case $b_0 = 1$, since $X_0^2, X_0Y_0, \dots, X_0Y_m, Y_0^2, X_1Y_0, \dots, X_nY_0 \in F^\perp$ we get

$$(\alpha_1 X_0X_1 + \ldots + \alpha_n X_0X_n + \beta_1 Y_0Y_1 + \ldots + \beta_m Y_0Y_m) \circ F = 0,$$

and so $\alpha_1 = \ldots = \alpha_n = \beta_1 = \ldots = \beta_m = 0$. Hence, $L = \alpha_0 X_0 + \beta_0 Y_0$ and $L(X_0 + Y_0) = (\alpha_0 X_0 + \beta_0 Y_0)(X_0 + Y_0) \in I_{\mathbb{X}}$.

Now consider the hyperplanes $X_0 = 0$ and $Y_0 = 0$.

By Lemma 4.6 we get

$$\begin{aligned} F^\perp + (X_0) &= (x_0 x_1^{a_1} \cdots x_n^{a_n})^\perp \cap (y_0 y_1^{b_1} \ldots y_m^{b_m})^\perp + (\Pi b_i! X_0 X_1^{a_1} \cdots X_n^{a_n} - \Pi a_i! Y_0 Y_1^{b_1} \ldots Y_m^{b_m}) + (X_0) = \\ &= (x_0 x_1^{a_1} \cdots x_n^{a_n})^\perp \cap (y_0 y_1^{b_1} \ldots y_m^{b_m})^\perp + (Y_0 Y_1^{b_1} \ldots Y_m^{b_m}) + (X_0) \subseteq \\ &\subseteq ((x_0 x_1^{a_1} \cdots x_n^{a_n})^\perp + (Y_0 Y_1^{b_1} \ldots Y_m^{b_m}, X_0)) \cap ((y_0 y_1^{b_1} \ldots y_m^{b_m})^\perp + (Y_0 Y_1^{b_1} \ldots Y_m^{b_m}, X_0)) = \\ &= (x_1^{a_1} \cdots x_n^{a_n})^\perp \cap ((y_0 y_1^{b_1} \ldots y_m^{b_m})^\perp + (Y_0 Y_1^{b_1} \ldots Y_m^{b_m})). \end{aligned}$$

Hence

$$\begin{aligned} &\text{length } T/(F^\perp + (X_0)) \\ &\geq \text{length } T/(x_1^{a_1} \cdots x_n^{a_n})^\perp + \text{length } T/((y_0 y_1^{b_1} \ldots y_m^{b_m})^\perp + (Y_0 Y_1^{b_1} \ldots Y_m^{b_m})) - 1 \\ &= \Pi_{i \geq 1}(a_i + 1) + 2\Pi_{i \geq 1}(b_i + 1) - 2 = \mathrm{rk} F + \Pi_{i \geq 1}(b_i + 1) - 2 > \mathrm{rk} F. \end{aligned}$$

It follows that

$$\text{length } T/(F^\perp + (X_0)) > \mathrm{rk} F.$$

Analogously we have

$$\text{length } T/(F^\perp + (Y_0)) > \mathrm{rk} F.$$

So X_0 and Y_0 are not zero divisors for $I_{\mathbb{X}}$. Hence there are points of $\mathbb{X}$ lying both on the hyperplane $X_0 = 0$ and on $Y_0 = 0$.

Since, by [CCC15], there are no points of $\mathbb{X}$ on the linear space defined by the ideal (X_0, Y_0), and since $(\alpha_0 X_0 + \beta_0 Y_0)(X_0 + Y_0) \in I_{\mathbb{X}}$, it follows that $\alpha_0 = \beta_0 = 0$. Thus $I_{\mathbb{X}} : (X_0 + Y_0) + (X_0 + Y_0)$ contains only the linear form $X_0 + Y_0$, and the Claim is proved also in case $b_0 = 1$. □

Now, the idea is to show that $I_{\mathbb{X}} : (X_0 + Y_0) + (X_0 + Y_0)$ differs from $F^\perp : (X_0 + Y_0) + (X_0 + Y_0)$ also in degree $d - 1$. From this, and the Claim above, it would follow that the cardinality of $\mathbb{X}'$ is actually $\mathrm{rk} F$ and then we have no points of $\mathbb{X}$ over the hyperplane $X_0 + Y_0 = 0$.

Consider first the case $b_0 > 1$. In this case, since (see the proof of Lemma 4.7)

$$F^\perp : (X_0 + Y_0) + (X_0 + Y_0) = ((x_1^{a_1} \cdots x_n^{a_n})^\perp + (X_1^{a_1} \cdots X_n^{a_n})) \cap ((y_0^{b_0 - 1} y_1^{b_1} \cdots y_m^{b_m})^\perp + (Y_0)),$$

hence in degree $d-1$ we have that $F^\perp : (X_0+Y_0)+(X_0+Y_0) = T_{d-1}$, the whole vector space. We will prove that $(I_{\mathbb{X}} : (X_0+Y_0)+(X_0+Y_0))_{d-1} \neq T_{d-1}$. Since, from the Claim, $I_{\mathbb{X}} : (X_0+Y_0)+(X_0+Y_0)$ differs from $F^\perp : (X_0 + Y_0) + (X_0 + Y_0)$, then $|\mathbb{X}'| \geq 1 +$ length $T/(F^\perp : (X_0 + Y_0) + (X_0 + Y_0)) = \mathrm{rk} F - 1$. Hence there is at most one point of $\mathbb{X}$, say P, lying on the hyperplane $X_0 + Y_0 = 0$. Since there are no points on the linear space (X_0, Y_0), we can write $P = [1, u_1, \dots, u_n, -1, v_1, \dots, v_m]$.

Let

$$H = X_1^{a_1} \cdots X_n^{a_n} - u_1^{a_1} \cdots u_n^{a_n} X_0^{d-1}.$$

If we assume, by contradiction, that $I_{\mathbb{X}} : (X_0+Y_0)+(X_0+Y_0)$ contains all the forms of degree $d-1$, we have that

$$H \in I_{\mathbb{X}} : (X_0+Y_0)+(X_0+Y_0),$$

that is,

$$H+(X_0+Y_0)G \in I_{\mathbb{X}} : (X_0+Y_0)$$

for some $G \in T_{d-2}$.

Since $H+(X_0+Y_0)G$ vanishes at P and at the points of $\mathbb{X}'$, we actually have that

$$H+(X_0+Y_0)G \in I_{\mathbb{X}} \subset F^{\perp},$$

and from this

$$(H+(X_0+Y_0)G) \circ F = 0.$$

But

$$(X_1^{a_1}\cdots X_n^{a_n} - u_1^{a_1}\cdots u_n^{a_n} X_0^{d-1} + (X_0+Y_0)G) \circ F =$$
$$= \Pi a_i! x_0 + G \circ (x_1^{a_1}\cdots x_n^{a_n} + y_0^{b_0-1} y_1^{b_1}\ldots y_m^{b_m}) = 0,$$

and this is impossible, since $G \circ (x_1^{a_1}\cdots x_n^{a_n} + y_0^{b_0-1} y_1^{b_1}\ldots y_m^{b_m})$ cannot be $-\Pi a_i! x_0$.

Now let $b_0 = 1$. In this case we have

$$F^{\perp} : (X_0+Y_0)+(X_0+Y_0) = (x_1^{a_1}\cdots x_n^{a_n} + y_1^{b_1}\cdots y_m^{b_m})^{\perp} + (X_0, Y_0),$$

hence in degree $d-1$

$$\dim(F^{\perp} : (X_0+Y_0)+(X_0+Y_0))_{d-1} = \dim T_{d-1} - 1.$$

Since, from the Claim, $I_{\mathbb{X}} : (X_0+Y_0)+(X_0+Y_0)$ differs from $F^{\perp} : (X_0+Y_0)+(X_0+Y_0)$, then $|\mathbb{X}'| \geq 1 +$ length $T/(F^{\perp} : (X_0+Y)+(X_0+Y_0)) = \mathrm{rk}F - 1$. Hence there is at most one point of $\mathbb{X}$, say P, lying on the hyperplane $X_0+Y=0$. Since there are no points on the linear space (X_0, Y_0), we can assume that $P = [1, u_1, \ldots, u_n, -1, v_1, \ldots, v_m]$.

Let

$$H_1 = X_1^{a_1}\cdots X_n^{a_n} - u_1^{a_1}\cdots u_n^{a_n} X_0^{d-1},$$
$$H_2 = Y_1^{b_1}\cdots Y_m^{b_m} - v_1^{b_1}\cdots v_m^{b_m} Y_0^{d-1}.$$

We will prove that $H_1 \notin I_{\mathbb{X}} : (X_0+Y_0)+(X_0+Y_0)$. In fact, if $H_1 \in I_{\mathbb{X}} : (X_0+Y_0)+(X_0+Y_0)$ we have

$$H_1+(X_0+Y_0)G_1 \in I_{\mathbb{X}} : (X_0+Y_0)$$

for some $G_1 \in T_{d-2}$. But $H_1+(X_0+Y_0)G_1$ vanishes at P and at the points of $\mathbb{X}'$, so we have

$$H_1+(X_0+Y_0)G_1 \in I_{\mathbb{X}} \subset F^{\perp},$$

and from this

$$(H_1+(X_0+Y_0)G_1) \circ F = 0.$$

But

$$(H_1+(X_0+Y_0)G_1) \circ F = (X_1^{a_1}\cdots X_n^{a_n} - u_1^{a_1}\cdots u_n^{a_n} X_0^{d-1} + (X_0+Y_0)G_1) \circ F =$$
$$= \Pi a_i! x_0 + G_1 \circ (x_1^{a_1}\cdots x_n^{a_n} + y_0^{b_0-1} y_1^{b_1}\ldots y_m^{b_m}) = 0,$$

and this is impossible, since $G_1 \circ (x_1^{a_1}\cdots x_n^{a_n} + y_0^{b_0-1} y_1^{b_1}\ldots y_m^{b_m})$ cannot be $-\Pi a_i! x_0$.

Analogously we can show that $H_2 \notin I_{\mathbb{X}} : (X_0 + Y_0) + (X_0 + Y_0)$. Since H_1 and H_2 are linearly independent forms of degree $d-1$, and

$$H_1, H_2 \notin (I_{\mathbb{X}} : (X_0 + Y_0) + (X_0 + Y_0))_{d-1},$$

then

$$\dim(I_{\mathbb{X}} : (X_0 + Y_0) + (X_0 + Y_0))_{d-1} \leq \dim T_{d-1} - 2.$$

It follows that

$$(I_{\mathbb{X}} : (X_0 + Y_0) + (X_0 + Y_0))_{d-1} \neq (F^{\perp} : (X_0 + Y_0) + (X_0 + Y_0))_{d-1}.$$

□

Proof of Theorem 4.5. We know by Lemma 4.8, that the Waring locus of F is contained in the union of the two hyperplanes $X_0 = 0$ and $Y_0 = 0$. Moreover, it is easy to check that, given $\mathbb{X}$ an apolar set of F, we have exactly $(a_1 + 1) \cdots (a_n + 1)$ points of $\mathbb{X}$ on the hyperplane $y_0 = 0$ and $(b_1 + 1) \cdots (b_n + 1)$ points of $\mathbb{X}$ on the hyperplane $x_0 = 0$. Then, the claim follows from Remark 1. □

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