Rendiconti Seminario Matematico Univ. Pol. Torino
Workshop for Sergio Console
Vol. 74, 1 (2016), 213 – 233

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BJÖRLING TYPE PROBLEMS FOR ELASTIC SURFACES

To the memory of our friend Sergio Console

Abstract. In this survey we address the Björling problem for various classes of surfaces associated to the Euler–Lagrange equation of the Helfrich elastic energy subject to volume and area constraints.

Introduction

The equilibrium configurations of elastic surfaces, such as lipid bilayers in biological membranes, arise as critical points of the Helfrich energy functional, subject to area and volume constraints [19, 36, 48, 79, 83]. The corresponding Euler–Lagrange equation is known as the *Ou-Yang–Helfrich equation*, or the *membrane shape equation* [69, 72, 73]. A number of different important classes of surfaces, including minimal, constant mean curvature (CMC), and Willmore surfaces, are governed by nonlinear partial differential equations which are obtained as special cases of the Ou-Yang–Helfrich equation. In this paper, we address the Björling problem for various classes of surfaces associated to special reductions of the Ou-Yang–Helfrich equation.

Section 1 introduces the Helfrich functional and its associated Euler–Lagrange equation. It then discusses some of its most important reductions as well as the relations among the corresponding classes of surfaces.

Section 2 recalls the classical Björling problem for minimal surfaces and outlines the recently solved Björling type problems for non-minimal CMC surfaces and for Willmore surfaces [12, 15]. The solutions of these problems all ultimately rely on the harmonicity of a suitable Gauss map and hence on the possibility of exploiting the techniques from integrable system theory.

Section 3 discusses a Björling problem for equilibrium elastic surfaces, that is, for surfaces in Euclidean space whose mean curvature function H satisfies the shape equation

$$\Delta H = \Phi(a,c),$$

where Δ denotes the Laplace–Beltrami operator of the surface and Φ is a real analytic symmetric function of the principal curvatures a and c. Contrary to the previous cases, this equation is not known to be related to harmonic map theory or to other integrable systems, so these approaches do not apply in principle. In this case, the techniques

*Authors were partially supported by PRIN 2010-2011 “Varietà reali e complesse: geometria, topologia e analisi armonica”; and by the GNSAGA of INDAM. The first author also acknowledges support from Spanish MEC-Feder research project MTM2014-52368-P and the third author from FIRB 2008 “Geometria Differenziale Complessa e Dinamica Olomorfa”.

rely upon the Cartan–Kähler theory of Pfaffian differential systems and the method of moving frames [17, 21, 33, 34, 40].

Section 4 presents some examples.

Acknowledgments. The authors would like to thank the referee for useful comments and suggestions.

1. The Helfrich energy and the shape equation

1.1. The Helfrich functional

The Helfrich functional (Canham [19], Evans [27], Helfrich [36]) for a compact oriented surface S embedded in $\mathbb{R}^3$ is defined by

$$\mathcal{H}(S) = \mathfrak{b}\int_S (H-c_0)^2 dA + \mathfrak{c}\int_S K dA, \tag{1}$$

where

- dA is the *area element* of the surface;
- $H = (a+c)/2$ is the *mean curvature* of S;
- $K = ac$ is the *Gauss curvature* of S;
- a, c denote the *principal curvatures* of S;
- $\mathfrak{b}$, $\mathfrak{c} \in \mathbb{R}$ are the *bending rigidities*, constants depending on the material;
- $c_0 \in \mathbb{R}$ is the *spontaneous curvature*.

Physically, the formula for $\mathcal{H}$ follows from Hooke's law [44]. The Helfrich functional models the *bending elastic energy* of biological membranes formed by a double layer of phospholipids. In water, these molecules spontaneously aggregate forming a closed bilayer which can be regarded as a surface S embedded in $\mathbb{R}^3$ (*elastic surface*). The spontaneous curvature c_0 accounts for an asymmetry in the layers. The constants $\mathfrak{b}$ and $\mathfrak{c}$ are material-dependent parameters expressing bending energies.

1.2. The constrained Helfrich functional

There are two natural constraints associated in general with the membrane S: (1) the total area $\mathcal{A}(S)$ should be fixed and (2) the enclosed volume $\mathcal{V}(S)$ should be fixed.

The equilibrium configurations of a bilayer vesicle modeled by an elastic surface S with fixed surface area $\mathcal{A}(S)$ and enclosed volume $\mathcal{V}(S)$ are determined by minimization of

$$\begin{aligned} \mathcal{F}(S) &= \mathcal{H}(S) + \mathfrak{a}\,\mathcal{A}(S) + p\,\mathcal{V}(S) \\ &= \int_S \left[\mathfrak{a} + \mathfrak{b}(H-c_0)^2 + \mathfrak{c}K\right] dA + p\,\mathcal{V}(S), \end{aligned} \tag{2}$$

where $\mathfrak{a} \in \mathbb{R}$ is a constant expressing the *surface lateral tension* (stretching), and $p \in \mathbb{R}$ is a constant, called *pressure*, which indicates the difference between outside and inside pressure. The tension $\mathfrak{a}$ and the pressure p play the the role of Lagrange multipliers for the constraints on area and volume.

1.3. The shape equation

The Euler–Lagrange equation for the constrained Helfric functional $\mathcal{F}$, computed by Ou-Yang and Helfrich [72, 73], is given by

$$\mathfrak{b}\left\{\Delta H + 2H(H^2 - K)\right\} - 2\left(\mathfrak{a} + \mathfrak{b}c_0^2\right)H + 2\mathfrak{b}c_0 K - p = 0, \tag{3}$$

where Δ denotes the Laplace–Beltrami operator of the induced metric on S.

REMARK 1.1. Observe that $\mathfrak{c}$ does not enter into the Euler-Lagrange equation; indeed, $2\mathfrak{c}\pi\chi(S) = \mathfrak{c}\int_S K dA$. Thus, for any fixed topology, it can be neglected, and

$$\mathcal{F}(S) = \int_S \left[\mathfrak{a} + \mathfrak{b}(H - c_0)^2\right] dA + p\mathcal{V}(S). \tag{4}$$

DEFINITION 1.1. *We call $S \subset \mathbb{R}^3$ an equilibrium surface if it satisfies the fourth order nonlinear PDE*

$$\Delta H = \Phi(a, c), \tag{5}$$

where Φ is a real analytic symmetric function of the principal curvatures a and c. The equation (5) is referred to as the shape equation.

1.4. Reductions of $\mathcal{F}(S)$ and related examples

There are several important reductions of the constrained Helfrich functional $\mathcal{F}$, and hence of the corresponding Ou-Yang–Helfrich equation (3).

Minimal surfaces

If $\mathfrak{b} = p = 0$, $\mathcal{F}$ reduces to the *area functional* $\mathcal{A}(S)$, which in turn leads to the theory of minimal surfaces.

CMC surfaces

If $\mathfrak{b} = 0$, $\mathcal{F}$ reduces to the area functional $\mathcal{A}(S)$, with a volume constraint, which leads to the theory constant mean curvature (CMC) surfaces.

Willmore surfaces

If $\mathfrak{c}_0 = p = 0$, $\mathfrak{b} = 1$, and $\mathfrak{a} = 1, 0, -1$ is interpreted as the curvature of the simply connected 3-dimensional space forms S^3, $\mathbb{R}^3$, and H^3, respectively, $\mathcal{F}$ reduces to the *Willmore functional*

$$\mathcal{W}(S) = \int_S (\mathfrak{a} + H^2) dA,$$

while the Ou-Yang–Helfrich equation reduces to the Thomsen–Shadow equation (cf. [39, 80])

$$\Delta H + 2H(H^2 - (K - \mathfrak{a})) = 0.$$

The critical points of the Willmore functional are the well-known Willmore surfaces. The Willmore functional $\mathcal{W}(S)$ is conformally invariant, that is, $\mathcal{W}(S) = \mathcal{W}(F(S))$ for any conformal transformation F of the ambient space. This property has been fundamental in the study of Willmore surfaces and especially in the proof of the recently solved Willmore conjecture by Marques and Neves [49].

Elastic curves

Another natural reduction of the constrained Helfrich energy is given by the classical bending energy of a curve $\gamma \subset \mathbb{R}^2$, the *Elastica functional*,

$$\mathcal{E}(\gamma) = \int_\gamma \kappa^2 ds.$$

In general, see, for instance, [46], a curve $\gamma(s)$ parametrized by arclength with curvature κ in a space form M^2 of sectional curvature G is said to be a *free elastic* curve if it is critical for the functional

$$\mathcal{E}(\gamma) = \int_\gamma \kappa^2(s) ds. \tag{6}$$

The corresponding Euler–Lagrange equation is

$$2\kappa'' + \kappa^3 + 2G\kappa = 0.$$

A curve γ is said an *elastic* curve if it is critical for (6) with the integral constraint

$$L(\gamma) = \int_\gamma ds = \ell,$$

i.e., the curve γ has constant length ℓ. In this case, the Euler–Lagrange equation is

$$2\kappa'' + \kappa^3 + 2(\mu + G)\kappa = 0.$$

REMARK 1.2. The functional $\mathcal{F}(S)$ became important in the study of biconcave shape of red blood cells [23]. In this respect, the Willmore functional $\mathcal{W}(S)$ is not a good model, since the unique minimum of $\mathcal{W}$ for topologically spherical vesicles is the round sphere [71].

1.5. Minimal, Willmore and CMC surfaces

Let $\widehat{S} \subset S^3$ be a surface in the 3-sphere and let $S \subset \mathbb{R}^3$ be its stereographic projection to $\mathbb{R}^3$ from a pole not in $\widehat{S}$. Since $\mathcal{W}$ is conformally invariant, i.e., $\mathcal{W}(\widehat{S}) = \mathcal{W}(S)$, if $\widehat{S}$ is minimal, and hence Willmore, we have that also S is a Willmore surface in $\mathbb{R}^3$.

In particular, according to a result of Lawson [47] asserting that every compact, orientable surface can be minimally imbedded in S^3, it follows that there exist compact Willmore surfaces of every genus embedded in $\mathbb{R}^3$.

Minimal surfaces in space forms are *isothermic*, that is, away from umbilic points they locally admit curvature line coordinates which are conformal (isothermal) for the induced metric. Besides minimal surfaces, examples of isothermic surfaces also include CMC surfaces and surfaces of revolution. Interestingly enough, isothermic surfaces form a Möbius invariant class of surfaces.

By a classical theorem of Thomsen [38, 42, 80], a surface is Willmore and isothermic if and only if it is minimal in some 3-dimensional space form. Thus, minimal immersions are the only CMC Willmore surfaces in a given 3-dimensional space form.

Although non-minimal CMC surfaces in space forms are not Willmore, they are *constrained Willmore*, that is, they are critical for the Willmore functional under compactly supported infinitesimal conformal variations [9]. In contrast to Thomsen's theorem, a constrained Willmore surface which is isothermic need not have constant mean curvature. An example of an isothermic, constrained Willmore surface that does not have constant mean curvature in some space form was provided by Burstall [9]; this is given by a cylinder over a plane curve. However, as proven by Richter [77], an analogue of Thomsen's theorem holds within the class of tori.

1.6. Willmore surfaces of revolution and elastic curves

Let γ be a regular curve in the *hyperbolic plane* H^2, where H^2 is represented by the upper half-plane above the x_1-axis in the x_1x_2-plane of $\mathbb{R}^3$. If $S_\gamma \subset \mathbb{R}^3$ is the *surface of revolution* obtained by revolving the profile curve γ about the x_1-axis, then the Willmore functional $\mathcal{W}$ reduces to

$$\mathcal{W}(S) = \int_S H^2 = \frac{\pi}{2}\int_\gamma \kappa^2,$$

where κ is the hyperbolic curvature of γ (cf. [18, 33, 45]). Thus, the surface S_γ is Willmore if and only if γ is a free elastic curve if and only if

$$\kappa'' + \frac{1}{2}\kappa^3 - \kappa = 0.$$

The proof follows easily from the *Principle of Symmetric Criticality* of Palais [74]. It follows that Willmore surfaces of revolution are minimal in some space form.

According to Langer and Singer [45], the length of γ determines the conformal type of S_γ. Moreover, S_γ is constrained Willmore if and only if γ is an elastic curve if

and only if
$$\kappa'' + \frac{1}{2}\kappa^3 + (\mu - 1)\kappa = 0.$$

1.7. Pinkall's Willmore tori

A *Hopf torus* $S_\gamma := \pi^{-1}(\gamma)$ is the inverse image under the Hopf fibration $\pi : S^3 \to S^2$ of a closed curve γ immersed in S^2. Pinkall [75] used Hopf tori to construct a new infinite series of compact embedded Willmore surfaces in $\mathbb{R}^3$ which are not conformally equivalent to a minimal immersion.

For a Hopf torus S_γ, we have that $K = 0$ and $H = -\kappa$, where κ denotes the curvature of γ in S^2.

Moreover, S_γ is critical for $\mathcal{W}$ if and only if γ is elastic in S^2, that is, if and only if $\kappa'' + \frac{1}{2}\kappa^3 + \kappa = 0$. Except for the Clifford torus, none of Pinkall's Willmore tori are conformally equivalent to a minimal immersion in space forms [42, 75].

S_γ is critical for $\mathcal{W}$ with fixed area and volume (Helfrich model) if and only if γ is critical for $\int_\gamma \kappa^2$ with fixed length and enclosed area if and only if
$$\kappa'' + \frac{1}{2}\kappa^3 + (\mu + 1)\kappa + \lambda = 0.$$
For further details on this, we refer the reader to the recent work of L. Heller [37].

REMARK 1.3. The presence of the *spontaneous curvature* c_0 combined with the *area and volume constraints* implies that $\mathcal{F}$ is not conformally invariant. Thus, several analytic methods used for the study of the Willmore functional, including Simon's regularity, and the existence of minimizers under fixed conformal class, cannot be employed. Additional information on these topics can be found in the lecture notes of Kuwert and Schätzle [43], and the bibliography therein. In the literature, only few explicit solutions of the shape equation are known: axisymmetric surfaces of spherical and toroidal topology, surfaces of biconcave shape [20, 70, 72, 73, 79]. The existence of global minimizers of $\mathcal{F}$ is known only for very special classes of surfaces, e.g., axisymmetric surfaces and biconcave shaped surfaces [22].

2. Geometric Cauchy problems

2.1. The classical Björling problem

The classical Björling problem for minimal surfaces reads as follows [8, 24, 69].

Let (α, N) be a pair consisting of a real analytic curve $\alpha : J \to \mathbb{R}^3$ parametrized by arclength,* where $J \subset \mathbb{R}$ is an open interval, and of a real analytic unit vector field $N : J \to \mathbb{R}^3$ along α, such that $\langle \alpha'(x), N(x) \rangle = 0$, for all $x \in J$. The *Björling problem* consists in finding a minimal immersion $f : \Sigma \to \mathbb{R}^3$ of some domain $\Sigma \subset \mathbb{R}^2$ with $J \subset \Sigma$, such that the following conditions hold true:

*Actually, α need not be parametrized by arclength.

1. $f(x,0) = \alpha(x)$, for $x \in J$,
2. $n(x,0) = N(x)$, for $x \in J$,

where n denotes the unit normal (Gauss map) of f, $n : \Sigma \to \mathbb{R}^3$.

The Björling problem was posed and solved by E. G. Björling in 1844 [6] as a special instance of the general theorem of Cauchy–Kovalevskaya, from which one expects to find a uniquely determined solution to the problem. Following H. A. Schwarz [78], vol. 1, pp. 179–89, such a unique solution can be given by an explicit representation formula in terms of the prescribed pair (α, N), namely

$$f(x,y) = \Re\left\{\widehat{\alpha}(z) - i\int_{z_0}^{z} \widehat{N}(w) \times \widehat{\alpha}'(w)dw\right\}, \quad z = x+iy,$$

where $\widehat{\alpha}(z)$ and $\widehat{N}(z)$ denote the holomorphic extensions of $\alpha(x)$ and $N(x)$, respectively. The solution to Björling's problem can be understood as follows:

- The Gauss map of a minimal surface is *holomorphic*.
- The *Weierstrass representation* gives a formula for the surface in terms of holomorphic data.
- It suffices to know the data along a curve.

REMARK 2.1. In the mid nineteenth century, Bonnet observed that the solution to Björling's problem permits also the determination of minimal surfaces containing a given curve as: (1) a *geodesic*: the normal to the surface is the principal normal vector of the geodesic as a space curve; (2) an *asymptotic line*: the normal to the surface is the binormal vector to the curve; (3) or a *curvature line* (use Joachimsthal's theorem).

Some modern, more sophisticated uses of the above classical formula for the study of minimal surfaces can be found, for instance, in [28, 50, 53].

2.2. Björling's problem for other surface classes

Within the above circle of ideas, the Björling problem has been studied for other classes of surfaces admitting a (holomorphic) Weierstrass representation, e.g.,

- the class of CMC 1 surfaces in hyperbolic 3-space using Bryant's holomorphic representation [28];
- the class of minimal surfaces in a three-dimensional Lie group [52] using previous work of [51].

The problem has been investigated in several different geometric situations and for other surface classes, including surfaces of constant curvature, affine spheres, and time-like surfaces (see, for instance, [1, 3, 4, 10, 11, 13, 14, 31, 29, 30] and the literature therein).

What about the Björling problem for non-minimal CMC surfaces, or Willmore surfaces?

2.3. Björling's problem for CMC surfaces

Let $f : M \to \mathbb{R}^3$ be a CMC H immersion with $H \neq 0$. The Gauss map $n : \Sigma \to S^2$ of $f : M \to \mathbb{R}^3$ is not holomorphic. However, it is well-known that $n : \Sigma \to S^2 = \mathrm{SU}(2)/S^1 \subset \mathrm{SU}(2)$ is a harmonic map.

The condition that n is harmonic is equivalent to the existence of a S^1-family of $\mathfrak{su}(2)$-valued 1-forms α_λ satisfying the Maurer–Cartan condition, for all $\lambda \in S^1$. Accordingly, by the work of Pohlmeyer [76], Zakharov–Shabat [86], and Uhlenbeck [82], the Gauss map has a representation as a holomorphic map into a *loop group*.

In 2010, Brander and Dorfmeister [12] solved the Björling problem for CMC surfaces using the loop group formulation of CMC surfaces, which in turn is based on the Dorfmeister–Pedit–Wu construction of harmonic maps from a surface to a compact symmetric space [25].

2.4. Björling's problem for Willmore surfaces

For Willmore surfaces, there is not a unique solution to the Björling problem as stated above. This has to do with the fact that the Willmore functional is Möbius invariant.

We recall that Möbius geometry is the conformal geometry of 3-sphere

$$S^3 \cong \mathbb{P}(\mathfrak{N}^{4,1}),$$

viewed as the projectivization of the *null cone* $\mathfrak{N}^{4,1}$ of Minkowski 4-space $\mathbb{R}^{4,1}$. A map $y : \Sigma \to S^3$ is lifted to a map Y into $\mathfrak{N}^{4,1}$, so that $y = [Y]$. In this case, the role of the Gauss map is now played by the *conformal Gauss map*

$$\psi : \Sigma \to S^{3,1} \cong \{\text{oriented 2-spheres of } S^3\}.$$

Away from umbilic points, the map ψ is a spacelike immersion which is orthogonal to Y and dY. In other words, this means that y envelopes ψ.

Now, a surface $y : \Sigma \to S^3$ is Willmore if and only if its conformal Gauss map ψ is harmonic into $S^{3,1}$. For a Willmore surface y, there is a *dual Willmore* $\hat{y}$ with the same conformal Gauss map ψ. The dual pair $(y, \hat{y})$ are the two envelopes of ψ, viewed as a 2-parameter family of spheres in S^3 (see [7, 16, 38, 42] and [66] for more details on the geometry of the conformal Gauss map).

A Cauchy problem for Willmore surfaces

A suitable initial value problem for Willmore surfaces is described as follows. Given a real analytic *curve of spheres* $\psi_0(x) : J \to S^{3,1}$, with enveloping curves $[Y_0]$, $[\widehat{Y}_0]$, satisfying $\langle Y_0, Y_0\rangle = \langle \widehat{Y}_0, \widehat{Y}_0\rangle = 0$, $\langle Y_0, \widehat{Y}_0\rangle = -1$, find a pair of dual Willmore surfaces

$$y = [Y],\, \hat{y} = [\widehat{Y}] : \Sigma \to S^3, \quad J \subset \Sigma,$$

such that:

- $Y(x,0) = Y_0(x)$, for $x \in J$,
- $\widehat{Y}(x,0) = \widehat{Y}_0(x)$, for $x \in J$,
- $\psi(x,0) = \psi_0(x)$, for $x \in J$,

being $\psi : \Sigma \to S^{3,1}$ the conformal Gauss map of y, $y : \Sigma \to S^3$.

Using the harmonicity of the conformal Gauss map, Helein [35] gives a Weierstrass type representation of Willmore immersions using previous work of Bryant [16] on Willmore surfaces and an extension of the Dorfmeister–Pedit–Wu method.

In 2014, Brander and Wang [15] solved the Björling problem for Willmore surfaces using Helein's loop group formulation.

REMARK 2.2. In the above situations, the theory of harmonic maps provides a unifying framework. More precisely:

- Harmonicity of a suitable Gauss map characterizes previous examples.
- They may be unified by application of the theory of harmonic maps.
- Such harmonic maps comprise an integrable system with Lax representation, spectral deformations, algebro-geometric solutions, etc.
- Harmonic maps and integrable systems provide a conceptual explanation for the solution of the above geometric Cauchy problems.

3. A Björling problem for equilibrium elastic surfaces

In this section we address the question of existence and uniqueness of a suitably formulated geometric Björling problem for equilibrium surfaces, i.e., surfaces satisfying the shape equation $\Delta H = \Phi(a,c)$. The position of the problem and its solution is given throughout this section.

THEOREM 3.1 ([41]). *Given:*

1. ***a real analytic curve*** $\alpha : J \to \mathbb{R}^3$, *with* $|\alpha'(x)| = 1$, *Frenet frame* $T = \alpha'$, N, B, *curvature* $\kappa(x) \neq 0$, *and torsion* $\tau(x)$*;*
2. ***a unit normal*** $W_0 = N(x_0)\cos a_0 + B(x_0)\sin a_0$, ***at*** $x_0 \in J$*;*
3. ***two real analytic functions*** h, $h^W : J \to \mathbb{R}$, *such that* h *satisfies*

$$h + \kappa \sin\left(-\int_{x_0}^{x} \tau(u)du + a_0\right) < 0, \tag{7}$$

then, there exists a real analytic immersion $f : \Sigma \to \mathbb{R}^3$, *where* $\Sigma \subset \mathbb{R}^2$ *is an open neighborhood of* $J \times \{0\}$, *with curvature line coordinates* (x,y), *such that:*

(a) *the mean curvature* H *of* f *satisfies* $\Delta H = \Phi(a,c)$*;*
(b) $f(x,0) = \alpha(x)$ *for all* $x \in J$, *and* α *is a curvature line of* f*;*

(c) the tangent plane to f at $f(x_0,0)$ is spanned by $T(x_0)$ and W_0;
(d) $H|_J = h$ and $\frac{\partial H}{\partial y}\big|_J = h^W$.

If $\widehat{f}:\widehat{\Sigma}\to\mathbb{R}^3$ is any other principal immersion satisfying the above conditions, then $f(\Sigma\cap\widehat{\Sigma}) = \widehat{f}(\Sigma\cap\widehat{\Sigma})$, i.e., f is unique up to reparametrizations.

REMARK 3.1. In Theorem 3.1, the curve α turns out to be a curvature line of the solution surface. This has to do with the fact that the normal vector field W along α such that $W(x_0) = W_0$ is chosen to be a (relatively) parallel field in the sense of Bishop [5] (see Section 3.3). A more general problem would be that of considering the shape equation together with a curve α, a generic unit normal W along the curve, and some other objects along α that determine all possible data up to order three that one can consider on the surface along α. The problem is then whether there exists a unique equilibrium surface meeting these data. The solution of this more general problem is likely to imply Theorem 3.1 as a special case.

As opposed to the cases discussed above, the shape equation is not known to be related to harmonic map theory, nor to other integrable systems. So these approaches do not apply in principle. Also, no explicit representation formulae are known for equilibrium surfaces that could be used for solving the Cauchy problem in geometric terms as in the classical Björling problem.

Our techniques will rely upon the Cartan–Kähler theory of Pfaffian differential systems and the method of moving frames (for a similar approach to the integrable system of Lie-minimal surfaces and other systems in submanifold geometry, as well as to several geometric variational problems within the general scheme of Griffiths' formalism of the calculus of variations [33], we refer to [55, 57, 58, 59, 61, 63, 64] and [26, 56, 60, 62, 65, 67]).

The main steps in the proof of Theorem 3.1 are the following:

1. Construction of a Pfaffian differential system (PDS) whose integral manifolds are canonical lifts of principal frames along surfaces satisfying the shape equation.
2. Analysis of the PDS to conclude that it is in involution by Cartan's test. Hence the Cartan–Kähler theorem yields the existence and uniqueness of an integral surface containing a given integral curve.
3. Construction of integral curves from the given initial data.
4. Geometric interpretation of the construction and of the existence result to determine the equilibrium surface.

3.1. The Pfaffian system of equilibrium surfaces

Structure equations of $\mathbb{E}(3)$

The Euclidean group $\mathbb{E}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3) \subset \mathrm{GL}(4,\mathbb{R})$ acts transitively on $\mathbb{R}^3$ by $(\mathbf{x},A)y = \mathbf{x} + Ay$. An element $(\mathbf{x},A)$ of $\mathbb{E}(3)$ is a frame A_1,A_2,A_3 at $\mathbf{x}$, where A_i denotes the

ith column of A. By regarding $\mathbf{x}$ and A_i as $\mathbb{R}^3$-valued maps on $\mathbb{E}(3)$, there are unique 1-forms θ^i, θ^i_j on $\mathbb{E}(3)$, $i,j=1,2,3$, such that

$$\begin{cases} d\mathbf{x} = \sum \theta^i A_i, \\ dA_i = \sum \theta^j_i A_j, \quad \theta^j_i = -\theta^i_j, \quad i=1,2,3. \end{cases}$$

The Maurer-Cartan forms θ^i, θ^i_j, $i,j=1,2,3$, of $\mathbb{E}(3)$ satisfy the *structure equations*

$$\begin{cases} d\theta^i = -\sum \theta^i_j \wedge \theta^j, \\ d\theta^i_j = -\sum \theta^i_k \wedge \theta^k_j. \end{cases} \tag{8}$$

Structure equations of the principal frame bundle

Let $f : X^2 \to \mathbb{R}^3$ be an immersed surface with unit normal field n, such that f does not have umbilic points. A *principal adapted frame* is a mapping

$$(f,(A_1,A_2,A_3)) : U \subset X \to \mathbb{E}(3),$$

such that, for each $\zeta \in X$, $\{A_1(\zeta),A_2(\zeta),A_3(\zeta)\}$ is a orthonormal basis of $T_{f(\zeta)}\mathbb{R}^3$, such that $A_1(\zeta),A_2(\zeta)$ are principal directions and $A_3 = n$.

These conditions easily imply that (θ^1,θ^2) induces a coframe field in X, and $\theta^3 = 0$. Moreover, we can express

$$\theta^3_1 = a\theta^1, \quad \theta^3_2 = c\theta^2, \quad \theta^2_1 = p\theta^1 + q\theta^2,$$

where a and c are the principal curvatures associated to the directions A_1 and A_2, respectively, which are assumed to satisfy $a > c$ without loss of generality, and p and q are smooth functions, the Christoffel symbols of f with respect to the coframe (θ^1,θ^2). Note that the first and second fundamental forms read $\mathrm{I} = df \cdot df = \theta^1\theta^1 + \theta^2\theta^2$ and $\mathrm{II} = -df \cdot dn = a\theta^1\theta^1 + c\theta^2\theta^2$, respectively.

The structure equations (8) in this setting give

$$\begin{cases} d\theta^1 = p\theta^1 \wedge \theta^2, \\ d\theta^2 = q\theta^1 \wedge \theta^2, \end{cases}$$

as well as the Gauss equation

$$dp \wedge \theta^1 + dq \wedge \theta^2 + (ac + p^2 + q^2)\theta^1 \wedge \theta^2 = 0, \tag{9}$$

and the Codazzi equations

$$\begin{cases} da \wedge \theta^1 + p(c-a)\theta^2 \wedge \theta^1 = 0, \\ dc \wedge \theta^2 + q(c-a)\theta^1 \wedge \theta^2 = 0. \end{cases} \tag{10}$$

REMARK 3.2. In order to simplify the notation in the sequel, given a smooth function $g : X \to \mathbb{R}$, let us write $dg = g_1\theta^1 + g_2\theta^2$, where $g_1, g_2 : X \to \mathbb{R}$ could be considered the "partial derivatives" of g with respect to θ^1, θ^2. It is easily seen that the mixed partials satisfy $g_{12} - g_{21} = pg_1 + qg_2$. Using the relation

$$(\Delta g)\,\theta^1 \wedge \theta^2 = d * dg = d(-g_2\theta^1 + g_1\theta^2),$$

which defines the Laplace–Beltrami operator Δ with respect to I, we find that

$$\Delta g = g_{11} + g_{22} + qg_1 - pg_2.$$

In this formalism Gauss and Codazzi equations (9) and (10) are writen as

$$p_2 - q_1 = ac + p^2 + q^2, \qquad \begin{cases} a_2 = -p(c-a), \\ c_1 = -q(c-a), \end{cases}$$

so we will define the auxiliary function $r = \frac{1}{2}(p_2 + q_1)$, in order to express

$$p_2 = \ r + \frac{1}{2}(ac + p^2 + q^2), \quad q_1 = \ r - \frac{1}{2}(ac + p^2 + q^2).$$

Differentiating the Codazzi equations yields

$$\begin{cases} a_{21} = (p_1 - pq)(a-c) + pa_1, \quad a_{12} = 2pa_1 + p_1(a-c), \\ a_{22} = \left(r + \frac{1}{2}(ac + p^2 + q^2)\right)(a-c) + p^2(a-c) - pc_2, \\ c_{21} = q_2(a-c) - 2qc_2, \quad c_{12} = (q_2 + pq)(a-c) - qc_2, \\ c_{11} = \left(r - \frac{1}{2}(ac + p^2 + q^2)\right)(a-c) + qa_1 - q^2(a-c), \end{cases}$$

so $\Delta H = \frac{1}{2}\Delta(a+c)$ is expressed in terms of the invariants p, q, a, c, r, a_1, c_2 by

$$\Delta H = \frac{1}{2}\,(a_{11} + c_{22}) - r(c-a) + qa_1 - pc_2.$$

Therefore, f satisfies the shape equation $\Delta H = \Phi(a,c)$ if and only if

$$a_{11} + c_{22} = 2\left(\Psi(p,q,a,c,a_1,c_2) + r(c-a)\right),$$

where

$$\Psi(p,q,a,c,a_1,c_2) = \Phi(a,c) + pc_2 - qa_1.$$

The important point here is that Ψ is not a function of r.

The Pfaffian system of principal frames

On the manifold

$$Y_{(1)} = \mathbb{E}(3) \times \left\{(p,q,a,c) \in \mathbb{R}^4 \,|\, a - c > 0\right\},$$

consider (I_1, Ω), the Pfaffian differential system (PDS) differentially generated by the 1-forms

$$\begin{aligned} &\alpha^1 = \theta^3, && \alpha^2 = \theta^2_1 - p\theta^1 - q\theta^2, \\ &\alpha^3 = \theta^3_1 - a\theta^1, && \alpha^4 = \theta^3_2 - c\theta^2, \end{aligned} \tag{11}$$

with *independence condition* $\Omega = \theta^1 \wedge \theta^2 \neq 0$.

Using the compatibility conditions for a surface in $\mathbb{R}^3$, we have the following.

LEMMA 3.1. *The* integral manifolds *of* (I_1, Ω) *are the smooth maps*

$$F_{(1)} := (F, A, p, q, a, c) : X \to Y_{(1)}$$

defined on an oriented connected surface X, such that:

- $F : X \to \mathbb{R}^3$ *is an* umbilic free *smooth immersion;*
- $(F, A) = (F, (A_1, A_2, A_3)) : X \to \mathbb{E}(3)$ *is a* principal frame *along F; a, c are the* principal curvatures *and p, q are the* Christoffel symbols *of F*.

Modulo the algebraic ideal generated by $\{\alpha^1, \alpha^2, \alpha^3, \alpha^4\}$, it is easy to get

$$\begin{cases} d\theta^1 \equiv p\theta^1 \wedge \theta^2, \quad d\theta^2 \equiv q\theta^1 \wedge \theta^2, \\ d\alpha^1 \equiv 0, \\ d\alpha^2 \equiv -dp \wedge \theta^1 - dq \wedge \theta^2 - (ac + p^2 + q^2)\theta^1 \wedge \theta^2, \\ d\alpha^3 \equiv -da \wedge \theta^1 + p(c-a)\theta^1 \wedge \theta^2, \\ d\alpha^4 \equiv -dc \wedge \theta^2 - q(c-a)\theta^1 \wedge \theta^2. \end{cases}$$

Nonetheless, the equations $\alpha_j = 0$ also imply the equations $d\alpha_j = 0$, which uncover additional integrability conditions. The prolongation of (I_1, Ω) is the system with these new 1-forms. To prolong (I_1, Ω), we define

$$Y_{(2)} = Y_{(1)} \times \left\{ (p_1, q_2, r, a_1, c_2, a_{11}, c_{22}) \in \mathbb{R}^7 \right\},$$

where $(p_1, q_2, r, a_1, c_2, a_{11}, c_{22})$ are the new fiber coordinates, and consider the PDS (I_2, Ω), differentially generated by

$$\begin{cases} \alpha^1, \alpha^2, \alpha^3, \alpha^4 \text{ defined in (11)}, \\ \beta^1 = dp - p_1\theta^1 - (r + \frac{1}{2}(ac + p^2 + q^2))\theta^2, \\ \beta^2 = dq - (r - \frac{1}{2}(ac + p^2 + q^2))\theta^1 - q_2\theta^2, \\ \gamma^1 = da - a_1\theta^1 + p(c-a)\theta^2, \\ \gamma^2 = dc + q(c-a)\theta^1 - c_2\theta^2, \\ \delta^1 = da_1 - a_{11}\theta^1 + (-2a_1p + (c-a)p_1)\theta^2, \\ \delta^2 = dc_2 + (2c_2q + (c-a)q_2)\theta^1 - c_{22}\theta^2. \end{cases} \tag{12}$$

Since the new 1-forms represent the geometric equations for a surface as derived in Section 3.1, from Lemma 3.1 we deduce the following.

LEMMA 3.2. *The* integral manifolds *of* (I_2, Ω) *are the smooth maps* $F_{(2)} : X \to Y_{(2)}$,

$$F_{(2)} = (F, A, p, q, a, c, p_1, q_2, r, a_1, c_2, a_{11}, c_{22}),$$

where $(F, A, p, q, a, c) : X \to Y_{(1)}$ *is an* integral manifold *of* (I_1, Ω)*, and the rest of variables represent the corresponding geometric quantities of the immersion* $F : X \to \mathbb{R}^3$.

The PDS of equilibrium surfaces

Let $Y_* \subset Y_{(2)}$ be the 16-dimensional submanifold (analytic subvariety) of $Y_{(2)}$ defined by the shape equation

$$a_{11} + c_{22} = 2\left[\Phi(a,c) + r(c-a) - qa_1 + pc_2\right] \iff \Delta H = \Phi(a,c).$$

We choose fiber coordinates $p,q,a,c,p_1,q_2,r,a_1,c_2$, where ℓ is defined by

$$\begin{cases} a_{11} = \ \ell + r(c-a) + \Psi(p,q,a,c,a_1,c_2), \\ c_{22} = -\ \ell + r(c-a) + \Psi(p,q,a,c,a_1,c_2). \end{cases}$$

The *PDS of equilibrium surfaces* (I_*,Ω) is defined as (I_2,Ω) restricted to Y_*. By Lemma 3.2, the integral manifolds of (I_*,Ω) are the prolongations $F_{(2)}$ of umbilic free immersions $F : X \to \mathbb{R}^3$ satisfying the shape equation $\Delta H = \Phi(a,c)$.

3.2. Involution of the PDS of equilibrium surfaces

Let $V_p(I_*)$, the variety of p-dimensional integral elements of I_*, $p = 1,2$, which is a contained in $G_p(TY_*)$, the Grassmannian bundle of p-dimensional subspaces of TY_*. By construction, (I_*,Ω) is differentially generated by

$$\begin{cases} \alpha^1,\alpha^2,\alpha^3,\alpha^4,\beta^1,\beta^2,\gamma^1,\gamma^2 \text{ defined in (12)}, \\ \delta^1 = da_1 - (\ell + r(c-a) + \Psi)\,\theta^1 + (p_1(c-a) - 2a_1 p)\,\theta^2, \\ \delta^2 = dc_2 + (q_2(c-a) + 2c_2 q)\,\theta^1 + (\ell - r(c-a) - \Psi)\,\theta^2. \end{cases}$$

Modulo the ideal generated by $\{\alpha^1,\alpha^2,\alpha^3,\alpha^4,\beta^1,\beta^2,\gamma^1,\gamma^2,\delta^1,\delta^2\}$, we get

$$\begin{cases} d\alpha^j \equiv 0, \quad j = 1,2,3,4, \\ d\gamma^a \equiv 0, \quad a = 1,2, \\ d\beta^1 \equiv -dp_1 \wedge \theta^1 - dr \wedge \theta^2 - B^1\theta^1 \wedge \theta^2, \\ d\beta^2 \equiv -dr \wedge \theta^1 - dq_2 \wedge \theta^2 - B^2\theta^1 \wedge \theta^2, \\ d\delta^1 \equiv -(d\ell + (c-a)dr) \wedge \theta^1 + (c-a)dp_1 \wedge \theta^2 - D^1\theta^1 \wedge \theta^2, \\ d\delta^2 \equiv (c-a)dq_2 \wedge \theta^1 + (d\ell - (c-a)dr) \wedge \theta^2 + D^2\theta^1 \wedge \theta^2. \end{cases} \tag{13}$$

where B^1,B^2,D^1,D^2 are some real analytic functions of the fiber coordinates. Thus (I_*,Ω) is differentially generated by $\{\alpha^1,\alpha^2,\alpha^3,\alpha^4,\beta^1,\beta^2,\gamma^1,\gamma^2,\delta^1,\delta^2\}$ together with the 1-forms in the right-hand sides of the last four equations of (13), which are linearly independent provided that $c \neq a$.

Given $p = 1,2$, let us consider the reduced Cartan character s'_p, defined as the maximum rank of the polar equations associated to an element in $G_p(TY_*)$. From the above discussion, it is not difficult to compute $s'_1 = 4$ and $s'_2 = 0$, so $s'_1 + 2s'_2 = 4$, which coincides with the degree of indeterminacy of the system given by Equation (13). The reader is referred to the monograph [34] for a clear and comprehensive explanation of this technique.

This means that (I,Ω) passes Cartan's test, i.e., the system is in involution. Given $E_1 \in V_1(I_*)$ at some $m \in Y_*$, its *polar* or *extension space* is given by

$$H(E_1) := \{u \in T_mY_* \,:\, E_1 + \mathbb{R}u \text{ is an integral element}\}.$$

Since $\dim H(E_1) = 2$ and the fibers of $V_2(I_*)$ are affine linear subspaces of $G_2(T_mY_*)$, it follows that $H(E_1)$ is the unique element of $V_2(I_*)$ containing E_1. From Cartan–Kähler theory, we get the following existence and uniqueness result:

LEMMA 3.3.

1. (I_*,Ω) *is in involution and its solutions depend on four functions in one variable.*
2. *For every 1-dimensional real analytic integral manifold* $\mathcal{A} \subset Y_*$ *there exists a unique real analytic 2-dimensional integral manifold* $\mathcal{X} \subset Y_*$ *through* $\mathcal{A}$.

3.3. Sketch of the proof of Theorem 3.1

Let us consider the *Cauchy data* $\left(\alpha, x_0, W_0, h, h^W\right)$, as in the statement of Theorem 3.1. Let (T,N,B) be a Frenet frame along the real analytic curve α, and denote by κ and τ the curvature and torsion of $\alpha : J \to \mathbb{R}^3$, respectively, determined by the Frenet-Serret equations $T' = \kappa N$, $N' = -kT + \tau B$ and $B' = -\tau N$. We define normal vector fields along α as

$$W(x) = \cos s(x)N(x) + \sin s(x)B(x),$$
$$JW(x) = -\sin s(x)N(x) + \cos s(x)B(x),$$

where the auxiliary real analytic function $s : J \to \mathbb{R}$ is given by

$$s(x) := -\int_{x_0}^{x} \tau(u)du + a_0,$$

so W extends the vector W_0 at x_0 (i.e., $W(x_0) = W_0$), and

$$\mathcal{G} = (\alpha, T, W, JW) : J \to \mathbb{E}(3).$$

is a orthonormal frame field along α (note that (T,W,JW) is nothing but a parallel frame along α, see [5]). Frenet-Serret equations easily yield

$$\frac{d\mathcal{G}}{dx} = \mathcal{G}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\mathfrak{p} & -\mathfrak{a} \\ 0 & \mathfrak{p} & 0 & 0 \\ 0 & \mathfrak{a} & 0 & 0 \end{pmatrix}, \quad \text{where} \begin{cases} \mathfrak{p} = \kappa\cos s(x), \\ \mathfrak{a} = -\kappa\sin s(x). \end{cases}$$

Next we define all the variables associated to the Cauchy data in order to produce a 1-dimensional integral element of I_*:

$$\begin{array}{ll} \mathfrak{c} = \mathfrak{a} - 2(h + \kappa\sin s(x)), & \mathfrak{q} = -\frac{1}{\mathfrak{c}-\mathfrak{a}}\frac{d\mathfrak{c}}{dx}, \\ \mathfrak{a}_1 = \frac{d\mathfrak{a}}{dx}, & \mathfrak{c}_2 = 2h^W + \mathfrak{p}(\mathfrak{c}-\mathfrak{a}), \\ \mathfrak{p}_1 = \frac{d\mathfrak{p}}{dx}, & \mathfrak{q}_2 = -\frac{1}{\mathfrak{c}-\mathfrak{a}}\left(\frac{d\mathfrak{c}_2}{dx} + 2\mathfrak{c}_2\mathfrak{q}\right), \\ \mathfrak{r} = \frac{d\mathfrak{q}}{dx} + \frac{1}{2}(\mathfrak{a}\mathfrak{c} + \mathfrak{p}^2 + \mathfrak{q}^2), & \mathfrak{l} = \frac{d^2\mathfrak{a}}{dx^2} - \mathfrak{r}(\mathfrak{c}-\mathfrak{a}) - \Psi(\mathfrak{p},\mathfrak{q},\mathfrak{a}_1,\mathfrak{c}_2). \end{array}$$

Note that the condition (7) in the statement is equivalent to $\mathfrak{a} > \mathfrak{c}$. It is straightforward that the curve $\mathcal{A} : J \to Y_*$ mapping x to $(\mathcal{G}, \mathfrak{p}, \mathfrak{q}, \mathfrak{a}, \mathfrak{c}, \mathfrak{p}_1, \mathfrak{q}_2, \mathfrak{r}, \mathfrak{a}_1, \mathfrak{c}_2, \mathfrak{l})$ at x parametrizes an integral curve $\mathfrak{U}$ of I_*, such that $\theta^1 = dx$, $\theta^2 = 0$, and

$$\begin{cases} p \circ \mathcal{A} = \mathfrak{p}, & q \circ \mathcal{A} = \mathfrak{q}, \\ a \circ \mathcal{A} = \mathfrak{a}, & c \circ \mathcal{A} = \mathfrak{c}, \\ p_1 \circ \mathcal{A} = \mathfrak{p}_1, & q_2 \circ \mathcal{A} = \mathfrak{q}_2, \\ r \circ \mathcal{A} = \mathfrak{r}, & a_1 \circ \mathcal{A} = \mathfrak{a}_1, \\ c_2 \circ \mathcal{A} = \mathfrak{c}_2, & \ell \circ \mathcal{A} = \mathfrak{l}. \end{cases}$$

Lemma 3.3 guarantees the existence and uniqueness of an integral surface $\mathcal{X} \subset Y_*$ containing the canonical integral curve $\mathfrak{U}$. From Section 3.1, we get that the first component $\mathbf{x}$ of $(\mathbf{x}, A, p, q, a, c, p_1, q_2, r, a_1, c_2, \ell) \in \mathcal{X}$ can be parametrized by a real analytic umbilic free immersion $F : \Sigma \to \mathbb{R}^3$, which is a solution to the shape equation $\Delta H = \Phi(a, c)$ and whose prolongation $F_{(2)}$ coincides with the inclusion $\iota : \mathcal{X} \to Y_*$.

Here, the extension result tells us that $\Sigma \subset \mathbb{R}^2$ can be taken as an open neighborhood of $J \times \{0\}$, such that $F(x, 0) = \alpha(x)$ for all $x \in J$. Since $\mathcal{X}$ has initial condition $\mathfrak{U} \subset \mathcal{X}$ and $\mathcal{A}^*(\theta^2) = 0$, α is a curvature line of F. Moreover,

- $F_*(T_{\mathcal{A}(x_0)}(\mathcal{X})) = \mathrm{span}(\{A_1(\alpha(x_0)), A_2(\alpha(x_0))\}) = \mathrm{span}(\{T(x_0), W(x_0)\})$,
- $H \circ \mathcal{A} = \frac{1}{2}(\mathfrak{a} + \mathfrak{c}) = h$,
- $dH|_{\mathcal{A}(x_0)} \equiv \frac{1}{2}(\mathfrak{c}_2 - \mathfrak{p}(\mathfrak{c} - \mathfrak{a}))\theta^2|_{\mathcal{A}(x_0)} = h^W \theta^2|_{\mathcal{A}(x_0)} \pmod{\theta^1|_{\mathcal{A}(x_0)}}$.

Hence F satisfies conditions (b), (c) and (d) in the statement. It remains to prove the uniqueness of F, which follows from the uniqueness in Lemma 3.3.

4. Examples: Helfrich cylinders

Let $S \subset \mathbb{R}^3(x_1, x_2, x_3)$ be a cylinder over a simple closed curve $\alpha \subset \mathbb{R}(x_1, x_2)$, with curvature $\kappa(x)$ and with generating lines parallel to the x_3-axis. It then follows that $H = -\kappa/2$ and $K = 0$. The cylinder S satisfies $\Delta H = \Phi(a, c)$ if and only if $\kappa'' = -2\Phi(-\kappa, 0)$. Given the initial data:

- $\alpha \subset \mathbb{R}^2(x_1, x_2) \subset \mathbb{R}^3$, a convex simple closed plane curve with signed curvature κ satisfying $\kappa'' = -2\Phi(-\kappa, 0)$;
- a point $\alpha(x_0)$ and the unit normal vector $W_0 = -e_3$ (it corresponds to $a_0 = -\pi/2$);
- $h = -\kappa/2$ and $h^W = 0$ (the integral condition is satisfied),

S is the unique equilibrium surface determined by such initial data.

A Helfrich cylinder is a cylindrical surface S satisfying the Ou-Yang–Helfrich equation (3). In this case,

- the curvature $\kappa(x)$ of $\alpha \subset \mathbb{R}^2(x_1, x_2)$ must satisfy

$$\kappa'' + \frac{1}{2}\kappa^3 - \nu\kappa - \frac{2p}{\mathfrak{b}} = 0, \quad \nu := 2(\mathfrak{a} + kc_0^2)/\mathfrak{b}. \tag{14}$$

- Differentiating (14), yields

$$\kappa''' + \frac{3}{2}\kappa^2\kappa' - \nu\kappa' = 0.$$

- $\kappa(x,t) = \kappa(x+\nu t)$ is a traveling wave solution of the *modified Korteveg–de Vries (mKdV) equation* ([32, 68])

$$\kappa_t = \frac{3}{2}\kappa^2\kappa_x + \kappa_{xxx}.$$

- α moves without changing its shape when its curvature evolves according to the mKdV equation (congruence curve).
- (14) has a first integral,

$$(\kappa')^2 + \frac{1}{4}(\kappa^4 + w_2\kappa^2 + w_1\kappa + w_0) = 0. \tag{15}$$

- If the pressure p vanishes, S is a *Willmore cylinder* and α is a closed elastic curve (lemniscates): it has self-intersections.
- *Closedness and embeddedness* of curves satisfying (15) have been studied by Vassilev, Djondjorov, and Mladenov [84], and Musso [54].

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AMS Subject Classification: 53C42, 58A17, 76A20

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Lavoro pervenuto in redazione il 16.05.2016.