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Homogenization of networks in domains with oscillating boundaries

Andrea Braides* and Valeria Chiadò Piat†

In memoriam V.V. Zhikov

Abstract

We consider the asymptotic behaviour of integral energies with convex integrands defined on one-dimensional networks contained in a region of the three-dimensional space with a fast-oscillating boundary as the period of the oscillation tends to zero, keeping the oscillation themselves of fixed size. The limit energy, obtained as a Γ-limit with respect to an appropriate convergence, is defined in a 'stratified' Sobolev space and is written as an integral functional depending on all, two or just one derivative, depending on the connectedness properties of the sublevels of the function describing the profile of the oscillations. In the three cases, the energy function is characterized through an usual homogenization formula for p-connected networks, a homogenization formula for thin-film networks and a homogenization formula for thin-rod networks, respectively.

Keywords: networks, homogenization, thin structures, p-connectedness, Γ-convergence

1 Introduction

We will consider energies defined in a portion of the three-dimensional physical space delimited by a corrugated surface, whose overall behaviour is determined by oscillations which have an amplitude at the same scale of the linear dimensions of the sample. This geometric setting can be modelled by introducing a profile function g and a typical oscillation length scale $\varepsilon > 0$, which is instead assumed to be small. The function $g : \mathbb{R}^2 \to [0, 1]$ is supposed to be periodic. Taken a set ω in $\mathbb{R}^2$ as a basis, the domains of our energies can be written as

$$\Omega_\varepsilon = \Big\{x \in \mathbb{R}^3 : 0 < x_3 < g\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big)\Big\} \subset \omega \times (0, 1).$$

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Note that while the frequency of the profile increases as $\varepsilon \to 0$, its height remains constant. A popular choice is when Ω_ε describes a periodic array of rods with basis on a three-dimensional plate, in which case we may take g equal to 1 on a union of periodic disjoint sets and another constant otherwise (see e.g. the recent paper [13]). We will consider a general g as in [3, 14, 12] with the requirement that the sublevel sets

$$\Sigma(z) = \{(x_1, x_2) \in \mathbb{R}^2 : z < g(x_1, x_2)\}$$

satisfy the following conditions: there exist $0 < z_1 \leq z_2 \leq 1$ such that

(i) $\Sigma(z)$ is connected for $0 \leq z < z_1$;

(ii) $\Sigma(z)$ is disconnected and consists of a disjoint union of connected sets periodic in the x_2 direction differing by a translation in the x_1-direction for $z_1 < z < z_2$;

(iii) $\Sigma(z)$ is the union of disjoint compact connected components for $z_2 < z < 1$, each two differing by a translation..

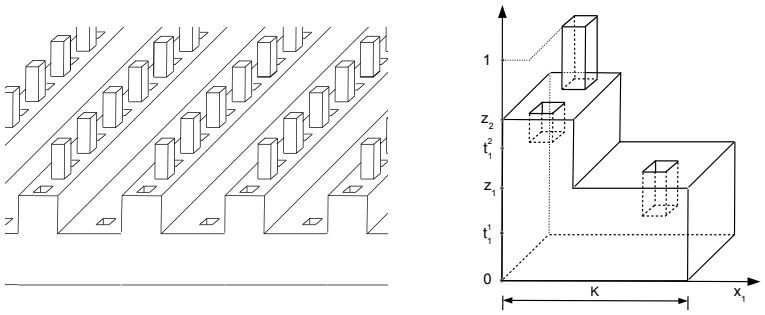


Figure 1: A set Ω_ε and the graph of its g on the periodicity cell

The conditions are satisfied in the case of rods on a plate, for which in particular $\Sigma(z) = \mathbb{R}^2$ for $z < z_1$ and $z_1 = z_2$ so that the set of z satisfying (ii) is empty. A set Ω_ε satisfying the conditions above is pictured on the left-hand side of Fig. 1, in which case g is piecewise constant and the corresponding graph in its period $(0, K)^2$ is represented on the right-hand side picture in Fig. 1. In this case the sublevel sets are not constant with z in the three cases (i)–(iii). Indeed, in the notation introduced in the figure, which is in accordance with that used in the following, in this example $\Sigma(z) = \mathbb{R}^2$ for $z \leq t_1^1$ and $\Sigma(z)$ is a connected set obtained by removing a periodic array of squares from $\mathbb{R}^2$ for $t_1^1 < z \leq z_1$, while $\Sigma(z)$ is a periodic set of stripes for $z_1 < z \leq t_1^2$, from which a periodic array of squares is removed for $t_1^2 < z \leq z_2$.

Note that if we normalize g so that $\sup g = 1$, as in the examples above, then the sets Ω_ε "invade" $\omega \times (0, 1)$ in the sense that the weak limit of their characteristic functions has a strictly positive Lebesgue density on the whole $\omega \times (0, 1)$. This observation allows us to regard limits of problems defined on Ω_ε as defined in the whole $\omega \times (0, 1)$. This is made more precise by looking at particular energies defined on Ω_ε.

In the spirit of [8, 16, 19] (see also [17, 18]), in the present paper we will examine the behaviour of the restrictions to Ω_ε of energies depending on a measure defined on

a lower-dimensional set. We will deal with the prototypical case of one-dimensional networks, and explicitly only when the measure is a scaling of the one-dimensional Hausdorff measure restricted to the canonical cubic network; i.e., to

$$\mathcal{N} = \{x \in \mathbb{R}^3 : x_i \in \mathbb{Z} \text{ for at least two indices } i \in \{1,2,3\}\},$$

but we will keep our analysis as general as possible so that other one-dimensional networks can be considered without essentially changing statements and proofs. The corresponding energies are of the form

$$F_\varepsilon(u) = \varepsilon^2 \int_{\Omega_\varepsilon \cap \varepsilon\mathcal{N}} f\Big(\frac{x}{\varepsilon}, Du\Big) d\mathcal{H}^1 \tag{1}$$

defined on $u \in C^1(\omega \times (0,1))$. The function f is supposed to be 1-periodic and satisfy standard p-growth conditions with $p > 1$. The scaling factor ε^2 is explained by noting that the total length of $\Omega_\varepsilon \cap \varepsilon\mathcal{N}$ is of the order of $\frac{1}{\varepsilon^2}$.

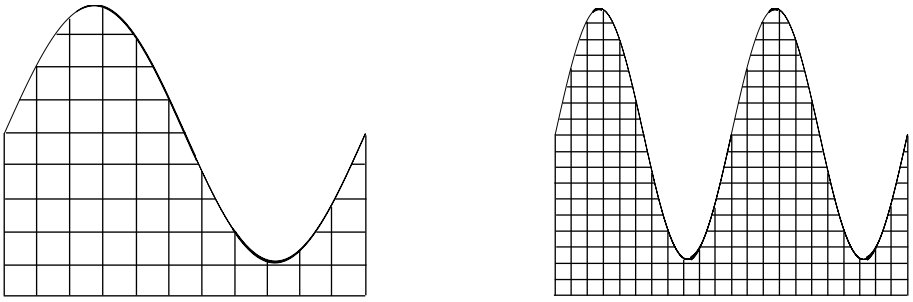


Figure 2: Cross-section of a set Ω_ε and the corresponding $\Omega_{\varepsilon/2}$

We consider a K-periodic profile function g with K integer. This assumption implies that Ω_ε is εK-periodic in the directions x_1 and x_2, but note that each Ω_ε has a different structure in the third direction. In Fig. 2 we picture the cross-section of Ω_ε for some ε and $\varepsilon/2$, respectively.

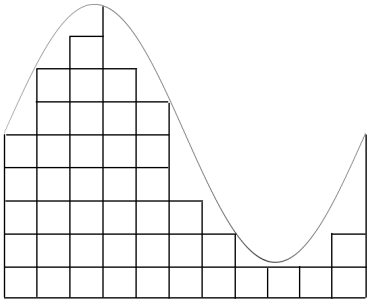


Figure 3: Cross-section of the "backbone network" of a set Ω_ε

The simple geometry of the network allows to clarify the asymptotic analysis of the energies F_ε as $\varepsilon \to 0$. Indeed, note first that, upon locally optimizing recovery

sequences, we may consider the "backbone structure" obtained by removing from the set $\Omega_\varepsilon \cap \varepsilon\mathcal{N}$ all one-dimensional segments with only one endpoint in $\mathbb{Z}^3$; i.e., in place of that set, we may consider the set $\mathcal{S}_\varepsilon$ that contains all segments in the coordinate direction with both endpoints in $\Omega_\varepsilon \cap \varepsilon\mathbb{Z}^3$. The cross-section of such a set corresponding to Ω_ε as in Fig. 2 is reproduced in Fig. 3. Now, since the two-dimensional sections of $\mathcal{S}_\varepsilon$ with $x_3 \in \varepsilon\mathbb{Z}$ are decreasing with x_3 and may have only a finite number of geometries, there are a finite number of points $\{s_j : j = 1, \dots, n\}$ with $s_{i-1} < s_i$, and periodic one-dimensional networks S_j in $\mathbb{R}^2$ such that $\mathcal{S}_\varepsilon$ locally coincides with $\varepsilon(S_j \times \mathbb{Z})$ on $\omega \times (s_{j-1}, s_j)$ (we set $s_0 = 0$).

We assume that the two-dimensional traces of the backbone structure at different values of the third component $x_3 = z \in \varepsilon\mathbb{Z}$ inherit the corresponding connected properties of $\Sigma(z)$. Note that, if $z_1 > 0$ then the measures $\varepsilon^2 d\mathcal{H}^1$ in the energies (1), when restricted to the sets $\varepsilon(S_j \times \mathbb{Z}) \cup ((\mathbb{Z}^2 \cap S_j) \times \mathbb{R})$, satisfy the p-connectedness hypotheses of [8, 16, 19] (see Section 2.3) for $s_j < z_1$, thanks to condition (i) on $\Sigma(z)$. Hence, the energies satisfy an equi-coerciveness property that guarantee that their Γ-limit is defined in a Sobolev space $W^{1,p}$ and that they can be homogenized and represented as an integral with some energy function $f^j_{\rm hom} = f^j_{\rm hom}(Du)$. As a consequence functions in the domain of the Γ-limit of F_ε satisfy

$$u \in W^{1,p}(\omega \times (0, z_1)). \tag{2}$$

Conditions (ii) and (iii) on $\Sigma(z)$ guarantee on one hand that the energies are coercive with respect some weak-L^p type convergence, and the domain of the Γ-limit is composed of functions u with

$$\partial_2 u \in L^p(\omega \times (0, z_2)), \qquad \partial_3 u \in L^p(\omega \times (0, 1)). \tag{3}$$

Hence, the domain of the Γ-limit is exactly the space $X_p(\omega)$ of the functions $u \in L^p(\omega \times (0,1))$ such that the partial derivatives in the sense of distributions satisfy conditions (2) and (3) above. The determination of the Γ-limit in this domain can be obtained by again examining the behaviour of functionals (1), with $\varepsilon\mathcal{N}$ replaced by $\varepsilon(S_j \times \mathbb{Z}) \cup ((\mathbb{Z}^2 \cap S_j) \times \mathbb{R})$. If $z_1 < s_j < z_2$ then such energies can be regarded as defined on disconnected networks of *thin-film type* with average normal in the x_1 direction, whose dimensionally reduced limit depends only on the derivative in the x_2 and x_3 directions and is described by an energy function $f^j_{\rm hom} = f^j_{\rm hom}(\partial_2 u, \partial_3 u)$. Finally, if $z_2 < s_j < z_3$ then such energies can be regarded as defined on disconnected networks of *thin-rod type* oriented in the x_3 direction and is described by an energy function $f^j_{\rm hom} = f^j_{\rm hom}(\partial_3 u)$. For the proof of these two results we assume that f is convex in the gradient variable as a technical hypothesis. By proving that optimal sequences in each $\omega \times (s_{j-1}, s_j)$ are compatible with a global construction, we finally show that the Γ-limit can be represented as

$$\begin{aligned} F_{\rm hom}(u) \;=\; & \sum_{j \in \{s_j \le z_1\}} \int_{\omega \times (s_{j-1}, s_j)} f^j_{\rm hom}(Du)\, dx + \sum_{j \in \{z_1 < s_j \le z_2\}} \int_{\omega \times (s_{j-1}, s_j)} f^j_{\rm hom}(\partial_2 u, \partial_3 u)\, dx \\ & + \sum_{j \in \{z_2 < s_j \le 1\}} \int_{\omega \times (s_{j-1}, s_j)} f^j_{\rm hom}(\partial_3 u)\, dx \end{aligned} \tag{4}$$

for all $u \in X_p(\omega)$. We note that the result in its general form can be stated and proved without changes for more general networks satisfying p-connectedness assumptions (see [8]).

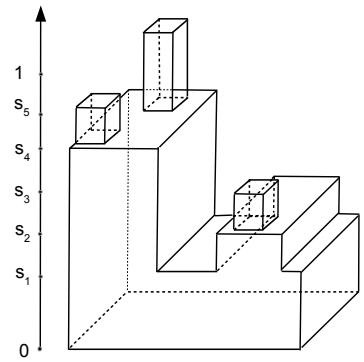


Figure 4: Graph of g with $\Sigma(z)$ not satisfying hypotheses (ii) and (iii)

We remark that $\Sigma(z)$ may not satisfy assumptions (i)–(iii) for a general g. In Fig. 4 we picture such a piecewise-constant g. Note that for $s_1 < z \leq s_2$ we have two different types of infinite connected components in the x_2 direction, for $s_2 < z \leq s_3$ both infinite and compact connected components, and for $s_4 < z \leq s_5$ two different types of compact connected components. The outline of the proof described above can be adapted to such functions, taking into account that thin-film and thin-rod arguments may have to be used for different underlying sets on $\omega \times (s_{j-1}, s_j)$ at the same time, with a heavier notation. We note moreover that assumption (ii) can be generalized to $\Sigma(z)$ consisting of the union of connected sets periodic in some direction v differing by a translation. In this case the derivative $\partial_2 u$ must be substituted with the partial derivative of u in the direction v in the definition of $X_p(\omega)$ and the subsequent computation of the Γ-limit.

Finally, it must be remarked that our result can be seen as an improvement of the corresponding convergence theorem in [3] since here a weaker topology is used, with respect to which the functionals are equicoercive, and not the strong L^p-convergence used therein, but this stronger convergence result is allowed by the additional convexity hypothesis on f.

2 Notation and statement of the problem

With $\lfloor t \rfloor$ we denote the integer part of $t \in \mathbb{R}$. If $x \in \mathbb{R}^3$, we set $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \lfloor x_3 \rfloor)$. We also write $(x_1, x_2, x_3) = (\widehat{x}, x_3)$, when needed. For any $y, y' \in \mathbb{R}^3$ (or $\mathbb{R}^2$) we denote by $[y, y']$ the (straight-line) segment with end points y, y', and by $y \cdot y'$ their scalar product.

The letter c will denote a generic strictly positive constant.

We refer e.g. to [10] for weak semicontinuity and strong continuity properties of

functionals of the form

$$\int_\Omega f(x, Du)\, dx$$

defined on standard Sobolev spaces, with f convex in the gradient variable and satisfying standard growth conditions. Such properties will be used in the sequel without further reference.

2.1 The geometrical setting: networks

We denote by $\mathcal{N}$ the one-dimensional cubic network generated by $\mathbb{Z}^3$, i.e.,

$$\mathcal{N} \;=\; \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \text{at least two components of } x \text{ are in } \mathbb{Z}\}. \quad (5)$$

The profile function. We consider a lower-semicontinuous function $g = g(\widehat{y}) = g(y_1, y_2) = \mathbb{R}^2 \to [0, 1]$, and suppose that g is $(0, K)^2$-periodic for some $K \in \mathbb{N} \setminus \{0\}$, and $\min g > 0$.

The network. For any $z \in \mathbb{R}$ we denote by $S(z)$ the subset of $\mathbb{R}^2$

$$S(z) = \bigcup \{[y, y'] :\ y, y' \in \mathbb{Z}^2,\ |y - y'| = 1,\ g(w) \geq z \text{ if } w \in [y, y']\}. \quad (6)$$

We assume that $S(z)$ satisfies the analog of the assumptions that we described for $\Sigma(z)$ in the Introduction. More precisely, there exist $0 < z_1 \leq z_2 \leq 1$ such that

(a) if $0 \leq z \leq z_1$, then $S(z)$ is connected

(b) if $z_1 < z \leq z_2$, then $S_1(z) := S(z) \cap ([0, K) \times \mathbb{R})$ is connected and $S_1(z)$ is disconnected from $S_1(z) + K(i, 0)$ for any $i \in \mathbb{Z}$, $i \neq 0$.

(c) if $z_2 < z \leq 1$ then $S_2(z) := S(z) \cap [0, K)^2$ is connected and not empty and $S_2(z)$ is disconnected from $S_2(z) + K(i, j)$ for any $(i, j) \in \mathbb{Z}^2$, $(i, j) \neq (0, 0)$.

We note that the assumption $z_1 > 0$ is essential for Lemma 2.4. If $z_1 = z_2$ the problem reduces to cases (a) and (c), while if $z_2 = 1$ we have only cases (a) and (b).

Note that $z \mapsto S(z)$ is locally constant. Since $S(z)$ is K-periodic and decreasing with z, there exist t_i^j, $j = 1, 2, 3$, and $n_j \in \mathbb{N}$, such that

$$\begin{aligned} 0 &= t_0^1 < \ldots < t_{n_1}^1 = z_1 \\ z_1 &= t_0^2 < \ldots < t_{n_2}^2 = z_2 \\ z_2 &= t_0^3 < \ldots < t_{n_3}^3 = 1, \end{aligned} \quad (7)$$

and

$$S(z) = S_i^j \quad \text{is constant for } z \in (t_{i-1}^j, t_i^j). \quad (8)$$

2.2 Energies defined on networks

We introduce the one-dimensional cubic lattice $\varepsilon\mathcal{N}$ scaled by a small parameter $\varepsilon > 0$ and consider its intersection with the subgraph of the function g, namely

$$D_\varepsilon = \varepsilon\mathcal{N} \cap \Big\{x \in \mathbb{R}^3 : 0 < x_3 < g\Big(\frac{\widehat{x}}{\varepsilon}\Big)\Big\}. \tag{9}$$

We fix a bounded open set $\omega \subset \mathbb{R}^2$, with Lipschitz boundary, and a Carathéodory function $f = f(y,\xi) : \mathbb{R}^3 \times \mathbb{R}^3 :\to \mathbb{R}$, such that

(i) $f(\cdot,\xi)$ is $(0,K)^3$-periodic for any $\xi \in \mathbb{R}^3$;

(ii) $f(y,\cdot)$ is convex for all $y \in \mathbb{R}^3$

(iii) there exist $p > 1$ and $0 < c_1 \le c_2$,

$$c_1(|\xi|^p - 1) \le f(y,\xi) \le c_2(1 + |\xi|^p) \tag{10}$$

for all $y \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$. For simplicity, we also assume that

$$f(y,0) = 0 \quad \text{ for all } y \in \mathbb{R}^3. \tag{11}$$

Our aim is to study the asymptotic behaviour of minimum problems for the functional

$$F_\varepsilon(u) = \begin{cases} \varepsilon^2 \displaystyle\int_{(\omega\times(0,1))\cap D_\varepsilon} f\Big(\frac{x}{\varepsilon}, Du\Big)\, d\mathcal{H}^1 & \text{if } u \in \mathcal{C}^1(\mathbb{R}^3) \\ +\infty & \text{otherwise} \end{cases} \tag{12}$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure.

Following the approach by Γ-convergence [6, 7], we have to choose a suitable topology, and then compute the Γ-limit of F_ε with respect to that topology. The choice of the topology is driven by coerciveness requirements that allow to deduce the convergence of minimum problems, and is specified in the following section. Since the domain D_ε may contain small disconnected parts or small "appendices" that make the geometry more complex but do not influence the behaviour of these energies, it is convenient to restrict energies and functions to a more handy set, which is locally periodic. Indeed, as long as the computation of the Γ-limit is concerned, energies defined on functions whose domain is D_ε are equivalent to those defined on $\mathcal{S}_\varepsilon$, where

$$\mathcal{S}_\varepsilon = \bigcup\{[v,w] : v,w \in \varepsilon\mathbb{Z}^3,\ |v-w| = \varepsilon,\ [v,w] \subset D_\varepsilon\}.$$

This means that, the Γ-limits of $F_\varepsilon(u, D_\varepsilon)$ and $F_\varepsilon(u, \mathcal{S}_\varepsilon)$ are the same. One inequality is trivial since $\mathcal{S}_\varepsilon \subset D_\varepsilon$, while the converse inequality is due to the fact that, thanks to (11), recovery sequences on the larger set can be defined by extending the corresponding recovery sequences on $\mathcal{S}_\varepsilon$ as constants on each component of $D_\varepsilon \setminus \mathcal{S}_\varepsilon$.

With this observation in mind, we will consider the energies

$$F_\varepsilon(u) = \begin{cases} \varepsilon^2 \displaystyle\int_{(\omega\times(0,1))\cap \mathcal{S}_\varepsilon} f\Big(\frac{x}{\varepsilon}, Du\Big) d\mathcal{H}^1 & \text{if } u \in \mathcal{C}^1(\mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases} \tag{13}$$

Note that such energies can also be seen as a particular case of those in (12) choosing piecewise-constant g such that D_ε coincides with $\mathcal{S}_\varepsilon$. Note in addition that these energies can be defined in spaces of Sobolev functions defined on $\mathcal{S}_\varepsilon$ with respect to the $\mathcal{H}^1$ measure as in [4] (see also [8, 5] for the notation in that setting).

2.3 Continuum convergence of functions on networks

We introduce the measures $\mu_\varepsilon = \varepsilon^2\mathcal{H}^1|\mathcal{S}_\varepsilon$; i.e., defined by

$$\mu_\varepsilon(A) = \varepsilon^2\mathcal{H}^1(A \cap \mathcal{S}_\varepsilon), \tag{14}$$

Their weak* limit is absolutely continuous with respect to the three-dimensional Lebesgue measure $\mathcal{L}^3$; more precisely, we have

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu(x_3)\mathcal{L}^3 \tag{15}$$

and

$$\mu(z) = \eta(z) + \frac{1}{K^2}\mathcal{H}^1(S(z)\cap[0,K]^2), \text{ where } \eta(z) = \frac{1}{K^2}\#\{S(z)\cap\{1,\dots,K\}^2\}. \tag{16}$$

Definition 2.1. *We say that* $u_\varepsilon \xrightarrow{\mu} u$ *as* $\varepsilon \to 0$ *if and only if* $u_\varepsilon\mu_\varepsilon \overset{*}{\rightharpoonup} u\mu\mathcal{L}^3$ *as* $\varepsilon \to 0$*, with* μ *as in* (15)*.*

We will use this definition of convergence, which we will simply denote as $u_\varepsilon \to u$ if no confusion may arise, in the computation of Γ-limits. This is justified by (14) and the compactness Lemma 2.4 below.

We remark that the restriction of μ_ε to (compact subsets of) $\omega \times (t_{i-1}^j, t_i^j)$ can be as a scaled version of a periodic measure. Namely,

$$\mu_\varepsilon(A) = \varepsilon^3\mu_i^j\Big(\frac{1}{\varepsilon}A\Big)$$

for A compactly contained in $\omega \times (t_{i-1}^j, t_i^j)$, where

$$\mu_i^j = \mathcal{H}^1|(S_i^j \times \mathbb{Z}) \cup ((\mathbb{Z}^2 \cap S_i^j) \times \mathbb{R}).$$

These measures satisfy connectedness conditions (see [15]) that allow to apply some homogenization results for singular structures, e.g. those of [8, 16, 19].

Remark 2.2 (p-connectedness)**.** If $j = 1$ then each measure μ_i^1 satisfies the following conditions.

1) Coerciveness: there exist two constants $c_0 > 0$ and $\delta \geq 0$ such that

$$|u^w - u^{w'}|^p \leq c_0 \int_{(Y^w \cup Y^{w'})+(-\delta,\delta)^3} |Du|^p \, d\mu_i^1$$

for every $w, w' \in \mathbb{Z}^3$ with $|w - w'| = 1$ and for every $u \in C^1(\mathbb{R}^3)$, where

$$u^w = \frac{1}{\mu(Y)} \int_{w+Y} u \, d\mu_i^1.$$

2) Poincaré-Wirtinger's inequality: there exist two constants $c = c(p) > 0$, $\delta \geq 0$ such that

$$\int_Y |u - \overline{u}|^p \, d\mu_i^1 \leq c \int_{(-\delta, 1+\delta)^3} |Du|^p \, d\mu_i^1,$$

for every $u \in C^1(\mathbb{R}^3)$, where $\overline{u} = \dfrac{1}{\mu(Y)} \displaystyle\int_Y u \, d\mu_i^1$.

Remark 2.3 (an auxiliary convergence)**.** We will also use the auxiliary convergence. $u_\varepsilon \xrightarrow{\eta} u$ defined as $u_\varepsilon \eta_\varepsilon \overset{*}{\rightharpoonup} u\eta\mathcal{L}^3$ as $\varepsilon \to 0$, with η as in (16) and $\eta_\varepsilon = \varepsilon^3 \displaystyle\sum_{x \in \mathbb{Z}^3 \cap \mathcal{S}_\varepsilon} \delta_x$, where δ_x denotes the Dirac delta at x.

Lemma 2.4. *If $F_\varepsilon(u_\varepsilon) \leq c < +\infty$ for all $\varepsilon > 0$, then, for every $\varepsilon_j \to 0$ there exists a subsequence (not relabeled) and a function u such that, $u_{\varepsilon_j} \xrightarrow{\mu} u$ up to addition of a constant and*

$$u, \partial_3 u \in L^p(\omega \times (0,1)), \quad \partial_1 u \in L^p(\omega \times (0, z_1)), \quad \partial_2 u \in L^p(\omega \times (0, z_2)). \tag{17}$$

By statement (17), we mean that the distributional partial derivative of u is (identified with) an L^p function in the corresponding space.

Proof. We first note that by an application of a Poincaré-Wirtinger inequality on the network, the convergences $u_\varepsilon \xrightarrow{\mu} u$ and $u_\varepsilon \xrightarrow{\eta} u$ are equivalent on functions with equibounded energy.

Moreover, on $\omega \times (0, \min g)$ the measure η_ε is simply the sum of Dirac deltas at points in $\varepsilon\mathbb{Z}^3$ and a bound on F_ε implies a bound on the p-norm of difference quotients between nearest neighbours in $\varepsilon\mathbb{Z}^3$ there. Hence the functions u_ε can be interpreted as $W^{1,p}(\omega \times (0, \min g))$-functions with equibounded L^p-norm of the gradient, which assures that there exist constants c_ε such that $u_\varepsilon + c_\varepsilon$ converge in $L^p(\omega \times (0, \min g))$, up to extraction of a subsequence. Note that this also holds on $\omega \times (0, z_1)$, using an extension argument as in [1]. However, the compactness on $\omega \times (0, \min g)$ is sufficient for our argument.

By hypothesis (c) on $S(z)$ we may suppose, up to an integer translation, that $g(0,0) = 1$. By an application of a Poincaré-Wirtinger inequality on the network, the convergences $u_\varepsilon \xrightarrow{\mu} u$ and $u_\varepsilon \xrightarrow{\eta} u$ are equivalent to the weak convergence of the functions

$$\overline{u}_\varepsilon(x) = u_\varepsilon\Big(\Big\lfloor \frac{\hat{x}}{\varepsilon K} \Big\rfloor \varepsilon K, x_3\Big),$$

obtained only considering the functions u_ε on a periodic array of vertical sections on which the energies are coercive, to u in $L^p(\omega \times (0,1))$.

This characterization immediately shows that $\partial_3 u \in L^p(\omega \times (0,1))$. Indeed for all test functions $\phi \in C_0^\infty(\omega \times (0,1))$ we have (denoting D_3 the partial derivative in the sense of distributions on $\omega \times (0,1)$)

$$\begin{aligned} \langle D_3 u, \phi \rangle &= -\int_{\omega \times (0,1)} u \, \partial_3 \phi \, dx \\ &= -\lim_{\varepsilon \to 0} \int_{\omega \times (0,1)} \overline{u}_\varepsilon \, \partial_3 \phi \, dx = -\int_{\omega \times (0,1)} U \partial_3 \phi \, dx, \end{aligned}$$

where U is the weak limit of a (possibly further) subsequence of $\{\partial \overline{u}_\varepsilon\}$. Note again that this sequence is precompact in $L^p(\omega \times (0,1))$. A similar argument also shows that $\partial_2 u \in L^p(\omega \times (0, z_2))$.

Finally, since our energies satisfy the periodicity and p-connectedness hypotheses of [8, 16, 19] on each subset $\omega \times (t^1_{i-1}, t^1_i)$, we can conclude the existence of a subsequence such that, up to translations $u_\varepsilon \mu_\varepsilon \overset{*}{\rightharpoonup} u\mu\, \mathcal{L}^3$ on $\omega \times (0, z_1)$ and $u \in W^{1,p}(\omega \times (0, z_1))$. □

Definition 2.5 ("stratified Sobolev space")**.** We define the space $X_p(\omega)$ as the space of functions $u \in L^p(\omega \times (0,1))$ whose distributional derivatives satisfy (17).

3 Definition of homogenized energy densities

We begin by noting that on $\omega \times (t^j_{i-1}, t^j_i)$ the set $\mathcal{S}_\varepsilon$ coincides with $\varepsilon \mathcal{S}^j_i$, where

$$\mathcal{S}^j_i = \big((S^j_i \times \mathbb{Z}) \cup ((\mathbb{Z}^2 \cap S^j_i) \times \mathbb{R})\big).$$

Our energies, restricted to such sets, can be homogenized using known results or adapting techniques developed for standard functionals. We have three different homogenization results depending on j.

Theorem 3.1. *For all $a < b$, let $F^{1,i}_\varepsilon(\cdot, (a,b))$ be defined by*

$$F^{1,i}_\varepsilon(u, (a,b)) = \varepsilon^2 \int_{(\omega \times (a,b)) \cap \varepsilon \mathcal{S}^1_i} f\Big(\frac{x}{\varepsilon}, Du\Big)\, d\mathcal{H}^1 \tag{18}$$

for $u \in C^1(\omega \times (a,b))$. Then $F^{1,i}_\varepsilon(\cdot, (a,b))$ Γ-converge, with respect to the convergence $u_\varepsilon \xrightarrow{\mu} u$ to the functional

$$F^{1,i}_{\rm hom}(u, (a,b)) = \int_{\omega \times (a,b)} f^{1,i}_{\rm hom}(Du) dx \tag{19}$$

with domain $W^{1,p}(\omega \times (a,b))$, where the convex function $f^{1,i}_{\rm hom} : \mathbb{R}^3 \to \mathbb{R}$ satisfies

$$\begin{aligned} f^{1,i}_{\rm hom}(\xi) &= \lim_{T \to +\infty} \frac{1}{T^3} \inf\Big\{ \int_{[0,T]^3 \cap \mathcal{S}^1_i} f(y, Du)\, d\mathcal{H}^1 : u(x) = \xi \cdot x \\ &\qquad\qquad \text{in a neighbourhood of } \partial [0,T]^3 \Big\} \qquad (20) \\ &= \frac{1}{K^3} \inf\Big\{ \int_{[0,K)^3 \cap \varepsilon \mathcal{S}^1_i} f(y, Du)\, d\mathcal{H}^1 : u(x) - \xi \cdot x \ K\text{-periodic} \Big\}. \qquad (21) \end{aligned}$$

and a growth condition

$$a_1(|\xi|^p - 1) \le f^{1,i}_{\rm hom}(\xi) \le a_2(1 + |\xi|^p) \text{ for all } \xi \in \mathbb{R}^3 \tag{22}$$

with $0 < a_1 \le a_2$.

Proof. The theorem follows from [8] remarking that $\mathcal{S}_i^1$ satisfies the p-connectedness assumptions therein, cell-formula (21) following from (20) and the convexity of f by a standard averaging argument (see e.g. [10] Section 14.3.1). □

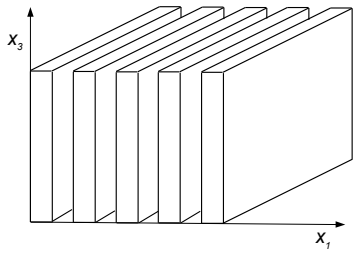


Figure 5: A thin-film profile

We now examine the case $j = 2$, corresponding to $S(z)$ being "connected in the direction x_2". A pictorial representation for the corresponding geometry (pictured as a continuum) is contained in Fig. 5

Theorem 3.2 (thin-film homogenization)**.** *For all $a < b$, let $F_\varepsilon^{2,i}(\cdot,(a,b))$ be defined by*

$$F_\varepsilon^{2,i}(u,(a,b)) = \varepsilon^2 \int_{(\omega\times(a,b))\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon}, Du\Big)\, d\mathcal{H}^1 \tag{23}$$

for $u \in C^1(\omega\times(a,b))$. Then $F_\varepsilon^{2,i}(\cdot,(a,b))$ Γ-converge, with respect to the convergence $u_\varepsilon \xrightarrow{\mu} u$ to the functional

$$F_{\rm hom}^{2,i}(u,(a,b)) = \int_{\omega\times(a,b)} f_{\rm hom}^{2,i}(\partial_2 u, \partial_3 u)dx \tag{24}$$

with domain the space of functions in $L^p(\omega\times(a,b))$ such that the distributional derivatives with respect to x_2 and x_3 belong to $L^p(\omega\times(a,b))$, where the convex function $f_{\rm hom}^{2,i}:\mathbb{R}^2\to\mathbb{R}$ satisfies

$$\begin{aligned} f_{\rm hom}^{2,i}(\xi_2,\xi_3) &= \lim_{T\to+\infty}\frac{1}{T^2K}\inf\Big\{\int_{([0,K)\times[0,T]^2)\cap\mathcal{S}_i^2} f(y,Du)\,d\mathcal{H}^1 : \\ &\qquad u(x)=\xi_2x_2+\xi_3x_3+v_0(x_1)\ \textit{in a neighbourhood of}\ [0,K)\times\partial[0,T]^2\Big\} \qquad (25)\\ &= \frac{1}{K^3}\inf\Big\{\int_{[0,K)^3\cap\mathcal{S}_i^2} f(y,Du)\,d\mathcal{H}^1 : u(x)-\xi_2x_2-\xi_3x_3\ K\textit{-periodic in } x_2,x_3\Big\}, \qquad (26)\end{aligned}$$

where v_0 is any fixed smooth function. and a growth condition

$$a_1(|\widehat{\xi}|^p-1) \le f_{\rm hom}^{2,i}(\xi_2,\xi_3) \le a_2(1+|\widehat{\xi}|^p) \quad \textit{for all } \widehat{\xi}=(\xi_2,\xi_3)\in\mathbb{R}^2 \tag{27}$$

with $0 < a_1 \le a_2$.

Proof. We preliminarily note that, fixed V open set of $\mathbb{R}^2$, the functionals

$$G_\varepsilon^{2,i}(v,V) = \frac{\varepsilon}{K}\int_{([0,\varepsilon K)\times V)\cap \varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon}, Du\Big)\, d\mathcal{H}^1 \tag{28}$$

can be considered as energies on "thin-film networks", and we may consider their Γ-limit in a dimension-reduction setting, where the "thin-film convergence" $v_\varepsilon \to v$ to a function $v \in W^{1,p}(V)$ may be defined as the convergence of

$$\sum_{z\in\varepsilon\mathcal{S}_i^2\cap(\varepsilon[0,K)\times V)} \varepsilon^2 v_\varepsilon(z)\delta_z \overset{*}{\rightharpoonup} \zeta_i^2 v(x_2,x_3)\mathcal{H}^2|\{x : x_1 = 0, \widehat{x}\in V\},$$

and the constant ζ_i^2 is defined by $\displaystyle\sum_{z\in\varepsilon\mathcal{S}_i^2\cap(\varepsilon[0,K)\times\mathbb{R}^2)} \varepsilon^2\delta_z \overset{*}{\rightharpoonup} \zeta_i^2\mathcal{H}^2|\{x_1 = 0\}$. The Γ-limit of $G_\varepsilon^{2,i}$ with respect to this convergence is then given by

$$G_{\hom}^{2,i}(v,V) = \int_V f_{\hom}^{2,i}(Dv)\, dx. \tag{29}$$

A general result for thin films depending on general measures is not present in the literature, and will be contained in a forthcoming paper [9]. In the setting of the present paper, the result can be obtained in parallel with those of [2] (discrete thin films with flat profile) and [11] (elastic thin films with varying profile). In those papers the limit energy is described as

$$\begin{aligned} f_{\hom}^{2,i}(\xi_2,\xi_3) \;=\; & \lim_{T\to+\infty}\frac{1}{T^2K}\inf\Big\{\int_{([0,K)\times[0,T]^2)\cap\mathcal{S}_i^2} f(y,Du)\, d\mathcal{H}^1 \\ & : u(x) - \xi_2x_2 - \xi_3x_3 \ T\text{-periodic in } x_2,x_3\Big\}. \end{aligned} \tag{30}$$

From this, formula (26) follows from the convexity of f by the same argument as in [10] Section 14.3.1 using translations in the x_1-x_3 directions. Note that test functions for (25) are also test functions for (30), from which one inequality between the formulas holds. Conversely, we may consider test functions for (26) extended by periodicity to obtain test functions for (25) by a cut-off argument with $\xi_2x_2+\xi_3x_3+v_0(x_1)$ in a neighbourhood of $[0,K)\times\partial([0,T]^2)$ to eventually obtain the equality between all formulas. More precisely, with fixed $\eta>0$ let $u : [0,K)\times\mathbb{R}^2 \to \mathbb{R}$ be such that $u(x) - \xi_2x_2 - \xi_3x_3$ is K-periodic in x_2, x_3 and

$$\frac{1}{K^3}\int_{[0,K)^3\cap\mathcal{S}_i^2} f(y,Du)\, d\mathcal{H}^1 < f_{\hom}^{2,i}(\xi_2,\xi_3) + \eta.$$

We also denote $P_{KT} := [0,K)\times[0,T]^2$ and, for $S>0$,

$$A_T^S = [0,K)\times([0,T]^2\setminus[S,T-S]^2).$$

In order to replace u by u_T admissible function for formula (25), such that

$$\frac{1}{T^2K}\int_{P_{KT}\cap\mathcal{S}_i^2} f(y,Du_T)\, d\mathcal{H}^1 \le \frac{1}{K^3}\int_{[0,K)^3\cap\mathcal{S}_i^2} f(y,Du)\, d\mathcal{H}^1 + \eta + o(1) \tag{31}$$

as $T \to +\infty$, we choose a smooth cut-off function $\varphi = \varphi_T : [0,T]^2 \to [0,1]$, such that $\varphi = 1$ in $[0,T]^2 \setminus [K, T-K]^2$ and $\varphi = 0$ in $[2K, T-2K]^2$, and we set

$$u_T(x) = u(x) + \varphi_T(\widehat{x})\big(v_0(x_1) + \widehat{\xi}\cdot\widehat{x} - u(x)\big).$$

Note that $u_T - \widehat{\xi}\cdot\widehat{x} = v_0$ on A_T^K.

We can split the integral on P_{KT} as

$$\begin{aligned}\frac{1}{T^2K}\int_{P_{KT}\cap\mathcal{S}_i^2} f(y,Du_T)\,d\mathcal{H}^1 &= \frac{1}{T^2K}\int_{(P_{KT}\setminus A_T^{2K})\cap\mathcal{S}_i^2} f(y,Du)\,d\mathcal{H}^1\\ &+\frac{1}{T^2K}\int_{\mathcal{S}_i^2\cap A_T^K} f(y,Dv_0+(0,\widehat{\xi}))\,d\mathcal{H}^1 + \frac{1}{T^2K}\int_{(A_T^{2K}\setminus A_T^K)\cap\mathcal{S}_i^2} f(y,Du_T)\,d\mathcal{H}^1\\ &=: A+B+C\end{aligned}$$

The first integral can be estimated using the periodicity of u as

$$A \le \frac{1}{K^3}\int_{[0,K)^3\cap\mathcal{S}_i^2} f(y,Du)\,d\mathcal{H}^1 < f_{\rm hom}^{2,i}(\xi_2,\xi_3) + \eta.$$

As for the second integral, using the upper bound (10) for $f(y,\xi)$, and estimating the number of disjoint periodicity cubes of side length K which are contained in A_T^K by cKT, we have

$$B \le \frac{c}{T}\int_{[0,K)^3}\Big(1+|Dv_0|^p+|\widehat{\xi}|^p\Big)\,d\mathcal{H}^1 \le \frac{c}{T}.$$

As for the third integral, combining the above arguments, we have

$$\begin{aligned}C &\le \frac{c}{T^2K}\int_{\mathcal{S}_i^2\cap(A_T^{2K}\setminus A_T^K)}\Big(1+|v_0|^p+|\widehat{\xi}\cdot\widehat{y}-u|^p+|Dv_0|^p+|(0,\widehat{\xi})-Du|^p\Big)d\mathcal{H}^1\\ &\le \frac{c}{T}(1+\eta)\end{aligned}$$

Summing up the above three estimates, we have (31).

In order to prove a lower bound for $F_{\rm hom}^{2,i}$, we first treat the case when ω is a square. For the sake of notational simplicity we suppose that $\omega = (0,1)^2$. Let $u_\varepsilon \to u$ and let $\rho > 0$. The argument of the proof is to subdivide $(0,1)^2$ into stripes of width ρ in the x_1 direction, and examine the average behaviour of u_ε on each of these stripes averaging on substripes of width εK; finally, by letting $\rho \to 0$ we obtain a lower bound for $F_{\rm hom}^{2,i}(u)$. Again, we may suppose that $1/\rho \in \mathbb{N}$ and $\rho/\varepsilon K \in \mathbb{N}$ to ease notation, so that we do not have remainders in this subdivision process.

For $k \in \{1,\dots,\frac{1}{\rho}\}$ we define $u_\varepsilon^{\rho,k} : [0,\varepsilon K)\times(0,1)\times(a,b) \to \mathbb{R}$ by

$$u_\varepsilon^{\rho,k}(x) = \frac{\varepsilon K}{\rho}\sum_{l=1}^{\frac{\varepsilon K}{\rho}} u_\varepsilon(x_1+(k-1)\rho+(l-1)\varepsilon K, x_2, x_3)\,.$$

By the convexity of f we have

$$\begin{aligned}
&\frac{\varepsilon K}{\rho}\sum_{l=1}^{\frac{\varepsilon K}{\rho}}\int_{((k-1)\rho+[0,\varepsilon K))\times(0,1)\times(a,b))\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon},Du_\varepsilon\Big)\,d\mathcal{H}^1\\
=\;&\frac{\varepsilon K}{\rho}\sum_{l=1}^{\frac{\varepsilon K}{\rho}}\int_{[0,\varepsilon K)\times(0,1)\times(a,b))\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon},Du_\varepsilon(x_1+(k-1)\rho+(l-1)\varepsilon K,x_2,x_3)\Big)\,d\mathcal{H}^1\\
\geq\;&\int_{[0,\varepsilon K)\times(0,1)\times(a,b))\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon},Du_\varepsilon^{\rho,k}\Big)\,d\mathcal{H}^1. \qquad (32)
\end{aligned}$$

The sequence $\{u_\varepsilon^{\rho,k}\}$ thin-film converges as $\varepsilon\to 0$ to the function $u^{\rho,k}\in W^{1,p}((0,1)\times(a,b))$ defined by

$$u^{\rho,k}(x_2,x_3)=\frac{1}{\rho}\int_{(k-1)\rho}^{k\rho} u(t,x_2,x_3)\,dt. \qquad (33)$$

We also set

$$u^\rho(x_1,x_2,x_2)=u^{\rho,k}(x_2,x_3)\quad\text{ if }\quad(k-1)\rho\leq x_1<k\rho. \qquad (34)$$

Note that u^ρ is in the domain of $F_{\rm hom}^{2,i}$.

By (32) and the thin-film convergence to (29) we then have

$$\begin{aligned}
&\liminf_{\varepsilon\to0}\varepsilon^2\int_{((0,1)^2\times(a,b))\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon},Du_\varepsilon\Big)\,d\mathcal{H}^1\\
&\quad=\liminf_{\varepsilon\to0}\sum_{k=1}^{1/\rho}\sum_{l=1}^{\frac{\varepsilon K}{\rho}}\varepsilon^2\int_{((k-1)\rho+(l-1)\varepsilon K+[0,\varepsilon K))\times(0,1)\times(a,b))\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon},Du_\varepsilon\Big)\,d\mathcal{H}^1\\
&\quad\geq\liminf_{\varepsilon\to0}\sum_{k=1}^{1/\rho}\frac{\varepsilon\rho}{K}\int_{[0,\varepsilon K)\times(0,1)\times(a,b))\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon},Du_\varepsilon^{\rho,k}\Big)\,d\mathcal{H}^1\\
&\quad\geq\sum_{k=1}^{1/\rho}\rho\liminf_{\varepsilon\to0}\frac{\varepsilon}{K}\int_{[0,\varepsilon K)\times(0,1)\times(a,b))\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon},Du_\varepsilon^{\rho,k}\Big)\,d\mathcal{H}^1\\
&\quad\geq\sum_{k=1}^{1/\rho}\rho\int_{(0,1)\times(a,b)} f_{\rm hom}^{2,i}(\partial_2u^{\rho,k},\partial_3u^{\rho,k})\,dx_2\,dx_3\\
&\quad=\sum_{k=1}^{1/\rho}\int_{((k-1)\rho,k\rho)\times(0,1)\times(a,b)} f_{\rm hom}^{2,i}(\partial_2u^{\rho},\partial_3u^{\rho})\,dx\\
&\quad=\int_{(0,1)^2\times(a,b)} f_{\rm hom}^{2,i}(\partial_2u^{\rho},\partial_3u^{\rho})\,dx=F_{\rm hom}^{2,i}(u^\rho).
\end{aligned}$$

It suffices now to remark that $u^\rho\to u$ as $\rho\to0$ to obtain a lower bound. The case of a general ω is obtained by approximating ω from the interior by a union of squares, to which the previous argument is applied separately.

The upper bound can be proved by taking a target function u, which we may suppose smooth by a convolution argument and also defined outside $\omega \times (a,b)$ (see also the proof of Theorem 4.1 below). Again, we first treat the case when ω is a square, which we again assume to be $(0,1)^2$. We fix $\rho > 0$, which we may suppose being such that $1/\rho \in \mathbb{N}$ and $\rho/\varepsilon K \in \mathbb{N}$. The fist assumption is not a restriction, while the second one can be removed at the expense of some remainder term uniformly tending to 0 as $\varepsilon \to 0$. We consider the functions u^ρ and $u^{\rho,k}$ defined by (34) and (33), respectively. A recovery sequence$\{u_\varepsilon^\rho\}$ for u^ρ is then obtained, first considering for each k a recovery sequence $\{v_\varepsilon^{\rho,k}\}$ for $G^{2,i}_{\rm hom}(u^{\rho,k},(0,1)\times(a,b))$, and then defining

$$u_\varepsilon^\rho(x_1,x_2,x_3) = v_\varepsilon^{\rho,k}(x_1-(k-1)\rho-(l-1)\varepsilon K, x_2, x_3)$$

if $(k-1)\rho + (l-1)\varepsilon K \le x_1 < k\rho + l\varepsilon K$. Note that this function is not C^1, but may be modified to a C^1 function without changing the energy by the disconnectedness hypothesis on the network. We do not make this modification explicit. Since $F^{2,i}_{\rm hom}(u^\rho) = F^{2,i}_{\rm hom}(u)$ this proves the upper bound by approximation. The case of a general ω can be obtained by approximating it from the exterior by union of cubes, taking into account that u may be extended outside ω. □

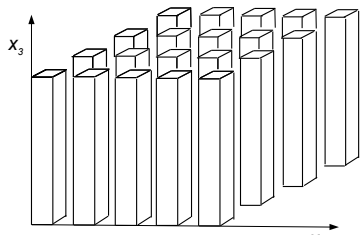


Figure 6: A thin-rod profile

We now examine the case $j = 3$, corresponding to $S(z)$ being the union of compact connected components. A pictorial representation for the corresponding geometry (pictured as a continuum) is contained in Fig. 6

Theorem 3.3 (thin-rod homogenization)**.** *For all $a < b$, let $F_\varepsilon^{3,i}(\cdot,(a,b))$ be defined by*

$$F_\varepsilon^{3,i}(u,(a,b)) = \varepsilon^2 \int_{(\omega\times(a,b))\cap\varepsilon\mathcal{S}_i^3} f\Big(\frac{x}{\varepsilon}, Du\Big)\, d\mathcal{H}^1 \tag{35}$$

for $u \in C^1(\omega\times(a,b))$. Then $F_\varepsilon^{3,i}(\cdot,(a,b))$ Γ-converge, with respect to the convergence $u_\varepsilon \xrightarrow{\mu} u$ to the functional

$$F^{3,i}_{\rm hom}(u,(a,b)) = \int_{\omega\times(a,b)} f^{3,i}_{\rm hom}(\partial_3 u)dx \tag{36}$$

with domain the space of functions in $L^p(\omega \times (a,b))$ such that the distributional derivative with respect to x_3 belongs to $L^p(\omega \times (a,b))$, where the convex function $f^{3,i}_{\rm hom} : \mathbb{R} \to \mathbb{R}$ satisfies

$$f^{3,i}_{\rm hom}(\xi_3) = \lim_{T\to+\infty} \frac{1}{TK^2} \inf\Big\{ \int_{([0,K)^2\times[0,T])\cap\mathcal{S}^3_i} f(y,Du)\,d\mathcal{H}^1 : \tag{37}$$

$$u(x) = \xi_3 x_3 + v_0(x_1,x_2) \text{ in a neighbourhood of } [0,K)^2 \times \partial[0,T] \Big\}$$

$$= \frac{1}{K^3} \inf\Big\{ \int_{[0,K)^3\cap\mathcal{S}^3_i} f(y,Du)\,d\mathcal{H}^1 : u(x) - \xi_3 x_3 \ K\text{-periodic in } x_3 \Big\}, \tag{38}$$

where v_0 is any fixed smooth function. Moreover, $f^{3,i}_{\rm hom}$ satisfies a growth condition

$$a_1(|\xi_3|^p - 1) \le f^{3,i}_{\rm hom}(\xi_3) \le a_2(1+|\xi_3|^p) \quad \text{for all } \xi_3 \in \mathbb{R} \tag{39}$$

with $0 < a_1 \le a_2$.

Proof. We preliminarily note that, fixed W open set of $\mathbb{R}$, the functionals

$$G^{3,i}_\varepsilon(v,W) = \frac{1}{K^2} \int_{(\varepsilon[0,K)^2\times W)\cap\varepsilon\mathcal{S}^3_i} f\Big(\frac{x}{\varepsilon}, Du\Big)\,d\mathcal{H}^1 \tag{40}$$

can be considered as "thin-rod networks", and we may consider their Γ-limit in a dimension-reduction setting, where the "thin-rod convergence" $w_\varepsilon \to w$ to a function $w \in W^{1,p}(W)$ may be defined as the convergence of

$$\sum_{i\in\varepsilon\mathcal{S}^3_i\cap(\varepsilon[0,K)\times W)} \varepsilon v_\varepsilon(i)\delta_i \overset{*}{\rightharpoonup} \zeta^3_i v(x_3)\mathcal{H}^2|\{x : x_1 = x_2 = 0, x_3 \in W\},$$

where the constant ζ^3_i is defined by $\displaystyle\sum_{i\in\varepsilon\mathcal{S}^3_i\cap(\varepsilon[0,K)\times\mathbb{R})} \varepsilon\delta_i \overset{*}{\rightharpoonup} \zeta^3_i\mathcal{H}^2|\{x : x_1 = x_2 = 0\}$.
The Γ-limit of $G^{3,i}_\varepsilon$ with respect to this convergence is then given by

$$G^{3,i}_{\rm hom}(v,V) = \int_V f^{3,i}_{\rm hom}(Dv)\,dx. \tag{41}$$

We still refer to [2] and [11] for similar convergence results, from which this one can be deduced. The equality of the two formulas for $f^{3,i}_{\rm hom}$ follows from the convexity of f as in the proof of Theorem 3.2.

The proof for the lower bound for $F^{3,i}_{\rm hom}$ may be obtained in a way similar to that of $F^{2,i}_{\rm hom}$. Again, we first treat the case when ω is a square. For the sake of notational simplicity we suppose that $\omega = (0,1)^2$. Let $u_\varepsilon \to u$ and let $\rho > 0$. The argument of the proof is to subdivide $(0,1)^2$ into squares of side length ρ, and examine the average behaviour of u_ε on each of these squares averaging on subsquares of side length εK; finally, by letting $\rho \to 0$ we obtain a lower bound for $F^{3,i}_{\rm hom}(u)$. Again, we may suppose that $1/\rho \in \mathbb{N}$ and $\rho/\varepsilon K \in \mathbb{N}$ to ease notation, so that we do not have remainders in this subdivision process.

For $k=(k_1,k_2)\in\{1,\dots,\frac{1}{\rho}\}^2$ we define $u_\varepsilon^{\rho,k}:[0,\varepsilon K)^2\times(a,b)\to\mathbb{R}$ by

$$u_\varepsilon^{\rho,k}(x)=\frac{\varepsilon^2K^2}{\rho^2}\sum_{l\in\{1,\dots,\frac{\varepsilon K}{\rho}\}^2}u_\varepsilon^{k,l}(x)\,,$$

where $u_\varepsilon^{k,l}(x)=u_\varepsilon(x_1+(k_1-1)\rho+(l_1-1)\varepsilon K,x_2+(k_2-1)\rho+(l_2-1)\varepsilon K,x_3)$. We also set

$$Q_\varepsilon^{k,l}=(k_1-1,k_2-1)\rho+((l_1-1,l_2-1)\varepsilon K+[0,\varepsilon K)^2.$$

By the convexity of f we have

$$\begin{aligned}
&\frac{\varepsilon^2K^2}{\rho^2}\sum_{l\in\{1,\dots,\frac{\varepsilon K}{\rho}\}^2}\int_{(Q_\varepsilon^{k,l}\times(a,b))\cap\varepsilon\mathcal{S}_i^3}f\Big(\frac{x}{\varepsilon},Du_\varepsilon\Big)\,d\mathcal{H}^1\\
=\;&\frac{\varepsilon^2K^2}{\rho^2}\sum_{l\in\{1,\dots,\frac{\varepsilon K}{\rho}\}^2}\int_{([0,\varepsilon K)^2\times(a,b))\cap\varepsilon\mathcal{S}_i^3}f\Big(\frac{x}{\varepsilon},Du_\varepsilon^{k,l}(x)\Big)\,d\mathcal{H}^1\\
\geq\;&\int_{([0,\varepsilon K)^2\times(a,b))\cap\varepsilon\mathcal{S}_i^3}f\Big(\frac{x}{\varepsilon},Du_\varepsilon^{\rho,k}\Big)\,d\mathcal{H}^1.
\end{aligned}\tag{42}$$

The sequence $\{u_\varepsilon^{\rho,k}\}$ thin-rod converges as $\varepsilon\to0$ to the function $u^{\rho,k}\in W^{1,p}(a,b)$ defined by

$$u^{\rho,k}(x_3)=\frac{1}{\rho^2}\int_{(k_1-1,k_2-1)+(0,\rho)^2}u(t_1,t_2,x_3)\,dt_1\,dt_2.$$

We also set $u^\rho(x_1,x_2,x_2)=u^{\rho,k}(x_3)$ if $(x_1,x_2)\in(k_1-1,k_2-1)+(0,\rho)^2$. Note that u^ρ is in the domain of $F_{\rm hom}^{3,i}$.

By (42) and the thin-rod convergence to (41) we then have

$$\begin{aligned}
&\liminf_{\varepsilon\to0}\varepsilon^2\int_{((0,1)^2\times(a,b))\cap\varepsilon\mathcal{S}_i^3}f\Big(\frac{x}{\varepsilon},Du_\varepsilon\Big)\,d\mathcal{H}^1\\
&\quad=\liminf_{\varepsilon\to0}\sum_{k\in\{1,\dots1/\rho\}^2}\sum_{l\in\{1,\dots,\frac{\varepsilon K}{\rho}\}^2}\varepsilon^2\int_{(Q_\varepsilon^{k,l}\times(a,b))\cap\varepsilon\mathcal{S}_i^3}f\Big(\frac{x}{\varepsilon},Du_\varepsilon\Big)\,d\mathcal{H}^1\\
&\quad\geq\liminf_{\varepsilon\to0}\sum_{k\in\{1,\dots1/\rho\}^2}\frac{\rho^2}{K^2}\int_{[0,\varepsilon K)^2\times(a,b))\cap\varepsilon\mathcal{S}_i^3}f\Big(\frac{x}{\varepsilon},Du_\varepsilon^{\rho,k}\Big)\,d\mathcal{H}^1\\
&\quad\geq\sum_{k\in\{1,\dots1/\rho\}^2}\rho^2\liminf_{\varepsilon\to0}\frac{1}{K^2}\int_{[0,\varepsilon K)^2\times(a,b))\cap\varepsilon\mathcal{S}_i^3}f\Big(\frac{x}{\varepsilon},Du_\varepsilon^{\rho,k}\Big)\,d\mathcal{H}^1\\
&\quad\geq\sum_{k\in\{1,\dots1/\rho\}^2}\rho^2\int_{(a,b)}f_{\rm hom}^{3,i}(\partial_3u^{\rho,k})\,dx_3\\
&\quad=\sum_{k\in\{1,\dots1/\rho\}^2}\int_{((k_1-1,k_2-1)+(0,\rho)^2)\times(a,b)}f_{\rm hom}^{3,i}(\partial_3u^\rho)\,dx\\
&\quad=\int_{(0,1)^2\times(a,b)}f_{\rm hom}^{3,i}(\partial_3u^\rho)\,dx=F_{\rm hom}^{3,i}(u^\rho).
\end{aligned}$$

It suffices now to remark that $u^\rho \to u$ as $\rho \to 0$ to obtain a lower bound. The case of a general ω is obtained by approximating ω from the interior by a union of squares, to which the previous argument is applied separately.

As in the proof for the thin-film case, the upper bound can be considered by taking a target function u which we may suppose smooth by a convolution argument, and a recovery sequence obtained by separately considering recovery sequences on each "thin-rod" section. The construction is completely analogous to that of the upper bound in Theorem 3.2, using the functions u^ρ, $u^{\rho,k}$ as defined above, and recovery sequences for $G^{3,i}_{\rm hom}(u^{\rho,k},(a,b))$. □

We are in the position to define the *homogenized energy density* of f using the definition of $f^{j,i}_{\rm hom}$ in the corresponding layer. More precisely, we define $f_{\rm hom} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ as

$$f_{\rm hom}(x_3,\xi) = \begin{cases} f^{1,i}_{\rm hom}(\xi) & \text{if } t^1_{i-1} < x_3 < t^1_i \\ f^{2,i}_{\rm hom}(\xi_2,\xi_3) & \text{if } t^2_{i-1} < x_3 < t^2_i \\ f^{3,i}_{\rm hom}(\xi_3) & \text{if } t^3_{i-1} < x_3 < t^3_i, \end{cases} \tag{43}$$

where t^j_i are defined in (7).

4 Statement and proof of the main result

We recall that $X_p = X_p(\omega)$ is defined as the space of all functions in $L^p(\omega\times(0,1))$ such that the partial derivatives in the sense of distributions satisfy $\partial_1 u \in L^p(\omega\times(0,z_1))$, $\partial_2 u \in L^p(\omega\times(0,z_2))$, and $\partial_3 u \in L^p(\omega\times(0,1))$, where z_1, z_2 are defined by the connectedness properties of the profile function g. For such a function, with an abuse of notation that may cause no ambiguity in the following, we denote by Du any measurable vector function (Ξ_1,Ξ_2,Ξ_3) such that

$$\Xi_1 = \partial_1 u \text{ a.e. in } \omega\times(0,z_1), \quad \Xi_2 = \partial_2 u \text{ a.e. in } \omega\times(0,z_2), \quad \Xi_3 = \partial_3 u \text{ a.e. in } \omega\times(0,1).$$

Theorem 4.1. *Let $f_{\rm hom}$ be defined by* (43)*, and, for given ω Lipschitz subset of $\mathbb{R}^2$, let F_ε be defined by* (13)*. Then F_ε Γ-converges with respect to the convergence $u_\varepsilon \xrightarrow{\mu} u$ to the functional*

$$F_{\rm hom}(u) = \int_{\omega\times(0,1)} f_{\rm hom}(x_3, Du)\,dx \tag{44}$$

for $u \in X_p(\omega)$.

Note that the integral in (44) can also be split as in (4).

Proof. In order to prove the lower bound, it suffices to remark that, for $u_\varepsilon \xrightarrow{\mu} u$, we have, using Theorems 3.1–3.3

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon\to 0} \sum_{j=1}^{3}\sum_{i=1}^{n_j} F_\varepsilon^{j,i}(u_\varepsilon,(t_{i-1}^j,t_i^j))\\ &\geq \sum_{j=1}^{3}\sum_{i=1}^{n_j} \liminf_{\varepsilon\to 0} F_\varepsilon^{j,i}(u_\varepsilon,(t_{i-1}^j,t_i^j))\\ &\geq \sum_{j=1}^{3}\sum_{i=1}^{n_j} F_{\rm hom}^{j,i}(u,(t_{i-1}^j,t_i^j)) = F_{\rm hom}(u).\end{aligned}$$

The upper bound can be proved by successive approximations. First, observe that, by an extension argument in the x_1-x_2 directions we may suppose that $u \in X_p(\omega')$ for some ω' compactly containing ω. In addition, for all $\zeta > 0$ a dilation argument defining

$$u_\zeta(x_1,x_2,x_3) = u\Big(x_1,x_2,(x_3+\zeta)\frac{z_1}{z_1+2\zeta}\Big)$$

gives $u_\zeta \in W^{1,p}(\omega'\times(-\zeta,z_1+\zeta))$, $\partial_2 u_\zeta \in L^p(\omega'\times(-\zeta,z_2+\zeta))$ and $\partial_3 u_\zeta \in L^p(\omega'\times(-\zeta,1+\zeta))$. Hence, up to a further mollification argument, we conclude that we may approximate any $u \in X_p(\omega)$ with functions u_ζ in $C^\infty(\omega\times(0,1))$ with $u_\zeta \to u$ in $L^p(\omega\times(0,1))$ as $\zeta\to 0$ and

$$\partial_1 u_\zeta \to \partial_1 u \in L^p(\omega\times(0,z_1)),\ \partial_2 u_\zeta \to \partial_2 u \in L^p(\omega\times(0,z_2)),\ \partial_3 u_\zeta \to \partial_3 u \in L^p(\omega\times(0,1)),$$

so that $F_{\rm hom}(u_\zeta) \to F_{\rm hom}(u)$ thanks to the upper bounds (22), (27), (39). Furthermore, by a triangulation argument, the same holds with suitable piecewise-affine u_ζ. It suffices then to prove the upper bound for u piecewise affine in $\omega\times(0,1)$.

We now exhibit a recovery sequence for u affine on a subset A of $\omega\times(0,1)$ where u is affine. More precisely, we suppose that

$$u(x) = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + q \text{ on } A.$$

A recovery sequence on the whole $\omega\times(0,1)$ will be obtained by patching up the constructions. Upon choosing some points of the underlying triangulation exactly at the levels $x_3 = t_i^j$, it is not restrictive to suppose that A is a simplex with interior contained in some $\omega\times(t_{i-1}^j,t_i^j)$; the construction will be different in the three cases $j=1,2$, or 3.

We first treat the case $j=1$. With fixed $r>0$ we choose $T>0$ such that $T\in K\mathbb{Z}$ and $v\in C^1((0,T)^3)$ such that

$$\frac{1}{T^3}\int_{[0,T]^3\cap\mathcal{S}_i^1} f(y,Dv)\,d\mathcal{H}^1 \leq f_{\rm hom}^{1,i}(\xi) + r \tag{45}$$

and $v(x) = \xi\cdot x$ in a neighbourhood of $\partial[0,T]^3$. Let $k+(0,\varepsilon T)^3$ be contained in A for some $k\in\varepsilon T\mathbb{Z}^3$. Then we define

$$u_\varepsilon(x) = \varepsilon\, v\Big(\frac{x-k}{\varepsilon}\Big) + k\cdot\xi + q.$$

Note that $u_\varepsilon(x) = u(x)$ in a neighbourhood of $k + \partial(0,\varepsilon T)^3$, and

$$\varepsilon^2 \int_{(k+[0,\varepsilon T]^3)\cap\varepsilon\mathcal{S}_i^1} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\, d\mathcal{H}^1 \le \varepsilon^3 T^3 (f_{\rm hom}^{1,i}(\xi) + r)$$

by (45). If we define $u_\varepsilon(x) = u(x)$ if $x \in A$ is not contained in any such cube $k + (0,\varepsilon T)^3$ then we have

$$\varepsilon^2 \int_{A\cap\varepsilon\mathcal{S}_i^1} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\, d\mathcal{H}^1 \le |A|(f_{\rm hom}^{1,i}(\xi) + r) + c\varepsilon T\,\mathcal{H}^2(\partial A)(1 + |\xi|^p), \tag{46}$$

the last term due to the contribution of the cubes intersecting ∂A.

In the case $j = 2$, choosing $v_0(x_1) = \xi_1 x_1$ in (25) we may suppose that for the same $T \in K\mathbb{Z}$ also $v \in C^1((0,K)\times(0,T)^2)$ exists such that

$$\frac{1}{T^2 K} \int_{([0,K)\times[0,T]^2)\cap\mathcal{S}_i^2} f(y, Dv)\, d\mathcal{H}^1 \le f_{\rm hom}^{2,i}(\xi) + r \tag{47}$$

and $v(x) = \xi_2 x_2 + \xi_3 x_3 + \xi_1 x_1$ in a neighbourhood of $\partial([0,K)\times[0,T]^2)$. Let $k + (0,\varepsilon K)\times(0,\varepsilon T)^2$ be contained in A for some $k \in \varepsilon(K\mathbb{Z}\times T\mathbb{Z}^2)$. Then we define

$$u_\varepsilon(x) = \varepsilon\, v\Big(\frac{x-k}{\varepsilon}\Big) + k\cdot\xi + q.$$

Note that $u_\varepsilon(x) = u(x)$ in a neighbourhood of $\partial(k + (0,\varepsilon K)\times(0,\varepsilon T)^2)$, and

$$\varepsilon^2 \int_{(k+[0,\varepsilon K)\times[0,\varepsilon T]^2)\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\, d\mathcal{H}^1 \le \varepsilon^3 K T^2 (f_{\rm hom}^{2,i}(\xi) + r)$$

by (47). Again, we define $u_\varepsilon(x) = u(x)$ if $x \in A$ is not contained in any such $k + (0,\varepsilon K)\times(0,\varepsilon T)^2$ then we have

$$\varepsilon^2 \int_{A\cap\varepsilon\mathcal{S}_i^2} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\, d\mathcal{H}^1 \le |A|(f_{\rm hom}^{2,i}(\xi_2,\xi_3) + r) + c\varepsilon T\mathcal{H}^2(\partial A)(1 + |\xi|^p), \tag{48}$$

the last term due to the contribution of the cubes intersecting ∂A.

In the case $j = 3$, choosing $v_0(x_1,x_2) = \xi_1 x_1 + \xi_2 x_2$ in (37) we may suppose that for the same $T \in K\mathbb{Z}$ also $v \in C^1((0,K)^2\times(0,T))$ exists such that

$$\frac{1}{TK^2} \int_{([0,K)^2\times[0,T])\cap\mathcal{S}_i^3} f(y, Dv)\, d\mathcal{H}^1 \le f_{\rm hom}^{3,i}(\xi) + r, \tag{49}$$

and $v(x) = \xi_3 x_3 + \xi_1 x_1 + \xi_2 x_2$ in a neighbourhood of $\partial([0,K)^2\times[0,T])$. Let $k + (0,\varepsilon K)^2\times(0,\varepsilon T)$ be contained in A for some $k \in \varepsilon(K\mathbb{Z}^2\times T\mathbb{Z})$. Then we define

$$u_\varepsilon(x) = \varepsilon\, v\Big(\frac{x-k}{\varepsilon}\Big) + k\cdot\xi + q.$$

Note that $u_\varepsilon(x) = u(x)$ in a neighbourhood of $\partial(k + (0,\varepsilon K)^2\times(0,\varepsilon T))$, and

$$\varepsilon^2 \int_{(k+[0,\varepsilon K)\times[0,\varepsilon T]^2)\cap\varepsilon\mathcal{S}_i^3} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\, d\mathcal{H}^1 \le \varepsilon^3 K T^2 (f_{\rm hom}^{3,i}(\xi_3) + r)$$

by (49). Again, we define $u_\varepsilon(x) = u(x)$ if $x \in A$ is not contained in any such $k + (0, \varepsilon K) \times (0, \varepsilon T)^2$ then we have

$$\varepsilon^2 \int_{A \cap \varepsilon \mathcal{S}_i^3} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\, d\mathcal{H}^1 \le |A|(f_{\rm hom}^{3,i}(\xi) + r) + c\varepsilon T\, \mathcal{H}^2(\partial A)(1 + |\xi|^p), \qquad (50)$$

the last term due to the contribution of the cubes intersecting ∂A.

We finally note that the constructions are compatible since $u_\varepsilon = u$ on each ∂A, so that u_ε is actually defined in the whole $\omega \times (0, 1)$. Gathering (46), (48) and (50), and letting $\varepsilon \to 0$ we obtain

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \le \int_{\omega \times (0,1)} f_{\rm hom}(x_3, Du)\, dx + \mathcal{H}^2(\omega)\, r,$$

which proves the upper bound by the arbitrariness of $r > 0$. □

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