Checking the consistency of solutions in decision-making problems with multiple weighted agents

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Abstract

A decision-making problem diffused in various practical contexts is that of aggregating multi-agent judgements into a consensus ordering, in the case the agents' importance is expressed through a set of weights. A crucial point in this aggregation is that the consensus ordering well reflects the input data, i.e., agents' judgements and importance. The scientific literature encompasses several aggregation techniques, even if it does not include a versatile tool for a quantitative assessment and comparison of their performance.

The aim of this paper is introducing a new indicator (p), which allows to verify the degree of consistency between consensus ordering and input data. This indicator is simple, intuitive and independent from the aggregation technique in use; for this reason, it can be applied to a variety of practical contexts and used to compare the results obtained through different aggregation techniques, when applied to a specific problem. The description is supported by various application examples.

Keywords: Multiple agents, Consensus ordering, Aggregation, Paired comparison, Consistency, Weight.

INTRODUCTION

A very general decision-making problem is that of aggregating multi-agent judgments, concerning different alternatives, into a consensus ordering. The adjective “consensus” indicates that this ordering should reflect agents’ judgements as much as possible, even in the presence of divergences. Summarizing, the problem is characterized by the following elements (see the scheme in Fig. 1):

- A set of alternatives to be prioritized (a, b, c, d, e, ...);
- A set of decision-making agents[1] (D_1, D_2, …, D_M) expressing their opinion on the alternatives, through various possible forms (e.g., paired-comparison judgements, evaluations/measurements on ordinal/interval/ratio scales, linear/partial preference orderings, etc.);
- An importance hierarchy of agents, which is usually expressed through a set of weights or *importances* (r_1, r_2, …, r_M), defined on a ratio scale by a team of experts (Vora et al., 2014; Ngan et al., 2016);
- A consensus ordering of the alternatives, which represents the output of the problem.

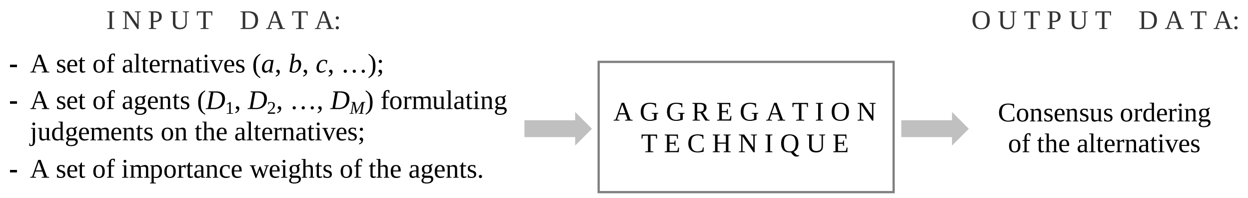

Fig. 1. Input/output data concerning the general problem of the aggregation of multi-agent judgements on a set of alternatives, into a consensus ordering.

The problem of interest is quite old and has been studied in various fields, stimulating the development of a variety of aggregation techniques (Von Neumann and Morgenstern, 1944; Fine and Fine, 1974; Fishburn, 1974; Hwang and Lin, 1987; Keeney and Raiffa, 1993). For example, in the field of *social choice* and *voting theory*, the authors recall the method by Condorcet and that by Borda (Borda, 1781; Franceschini et al., 2007); in the field of multicriteria decision making, the Electre (Figueira et al., 2005), Promethee (Brans and Mareschal, 2005) or AHP (Saaty, 1980) methods, in the field of the *Internet intelligent agents*, that by Yager (2001), etc..

Each of these techniques has its *pro* and *contra*; based on this consideration, an interesting question may arise: *For a generic decision-making problem, how could the best aggregation technique(s) be identified*? It is probably impossible to answer this question, since the “true” solution for a generic problem is not known *a priori* (Figueira et al., 2005; Cook, 2006). Nevertheless, the fact remains that one technique may be more or less appropriate than one other depending on: (i) the practical

[1] By a decision-making agent, we will consider any of a wide variety of different types of entities; examples could be human beings, individual criteria in a multicriteria decision-making process, intelligent entities in the field of artificial intelligence, etc..

purpose of the aggregation (for example, isolating the best alternative or a limited number of excellent alternatives, excluding the worst alternatives, defining a complete ranking, etc.), (ii) the form in which the agents' judgments and/or the importance hierarchy are expressed, and (iii) the ability to encourage the involvement and participation of decision-makers in constructing a shared solution (Zopounidis and Pardalos, 2010).

Also, techniques can be differentiated on the basis of various functional aspects, such as: degree of simplicity of evaluations/elaborations required by the agents, efficiency in the use of the information available, computational complexity, etc.. For a detailed description of these aspects, see the relevant scientific literature and extensive reviews (Arrow, 1950; Brans and Mareschal, 2005; Herrera-Viedma et al., 2014).

The rest of this paper, will not focus on the different aggregation techniques, which will be considered as "black boxes" able to transform the agents' judgments (input) into a consensus ordering (output). On the other hand, the paper is aimed at introducing a new indicator, denominated *p*, which allows a rough verification of the consistency of the solution provided by a certain aggregation technique, in a simple and intuitive way. This type of verification is important for at least two reasons:

1. Despite its seemingly simplicity, the aggregation problem of interest is surprisingly complicated, since it has been demonstrated that finding the "optimal" consensus ordering (assuming that it is possible) is NP-hard under certain conditions (Dwork et al., 2001). In order not to make the problem computationally burdensome, it can be convenient to use relatively simple and user-friendly techniques, as long as they are accompanied by practical tools for verifying the effectiveness of the solution (Nuray and Can, 2006; Akriditis et al. , 2011);
2. A verification tool may assist in selecting the best suited aggregation techniques to a specific decision problem.

In the scientific literature various verification tools have been proposed. A common feature is that they use some measures of correlation/similarity to compare the fused ordering with agents' judgements (Ng and Kantor, 2000; Wu and McClean, 2006). For example, popular statistics are the Spearman's rank, the Kendall's tau, and measures of likelihood/distance. Unfortunately, these tools are generally designed for verifying the solution of specific aggregation techniques and, for this reason, their range of application is limited by several aspects, such as:

- the form in which agent's judgements are expressed;
- the degree of "completeness" of judgments; for example, many techniques are not applicable when some alternatives are omitted or incomparable between each other;
- the form in which the importance hierarchy of the agents is expressed.

The proposed indicator is relatively versatile and practical, as it will be shown in the paper. Other not-so-dissimilar tools were proposed for more specific decision-making problems, in which the agents' importance is expressed in the form of a rank-ordering (Franceschini and Maisano, 2015a; Franceschini and Maisano, 2017).

The remainder of the paper is organized in three sections. The section "Description of the New Indicator" introduces the indicator *p*, focusing on its construction and practical use. The section "Application Examples" exemplifies the application of *p* in three different types of decision-making problems. The section "Discussion" presents a detailed discussion of the new indicator, focussing on its strengths and limitations. Additional information is contained in the appendix.

DESCRIPTION OF THE NEW INDICATOR

Before getting into the discussion of *p*, the authors anticipate that it is virtually adaptable to a generic aggregation technique, since it is mainly based on the comparison between (i) the paired-comparison relationships derived from one agent's judgements and (ii) those derived from the consensus ordering. The decision of using paired-comparison relationships is motivated by several reasons:

1. They allow to express the preference between two alternatives in a natural and intuitive way;
2. They can be derived from most of the forms in which agent's judgments are typically expressed – provided that they admit relationships of ordering among the alternatives; Fig. 2 shows several examples of paired-comparison relationships resulting from different types of judgments. This feature makes *p* potentially adaptable to a large amount of practical contexts;
3. They may also be obtained in the case some judgments are incomplete, i.e., they do not include all the alternatives (e.g., see case (ii) in Fig. 2) or admit incomparabilities between some alternatives (e.g., see case (v) in Fig. 2).

Form of the judgements		Paired-comparison relationships
(i) Evaluations on an ordinal scale a = Excellent b = Good c = Poor d = Poor	➡	$a > b,\ a > c,\ a > d,\ b > c,\ b > d,\ c \sim d$
(ii) Evaluations on an interval/ratio scale a = 1.52 b = Omitted judgement c = 5.18 d = 3.12	➡	$a \parallel b,\ c > a,\ d > a,\ c \parallel b,\ d \parallel b,\ c > d$
(iii) Paired-comparison relationships on a ratio scale $a = 2 \cdot b$ $a = 4 \cdot c$ $a = 2/3 \cdot d$ $b = 2 \cdot c$ $b = 1/3 \cdot d$ $c = 1/2 \cdot b$	➡	$a > b,\ a > c,\ d > a,\ b > c,\ d > b,\ d > c$
(iv) Linear ordering $a > (b \sim c) > d$	➡	$a > b,\ a > c,\ a > d,\ b \sim c,\ b > d,\ c > d$
(v) Partial ordering $a > [(b \sim c) \parallel d]$	➡	$a > b,\ a > c,\ a > d,\ b \sim c,\ b \parallel d,\ c \parallel d$

Fig. 2. Transformation of judgments expressed in various forms, into paired-comparison relationships. Alternatives of interest are *a*, *b*, *c* and *d*; symbols ">", "~" and "||" respectively mean "strictly preferred to", "indifferent to" and "incomparable to".

The authors are aware that transforming judgments into paired-comparisons can cause a loss of information, especially when these judgments are expressed on cardinal (i.e., *interval* or *ratio*) scales (Luce and Raiffa, 1957). However, we believe that this is the price to pay for making the proposed indicator applicable to a wide variety of practical contexts.

A preliminary operation for determining p is the construction of a table, which contains the paired-comparisons obtained from the agents' judgments and the consensus ordering. For the purpose of example, consider a decision-making problem in which $m = 4$ agents formulate their judgments concerning $n = 5$ alternatives (a, b, c, d and e). No matter in what form judgements are expressed, as long as they can be turned into paired-comparison relationships like $a>b$, $b>a$, $a\sim b$ or $a||b$, where symbols ">", "~" and "||" respectively mean "strictly preferred to", "indifferent to" and "incomparable to". Through some aggregation technique (no matter what), it is assumed that agents' judgements are aggregated into a consensus ordering $d>a>b>c>e$.

Agents' judgements are then transformed into the sets of paired-comparison relationships reported in Tab. 1(a). Likewise agents' judgements, the consensus ordering is also transformed into a set of paired-comparison relationships (see the last column of Tab. 1(a)).

Tab. 1. (a) Table of paired-comparison data related to agents' judgments and consensus ordering (i.e., $d>a>b>c>e$), in a fictitious decision problem; agents are sorted decreasingly with respect to their importance (r_j). (b) Corresponding consistency table.

(a)

Paired comparison	From agents' judgements D_1	D_2	D_3	D_4	From consensus ordering
a, b	$a > b$	$a \| b$	$a > b$	$b > a$	$a > b$
a, c	$a > c$	$a > c$	$a > c$	$c > a$	$a > c$
a, d	$a \sim d$	$d > a$	$a > d$	$a \| d$	$d > a$
a, e	$a > e$	$a > e$	$a > e$	$a \| e$	$a > e$
b, c	$b > c$	$b > c$	$b > c$	$b > c$	$b > c$
b, d	$d > b$	$d > b$	$b \sim d$	$b \| d$	$d > b$
b, e	$b > e$	$b > e$	$b > e$	$b > e$	$b > e$
c, d	$d > c$	$d > c$	$d > c$	$c \| d$	$d > c$
c, e	$c > e$	$e > c$	$c \sim e$	$c > e$	$c > e$
d, e	$d > e$	$d > e$	$d > e$	$d \| e$	$d > e$
r_j	0.40	0.30	0.15	0.15	

(b)

Paired comparison	D_1	D_2	D_3	D_4
a, b	1	N/A	1	0
a, c	1	1	1	0
a, d	0.5	1	0	N/A
a, e	1	1	1	N/A
b, c	1	1	1	1
b, d	1	1	0.5	N/A
b, e	1	1	1	1
c, d	1	1	1	N/A
c, e	1	0	0.5	1
d, e	1	1	1	N/A
c_j	10	9	10	5
x_j	9.5	8	8	3
p_j	95%	88.9%	80%	60%
w_j	4	2.7	1.5	0.75

Note: r_j is the agent's importance; c_j is the number of "usable" paired-comparisons; x_j is the agent's total score; $p_j = x_j/c_j$ is the portion of consistent paired-comparison relationships; $w_j = r_j \cdot c_j$ is the agent's weight.

Each j-th agent is associated with an indicator of importance (r_j) and an indicator (c_j) corresponding to the number of paired-comparisons *usable* for evaluating the compatibility between agents' judgments and consensus ordering. Conventionally, the adjective "usable" denotes a paired comparison not producing any relationship of incomparability ("||"), neither in the agent's judgements, nor in the consensus ordering, i.e., only relationships of strict preference (">") or indifference ("~"). Obviously, $c_j \leq C_2^n \quad \forall$ j-th agent, being C_2^n the total number of paired-comparisons for n generic alternatives (e.g., 10 in this specific example, since $n = 5$).

Subsequently, it is constructed a “consistency table” (in Tab. 1(b)), which turns the paired-comparison relationships of each agent into scores, according to the convention in Tab. 2. The conventional choice of assigning 0.5 points in the case of weak consistency is justified by the fact that this is the intermediate case between that one of full consistency (with score 1) and that of inconsistency (with score 0). Through a sensitivity analysis, it was found that small variations in this score (e.g., using 0.3 or 0.7 instead of 0.5) have little impact on the resulting consistency indicators.

The consistency table also reports the sum of the scores (x_j) relating to each *j*-th agent.

Tab. 2. Scores related to paired-comparison relationships of one agent, in the construction of the consistency table.

Case	Score
1. Full consistency, i.e., identical relationship of strict preference (“>”) or indifference (“~”); e.g., when comparing the relationship $a > b$ with itself.	1
2. Weak consistency, i.e., consistency with respect to a weak preference relationship only (“> or ~”, i.e., strict preference or indifference); e.g., when comparing the relationship $a > b$ with $a \sim b$.	0.5
3. Inconsistency (with respect to both strict and weak preference relationships); e.g., when comparing the relationship $a > b$ with $b > a$.	0
4. Incomparability between the two alternatives in the agent judgments and/or in the fused ordering.	N/A

Next, the portion of consistent paired-comparisons can be calculated for each *j*-th agent as:

$$p_j = \frac{x_j}{c_j}, \qquad (1)$$

where: x_j is the total score related to the *j*-th agent;

c_j is the number of usable paired comparisons related to the *j*-th agent.

It can be noticed that the non-usable paired comparisons do not influence the evaluation of consistency, since they have no contribution neither in the x_j nor in the c_j terms.

Next, the p_j values of the different agents can be aggregated into *p*, by means of a weighted average based on a set of weights (w_j), defined as:

$$w_j = r_j \cdot c_j \,. \qquad (2)$$

The proportionality with respect to the c_j values is justified by the fact that it seems reasonable that the most consolidated p_j values, i.e., those calculated using a larger number of paired comparisons, contribute more to *p*. The proportionality with respect to the r_j values is justified by the fact that it is expected that the higher the importance of agents, the higher the consistency between consensus ordering and judgements. In other words, this condition does not entail that the judgements by the less important agents should not to be consistent with the consensus ordering, but just means that the consistency of the more important agents counts more.

Based on the above considerations, the synthetic indicator *p* is defined as:

$$p = \sum_{j=1}^{m} \left(p_j \cdot \frac{w_j}{\sum_{j=1}^{m} w_j} \right) = \sum_{j=1}^{m} \left(p_j \cdot \frac{(r_j \cdot c_j)}{\sum_{j=1}^{m} (r_j \cdot c_j)} \right), \qquad p \in [0,1]. \tag{3}$$

By applying Eq. 2 to the data in Tab. 1(b), it can be obtained p = 87.7%. This relatively high value indicates that the aggregation produces a consensus ordering reflecting the agents' judgements quite well. This impression is confirmed by the p_j-chart in Fig. 3, which represents the profile of the agents' p_j values, sorted in terms of importance (r_j): the consensus ordering reflects the hierarchy of importance of the agents relatively well, since p_j values tend to decrease as the r_j values decrease; this means that the more important the agents, the higher the consistency between consensus ordering and judgements.

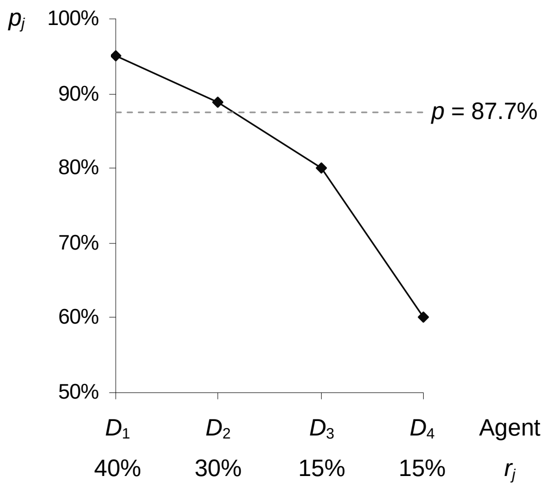

Fig. 3. p_j-chart relating to the decision-making problem in Tab. 1.

The judgments by D_4 are not very consistent with the consensus ordering (see the relatively low value of p_4 = 60%); however this inconsistency does not affect p significantly, since D_4 (along with D_3) is the least important agent and it provides a limited number of usable paired comparisons (i.e., c_4 = 4) for the consistency assessment.

It is worth noticing that, in the case of *full democracy*, i.e., when all agents are equi-important ($r_j = r \;\forall\, j$-th agent), p corresponds to the overall portion of consistent paired comparisons (among the usable ones), related to the totality of the agents.

APPLICATION EXAMPLES

This section illustrates the potential of p when (i) verifying the consistency of the solution provided by a generic aggregation technique and (ii) comparing the solutions obtained from different techniques, which are applied to the same decision-making problem. As an example, consider a specific multicriteria decision-making problem aimed at identifying the best among six different locations (i.e., the alternatives of the problem *a*, *b*, *c*, *d*, *e* and *f*), in which to install a new manufacturing plant. The six alternatives are evaluated on the basis of 5 criteria (D_1 to D_5), i.e., the agents of the problem. Each *j*-th agent is associated with an indicator (r_j) reflecting its relative importance. Tab. 3 describes the various agents and their typology of judgements. Two of the five agents (i.e., D_1 and D_5) have a negative preference sense, since the degree of preference increases/decreases as the judgement value decreases/increases. The agents' judgments are reported in Tab. 4.

Having defined the problem, the goal is to aggregate agents' judgements into a single consensus ordering. This aggregation will be performed through three techniques based on very different principles and structures: (i) the Electre-II method (Figueira et al., 2005), (ii) the Borda's count (Borda, 1781), and (iii) the algorithm by Yager (2001). The authors are aware that much more advanced techniques can be found in the literature (Herrera-Viedma et al., 2014); however, these three ones were chosen not to overcomplicate the discussion.

Tab. 3. Description of the agents/criteria of a fictitious decision-making problem concerning the selection of the best location where to install a new manufacturing plant.

Agent description	Form of the judgement	Unit	Preference sense	r_j
D_1 – Mean distance from the suppliers of raw materials	Measurement on a ratio scale	[km]	Negative	35%
D_2 – Average technical skills of labour	Evaluation on a 3-level ordinal scale (Low, Medium, High)	N/A	Positive	20%
D_3 – No. of typologies of available energy sources (gas, oil, renewables, electricity, etc.)	Count on an absolute scale	[-]	Positive	20%
D_4 – No. of available transport infrastructures (roads, highways, railways, ports, airports, etc.)	Count on an absolute scale	[-]	Positive	15%
D_5 – Average hourly cost of labour	Measurement on a ratio scale	[€]	Negative	10%

Tab. 4. Judgments on the alternatives of interest (*a, b, c, d, e, f*) by five agents (D_1 to D_5). In the last row, the judgements related to each agent are turned into the corresponding (linear) preference orderings.

Agent		D_1	D_2	D_3	D_4	D_5
r_j		35%	20%	20%	15%	10%
Unit		[km]	N/A	[-]	[-]	[€]
Preference sense		Negative	Positive	Positive	Positive	Negative
Alternatives	*a*	20	H	4	3	24
	b	140	L	3	2	20
	c	80	H	3	4	24
	d	200	M	2	3	14
	e	120	M	4	3	16
	f	140	M	1	2	20
Preference ordering		*a*>*c*>*e*>(*b*~*f*)>*d*	(*a*~*c*)>(*d*~*e*~*f*)>*b*	(*a*~*e*)>(*b*~*c*)>*d*>*f*	*c*>(*a*~*d*~*e*)>(*b*~*f*)	*d*>*e*>(*b*~*f*)>(*a*~*c*)

These techniques generate the three consensus orderings reported in Tab. 5. The three subsections in the appendix contain a simplified illustration of each of the three techniques and the way the relevant fused orderings are constructed.

The consistency of the fused orderings will be assessed using *p*. To facilitate the construction of *p*, it is convenient to turn the input judgements into corresponding preference orderings (see the last row of Tab. 4) and then into paired-comparison data (see

Tab. 6). In this specific case, preference orderings do not include omissions or incomparabilities between the alternatives, therefore they do not result into any paired-comparison relationship of incomparability ("||") .

Tab. 5. Consensus orderings resulting from the application of three aggregation techniques to the decision-making problem summarized in Tab. 3.

Technique	Consensus ordering
Electre-II method	$a > c > e > [(b \sim f) \parallel d]$
Borda's count	$a > c > e > d > b > f$
Yager's algorithm	$e > (a \sim c) > (b \sim f) > d$

Tab. 6. Paired-comparison data related to the judgments by the agents of interest (Tab. 4) and three consensus orderings (Tab. 5).

Paired	From agents' judgements					From consensus orderings		
comparison	D_1	D_2	D_3	D_4	D_5	Electre-II	Borda's count	Yager's algorithm
a, b	$a > b$	$a > b$	$a > b$	$a > b$	$b > a$	$a > b$	$a > b$	$a > b$
a, c	$a > c$	$a \sim c$	$a > c$	$c > a$	$a \sim c$	$a > c$	$a > c$	$a \sim c$
a, d	$a > d$	$a > d$	$a > d$	$a \sim d$	$d > a$	$a > d$	$a > d$	$a > d$
a, e	$a > e$	$a > e$	$a \sim e$	$a \sim e$	$e > a$	$a > e$	$a > e$	$e > a$
a, f	$a > f$	$a > f$	$a > f$	$a > f$	$f > a$	$a > f$	$a > f$	$a > f$
b, c	$c > b$	$c > b$	$b \sim c$	$c > b$	$b > c$	$c > b$	$c > b$	$c > b$
b, d	$b > d$	$d > b$	$b > d$	$d > b$	$d > b$	$b \parallel d$	$d > b$	$b > d$
b, e	$e > b$	$e > b$	$e > b$	$e > b$	$e > b$	$e > b$	$e > b$	$e > b$
b, f	$b \sim f$	$f > b$	$b > f$	$b \sim f$	$b \sim f$	$b \sim f$	$b > f$	$b \sim f$
c, d	$c > d$	$c > d$	$c > d$	$c > d$	$d > c$	$c > d$	$c > d$	$c > d$
c, e	$c > e$	$c > e$	$e > c$	$c > e$	$e > c$	$c > e$	$c > e$	$e > c$
c, f	$c > f$	$c > f$	$c > f$	$c > f$	$f > c$	$c > f$	$c > f$	$c > f$
d, e	$e > d$	$d \sim e$	$e > d$	$d \sim e$	$d > e$	$e > d$	$e > d$	$e > d$
d, f	$f > d$	$d \sim f$	$d > f$	$d > f$	$d > f$	$d \parallel f$	$d > f$	$f > d$
e, f	$e > f$	$e \sim f$	$e > f$	$e > f$	$e > f$	$e > f$	$e > f$	$e > f$

Likewise preference orderings, the three consensus orderings can be turned into paired-comparison data (see the last three columns in

Tab. 6). It is worthwhile remarking that the ordering resulting from the application of Electre-II (i.e., $a>c>e>[(b\sim f)\|d]$) is *partial* (not *linear*), as some of the alternatives are incomparable with each other (i.e., $b\|d$ and $d\|f$) (Nederpelt and Kamareddine, 2004).

The three following subsections present the construction of the indicator *p*, related to each of the three techniques proposed; the fourth one presents a comparison between the solutions generated by the three techniques, by means of the p_j-chart.

Case of the Electre-II Method

The construction of the consensus ordering through the Electre-II method is illustrated in the section "Description and Application of the Electre-II Method" (in the appendix). Tab. 7 contains the corresponding consistency table (compare it with the data in

Tab. 6). The scores related to (*b*, *d*) and (*d*, *f*) are replaced with "N/A", because of the incomparabilities in the corresponding paired-comparison relationships from the consensus ordering. The total number of usable paired comparisons is therefore $c_j = C_2^{n=6} - 2 = 13 \ \forall j$-th agent.

The p_j values related to the five agents are reported at the bottom of Tab. 7. The relatively high values of these indicators denote a certain degree of consistency between consensus ordering and agents' judgements. The p_j values relating to the first four agents in terms of importance are relatively high (all larger than 80%). Instead, the p_j value relating to D_5 is very low (p_5 = 26.9%); this aspect is due to the negative correlation between this criterion (i.e., average hourly cost of labour) and the other ones (see Tab. 7).

Consistency between consensus ordering and agents' judgements looks good, as denoted by the relatively high value of p = 82.9% (obtained by applying Eq. 3). Furthermore, the agents' importance hierarchy seems to be reflected quite well, since the p_j values tend to decrease as the r_j value decrease.

Tab. 7. Consistency table related to the solution provided by the Electre-II method (i.e., *a*>*c*>*e*>[(*b*~*f*)||*d*]) and the judgements in

Tab. 6.

Paired comparison	D_1	D_2	D_3	D_4	D_5	
a, *b*	1	1	1	1	0	
a, *c*	1	0.5	1	0	0.5	
a, *d*	1	1	1	0.5	0	
a, *e*	1	1	0.5	0.5	0	
a, *f*	1	1	1	1	0	
b, *c*	1	1	0.5	1	0	
b, *d*	N/A	N/A	N/A	N/A	N/A	
b, *e*	1	1	1	1	1	
b, *f*	1	0.5	0.5	1	1	
c, *d*	1	1	1	1	0	
c, *e*	1	1	0	1	0	
c, *f*	1	1	1	1	0	
d, *e*	1	0.5	1	0.5	0	
d, *f*	N/A	N/A	N/A	N/A	N/A	
e, *f*	1	0.5	1	1	1	
r_j	0.35	0.20	0.20	0.15	0.10	
c_j	13	13	13	13	13	
x_j	13	11	10.5	10.5	3.5	
p_j	100.0%	84.6%	80.8%	80.8%	26.9%	
w_j	4.6	2.6	2.6	2.0	1.3	
$p_j \cdot (w_j / \Sigma w_j)$	0.350	0.169	0.162	0.121	0.027	p = 82.9%

Case of the Borda's Count

The section "Description and Application of the Borda's Count" (in the appendix) illustrates the construction of the consensus ordering when applying the Borda's count. The three top-positions of the resulting ordering (i.e., *a*>*c*>*e*>*d*>*b*>*f*) coincide with those in the solution by the Electre-II method (see Tab. 5), while there are some differences in the lower positions. The corresponding consistency table, shown in Tab. 8, is slightly different from that in Tab. 7, due to some variations in the rows corresponding to (d, b), (b, f) and (d, f). The p value (77.2%) is slightly lower than in the case of Electre-II, denoting a certain deterioration in terms of consistency.

Tab. 8. Consistency table related to the solution provided by the Borda's count (i.e., *a*>*c*>*e*>*d*>(*b*~*f*) and the judgements in

Tab. 6.

Paired comparison	D_1	D_2	D_3	D_4	D_5	
a, *b*	1	1	1	1	0	
a, *c*	1	0.5	1	0	0.5	
a, *d*	1	1	1	0.5	0	
a, *e*	1	1	0.5	0.5	0	
a, *f*	1	1	1	1	0	
b, *c*	1	1	0.5	1	0	
b, *d*	0	1	0	1	1	
b, *e*	1	1	1	1	1	
b, *f*	0.5	0	1	0.5	0.5	
c, *d*	1	1	1	1	0	
c, *e*	1	1	0	1	0	
c, *f*	1	1	1	1	0	
d, *e*	1	0.5	1	0.5	0	
d, *f*	0	0.5	1	1	1	
e, *f*	1	0.5	1	1	1	
r_j	0.35	0.20	0.20	0.15	0.10	
c_j	15	15	15	15	15	
x_j	12.5	12	12	12	5	
p_j	83.3%	80.0%	80.0%	80.0%	33.3%	
w_j	5.3	3.0	3.0	2.3	1.5	
$p_j \cdot (w_j / \Sigma w_j)$	0.292	0.160	0.160	0.120	0.033	p = 76.5%

Case of the Yager's Algorithm

This technique can be applied when (i) the agents' judgements are expressed through linear preference orderings (Nederpelt and Kamareddine, 2004) and (ii) the agents' importance ranking is expressed through a further linear ordering (Yager, 2001; Franceschini et al., 2016). In this specific case, the authors used the agents' preference orderings reported at the bottom of Tab. 4 and the importance rank-ordering D_1>(D_2~D_3)>D_4>D_5, which was obtained from the agents' r_j values (in Tab. 3).

The Yager's algorithm resulted in the consensus ordering *e*>(*a*~*c*)>(*b*~*f*)>*d*, the construction of which is described in the section "Description and Application of the Yager's Algorithm" (in the appendix). Tab. 9 contains the corresponding consistency table. The Yager's algorithm does not produce a perfect solution in terms of consistency with agents' judgements, as proven by the relatively low p value (72.5%). This result is not surprising, as this technique was criticized by Wang (2007) and Franceschini et al. (2014) for generating consensus orderings, which are often inconsistent with agents' judgments.

Tab. 9. Consistency table related to the solution provided by the Yager's algorithm (i.e., *e*>(*a*~*c*)>(*b*~*f*)>*d*)) and the judgements in

Tab. 6.

Paired comparison	D_1	D_2	D_3	D_4	D_5	
a, b	1	1	1	1	0	
a, c	0.5	1	0.5	0.5	1	
a, d	1	1	1	0.5	0	
a, e	0	0	0.5	0.5	1	
a, f	1	1	1	1	0	
b, c	1	1	0.5	1	0	
b, d	1	0	1	0	0	
b, e	1	1	1	1	1	
b, f	1	0.5	0.5	1	1	
c, d	1	1	1	1	0	
c, e	0	0	1	0	1	
c, f	1	1	1	1	0	
d, e	1	0.5	1	0.5	0	
d, f	1	0.5	0	0	0	
e, f	1	0.5	1	1	1	
r_j	0.35	0.20	0.20	0.15	0.10	
c_j	15	15	15	15	15	
x_j	12.5	10	12	10	6	
p_j	83.3%	66.7%	80.0%	66.7%	40.0%	
w_j	5.3	3.0	3.0	2.3	1.5	
$p_j \cdot (w_j / \Sigma w_j)$	0.292	0.133	0.160	0.100	0.040	p = 72.5%

Comparison Between the Three Solutions

A comparison between the solutions generated by the three aggregation techniques can be visualized by means of the p_j-chart in Fig. 4, which contains three corresponding profiles.

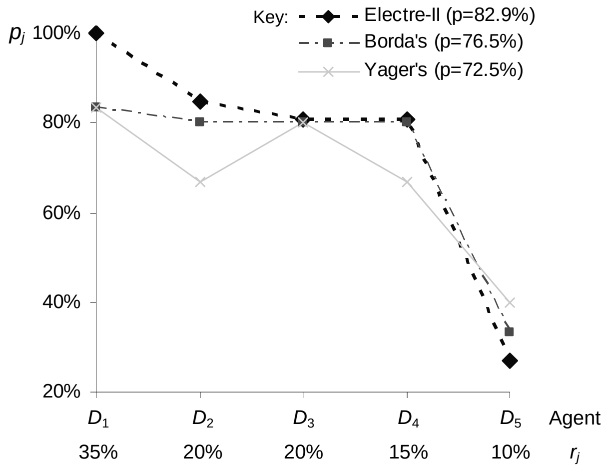

Fig. 4. p_j-chart concerning the three aggregation techniques in use, when applied to the decision-making problem in Tab. 4.

Despite the three profiles denote that the agents' importance hierarchy is generally respected (p_j values tend to decrease as r_j values decrease), it can be noticed that the solution by the Yager's algorithm is significantly poorer than the other ones. A comparison using the p values leads to the same conclusion.

DISCUSSION

The indicator p is a simple and intuitive tool for assessing the consistency between consensus ordering and agents' judgements, in a variety of multi-agent decision-making problems; the p_j-chart enriches the synthetic information given by p, showing the level of consistency between the solution and the judgements by each j-th agent.

Also, p is very versatile since it can be applied in the presence of incomparabilities between alternatives, both at the level of agents' judgments and consensus ordering. It could be applied even in the case in which the solution of the problem is expressed in forms that are different from a consensus ordering – such as measurements/assessments on ordinal/interval/ratio scales – as long as they can be transformed into paired-comparison relationships; possible examples are the solutions generated by the Thurstone's *Law of Comparative Judgements* (Thurstone, 1927; Franceschini and Maisano, 2015b) or the AHP method (Saaty, 1980), which are evaluations expressed on interval and ratio scales respectively.

The indicator p was presented as a *passive* tool for checking the consistency of the solution of one or more aggregation techniques, in a specific decision-making problem. Reversing this perspective, p could be used *actively*, for identifying the "optimal[2]" consensus ordering(s) for a specific problem, i.e., that one(s) for which p assumes the maximum possible value (i.e., p^{MAX}). The optimal consensus ordering(s) may be determined by: (i) automatically generating all the possible solutions to a specific problem – i.e., the possible partial orderings, without loops – through *ad hoc* software, (ii) determining the corresponding p for each of them, and (iii) selecting the solution(s) for which $p = p^{MAX}$. Knowing the optimal consensus ordering(s) and the corresponding p^{MAX} value may be useful for better assessing the performance of a certain technique, in a specific problem. For example, a technique that provides a solution with a p = 79%, in a problem where p^{MAX} = 80%, performs certainly better than a technique that provides a solution with p = 80%, in a specific problem where p^{MAX} = 98%. In fact, in problems where the agents' preference orderings have a high degree of similarity, the p^{MAX} value will tend to increase (since it will be easier to find a consensus ordering that "satisfies" all of the agents and their importance ranking), while in others in which the differences are greater, it will tend to decrease (since it will be more complicated to find a consensus ordering that "satisfies" all of the agents and their importance ranking). In this sense,

[2] The concept of "optimal" is relative to the conventions adopted when defining p.

p^{MAX} could be also seen as a measure of the degree of "dissimilarity" between the agents' preference orderings.

A limitation of p is that it requires the importance hierarchy of agents to be expressed by means of a set of importance indicators (r_j), defined on a ratio scale. When it is expressed in other forms, it would be necessary to introduce some (potentially questionable) transformations for determining the r_j values. For example, assuming that the agents' importance hierarchy is expressed by an importance rank-ordering, r_j values can be obtained using the agents' inverse rank positions. One way to avoid introducing questionable transformations is to define a variant of p, for specific problems in which r_j values are not available (Franceschini and Maisano, 2017).

Another limitation of p is inherent in the calculation and aggregation of the p_j values, based on potentially questionable conventions: e.g., (i) the scores for determining the x_j values reported in Tab. 2, (ii) the (weighted) additive model for synthesizing the p_j values, (iii) the direct proportionality of weights with respect to r_j and c_j values.

Future research will aim at complementing p with other practical tools for enriching the verification of the solutions generated by different aggregation techniques.

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APPENDIX

Description and Application of the Electre-II Method

This section provides a simplified description of the Electre-II method and its application to the agents' judgements in Tab. 4; for more information, see (Figueira et al., 2005).

In the initial phase of Electre-II, agents' judgements are turned into sets of paired-comparison relationships and then aggregated into a single set of outranking relationships (see Tab. A.1). The third-to-fifth columns of Tab. A.1 report the indicators W^+, $W^=$ and W^- (also denominated as "consistency scores"), i.e., the sums of the r_j values of the agents for which a generic paired comparison (a, b) results in the relationship $a > b$, $a = b$ and $b > a$ respectively. In detail, having defined J^+, $J^=$ and J^- as the sets of agents for which $a > b$, $a = b$ and $b > a$ respectively, the indicators W^+, $W^=$ and W^- are formally defined as:

$$
\begin{aligned}
W^+ &= \sum_{j \in J^+} r_j \quad \text{where} \quad J^+ = \{D_j : a > b\} \\
W^= &= \sum_{j \in J^=} r_j \quad \text{where} \quad J^= = \{D_j : a \sim b\}, \\
W^- &= \sum_{j \in J^-} r_j \quad \text{where} \quad J^- = \{D_j : b > a\}
\end{aligned}
\qquad \text{(A.1)}
$$

These indicators are then used to perform the so-called *concordance test*, which is based on the verification of both the conditions:

$$\begin{array}{l} \dfrac{W^{+}+W^{=}}{W} \geq c_{1} \\ \dfrac{W^{+}}{W^{-}} \geq c_{2} \end{array}, \qquad (A.2)$$

being $W = W^{+} + W^{=} + W^{-}$, and c_1 and c_2 two thresholds (generally) set to 0.7 and 1 respectively (Figueira et al., 2005). When, for a generic paired comparison (*a*, *b*) the concordance test is passed, it can be stated that *aOb*, where the symbol "*O*" denotes the outranking relationship. Results can be visualised in a graph (see Fig. A.1, step 1) in which vertices represent alternatives and edges joining two vertices represent the corresponding outranking relationship.

Tab. A.1. Determination of outranking relationships ("*O*") between pairs of alternatives, in the initial phase of the Electre-II method.

Paired comparison		Consistency scores W^{+}	$W^{=}$	W^{-}	Concordance test $(W^{+}+W^{=})/W$ $(\geq c_1)$		W^{+}/W^{-} $(\geq c_2)$		Outranking relationship
1	*a, b*	90	0	10	0.90	✓	9.00	✓	*aOb*
2	*a, c*	55	30	15	0.85	✓	3.67	✓	*aOc*
3	*a, d*	75	15	10	0.90	✓	7.50	✓	*aOd*
4	*a, e*	55	35	10	0.90	✓	5.50	✓	*aOe*
5	*a, f*	90	0	10	0.90	✓	9.00	✓	*aOf*
6	*b, a*	10	0	90	0.10	×	0.11	×	-
7	*b, c*	10	20	70	0.30	×	0.14	×	-
8	*b, d*	55	0	45	0.55	×	1.22	✓	-
9	*b, e*	0	0	100	0.00	×	0.00	×	-
10	*b, f*	20	60	20	0.80	✓	1.00	✓	*bOf*
11	*c, a*	15	30	55	0.45	×	0.27	×	-
12	*c, b*	70	20	10	0.90	✓	7.00	✓	*cOb*
13	*c, d*	90	0	10	0.90	✓	9.00	✓	*cOd*
14	*c, e*	70	0	30	0.70	✓	2.33	✓	*cOe*
15	*c, f*	90	0	10	0.90	✓	9.00	✓	*cOf*
16	*d, a*	10	15	75	0.25	×	0.13	×	-
17	*d, b*	45	0	55	0.45	×	0.82	×	-
18	*d, c*	10	0	90	0.10	×	0.11	×	-
19	*d, e*	10	35	55	0.45	×	0.18	×	-
20	*d, f*	45	20	35	0.65	×	1.29	✓	-
21	*e, a*	10	35	55	0.45	×	0.18	×	-
22	*e, b*	100	0	0	1.00	✓	$> c_2$	✓	*eOb*
23	*e, c*	30	0	70	0.30	×	0.43	×	-
24	*e, d*	55	35	10	0.90	✓	5.50	✓	*eOd*
25	*e, f*	80	20	0	1.00	✓	$> c_2$	✓	*eOf*
26	*f, a*	10	0	90	0.10	×	0.11	×	-
27	*f, b*	20	60	20	0.80	✓	1.00	✓	*fOb*
28	*f, c*	10	0	90	0.10	×	0.11	×	-
29	*f, d*	35	20	45	0.55	×	0.78	×	-
30	*f, e*	0	20	80	0.20	×	0.00	×	-

c_1 and c_2 have been set to 0.7 and 1 respectively.

The second phase of the Electre-II method is aimed at deriving a consensus ordering from the outranking relationships. To this purpose, the outranking relationships are turned into so-called *preorders*, either using a *top-down* or *bottom-up* procedure. In the top-down procedure, the preliminary step is to identify and eliminate possible circuits, i.e., those combinations of outranking relationships in which the transitivity property is violated – e.g., *aOb, bOc* and *cOa*. For each of the circuits identified, the corresponding outranking relationships are deleted and the alternatives are grouped in the same class and considered as indifferent (e.g., *a~b~c*). Next, the alternatives that are not outranked by any other alternative are determined; this defines the first class of the top-down procedure. The alternatives in that class are then deleted and the exploitation procedure is repeated until all alternatives have been classified to obtain a (top-down) preorder. The bottom-up procedure is analogous but it starts selecting the class of worst alternatives (alternatives outranking no other alternatives) and reiterating the procedure.

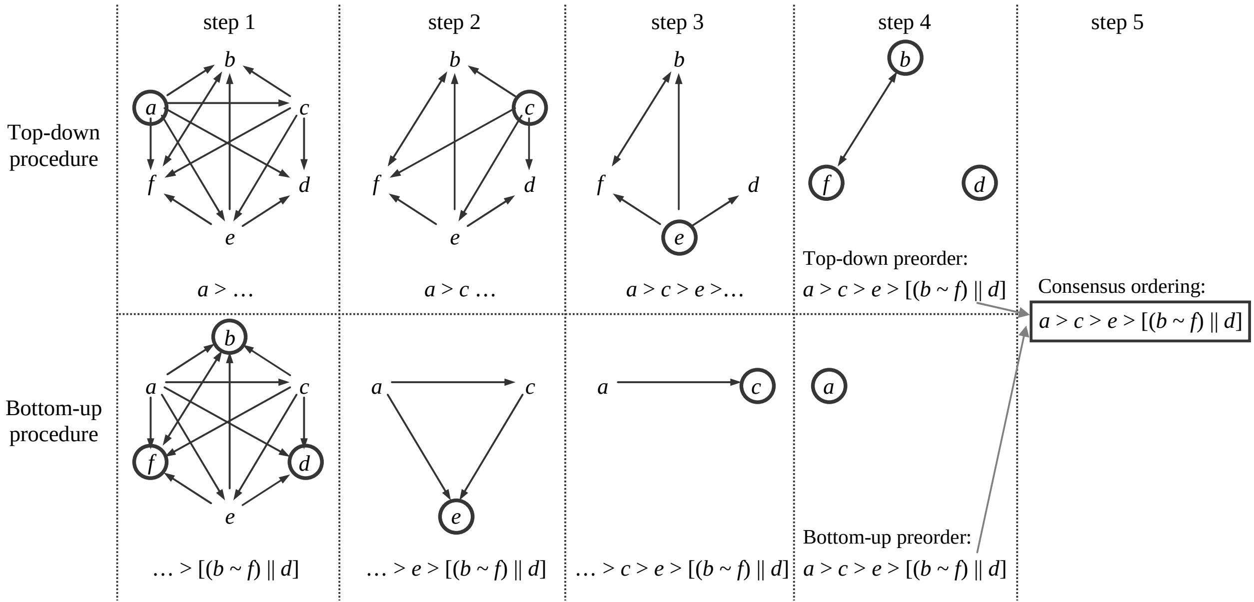

Fig. A.1. Basic steps for transforming the outranking graph (step 1) in a consensus ordering (step 5), in the second phase of the Electre-II methods; the alternatives selected in each step of the top-down and bottom-up procedure are circled. For further information, see (Figueroa et al., 2005).

The two procedures generate two preorders, which do not necessarily coincide; for example, an alternative which is not outranked and does not outrank any other alternative will appear as first on the list of the top-down preorder and it will appear as last in the bottom-up one. To handle this problem, it is necessary to build a single ordering with the combined result of the top-down and bottom-up preorders. Several ways have been suggested to perform this synthesis (Godsil and Royle, 2001). In this particular case, the top-down and bottom-up preorders are coincident, so there is no need for further synthesis (Figueroa et al., 2005).

Description and Application of the Borda's Count

The Borda's count consists of two basic steps: (i) turning the judgments of each (*j*-th) agent into a preference ordering, (ii) associating a score (k_{ij}) to the alternatives, corresponding to their rank-position in the preference ordering (see Tab. A.2), and (iii) synthesizing the scores obtained by each alternative, in an overall score – also known as Borda's score (B_i) – which corresponds to a weighted average based on the agents' r_j values. In formal terms, B_i is defined as:

$$B_i = \sum_{j=1}^{m} \left(r_j \cdot k_{ij} \right), \tag{A.3}$$

being:

r_j the relative importance of the j-th agent;

k_{ij} the rank-position of the i-th alternative in the preference ordering by the j-th agent;

m the total number of agents.

Tab. A.2 Application of the Borda's count to the decision-making problem in Tab. 3. The resulting consensus ordering (i.e., *a>c>e>d>b>f*) is obtained by ordering the alternatives decreasingly with respect to their B_i values.

Alternative	Rank-position in the preference orderings					B_i	Rank position in the consensus ordering
	D_1 35%	D_2 20%	D_3 20%	D_4 15%	D_5 10%		
a	1	1	1	2	5	155	1st
b	4	6	3	5	3	425	5th
c	2	1	3	1	5	215	2nd
d	6	3	5	2	1	410	4th
e	3	3	1	2	2	235	3rd
f	4	3	6	5	3	625	6th

The resulting consensus ordering is then constructed by ordering the alternatives in ascending order with respect to their B_i values. By applying the Borda's count to the input judgements in Tab. 3, the results shown in Tab. A.2 are obtained.

Description and Application of the Yager's Algorithm

The algorithm by Yager (2001) can be applied to specific decision-making problems in which:

- the agents' judgements are expressed in the form of linear orderings (Nederpelt and Kamareddine, 2004);
- the agents' preference orderings do not include any omitted or incomparable alternative;
- the agents' importance hierarchy is expressed in the form of a rank-ordering and not through the typical set of importance values.

Summarizing, the algorithm is based on the following four steps:

1. Transformation of the agents' linear orderings into preference vectors, according to the following convention: we place the alternatives as they appear in the ordering, with the most preferred one(s) in the top positions; if at any point $t > 1$ alternatives are tied (i.e., indifferent), we place them in the same element and then place the null set ("Null") in the next $t - 1$ lower positions. In this way, the total number of elements of a vector will coincide with the number alternatives. Tab. A.3 exemplifies the construction of the preference vectors from the orderings the last row of Tab. 4.

2. Transformation of the preference vectors into "reorganized" vectors. This transformation consists in (i) sorting the D_i vectors decreasingly with respect to the agents' importance and (ii) aggregating those with indifferent importance (e.g., D_2 and D_3 in the example) into a single vector. This aggregation is performed through a level-by-level union of the vector elements, where alternatives in elements with the same position are considered as indifferent. The resulting reorganized vectors are reported in Tab. A.4.
3. Definition of a sequence for the element-by-element reading of the reorganized vectors. The sequence is determined applying a lexicographical order based on two dimensions: (i) level of vector elements (from the bottom to the top) and (ii) importance of agents (in decreasing order). Tab. A.4 reports the full sequence numbers (S) associated with each element of the reorganized vectors.
4. Construction of the fused ordering: alternatives are progressively included into a gradual ordering, which is initially Null. A k-th alternative is included at the top of the gradual ordering, at the first occurrence in the element-by-element reading sequence. In the previous example, the resulting consensus ordering is: $e>(a\sim c)>(b\sim f)>d$. Tab. A.5 shows the step-by-step procedure for constructing the consensus ordering.

Tab. A.3. Construction of preference vectors related to the orderings by four fictitious agents (D_1 to D_5).

Agents		D_1	D_2	D_3	D_4	D_5
Orderings		$a>c>e>(b\sim f)>d$	$(a\sim c)>(d\sim e\sim f)>b$	$(a\sim e)>(b\sim c)>d>f$	$c>(a\sim d\sim e)>(b\sim f)$	$d>e>(b\sim f)>(a\sim c)$
Preference vectors	j	Elem.	Elem.	Elem.	Elem.	Elem.
	6	{a}	{a, c}	{a, e}	{c}	{d}
	5	{c}	Null	Null	{a, d, e}	{e}
	4	{e}	{d, e, f}	{b, c}	Null	{b, f}
	3	{b, f}	Null	Null	Null	Null
	2	Null	Null	{d}	{b, f}	{a, c}
	1	{d}	{b}	{f}	Null	Null

n=6 total alternatives are considered: a, b, c, d, e and f.
The agents' importance ordering is $D_1>(D_2\sim D_3)>D_4>D_5$.
j is the level of a certain element in the corresponding preference vector.

Tab. A.4. Reorganized vectors obtained by merging the preference vectors in Tab. A.3 and corresponding sequence numbers (S) of their elements.

Agents		(D_1)		(D_2 ~ D_3)		D_4		D_5	
Reorganized vectors	j	S	Elem.	S	Elem.	S	Elem.	S	Elem.
	6	21	{a}	22	{2a, c, e}	23	{c}	24	{d}
	5	17	{c}	18	Null	19	{a, d, e}	20	{e}
	4	13	{e}	14	{b, c, d, e, f}	15	Null	16	{b, f}
	3	9	{b, f}	10	Null	11	Null	12	Null
	2	5	Null	6	{d}	7	{b, f}	8	{a, c}
	1	1	{d}	2	{b, f}	3	Null	4	Null

Tab. A.5. Step-by-step construction of the fused ordering when applying the YA to the preference vectors in Tab. A.4.

Step (S)	Element	Residual alternatives	Gradual ordering
0	-	{a, b, c, d, e, f}	Null

1	{*d*}	{*a*, *b*, *c*, *e*, *f*}	*d*
2	{*b*, *f*}	{*a*, *c*, *e*}	(*b* ~ *f*) > *d*
3	Null	{*a*, *c*, *e*}	(*b* ~ *f*) > *d*
4	Null	{*a*, *c*, *e*}	(*b* ~ *f*) > *d*
5	Null	{*a*, *c*, *e*}	(*b* ~ *f*) > *d*
6	{*d*}	{*a*, *c*, *e*}	(*b* ~ *f*) > *d*
7	{*b*, *f*}	{*a*, *c*, *e*}	(*b* ~ *f*) > *d*
8	{*a*, *c*}	{*e*}	(*a* ~ *c*) > (*b* ~ *f*) > *d*
9	{*b*, *f*}	{*e*}	(*a* ~ *c*) > (*b* ~ *f*) > *d*
10	Null	{*e*}	(*a* ~ *c*) > (*b* ~ *f*) > *d*
11	Null	{*e*}	(*a* ~ *c*) > (*b* ~ *f*) > *d*
12	Null	{*e*}	(*a* ~ *c*) > (*b* ~ *f*) > *d*
13	{*e*}	Null	*e* > (*a* ~ *c*) > (*b* ~ *f*) > *d*
End	-	-	-