Compact Surfaces with No Bonnet Mate

COMPACT SURFACES WITH NO BONNET MATE

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Abstract. This note gives sufficient conditions (isothermic or totally non-isothermic) for an immersion of a compact surface to have no Bonnet mate.

1. Introduction

Consider a smooth immersion $\mathbf{x} : M \to \mathbf{R}^3$ of a connected, orientable surface M, with unit normal vector field $\mathbf{e}_3$. Its induced metric $I = d\mathbf{x} \cdot d\mathbf{x}$ and the orientation of M induced by $\mathbf{e}_3$ from the standard orientation of $\mathbf{R}^3$ induce a complex structure on M, which provides a decomposition into bidegrees of the second fundamental form II of $\mathbf{x}$ relative to $\mathbf{e}_3$,

$$-d\mathbf{e}_3 \cdot d\mathbf{x} = II = II^{2,0} + HI + II^{0,2}.$$

Here H is the mean curvature of $\mathbf{x}$ relative to $\mathbf{e}_3$ and $II^{2,0} = \overline{II^{0,2}}$ is the *Hopf quadratic differential* of $\mathbf{x}$. Relative to a complex chart (U, z) in M,

$$I = e^{2u} dz d\bar{z}, \quad II^{2,0} = \frac{1}{2} h e^{2u} dz dz, \tag{1}$$

where the *conformal factor* e^u, the *Hopf invariant* h, and the mean curvature H satisfy the *structure equations on U relative to z*,

$$-4e^{-2u} u_{z\bar{z}} = H^2 - |h|^2 \quad \text{Gauss equation}$$

$$(e^{2u} h)_{\bar{z}} = e^{2u} H_z \quad \text{Codazzi equation}$$

and the Gauss curvature is $K = H^2 - |h|^2$. See [JMN16, page 212].

In 1867 Bonnet [Bon67] began an investigation into the problem of whether there exist noncongruent immersions $\mathbf{x}, \tilde{\mathbf{x}} : M \to \mathbf{R}^3$ with the same induced metric, $I = \tilde{I}$, and the same mean curvature, $H = \tilde{H}$. This *Bonnet Problem* has been studied by Bianchi [Bia09], Graustein [Gra24], Cartan [Car42], Lawson–Tribuzy [LT81], Chern [Che85], Kamberov–Pedit–Pinkall [KPP98], Bobenko–Eitner [BE98, BE00], Roussos–Hernandez [RH90], Sabitov [Sab12], the present authors [JMN16], and many others cited in these references.

Definition 1. *An immersion $\mathbf{x} : M \to \mathbf{R}^3$ is Bonnet if there is a noncongruent immersion $\tilde{\mathbf{x}} : M \to \mathbf{R}^3$ such that $\tilde{I} = I$ and $\tilde{H} = H$. Then $\tilde{\mathbf{x}}$ is called a Bonnet mate of $\mathbf{x}$ and $(\mathbf{x}, \tilde{\mathbf{x}})$ form a Bonnet pair.*

A constant mean curvature (CMC) immersion $\mathbf{x} : M \to \mathbf{R}^3$, for which M is simply connected and $\mathbf{x}$ is not totally umbilic, admits a 1-parameter family of Bonnet mates, which are known as the associates of $\mathbf{x}$ [JMN16, Example 10.11, page

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302]. The local problem is thus to determine if an immersion $\mathbf{x}$ with nonconstant mean curvature has a Bonnet mate. By nonconstant mean curvature H we mean that $dH \neq 0$ on a dense, open subset of M.

Definition 2. *A Bonnet immersion $\mathbf{x} : M \to \mathbf{R}^3$ is proper if its mean curvature is nonconstant and there exist at least two noncongruent Bonnet mates.*

It is known [JMN16, page 211] that the umbilics of $\mathbf{x}$ are precisely the zeros of its Hopf quadratic differential $II^{2,0}$. For the following definitions we assume that $\mathbf{x}$ has no umbilics in the domain U. If (U, z) is a complex coordinate chart in M, then the local coefficient $e^{2u}h$ of $2II^{2,0}$ in U has the polar representation

$$e^{2u}h = e^{G+ig},$$

for a smooth function $G : U \to \mathbf{R}$ and a smooth map $e^{ig} : U \to \mathbf{S}^1$. The function $g : U \to \mathbf{R}$ is defined only locally, up to an additive integral multiple of 2π. If $w = w(z)$ is another complex coordinate in U, and if the invariants relative to it are denoted by $\hat{u}$ and $\hat{h}$, then

$$e^{2u}h = e^{2\hat{u}}\hat{h}(w')^2,$$

where $w' = \frac{dw}{dz}$ is a nowhere zero holomorphic function of z. Setting $e^{2\hat{u}}\hat{h} = e^{\hat{G}+i\hat{g}}$ on U, we find by an elementary calculation

$$g_{\bar{z}z} = \hat{g}_{\bar{z}z} \tag{2}$$

on U. The Laplace-Beltrami operator of (M, I) is given in the local chart (U, z) by $\Delta = 4e^{-2u}\frac{\partial^2}{\partial z \partial \bar{z}}$. We conclude from (2) that $\Delta g = \Delta \hat{g}$ on U, and therefore that Δg is a globally defined smooth function on M away from the umbilic points of $\mathbf{x}$.

Definition 3. *A surface immersion $\mathbf{x} : M \to \mathbf{R}^3$ is called isothermic if it has an atlas of charts $(U, (x, y))$ each of which satisfies $I = e^{2u}(dx^2 + dy^2)$ and $II = e^u(adx^2 + cdy^2)$* [JMN16, Definition 9.5, page 277].

Definition 3 is equivalent to the first statement of the following definition if there are no umbilics [JMN16, Corollary 9.14, page 280].

Definition 4. *An umbilic free immersion $\mathbf{x} : M \to \mathbf{R}^3$ of an oriented connected surface is isothermic if $\Delta g = 0$ identically on M. It is totally nonisothermic if $\Delta g \neq 0$ on a connected, open, dense subset of M.*

The following is known about the local situation. Suppose that $\mathbf{x} : M \to \mathbf{R}^3$ is an umbilic free immersion for which M is simply connected and possesses a complex coordinate $z : M \to \mathbf{C}$. Cartan [Car42] proved that if $\mathbf{x}$ is proper Bonnet, then it has a 1-parameter family of distinct mates [JMN16, Theorem 10.42, pages 340-342]. Graustein [Gra24] proved that if $\mathbf{x}$ is isothermic and Bonnet, then it is proper Bonnet. The present authors [JMN16, Theorem 10.13, pages 303-304] proved that if $\mathbf{x}$ is totally nonisothermic, then it has a unique Bonnet mate. This contrasts emphatically with the case when M is compact, as stated in item (2) of the following Theorem.

What is the global situation? Lawson–Tribuzy [LT81] proved that $\mathbf{x} : M \to \mathbf{R}^3$ cannot be proper Bonnet if M is compact. Since then the question whether there exist Bonnet pairs for a compact surface M of genus $g > 0$ has been open." Roussos–Hernandez [RH90] proved that $\mathbf{x} : M \to \mathbf{R}^3$ has no Bonnet mate if M is compact and $\mathbf{x}$ is a surface of revolution with nonconstant mean curvature. Sabitov [Sab12, Theorem 13, page 144] gives a sufficient condition preventing the existence of a Bonnet mate when the mean curvature is nonconstant and M is compact. He gives no geometric interpretation of his condition. It is known, and proved in the next section, that a necessary condition that $\mathbf{x}$ be Bonnet is that its set of umbilics is a discrete subset of M.

The goal of this paper is to prove the following result. It generalizes the Roussos–Hernandez result, since a surface of revolution is isothermic [JMN16, Example 9.7, page 277]. It also gives a geometrical clarification of the Sabitov result.

Theorem. *Let $\mathbf{x} : M \to \mathbf{R}^3$ be a smooth immersion with nonconstant mean curvature H of a compact, connected surface, and suppose that $\mathcal{D}$, the set of umbilics of $\mathbf{x}$, is a discrete subset of M.*

(1) *If $\mathbf{x} : M \setminus \mathcal{D} \to \mathbf{R}^3$ is isothermic, then $\mathbf{x} : M \to \mathbf{R}^3$ has no Bonnet mate.*
(2) *If $\mathbf{x} : M \setminus \mathcal{D} \to \mathbf{R}^3$ is totally nonisothermic, then $\mathbf{x} : M \to \mathbf{R}^3$ has no Bonnet mate.*

2. The deformation quadratic differential

From the Gauss equation above, the Hopf invariants h and $\tilde{h}$ relative to a complex coordinate z of two immersions with the same induced metric and the same mean curvatures must satisfy

$$|\tilde{h}| = |h|,$$

since $\tilde{u} = u$. Hence, the only possible difference in the invariants of two such immersions must be in the arguments of the complex valued functions h and $\tilde{h}$. Moreover, taking the difference of their Codazzi equations, we get

$$(e^{2u}\tilde{h} - e^{2u}h)_{\bar{z}} = e^{2u}(H_z - H_z) = 0,$$

at every point of the domain U of the complex coordinate z. This means that the function

$$F = e^{2u}(\tilde{h} - h) : U \to \mathbf{C}$$

is holomorphic.

Definition 5. *If $\mathbf{x}, \tilde{\mathbf{x}} : M \to \mathbf{R}^3$ are immersions that induce the same complex structure on M, then their deformation quadratic differential is*

$$\mathcal{Q} = \widetilde{II}^{2,0} - II^{2,0}.$$

If $\mathbf{x}$ and $\tilde{\mathbf{x}}$ have the same induced metric and mean curvature, then the expression for $\mathcal{Q}$ relative to a complex coordinate z is

$$\mathcal{Q} = \frac{1}{2}e^{2u}(\tilde{h} - h)dzdz = \frac{1}{2}F dzdz, \tag{3}$$

which shows that $\mathcal{Q}$ is a holomorphic quadratic differential on M, and

$$|F + e^{2u}h| = |e^{2u}\tilde{h}| = |e^{2u}h| \tag{4}$$

on U, since $|\tilde{h}| = |h|$. $\mathcal{Q}$ is identically zero on M if and only if $\tilde{h} = h$ in any complex coordinate system. Therefore, by Bonnet's Congruence Theorem, $\mathcal{Q} = 0$ if and only if the immersions $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are congruent in the sense that there exists a rigid motion $(\mathbf{y}, A) \in \mathbf{E}(3)$ such that $\tilde{\mathbf{x}} = \mathbf{y} + A\mathbf{x} : M \to \mathbf{R}^3$. Thus, an immersion $\tilde{\mathbf{x}} : M \to \mathbf{R}^3$ is a Bonnet mate of $\mathbf{x} : M \to \mathbf{R}^3$ if it induces the same metric and mean curvature and the deformation quadratic differential is not identically zero.

Proposition 6. *If an immersion* $\mathbf{x} : M \to \mathbf{R}^3$ *possesses a Bonnet mate* $\tilde{\mathbf{x}} : M \to \mathbf{R}^3$*, then the umbilics of* $\mathbf{x}$ *must be isolated and coincide with those of* $\tilde{\mathbf{x}}$*.*

Proof. Under the given assumptions, the holomorphic quadratic differential $\mathcal{Q}$ is not identically zero. Therefore, in any complex coordinate chart (U, z), we have $\mathcal{Q} = \frac{1}{2}F dz dz$, where F is a nonzero holomorphic function of z. Its zeros must be isolated. A point $m \in U$ is an umbilic of $\mathbf{x}$ if and only if $h(m) = 0$ if and only if $\tilde{h}(m) = 0$, by (4). In either case $F(m) = 0$ by (4). Therefore, the set of umbilic points is a subset of the set of zeros of $\mathcal{Q}$, which is a discrete subset of M. □

Let $\mathbf{x} : M \to \mathbf{R}^3$ be an immersion with a Bonnet mate $\tilde{\mathbf{x}} : M \to \mathbf{R}^3$. Let (U, z) be a complex coordinate chart in M and let h and $\tilde{h}$ be the Hopf invariants of $\mathbf{x}$ and $\tilde{\mathbf{x}}$, respectively, relative to z on U. Let $\mathcal{D}$ be the set of umbilics of $\mathbf{x}$, necessarily a discrete subset of M. On $U \setminus \mathcal{D}$ we have h never zero and

$$\tilde{h} = hA,$$

for a smooth function $A : U \setminus \mathcal{D} \to \mathbf{S}^1$, where $\mathbf{S}^1 \subset \mathbf{C}$ is the unit circle. On $U \setminus \mathcal{D}$ then, the difference of the Hopf differentials is the holomorphic quadratic differential

$$\mathcal{Q} = \widetilde{II}^{2,0} - II^{2,0} = II^{2,0}(A - 1).$$

This shows that $A : M \setminus \mathcal{D} \to \mathbf{S}^1$ is a well-defined smooth map on all of $M \setminus \mathcal{D}$.

Remark 7. *Under our assumption of nonconstant* H*, the map* A *cannot be constant, for otherwise* $II^{2,0}$ *would then be holomorphic and thus* H *would be constant by the Codazzi equation.*

Proposition 8 (Sabitov[Sab12])**.** *If an immersion* $\mathbf{x} : M \to \mathbf{R}^3$ *possesses a Bonnet mate* $\tilde{\mathbf{x}} : M \to \mathbf{R}^3$*, then the deformation quadratic differential* $\mathcal{Q}$ *of* $\mathbf{x}$ *is zero only at the umbilics of* $\mathbf{x}$*. Therefore,* $A : M \setminus \mathcal{D} \to \mathbf{S}^1$ *never takes the value* $1 \in \mathbf{S}^1$*.*

Proof. This is Theorem 1, pages 113ff of [Sab12]. He says the result is stated in [Bob08], but he believes the proof there is inadequate. Sabitov's proof uses results from the Hilbert boundary-value problem. The following proof is essentially the same as Sabitov's, but avoids use of the Hilbert boundary-value problem.

Seeking a contradiction, suppose $\mathcal{Q}(m_0) = 0$ for some point $m_0 \in M \setminus \mathcal{D}$. Since $\mathcal{Q}$ is holomorphic, and not identically zero, its zeros are isolated. Let (U, z) be a complex coordinate chart of $M \setminus \mathcal{D}$ centered at m_0, containing no other zeros of $\mathcal{Q}$, and such that $z(U)$ is an open disk of $\mathbf{C}$. Now $A(m_0) = 1$ and A is continuous, so we may assume U chosen small enough that A never takes the value -1 on U. Then there exists a smooth map $v : U \to \mathbf{R}$ such that $-\pi < v < \pi$ and $A = e^{iv}$ on U. Since $A = 1$ on U only at m_0, it follows that

$$v(U \setminus \{m_0\}) \subset (-\pi, 0) \text{ or } v(U \setminus \{m_0\}) \subset (0, \pi). \tag{5}$$

Let e^{2u} and h be the conformal factor and Hopf invariant of $\mathbf{x}$ relative to z. Then h never zero on U implies it has a polar representation $h = e^{f+ig}$, for some smooth functions $f, g : U \to \mathbf{R}$. Now $\mathcal{Q} = \frac{1}{2}F dz dz$, where

$$F = e^{2u}e^{f+ig}(e^{iv} - 1) = e^{2u+f}(e^{i(g+v)} - e^{ig}) : U \to \mathbf{C}$$

is holomorphic. Using the identity

$$e^{i(g+v)} - e^{ig} = e^{i(2g+v)/2}(e^{iv/2} - e^{-iv/2}) = 2ie^{i(g+v/2)} \sin(v/2),$$

we get

$$F = 2ie^{2u+f+i(g+v/2)}\sin(v/2)$$

on U. The contour integral of $d\log F$ about any circle in U centered at m_0 is $2\pi i$ times the number of zeros of F inside the circle. By assumption, this integral is not zero. But,

$$d\log F = d(2u + f + i(g + v/2)) + d\log(|\sin(v/2)|),$$

and the contour integral of the right hand side is zero, since these are exact differentials on $U \setminus \{m_0\}$. In fact, the values of $v/2$ on $U \setminus \{m_0\}$ lie entirely in $(0, \pi/2)$ or entirely in $(-\pi/2, 0)$, so $\sin(v/2)$ is never zero. This is the desired contradiction to our assumption that $\mathcal{Q}$ has a zero in $M \setminus \mathcal{D}$. □

As a consequence of this Proposition, the smooth map $A : M \setminus \mathcal{D} \to \mathbf{S}^1$ never takes the value $1 \in \mathbf{S}^1$, so there exists a smooth map

$$r : M \setminus \mathcal{D} \to (0, 2\pi) \subset \mathbf{R},$$

such that $A = e^{ir}$ on $M \setminus \mathcal{D}$.

3. Proof of the Theorem

Proof. Seeking a contradiction, we suppose that $\mathbf{x}$ possesses a Bonnet mate $\tilde{\mathbf{x}} : M \to \mathbf{R}^3$. Let $II^{2,0}$ and $\widetilde{II}^{2,0}$ be the Hopf quadratic differentials of $\mathbf{x}$ and $\tilde{\mathbf{x}}$, respectively. By the preceding propositions, the quadratic differential $\widetilde{II}^{2,0} - II^{2,0}$ is holomorphic on M, and on $M \setminus \mathcal{D}$

$$\widetilde{II}^{2,0} - II^{2,0} = II^{2,0}(e^{ir} - 1),$$

where the function $r : M\setminus\mathcal{D} \to (0, 2\pi)$ is smooth. Let (U, z) be a complex coordinate chart in $M\setminus\mathcal{D}$. Let h and e^u be the Hopf invariant and conformal factor of $\mathbf{x}$ relative to z. Then $h = e^{f+ig}$ on U, for some smooth functions $f : U \to \mathbf{R}$ and $e^{ig} : U \to \mathbf{S}^1$.

1). If $\mathbf{x}$ is isothermic, then $g_{\bar{z}z} = 0$ identically on U. Let $G = f + 2u : U \to \mathbf{R}$. Then $(e^{G+ig}(e^{ir} - 1))_{\bar{z}} = 0$ implies

$$r_{\bar{z}} = i(G + ig)_{\bar{z}}(1 - e^{-ir}) \tag{6}$$

on U. Applying ∂_z to this, and using that r_z is the complex conjugate of $r_{\bar{z}}$, we find

$$r_{\bar{z}z} = 0 \tag{7}$$

on U. Hence, $r : M \setminus \mathcal{D} \to (0, 2\pi)$ is a bounded harmonic function. Since the points of $\mathcal{D}$ are isolated and r is bounded, we know that r extends to a harmonic function on all of M. But then r must be constant, since M is compact. This contradicts our assumption of nonconstant H, by Remark 7.

2). If $\mathbf{x}$ is totally nonisothermic, we have either $\Delta g \leq 0$ or $\Delta g \geq 0$ on $M \setminus \mathcal{D}$. To be specific, let us suppose that $\Delta g \leq 0$ on $M \setminus \mathcal{D}$. Now (6) holds and by the proof of Theorem 10.13 on pages 303-304 of [JMN16], we have

$$e^{ir} = 1 + \frac{-2g_{\bar{z}z}}{D}(g_{\bar{z}z} + iL), \tag{8}$$

on U, where $L = |G_{\bar{z}} + ig_{\bar{z}}|^2 - G_{\bar{z}z}$ and $D = g_{\bar{z}z}^2 + L^2$. Applying ∂_z to (6) and using (8), we find

$$r_{\bar{z}z} = -2g_{\bar{z}z}, \tag{9}$$

on U. Therefore, $\Delta r = -2\Delta g \geq 0$ on $M \setminus \mathcal{D}$.

Recall [HK76, Def. §2.1, pages 40-41] that a function $v : V \to \mathbf{R} \cup \{-\infty\}$ on a domain $V \subset \mathbf{C}$ is *subharmonic* if

(1) $-\infty \leq v(z) < +\infty$ in V.
(2) v is upper semi-continuous in V. (This means that for any $c \in \mathbf{R}$, the set $\{z \in U : v(z) < c\}$ is open in V.)
(3) If z_0 is any point of V then there exist arbitrarily small positive values of R such that
$$v(z_0) \leq \frac{1}{2\pi R} \int_0^{2\pi} v(z_0 + Re^{it}) dt.$$

If v is of class C^2 in V, then v is subharmonic in V if and only if $v_{\bar{z}z} \geq 0$ in V [HK76, Example 3, page 41].

If M is a connected Riemann surface, we define a function $v : M \to \mathbf{R} \cup \{-\infty\}$ to be subharmonic if for any complex coordinate chart (U, z) of M, the local representative $v \circ z^{-1} : z(U) \to \mathbf{R}$ is subharmonic. This is well-defined by the Corollary to Theorem 2.8 on page 53 of [HK76].

We conclude from (9) that r is subharmonic on $M \setminus \mathcal{D}$. In the event that $\Delta g \geq 0$ on $M \setminus \mathcal{D}$, we conclude that $-r$ is subharmonic and continue as below with $-r$.

Suppose (U, z) is a complex coordinate chart centered at a point $m_0 \in \mathcal{D}$, and small enough that no other point of $\mathcal{D}$ lies in it. Then $r \circ z^{-1}$ is subharmonic on the open set $z(U) \setminus \{0\}$, so it extends uniquely to a subharmonic function on $z(U)$, by Theorem 5.8 on page 237 of [HK76]. It follows that r extends uniquely to a subharmonic function on M.

By Theorem 1.2 on page 4 of [HK76], if $v : V \to \mathbf{R} \cup \{-\infty\}$ is upper semi-continuous on a nonempty compact domain $V \subset \mathbf{C}$, then v attains its maximum on V; i.e., there exists $z_0 \in V$ such that $v(z) \leq v(z_0)$ for all $z \in V$. The same proof shows that this is true for an upper semi-continuous function on a compact Riemann surface. Thus, the subharmonic function $r : M \to \mathbf{R} \cup \{-\infty\}$ attains its maximum at some point $m_0 \in M$. Let (U, z) be a complex coordinate chart centered at m_0. Choose $R > 0$ such that the disk $D(0, R) = \{z \in \mathbf{C} : |z| \leq R\}$ is contained in $z(U)$. By the maximum principle for subharmonic functions [HK76, Theorem 2.3, page 47], $r \circ z^{-1}$ must be constantly equal to $r(m_0)$ on $D(0, R)$. It follows that
$$E = \{m \in M : r(m) = r(m_0)\}$$
is an open subset of M. But
$$E = M \setminus \{m \in M : r(m) < r(m_0)\}$$
is closed, since r is upper semi-continuous. We conclude that r is constant on M, which is our sought for contradiction, by Remark 7.

□

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