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# COMPACT SURFACES WITH NO BONNET MATE 

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#### Abstract

This note gives sufficient conditions (isothermic or totally nonisothermic) for an immersion of a compact surface to have no Bonnet mate.


## 1. Introduction

Consider a smooth immersion $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ of a connected, orientable surface $M$, with unit normal vector field $\mathbf{e}_{3}$. Its induced metric $I=d \mathbf{x} \cdot d \mathbf{x}$ and the orientation of $M$ induced by $\mathbf{e}_{3}$ from the standard orientation of $\mathbf{R}^{3}$ induce a complex structure on $M$, which provides a decomposition into bidegrees of the second fundamental form $I I$ of $\mathbf{x}$ relative to $\mathbf{e}_{3}$,

$$
-d \mathbf{e}_{3} \cdot d \mathbf{x}=I I=I I^{2,0}+H I+I I^{0,2}
$$

Here $H$ is the mean curvature of $\mathbf{x}$ relative to $\mathbf{e}_{3}$ and $I I^{2,0}=\overline{I I^{0,2}}$ is the Hopf quadratic differential of $\mathbf{x}$. Relative to a complex chart $(U, z)$ in $M$,

$$
\begin{equation*}
I=e^{2 u} d z d \bar{z}, \quad I I^{2,0}=\frac{1}{2} h e^{2 u} d z d z \tag{1}
\end{equation*}
$$

where the conformal factor $e^{u}$, the Hopf invariant $h$, and the mean curvature $H$ satisfy the structure equations on $U$ relative to $z$,

$$
\begin{aligned}
-4 e^{-2 u} u_{z \bar{z}} & =H^{2}-|h|^{2} \quad \text { Gauss equation } \\
\left(e^{2 u} h\right)_{\bar{z}} & =e^{2 u} H_{z} \quad \text { Codazzi equation }
\end{aligned}
$$

and the Gauss curvature is $K=H^{2}-|h|^{2}$. See [JMN16, page 212].
In 1867 Bonnet [Bon67] began an investigation into the problem of whether there exist noncongruent immersions $\mathbf{x}, \tilde{\mathbf{x}}: M \rightarrow \mathbf{R}^{3}$ with the same induced metric, $I=\tilde{I}$, and the same mean curvature, $H=\tilde{H}$. This Bonnet Problem has been studied by Bianchi [Bia09], Graustein [Gra24], Cartan [Car42], Lawson-Tribuzy [LT81], Chern [Che85], Kamberov-Pedit-Pinkall [KPP98], Bobenko-Eitner [BE98, BE00], Roussos-Hernandez [RH90], Sabitov [Sab12], the present authors [JMN16], and many others cited in these references.

Definition 1. An immersion $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ is Bonnet if there is a noncongruent immersion $\tilde{\mathbf{x}}: M \rightarrow \mathbf{R}^{3}$ such that $\tilde{I}=I$ and $\tilde{H}=H$. Then $\tilde{\mathbf{x}}$ is called a Bonnet mate of $\mathbf{x}$ and $(\mathbf{x}, \tilde{\mathbf{x}})$ form a Bonnet pair.

A constant mean curvature (CMC) immersion $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$, for which $M$ is simply connected and $\mathbf{x}$ is not totally umbilic, admits a 1-parameter family of Bonnet mates, which are known as the associates of $\mathbf{x}$ [JMN16, Example 10.11, page

[^0]302]. The local problem is thus to determine if an immersion $\mathbf{x}$ with nonconstant mean curvature has a Bonnet mate. By nonconstant mean curvature $H$ we mean that $d H \neq 0$ on a dense, open subset of $M$.
Definition 2. A Bonnet immersion $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ is proper if its mean curvature is nonconstant and there exist at least two noncongruent Bonnet mates.

It is known [JMN16, page 211] that the umbilics of $\mathbf{x}$ are precisely the zeros of its Hopf quadratic differential $I I^{2,0}$. For the following definitions we assume that $\mathbf{x}$ has no umbilics in the domain $U$. If $(U, z)$ is a complex coordinate chart in $M$, then the local coefficient $e^{2 u} h$ of $2 I I^{2,0}$ in $U$ has the polar representation

$$
e^{2 u} h=e^{G+i g}
$$

for a smooth function $G: U \rightarrow \mathbf{R}$ and a smooth map $e^{i g}: U \rightarrow \mathbf{S}^{1}$. The function $g: U \rightarrow \mathbf{R}$ is defined only locally, up to an additive integral multiple of $2 \pi$. If $w=w(z)$ is another complex coordinate in $U$, and if the invariants relative to it are denoted by $\hat{u}$ and $\hat{h}$, then

$$
e^{2 u} h=e^{2 \hat{u}} \hat{h}\left(w^{\prime}\right)^{2},
$$

where $w^{\prime}=\frac{d w}{d z}$ is a nowhere zero holomorphic function of $z$. Setting $e^{2 \hat{u}} \hat{h}=e^{\hat{G}+i \hat{g}}$ on $U$, we find by an elementary calculation

$$
\begin{equation*}
g_{\bar{z} z}=\hat{g}_{\bar{z} z} \tag{2}
\end{equation*}
$$

on $U$. The Laplace-Beltrami operator of $(M, I)$ is given in the local chart $(U, z)$ by $\Delta=4 e^{-2 u} \frac{\partial^{2}}{\partial z \partial \bar{z}}$. We conclude from (2) that $\Delta g=\Delta \hat{g}$ on $U$, and therefore that $\Delta g$ is a globally defined smooth function on $M$ away from the umbilic points of $\mathbf{x}$.

Definition 3. A surface immersion $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ is called isothermic if it has an atlas of charts $(U,(x, y))$ each of which satisfies $I=e^{2 u}\left(d x^{2}+d y^{2}\right)$ and $I I=$ $e^{u}\left(a d x^{2}+c d y^{2}\right)$ [JMN16, Definition 9.5, page 277].

Definition 3 is equivalent to the first statement of the following definition if there are no umbilics [JMN16, Corollary 9.14, page 280].

Definition 4. An umbilic free immersion $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ of an oriented connected surface is isothermic if $\Delta g=0$ identically on $M$. It is totally nonisothermic if $\Delta g \neq 0$ on a connected, open, dense subset of $M$.

The following is known about the local situation. Suppose that $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ is an umbilic free immersion for which $M$ is simply connected and possesses a complex coordinate $z: M \rightarrow \mathbf{C}$. Cartan [Car42] proved that if $\mathbf{x}$ is proper Bonnet, then it has a 1-parameter family of distinct mates [JMN16, Theorem 10.42, pages 340-342]. Graustein [Gra24] proved that if $\mathbf{x}$ is isothermic and Bonnet, then it is proper Bonnet. The present authors [JMN16, Theorem 10.13, pages 303-304] proved that if $\mathbf{x}$ is totally nonisothermic, then it has a unique Bonnet mate. This contrasts emphatically with the case when $M$ is compact, as stated in item (2) of the following Theorem.

What is the global situation? Lawson-Tribuzy [LT81] proved that $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ cannot be proper Bonnet if $M$ is compact. Since then the question whether there exist Bonnet pairs for a compact surface $M$ of genus $g>0$ has been open." RoussosHernandez [RH90] proved that $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ has no Bonnet mate if $M$ is compact and $\mathbf{x}$ is a surface of revolution with nonconstant mean curvature. Sabitov [Sab12,

Theorem 13, page 144] gives a sufficient condition preventing the existence of a Bonnet mate when the mean curvature is nonconstant and $M$ is compact. He gives no geometric interpretation of his condition. It is known, and proved in the next section, that a necessary condition that $\mathbf{x}$ be Bonnet is that its set of umbilics is a discrete subset of $M$.

The goal of this paper is to prove the following result. It generalizes the RoussosHernandez result, since a surface of revolution is isothermic [JMN16, Example 9.7, page 277]. It also gives a geometrical clarification of the Sabitov result.

Theorem. Let $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ be a smooth immersion with nonconstant mean curvature $H$ of a compact, connected surface, and suppose that $\mathcal{D}$, the set of umbilics of $\mathbf{x}$, is a discrete subset of $M$.
(1) If $\mathbf{x}: M \backslash \mathcal{D} \rightarrow \mathbf{R}^{3}$ is isothermic, then $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ has no Bonnet mate.
(2) If $\mathbf{x}: M \backslash \mathcal{D} \rightarrow \mathbf{R}^{3}$ is totally nonisothermic, then $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ has no Bonnet mate.

## 2. The deformation quadratic differential

From the Gauss equation above, the Hopf invariants $h$ and $\tilde{h}$ relative to a complex coordinate $z$ of two immersions with the same induced metric and the same mean curvatures must satisfy

$$
|\tilde{h}|=|h|,
$$

since $\tilde{u}=u$. Hence, the only possible difference in the invariants of two such immersions must be in the arguments of the complex valued functions $h$ and $\tilde{h}$. Moreover, taking the difference of their Codazzi equations, we get

$$
\left(e^{2 u} \tilde{h}-e^{2 u} h\right)_{\bar{z}}=e^{2 u}\left(H_{z}-H_{z}\right)=0
$$

at every point of the domain $U$ of the complex coordinate $z$. This means that the function

$$
F=e^{2 u}(\tilde{h}-h): U \rightarrow \mathbf{C}
$$

is holomorphic.
Definition 5. If $\mathbf{x}, \tilde{\mathbf{x}}: M \rightarrow \mathbf{R}^{3}$ are immersions that induce the same complex structure on $M$, then their deformation quadratic differential is

$$
\mathcal{Q}=\widetilde{I I}^{2,0}-I I^{2,0}
$$

If $\mathbf{x}$ and $\tilde{\mathbf{x}}$ have the same induced metric and mean curvature, then the expression for $\mathcal{Q}$ relative to a complex coordinate $z$ is

$$
\begin{equation*}
\mathcal{Q}=\frac{1}{2} e^{2 u}(\tilde{h}-h) d z d z=\frac{1}{2} F d z d z \tag{3}
\end{equation*}
$$

which shows that $\mathcal{Q}$ is a holomorphic quadratic differential on $M$, and

$$
\begin{equation*}
\left|F+e^{2 u} h\right|=\left|e^{2 u} \tilde{h}\right|=\left|e^{2 u} h\right| \tag{4}
\end{equation*}
$$

on $U$, since $|\tilde{h}|=|h|$. $\mathcal{Q}$ is identically zero on $M$ if and only if $\tilde{h}=h$ in any complex coordinate system. Therefore, by Bonnet's Congruence Theorem, $\mathcal{Q}=0$ if and only if the immersions $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are congruent in the sense that there exists a $\operatorname{rigid}$ motion $(\mathbf{y}, A) \in \mathbf{E}(3)$ such that $\tilde{\mathbf{x}}=\mathbf{y}+A \mathbf{x}: M \rightarrow \mathbf{R}^{3}$. Thus, an immersion $\tilde{\mathbf{x}}: M \rightarrow \mathbf{R}^{3}$ is a Bonnet mate of $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ if it induces the same metric and mean curvature and the deformation quadratic differential is not identically zero.

Proposition 6. If an immersion $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ possesses a Bonnet mate $\tilde{\mathbf{x}}: M \rightarrow$ $\mathbf{R}^{3}$, then the umbilics of $\mathbf{x}$ must be isolated and coincide with those of $\tilde{\mathbf{x}}$.

Proof. Under the given assumptions, the holomorphic quadratic differential $\mathcal{Q}$ is not identically zero. Therefore, in any complex coordinate chart $(U, z)$, we have $\mathcal{Q}=\frac{1}{2} F d z d z$, where $F$ is a nonzero holomorphic function of $z$. Its zeros must be isolated. A point $m \in U$ is an umbilic of $\mathbf{x}$ if and only if $h(m)=0$ if and only if $\tilde{h}(m)=0$, by (4). In either case $F(m)=0$ by (4). Therefore, the set of umbilic points is a subset of the set of zeros of $\mathcal{Q}$, which is a discrete subset of $M$.

Let $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ be an immersion with a Bonnet mate $\tilde{\mathbf{x}}: M \rightarrow \mathbf{R}^{3}$. Let $(U, z)$ be a complex coordinate chart in $M$ and let $h$ and $\tilde{h}$ be the Hopf invariants of $\mathbf{x}$ and $\tilde{\mathbf{x}}$, respectively, relative to $z$ on $U$. Let $\mathcal{D}$ be the set of umbilics of $\mathbf{x}$, necessarily a discrete subset of $M$. On $U \backslash \mathcal{D}$ we have $h$ never zero and

$$
\tilde{h}=h A
$$

for a smooth function $A: U \backslash \mathcal{D} \rightarrow \mathbf{S}^{1}$, where $\mathbf{S}^{1} \subset \mathbf{C}$ is the unit circle. On $U \backslash \mathcal{D}$ then, the difference of the Hopf differentials is the holomorphic quadratic differential

$$
\mathcal{Q}=\widetilde{I I}^{2,0}-I I^{2,0}=I I^{2,0}(A-1)
$$

This shows that $A: M \backslash \mathcal{D} \rightarrow \mathbf{S}^{1}$ is a well-defined smooth map on all of $M \backslash \mathcal{D}$.
Remark 7. Under our assumption of nonconstant $H$, the map $A$ cannot be constant, for otherwise $I I^{2,0}$ would then be holomorphic and thus $H$ would be constant by the Codazzi equation.
Proposition 8 (Sabitov[Sab12]). If an immersion $\mathbf{x}: M \rightarrow \mathbf{R}^{3}$ possesses a Bonnet mate $\tilde{\mathbf{x}}: M \rightarrow \mathbf{R}^{3}$, then the deformation quadratic differential $\mathcal{Q}$ of $\mathbf{x}$ is zero only at the umbilics of $\mathbf{x}$. Therefore, $A: M \backslash \mathcal{D} \rightarrow \mathbf{S}^{1}$ never takes the value $1 \in \mathbf{S}^{1}$.

Proof. This is Theorem 1, pages 113ff of [Sab12]. He says the result is stated in [Bob08], but he believes the proof there is inadequate. Sabitov's proof uses results from the Hilbert boundary-value problem. The following proof is essentially the same as Sabitov's, but avoids use of the Hilbert boundary-value problem.

Seeking a contradiction, suppose $\mathcal{Q}\left(m_{0}\right)=0$ for some point $m_{0} \in M \backslash \mathcal{D}$. Since $\mathcal{Q}$ is holomorphic, and not identically zero, its zeros are isolated. Let $(U, z)$ be a complex coordinate chart of $M \backslash \mathcal{D}$ centered at $m_{0}$, containing no other zeros of $\mathcal{Q}$, and such that $z(U)$ is an open disk of $\mathbf{C}$. Now $A\left(m_{0}\right)=1$ and $A$ is continuous, so we may assume $U$ chosen small enough that $A$ never takes the value -1 on $U$. Then there exists a smooth map $v: U \rightarrow \mathbf{R}$ such that $-\pi<v<\pi$ and $A=e^{i v}$ on $U$. Since $A=1$ on $U$ only at $m_{0}$, it follows that

$$
\begin{equation*}
v\left(U \backslash\left\{m_{0}\right\}\right) \subset(-\pi, 0) \text { or } v\left(U \backslash\left\{m_{0}\right\}\right) \subset(0, \pi) \tag{5}
\end{equation*}
$$

Let $e^{2 u}$ and $h$ be the conformal factor and Hopf invariant of $\mathbf{x}$ relative to $z$. Then $h$ never zero on $U$ implies it has a polar representation $h=e^{f+i g}$, for some smooth functions $f, g: U \rightarrow \mathbf{R}$. Now $\mathcal{Q}=\frac{1}{2} F d z d z$, where

$$
F=e^{2 u} e^{f+i g}\left(e^{i v}-1\right)=e^{2 u+f}\left(e^{i(g+v)}-e^{i g}\right): U \rightarrow \mathbf{C}
$$

is holomorphic. Using the identity

$$
e^{i(g+v)}-e^{i g}=e^{i(2 g+v) / 2}\left(e^{i v / 2}-e^{-i v / 2}\right)=2 i e^{i(g+v / 2)} \sin (v / 2)
$$

we get

$$
F=2 i e^{2 u+f+i(g+v / 2)} \sin (v / 2)
$$

on $U$. The contour integral of $d \log F$ about any circle in $U$ centered at $m_{0}$ is $2 \pi i$ times the number of zeros of $F$ inside the circle. By assumption, this integral is not zero. But,

$$
d \log F=d(2 u+f+i(g+v / 2))+d \log (|\sin (v / 2)|)
$$

and the contour integral of the right hand side is zero, since these are exact differentials on $U \backslash\left\{m_{0}\right\}$. In fact, the values of $v / 2$ on $U \backslash\left\{m_{0}\right\}$ lie entirely in $(0, \pi / 2)$ or entirely in $(-\pi / 2,0)$, so $\sin (v / 2)$ is never zero. This is the desired contradiction to our assumption that $\mathcal{Q}$ has a zero in $M \backslash \mathcal{D}$.

As a consequence of this Proposition, the smooth map $A: M \backslash \mathcal{D} \rightarrow \mathbf{S}^{1}$ never takes the value $1 \in \mathbf{S}^{1}$, so there exists a smooth map

$$
r: M \backslash \mathcal{D} \rightarrow(0,2 \pi) \subset \mathbf{R}
$$

such that $A=e^{i r}$ on $M \backslash \mathcal{D}$.

## 3. Proof of the Theorem

Proof. Seeking a contradiction, we suppose that $\mathbf{x}$ possesses a Bonnet mate $\tilde{\mathbf{x}}$ : $M \rightarrow \mathbf{R}^{3}$. Let $I I^{2,0}$ and $\widetilde{I I}^{2,0}$ be the Hopf quadratic differentials of $\mathbf{x}$ and $\tilde{\mathbf{x}}$, respectively. By the preceding propositions, the quadratic differential $\widetilde{I I}^{2,0}-I I^{2,0}$ is holomorphic on $M$, and on $M \backslash \mathcal{D}$

$$
\widetilde{I I}^{2,0}-I I^{2,0}=I I^{2,0}\left(e^{i r}-1\right)
$$

where the function $r: M \backslash \mathcal{D} \rightarrow(0,2 \pi)$ is smooth. Let $(U, z)$ be a complex coordinate chart in $M \backslash \mathcal{D}$. Let $h$ and $e^{u}$ be the Hopf invariant and conformal factor of $\mathbf{x}$ relative to $z$. Then $h=e^{f+i g}$ on $U$, for some smooth functions $f: U \rightarrow \mathbf{R}$ and $e^{i g}: U \rightarrow \mathbf{S}^{1}$.
1). If $\mathbf{x}$ is isothermic, then $g_{\bar{z} z}=0$ identically on $U$. Let $G=f+2 u: U \rightarrow \mathbf{R}$. Then $\left(e^{G+i g}\left(e^{i r}-1\right)\right)_{\bar{z}}=0$ implies

$$
\begin{equation*}
r_{\bar{z}}=i(G+i g)_{\bar{z}}\left(1-e^{-i r}\right) \tag{6}
\end{equation*}
$$

on $U$. Applying $\partial_{z}$ to this, and using that $r_{z}$ is the complex conjugate of $r_{\bar{z}}$, we find

$$
\begin{equation*}
r_{\bar{z} z}=0 \tag{7}
\end{equation*}
$$

on $U$. Hence, $r: M \backslash \mathcal{D} \rightarrow(0,2 \pi)$ is a bounded harmonic function. Since the points of $\mathcal{D}$ are isolated and $r$ is bounded, we know that $r$ extends to a harmonic function on all of $M$. But then $r$ must be constant, since $M$ is compact. This contradicts our assumption of nonconstant $H$, by Remark 7 .
2). If $\mathbf{x}$ is totally nonisothermic, we have either $\Delta g \leq 0$ or $\Delta g \geq 0$ on $M \backslash \mathcal{D}$. To be specific, let us suppose that $\Delta g \leq 0$ on $M \backslash \mathcal{D}$. Now (6) holds and by the proof of Theorem 10.13 on pages 303-304 of [JMN16], we have

$$
\begin{equation*}
e^{i r}=1+\frac{-2 g_{\bar{z} z}}{D}\left(g_{\bar{z} z}+i L\right) \tag{8}
\end{equation*}
$$

on $U$, where $L=\left|G_{\bar{z}}+i g_{\bar{z}}\right|^{2}-G_{\bar{z} z}$ and $D=g_{\bar{z} z}^{2}+L^{2}$. Applying $\partial_{z}$ to (6) and using (8), we find

$$
\begin{equation*}
r_{\bar{z} z}=-2 g_{\bar{z} z} \tag{9}
\end{equation*}
$$

on $U$. Therefore, $\Delta r=-2 \Delta g \geq 0$ on $M \backslash \mathcal{D}$.
Recall [HK76, Def. §2.1, pages 40-41] that a function $v: V \rightarrow \mathbf{R} \cup\{-\infty\}$ on a domain $V \subset \mathbf{C}$ is subharmonic if
(1) $-\infty \leq v(z)<+\infty$ in $V$.
(2) $v$ is upper semi-continuous in $V$. (This means that for any $c \in \mathbf{R}$, the set $\{z \in U: v(z)<c\}$ is open in V.)
(3) If $z_{0}$ is any point of $V$ then there exist arbitrarily small positive values of $R$ such that

$$
v\left(z_{0}\right) \leq \frac{1}{2 \pi R} \int_{0}^{2 \pi} v\left(z_{0}+R e^{i t}\right) d t
$$

If $v$ is of class $C^{2}$ in $V$, then $v$ is subharmonic in $V$ if and only if $v_{\bar{z} z} \geq 0$ in $V$ [HK76, Example 3, page 41].

If $M$ is a connected Riemann surface, we define a function $v: M \rightarrow \mathbf{R} \cup\{-\infty\}$ to be subharmonic if for any complex coordinate chart $(U, z)$ of $M$, the local representative $v \circ z^{-1}: z(U) \rightarrow \mathbf{R}$ is subharmonic. This is well-defined by the Corollary to Theorem 2.8 on page 53 of [HK76].

We conclude from (9) that $r$ is subharmonic on $M \backslash \mathcal{D}$. In the event that $\Delta g \geq 0$ on $M \backslash \mathcal{D}$, we conclude that $-r$ is subharmonic and continue as below with $-r$.

Suppose $(U, z)$ is a complex coordinate chart centered at a point $m_{0} \in \mathcal{D}$, and small enough that no other point of $\mathcal{D}$ lies in it. Then $r \circ z^{-1}$ is subharmonic on the open set $z(U) \backslash\{0\}$, so it extends uniquely to a subharmonic function on $z(U)$, by Theorem 5.8 on page 237 of [HK76]. It follows that $r$ extends uniquely to a subharmonic function on $M$.

By Theorem 1.2 on page 4 of [HK76], if $v: V \rightarrow \mathbf{R} \cup\{-\infty\}$ is upper semicontinuous on a nonempty compact domain $V \subset \mathbf{C}$, then $v$ attains its maximum on $V$; i.e., there exists $z_{0} \in V$ such that $v(z) \leq v\left(z_{0}\right)$ for all $z \in V$. The same proof shows that this is true for an upper semi-continuous function on a compact Riemann surface. Thus, the subharmonic function $r: M \rightarrow \mathbf{R} \cup\{-\infty\}$ attains its maximum at some point $m_{0} \in M$. Let $(U, z)$ be a complex coordinate chart centered at $m_{0}$. Choose $R>0$ such that the disk $D(0, R)=\{z \in \mathbf{C}:|z| \leq R\}$ is contained in $z(U)$. By the maximum principle for subharmonic functions [HK76, Theorem 2.3, page 47], $r \circ z^{-1}$ must be constantly equal to $r\left(m_{0}\right)$ on $D(0, R)$. It follows that

$$
E=\left\{m \in M: r(m)=r\left(m_{0}\right)\right\}
$$

is an open subset of $M$. But

$$
E=M \backslash\left\{m \in M: r(m)<r\left(m_{0}\right)\right\}
$$

is closed, since $r$ is upper semi-continuous. We conclude that $r$ is constant on $M$, which is our sought for contradiction, by Remark 7.

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