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# COMPACT SURFACES WITH NO BONNET MATE

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ABSTRACT. This note gives sufficient conditions (isothermic or totally non-isothermic) for an immersion of a compact surface to have no Bonnet mate.

## 1. INTRODUCTION

Consider a smooth immersion  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  of a connected, orientable surface  $M$ , with unit normal vector field  $\mathbf{e}_3$ . Its induced metric  $I = d\mathbf{x} \cdot d\mathbf{x}$  and the orientation of  $M$  induced by  $\mathbf{e}_3$  from the standard orientation of  $\mathbf{R}^3$  induce a complex structure on  $M$ , which provides a decomposition into bidegrees of the second fundamental form  $II$  of  $\mathbf{x}$  relative to  $\mathbf{e}_3$ ,

$$-de_3 \cdot d\mathbf{x} = II = II^{2,0} + HI + II^{0,2}.$$

Here  $H$  is the mean curvature of  $\mathbf{x}$  relative to  $\mathbf{e}_3$  and  $II^{2,0} = \overline{II^{0,2}}$  is the *Hopf quadratic differential* of  $\mathbf{x}$ . Relative to a complex chart  $(U, z)$  in  $M$ ,

$$(1) \quad I = e^{2u} dzd\bar{z}, \quad II^{2,0} = \frac{1}{2} h e^{2u} dzdz,$$

where the *conformal factor*  $e^u$ , the *Hopf invariant*  $h$ , and the mean curvature  $H$  satisfy the *structure equations on  $U$  relative to  $z$* ,

$$-4e^{-2u} u_{z\bar{z}} = H^2 - |h|^2 \quad \text{Gauss equation}$$

$$(e^{2u} h)_{\bar{z}} = e^{2u} H_z \quad \text{Codazzi equation}$$

and the Gauss curvature is  $K = H^2 - |h|^2$ . See [JMN16, page 212].

In 1867 Bonnet [Bon67] began an investigation into the problem of whether there exist noncongruent immersions  $\mathbf{x}, \tilde{\mathbf{x}} : M \rightarrow \mathbf{R}^3$  with the same induced metric,  $I = \tilde{I}$ , and the same mean curvature,  $H = \tilde{H}$ . This *Bonnet Problem* has been studied by Bianchi [Bia09], Graustein [Gra24], Cartan [Car42], Lawson–Tribuzy [LT81], Chern [Che85], Kamberov–Pedit–Pinkall [KPP98], Bobenko–Eitner [BE98, BE00], Roussos–Hernandez [RH90], Sabitov [Sab12], the present authors [JMN16], and many others cited in these references.

**Definition 1.** *An immersion  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  is Bonnet if there is a noncongruent immersion  $\tilde{\mathbf{x}} : M \rightarrow \mathbf{R}^3$  such that  $\tilde{I} = I$  and  $\tilde{H} = H$ . Then  $\tilde{\mathbf{x}}$  is called a Bonnet mate of  $\mathbf{x}$  and  $(\mathbf{x}, \tilde{\mathbf{x}})$  form a Bonnet pair.*

A constant mean curvature (CMC) immersion  $\mathbf{x} : M \rightarrow \mathbf{R}^3$ , for which  $M$  is simply connected and  $\mathbf{x}$  is not totally umbilic, admits a 1-parameter family of Bonnet mates, which are known as the associates of  $\mathbf{x}$  [JMN16, Example 10.11, page

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302]. The local problem is thus to determine if an immersion  $\mathbf{x}$  with nonconstant mean curvature has a Bonnet mate. By nonconstant mean curvature  $H$  we mean that  $dH \neq 0$  on a dense, open subset of  $M$ .

**Definition 2.** *A Bonnet immersion  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  is proper if its mean curvature is nonconstant and there exist at least two noncongruent Bonnet mates.*

It is known [JMN16, page 211] that the umbilics of  $\mathbf{x}$  are precisely the zeros of its Hopf quadratic differential  $II^{2,0}$ . For the following definitions we assume that  $\mathbf{x}$  has no umbilics in the domain  $U$ . If  $(U, z)$  is a complex coordinate chart in  $M$ , then the local coefficient  $e^{2u}h$  of  $2II^{2,0}$  in  $U$  has the polar representation

$$e^{2u}h = e^{G+ig},$$

for a smooth function  $G : U \rightarrow \mathbf{R}$  and a smooth map  $e^{ig} : U \rightarrow \mathbf{S}^1$ . The function  $g : U \rightarrow \mathbf{R}$  is defined only locally, up to an additive integral multiple of  $2\pi$ . If  $w = w(z)$  is another complex coordinate in  $U$ , and if the invariants relative to it are denoted by  $\hat{u}$  and  $\hat{h}$ , then

$$e^{2u}h = e^{2\hat{u}}\hat{h}(w')^2,$$

where  $w' = \frac{dw}{dz}$  is a nowhere zero holomorphic function of  $z$ . Setting  $e^{2\hat{u}}\hat{h} = e^{\hat{G}+i\hat{g}}$  on  $U$ , we find by an elementary calculation

$$(2) \quad g_{\bar{z}z} = \hat{g}_{\bar{z}z}$$

on  $U$ . The Laplace-Beltrami operator of  $(M, I)$  is given in the local chart  $(U, z)$  by  $\Delta = 4e^{-2u} \frac{\partial^2}{\partial z \partial \bar{z}}$ . We conclude from (2) that  $\Delta g = \Delta \hat{g}$  on  $U$ , and therefore that  $\Delta g$  is a globally defined smooth function on  $M$  away from the umbilic points of  $\mathbf{x}$ .

**Definition 3.** *A surface immersion  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  is called isothermic if it has an atlas of charts  $(U, (x, y))$  each of which satisfies  $I = e^{2u}(dx^2 + dy^2)$  and  $II = e^u(adx^2 + cdy^2)$  [JMN16, Definition 9.5, page 277].*

Definition 3 is equivalent to the first statement of the following definition if there are no umbilics [JMN16, Corollary 9.14, page 280].

**Definition 4.** *An umbilic free immersion  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  of an oriented connected surface is isothermic if  $\Delta g = 0$  identically on  $M$ . It is totally nonisothermic if  $\Delta g \neq 0$  on a connected, open, dense subset of  $M$ .*

The following is known about the local situation. Suppose that  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  is an umbilic free immersion for which  $M$  is simply connected and possesses a complex coordinate  $z : M \rightarrow \mathbf{C}$ . Cartan [Car42] proved that if  $\mathbf{x}$  is proper Bonnet, then it has a 1-parameter family of distinct mates [JMN16, Theorem 10.42, pages 340-342]. Graustein [Gra24] proved that if  $\mathbf{x}$  is isothermic and Bonnet, then it is proper Bonnet. The present authors [JMN16, Theorem 10.13, pages 303-304] proved that if  $\mathbf{x}$  is totally nonisothermic, then it has a unique Bonnet mate. This contrasts emphatically with the case when  $M$  is compact, as stated in item (2) of the following Theorem.

What is the global situation? Lawson–Tribuzy [LT81] proved that  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  cannot be proper Bonnet if  $M$  is compact. Since then the question whether there exist Bonnet pairs for a compact surface  $M$  of genus  $g > 0$  has been open.” Roussos–Hernandez [RH90] proved that  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  has no Bonnet mate if  $M$  is compact and  $\mathbf{x}$  is a surface of revolution with nonconstant mean curvature. Sabitov [Sab12,

Theorem 13, page 144] gives a sufficient condition preventing the existence of a Bonnet mate when the mean curvature is nonconstant and  $M$  is compact. He gives no geometric interpretation of his condition. It is known, and proved in the next section, that a necessary condition that  $\mathbf{x}$  be Bonnet is that its set of umbilics is a discrete subset of  $M$ .

The goal of this paper is to prove the following result. It generalizes the Roussos–Hernandez result, since a surface of revolution is isothermic [JMN16, Example 9.7, page 277]. It also gives a geometrical clarification of the Sabitov result.

**Theorem.** *Let  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  be a smooth immersion with nonconstant mean curvature  $H$  of a compact, connected surface, and suppose that  $\mathcal{D}$ , the set of umbilics of  $\mathbf{x}$ , is a discrete subset of  $M$ .*

- (1) *If  $\mathbf{x} : M \setminus \mathcal{D} \rightarrow \mathbf{R}^3$  is isothermic, then  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  has no Bonnet mate.*
- (2) *If  $\mathbf{x} : M \setminus \mathcal{D} \rightarrow \mathbf{R}^3$  is totally nonisothermic, then  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  has no Bonnet mate.*

## 2. THE DEFORMATION QUADRATIC DIFFERENTIAL

From the Gauss equation above, the Hopf invariants  $h$  and  $\tilde{h}$  relative to a complex coordinate  $z$  of two immersions with the same induced metric and the same mean curvatures must satisfy

$$|\tilde{h}| = |h|,$$

since  $\tilde{u} = u$ . Hence, the only possible difference in the invariants of two such immersions must be in the arguments of the complex valued functions  $h$  and  $\tilde{h}$ . Moreover, taking the difference of their Codazzi equations, we get

$$(e^{2u}\tilde{h} - e^{2u}h)_{\bar{z}} = e^{2u}(H_z - H_z) = 0,$$

at every point of the domain  $U$  of the complex coordinate  $z$ . This means that the function

$$F = e^{2u}(\tilde{h} - h) : U \rightarrow \mathbf{C}$$

is holomorphic.

**Definition 5.** *If  $\mathbf{x}, \tilde{\mathbf{x}} : M \rightarrow \mathbf{R}^3$  are immersions that induce the same complex structure on  $M$ , then their deformation quadratic differential is*

$$\mathcal{Q} = \widetilde{II}^{2,0} - II^{2,0}.$$

If  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  have the same induced metric and mean curvature, then the expression for  $\mathcal{Q}$  relative to a complex coordinate  $z$  is

$$(3) \quad \mathcal{Q} = \frac{1}{2}e^{2u}(\tilde{h} - h)dzdz = \frac{1}{2}Fdzdz,$$

which shows that  $\mathcal{Q}$  is a holomorphic quadratic differential on  $M$ , and

$$(4) \quad |F + e^{2u}h| = |e^{2u}\tilde{h}| = |e^{2u}h|$$

on  $U$ , since  $|\tilde{h}| = |h|$ .  $\mathcal{Q}$  is identically zero on  $M$  if and only if  $\tilde{h} = h$  in any complex coordinate system. Therefore, by Bonnet's Congruence Theorem,  $\mathcal{Q} = 0$  if and only if the immersions  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are congruent in the sense that there exists a rigid motion  $(\mathbf{y}, A) \in \mathbf{E}(3)$  such that  $\tilde{\mathbf{x}} = \mathbf{y} + A\mathbf{x} : M \rightarrow \mathbf{R}^3$ . Thus, an immersion  $\tilde{\mathbf{x}} : M \rightarrow \mathbf{R}^3$  is a Bonnet mate of  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  if it induces the same metric and mean curvature and the deformation quadratic differential is not identically zero.

**Proposition 6.** *If an immersion  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  possesses a Bonnet mate  $\tilde{\mathbf{x}} : M \rightarrow \mathbf{R}^3$ , then the umbilics of  $\mathbf{x}$  must be isolated and coincide with those of  $\tilde{\mathbf{x}}$ .*

*Proof.* Under the given assumptions, the holomorphic quadratic differential  $\mathcal{Q}$  is not identically zero. Therefore, in any complex coordinate chart  $(U, z)$ , we have  $\mathcal{Q} = \frac{1}{2}Fdzdz$ , where  $F$  is a nonzero holomorphic function of  $z$ . Its zeros must be isolated. A point  $m \in U$  is an umbilic of  $\mathbf{x}$  if and only if  $h(m) = 0$  if and only if  $\tilde{h}(m) = 0$ , by (4). In either case  $F(m) = 0$  by (4). Therefore, the set of umbilic points is a subset of the set of zeros of  $\mathcal{Q}$ , which is a discrete subset of  $M$ .  $\square$

Let  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  be an immersion with a Bonnet mate  $\tilde{\mathbf{x}} : M \rightarrow \mathbf{R}^3$ . Let  $(U, z)$  be a complex coordinate chart in  $M$  and let  $h$  and  $\tilde{h}$  be the Hopf invariants of  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$ , respectively, relative to  $z$  on  $U$ . Let  $\mathcal{D}$  be the set of umbilics of  $\mathbf{x}$ , necessarily a discrete subset of  $M$ . On  $U \setminus \mathcal{D}$  we have  $h$  never zero and

$$\tilde{h} = hA,$$

for a smooth function  $A : U \setminus \mathcal{D} \rightarrow \mathbf{S}^1$ , where  $\mathbf{S}^1 \subset \mathbf{C}$  is the unit circle. On  $U \setminus \mathcal{D}$  then, the difference of the Hopf differentials is the holomorphic quadratic differential

$$\mathcal{Q} = \widetilde{II}^{2,0} - II^{2,0} = II^{2,0}(A - 1).$$

This shows that  $A : M \setminus \mathcal{D} \rightarrow \mathbf{S}^1$  is a well-defined smooth map on all of  $M \setminus \mathcal{D}$ .

**Remark 7.** *Under our assumption of nonconstant  $H$ , the map  $A$  cannot be constant, for otherwise  $II^{2,0}$  would then be holomorphic and thus  $H$  would be constant by the Codazzi equation.*

**Proposition 8** (Sabitov[Sab12]). *If an immersion  $\mathbf{x} : M \rightarrow \mathbf{R}^3$  possesses a Bonnet mate  $\tilde{\mathbf{x}} : M \rightarrow \mathbf{R}^3$ , then the deformation quadratic differential  $\mathcal{Q}$  of  $\mathbf{x}$  is zero only at the umbilics of  $\mathbf{x}$ . Therefore,  $A : M \setminus \mathcal{D} \rightarrow \mathbf{S}^1$  never takes the value  $1 \in \mathbf{S}^1$ .*

*Proof.* This is Theorem 1, pages 113ff of [Sab12]. He says the result is stated in [Bob08], but he believes the proof there is inadequate. Sabitov's proof uses results from the Hilbert boundary-value problem. The following proof is essentially the same as Sabitov's, but avoids use of the Hilbert boundary-value problem.

Seeking a contradiction, suppose  $\mathcal{Q}(m_0) = 0$  for some point  $m_0 \in M \setminus \mathcal{D}$ . Since  $\mathcal{Q}$  is holomorphic, and not identically zero, its zeros are isolated. Let  $(U, z)$  be a complex coordinate chart of  $M \setminus \mathcal{D}$  centered at  $m_0$ , containing no other zeros of  $\mathcal{Q}$ , and such that  $z(U)$  is an open disk of  $\mathbf{C}$ . Now  $A(m_0) = 1$  and  $A$  is continuous, so we may assume  $U$  chosen small enough that  $A$  never takes the value  $-1$  on  $U$ . Then there exists a smooth map  $v : U \rightarrow \mathbf{R}$  such that  $-\pi < v < \pi$  and  $A = e^{iv}$  on  $U$ . Since  $A = 1$  on  $U$  only at  $m_0$ , it follows that

$$(5) \quad v(U \setminus \{m_0\}) \subset (-\pi, 0) \text{ or } v(U \setminus \{m_0\}) \subset (0, \pi).$$

Let  $e^{2u}$  and  $h$  be the conformal factor and Hopf invariant of  $\mathbf{x}$  relative to  $z$ . Then  $h$  never zero on  $U$  implies it has a polar representation  $h = e^{f+ig}$ , for some smooth functions  $f, g : U \rightarrow \mathbf{R}$ . Now  $\mathcal{Q} = \frac{1}{2}Fdzdz$ , where

$$F = e^{2u}e^{f+ig}(e^{iv} - 1) = e^{2u+f}(e^{i(g+v)} - e^{ig}) : U \rightarrow \mathbf{C}$$

is holomorphic. Using the identity

$$e^{i(g+v)} - e^{ig} = e^{i(2g+v)/2}(e^{iv/2} - e^{-iv/2}) = 2ie^{i(g+v/2)} \sin(v/2),$$

we get

$$F = 2ie^{2u+f+i(g+v/2)} \sin(v/2)$$

on  $U$ . The contour integral of  $d \log F$  about any circle in  $U$  centered at  $m_0$  is  $2\pi i$  times the number of zeros of  $F$  inside the circle. By assumption, this integral is not zero. But,

$$d \log F = d(2u + f + i(g + v/2)) + d \log(|\sin(v/2)|),$$

and the contour integral of the right hand side is zero, since these are exact differentials on  $U \setminus \{m_0\}$ . In fact, the values of  $v/2$  on  $U \setminus \{m_0\}$  lie entirely in  $(0, \pi/2)$  or entirely in  $(-\pi/2, 0)$ , so  $\sin(v/2)$  is never zero. This is the desired contradiction to our assumption that  $\mathcal{Q}$  has a zero in  $M \setminus \mathcal{D}$ .  $\square$

As a consequence of this Proposition, the smooth map  $A : M \setminus \mathcal{D} \rightarrow \mathbf{S}^1$  never takes the value  $1 \in \mathbf{S}^1$ , so there exists a smooth map

$$r : M \setminus \mathcal{D} \rightarrow (0, 2\pi) \subset \mathbf{R},$$

such that  $A = e^{ir}$  on  $M \setminus \mathcal{D}$ .

### 3. PROOF OF THE THEOREM

*Proof.* Seeking a contradiction, we suppose that  $\mathbf{x}$  possesses a Bonnet mate  $\tilde{\mathbf{x}} : M \rightarrow \mathbf{R}^3$ . Let  $II^{2,0}$  and  $\widetilde{II}^{2,0}$  be the Hopf quadratic differentials of  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$ , respectively. By the preceding propositions, the quadratic differential  $\widetilde{II}^{2,0} - II^{2,0}$  is holomorphic on  $M$ , and on  $M \setminus \mathcal{D}$

$$\widetilde{II}^{2,0} - II^{2,0} = II^{2,0}(e^{ir} - 1),$$

where the function  $r : M \setminus \mathcal{D} \rightarrow (0, 2\pi)$  is smooth. Let  $(U, z)$  be a complex coordinate chart in  $M \setminus \mathcal{D}$ . Let  $h$  and  $e^u$  be the Hopf invariant and conformal factor of  $\mathbf{x}$  relative to  $z$ . Then  $h = e^{f+ig}$  on  $U$ , for some smooth functions  $f : U \rightarrow \mathbf{R}$  and  $e^{ig} : U \rightarrow \mathbf{S}^1$ .

1). If  $\mathbf{x}$  is isothermic, then  $g_{\bar{z}z} = 0$  identically on  $U$ . Let  $G = f + 2u : U \rightarrow \mathbf{R}$ . Then  $(e^{G+ig}(e^{ir} - 1))_{\bar{z}} = 0$  implies

$$(6) \quad r_{\bar{z}} = i(G + ig)_{\bar{z}}(1 - e^{-ir})$$

on  $U$ . Applying  $\partial_z$  to this, and using that  $r_z$  is the complex conjugate of  $r_{\bar{z}}$ , we find

$$(7) \quad r_{\bar{z}z} = 0$$

on  $U$ . Hence,  $r : M \setminus \mathcal{D} \rightarrow (0, 2\pi)$  is a bounded harmonic function. Since the points of  $\mathcal{D}$  are isolated and  $r$  is bounded, we know that  $r$  extends to a harmonic function on all of  $M$ . But then  $r$  must be constant, since  $M$  is compact. This contradicts our assumption of nonconstant  $H$ , by Remark 7.

2). If  $\mathbf{x}$  is totally nonisothermic, we have either  $\Delta g \leq 0$  or  $\Delta g \geq 0$  on  $M \setminus \mathcal{D}$ . To be specific, let us suppose that  $\Delta g \leq 0$  on  $M \setminus \mathcal{D}$ . Now (6) holds and by the proof of Theorem 10.13 on pages 303-304 of [JMN16], we have

$$(8) \quad e^{ir} = 1 + \frac{-2g_{\bar{z}z}}{D}(g_{\bar{z}z} + iL),$$

on  $U$ , where  $L = |G_{\bar{z}} + ig_{\bar{z}}|^2 - G_{\bar{z}z}$  and  $D = g_{\bar{z}z}^2 + L^2$ . Applying  $\partial_z$  to (6) and using (8), we find

$$(9) \quad r_{\bar{z}z} = -2g_{\bar{z}z},$$

on  $U$ . Therefore,  $\Delta r = -2\Delta g \geq 0$  on  $M \setminus \mathcal{D}$ .

Recall [HK76, Def. §2.1, pages 40-41] that a function  $v : V \rightarrow \mathbf{R} \cup \{-\infty\}$  on a domain  $V \subset \mathbf{C}$  is *subharmonic* if

- (1)  $-\infty \leq v(z) < +\infty$  in  $V$ .
- (2)  $v$  is upper semi-continuous in  $V$ . (This means that for any  $c \in \mathbf{R}$ , the set  $\{z \in U : v(z) < c\}$  is open in  $V$ .)
- (3) If  $z_0$  is any point of  $V$  then there exist arbitrarily small positive values of  $R$  such that

$$v(z_0) \leq \frac{1}{2\pi R} \int_0^{2\pi} v(z_0 + Re^{it}) dt.$$

If  $v$  is of class  $C^2$  in  $V$ , then  $v$  is subharmonic in  $V$  if and only if  $v_{\bar{z}z} \geq 0$  in  $V$  [HK76, Example 3, page 41].

If  $M$  is a connected Riemann surface, we define a function  $v : M \rightarrow \mathbf{R} \cup \{-\infty\}$  to be subharmonic if for any complex coordinate chart  $(U, z)$  of  $M$ , the local representative  $v \circ z^{-1} : z(U) \rightarrow \mathbf{R}$  is subharmonic. This is well-defined by the Corollary to Theorem 2.8 on page 53 of [HK76].

We conclude from (9) that  $r$  is subharmonic on  $M \setminus \mathcal{D}$ . In the event that  $\Delta g \geq 0$  on  $M \setminus \mathcal{D}$ , we conclude that  $-r$  is subharmonic and continue as below with  $-r$ .

Suppose  $(U, z)$  is a complex coordinate chart centered at a point  $m_0 \in \mathcal{D}$ , and small enough that no other point of  $\mathcal{D}$  lies in it. Then  $r \circ z^{-1}$  is subharmonic on the open set  $z(U) \setminus \{0\}$ , so it extends uniquely to a subharmonic function on  $z(U)$ , by Theorem 5.8 on page 237 of [HK76]. It follows that  $r$  extends uniquely to a subharmonic function on  $M$ .

By Theorem 1.2 on page 4 of [HK76], if  $v : V \rightarrow \mathbf{R} \cup \{-\infty\}$  is upper semi-continuous on a nonempty compact domain  $V \subset \mathbf{C}$ , then  $v$  attains its maximum on  $V$ ; i.e., there exists  $z_0 \in V$  such that  $v(z) \leq v(z_0)$  for all  $z \in V$ . The same proof shows that this is true for an upper semi-continuous function on a compact Riemann surface. Thus, the subharmonic function  $r : M \rightarrow \mathbf{R} \cup \{-\infty\}$  attains its maximum at some point  $m_0 \in M$ . Let  $(U, z)$  be a complex coordinate chart centered at  $m_0$ . Choose  $R > 0$  such that the disk  $D(0, R) = \{z \in \mathbf{C} : |z| \leq R\}$  is contained in  $z(U)$ . By the maximum principle for subharmonic functions [HK76, Theorem 2.3, page 47],  $r \circ z^{-1}$  must be constantly equal to  $r(m_0)$  on  $D(0, R)$ . It follows that

$$E = \{m \in M : r(m) = r(m_0)\}$$

is an open subset of  $M$ . But

$$E = M \setminus \{m \in M : r(m) < r(m_0)\}$$

is closed, since  $r$  is upper semi-continuous. We conclude that  $r$  is constant on  $M$ , which is our sought for contradiction, by Remark 7. □

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