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Infinite S-expansion with ideal subtraction and some applications

D. M. Peñafiel[1,2,3,a)] and L. Ravera[2,3,b)]
[1]*Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile*
[2]*DISAT, Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy*
[3]*Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Torino, Via Pietro Giuria 1, 10125 Torino, Italy*

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According to the literature, the S-expansion procedure involving a finite semigroup is valid no matter what the structure of the original Lie (super)algebra is; however, when something about the structure of the starting (super)algebra is known and when certain particular conditions are met, the S-expansion method (with its features of resonance and reduction) is able not only to lead to several kinds of expanded (super)algebras but also to reproduce the effects of the standard as well as the generalized Inönü-Wigner contraction. In the present paper, we propose a new prescription for S-expansion, involving an infinite abelian semigroup $S^{(\infty)}$ and the subtraction of an infinite ideal subalgebra. We show that the subtraction of the infinite ideal subalgebra corresponds to a reduction. Our approach is a generalization of the finite S-expansion procedure presented in the literature, and it offers an alternative view of the generalized Inönü-Wigner contraction. We then show how to write the invariant tensors of the target (super)algebras in terms of those of the starting ones in the infinite S-expansion context presented in this work. We also give some interesting examples of application on algebras and superalgebras. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4991378]

I. INTRODUCTION

The contraction of finite dimensional Lie algebras is a tool that became very well-known, thanks to the paper of Inönü and Wigner (see Ref. 1 and also Ref. 2), in which the authors developed the so-called *Inönü-Wigner contraction*. This contraction procedure has then been successfully used in order to obtain different (super)algebras from the given ones. Recently, a generalization of the Inönü-Wigner contraction was obtained in Ref. 3 by rescaling not only the generators of a Lie superalgebra but also the arbitrary constants appearing in the components of the invariant tensor.

On the other hand, a method for expanding Lie algebras (*Lie algebras expansion*, also known as *power series expansion*) was introduced in Ref. 4 and then studied and applied in diverse studies to algebras and superalgebras (see Refs. 5–7 for further details). Subsequently, an alternative to the method of power series expansion, called *S-expansion* procedure, was developed (see Refs. 8–10) and applied in diverse scenarios, both in Mathematics and in Physics.

The S-expansion method replicates through the elements of a semigroup, the structure of the original algebra into a new one. The basis of the S-expansion consists, in fact, in combining the inner multiplication law of a discrete set S with the structure of a semigroup, with the structure constants of a Lie algebra $\mathfrak{g}$; the new, larger Lie algebra thus obtained is called *S-expanded algebra*, and it is written as $\mathfrak{g}_S = S \times \mathfrak{g}$. An important goal of the S-expansion procedure is that it allows us to write the invariant tensor of the S-expanded algebra from the knowledge of the invariant tensor of the original one.

a)diegomolina@udec.cl
b)lucrezia.ravera@polito.it

From the physical point of view, several (super)gravity theories have been extensively studied using the S-expansion approach, enabling numerous results over recent years (see Refs. 11–35). Recently, in Ref. 36, an analytic method for S-expansion was developed. This method gives the multiplication table(s) of the (abelian) set(s) involved in an S-expansion process for reaching a target Lie (super)algebra from a starting one, after having properly chosen the partitions over subspaces of the considered (super)algebras.

There are two facets applicable in the S-expansion method, which offer great manipulation on algebras, i.e., *resonance* (that transfers the structure of the semigroup to the target algebra, allowing us to control its structure with a suitable choice on the semigroup decomposition) and *reduction* (which plays a peculiar role in cutting the algebra properly).

The S-expansion procedure involving a *finite* semigroup is valid no matter what the structure of the original Lie (super)algebra is. However, when something about the structure of the starting (super)algebra is known and when certain particular conditions are met, the S-expansion method is able not only to lead to several kinds of expanded (super)algebras but also to reproduce the effects of the standard as well as the generalized Inönü-Wigner contraction (the latter can also be referred to as "Weimar-Woods generalized contraction" and involves higher powers of the contraction parameter). In Ref. 8, the authors explained how the information on the subspace structure of the original algebra can be used in order to find resonant subalgebras of the S-expanded algebra and discussed how this information can be put to use in a different way, namely, by extracting reduced algebras from the resonant subalgebra. By following this path, the generalized Inönü-Wigner contraction fits within their scheme and can thus be reproduced by S-expansion.

In the present paper, we develop a new prescription for the S-expansion procedure, which involves an infinite abelian semigroup and the removal of an (infinite) ideal subalgebra. We will call this procedure *infinite S-expansion*; it represents an extension and generalization of the finite one and it offers an alternative view of the generalized Inönü-Wigner contraction. Indeed, we will explicitly show how to reproduce a generalized Inönü-Wigner contraction by following our approach. We then explain how to reconstruct the components of the invariant tensor of the infinitely S-expanded algebra with ideal subtraction from those of the initial one. This is useful since it allows us to develop the dynamics and construct the Lagrangians of physical theories, starting from their algebraic structure.

Our alternative method differs from the one presented in Ref. 8 (that is the finite S-expansion) since it involves an infinite semigroup and the subtraction of an infinite ideal subalgebra. Obviously, the use of an infinite semigroup is not the only way for reaching the results we present in this paper (in fact, the examples of application we will consider are already well-known from the literature). However, in the present paper, we wish to describe an alternative path to expansion and contraction, in which, interestingly, the same results obtained in the context of finite S-expansion and Inönü-Wigner contraction [where, in particular, the latter does not change the dimension of the original (super)algebra] can be reproduced by acting on the original (super)algebra with an infinite abelian semigroup and by consequently subtracting an infinite ideal. The subtraction of the infinite ideal subalgebra here is crucial, since it allows us to obtain Lie (super)algebras with a finite number of generators, after having infinitely expanded the original Lie (super)algebras.

Furthermore, we will see that the prescription for infinite S-expansion with ideal subtraction leads to reduced algebras since the subtraction of the infinite ideal subalgebra can be viewed as a reduction involving an infinite number of semigroup elements that, together with the generators associated, play the role of "generating zeros." In this sense, the subtraction of the infinite ideal subalgebra can be viewed as a generalization of the 0_S-reduction. Let us observe that the role of "generating zeros" in the 0_S-reduction is played by the zero element, while in our method, it is the whole ideal subalgebra that plays this role; thus, the concept, in this sense, is slightly different. However, we will show that our method reproduces the same result of a 0_S-reduction.

This paper is organized as follows: In Sec. II, we give a review of the S-expansion procedure, with reduction and resonance, and of the Inönü-Wigner contraction. We also extend the concept of normal subgroup of a group to the concept of ideal subalgebra of an algebra since the latter will be useful in the rest of this work. In Sec. III, we briefly review the way in which a standard Inönü-Wigner contraction can be reproduced with a finite S-expansion procedure (with resonance and 0_S-reduction).

Then, we discuss how the authors of Ref. 8 used the information on the subspace structure of the original Lie algebra to extract reduced algebras from resonant subalgebras of S-expanded algebras, being able to reproduce, in this way, the generalized Inönü-Wigner contraction. Subsequently, we describe our prescription for infinite S-expansion with ideal subtraction and we explain how the generalized Inönü-Wigner contraction is reproduced in this context. Our method consists in first of all performing an S-expansion procedure involving an infinite abelian semigroup (which we will denote by $S^{(\infty)}$) and in consequently subtracting an (infinite) ideal subalgebra to the infinitely S-expanded one. We explicitly show that the subtraction of the infinite ideal subalgebra corresponds to a reduction. We also explain how to write the invariant tensors of the target (super)algebras in terms of those of the original ones in this context. In Sec. IV, we give some example of application on different (super)algebras, reproducing some results presented in the literature in the context of finite expansions and contractions, and we write the invariant tensors of some of the mentioned (super)algebras in terms of the invariant tensors of the starting ones. Section V contains a summary of our results, with comments and possible developments.

II. REVIEW OF *S*-EXPANSION, INÖNÜ-WIGNER CONTRACTION, AND IDEAL SUBALGEBRAS

The S-expansion of Lie (super)algebras through abelian semigroups consists in considering the direct product between an abelian semigroup S and a Lie algebra $\mathfrak{g}$: $S \times \mathfrak{g}$.

Under general conditions, relevant subalgebras can be systematically extracted from $S \times \mathfrak{g}$, like resonant algebras and reduced ones.

In the following, we give a review of the S-expansion, reduction, and resonance procedures[8] and of the Inönü-Wigner contraction process.[1,2]

We also extend the concept of normal subgroup of a group to the concept of ideal subalgebra of an algebra, which will be useful in the development of the (infinite) S-expansion prescription described in this paper.

A. *S*-expansion for an arbitrary semigroup *S*

The S-expansion procedure[8] consists in combining the structure constants of a Lie algebra $\mathfrak{g}$ with the inner multiplication law of a semigroup S, in order to define the Lie bracket of a new, S-expanded algebra $\mathfrak{g}_S = S \times \mathfrak{g}$.

Definition 1. Let $S = \{\lambda_\alpha\}$, with $\alpha = 1, \ldots, N$, be a finite, abelian semigroup with two-selector $K_{\alpha\beta}{}^{\gamma}$ defined by

$$K_{\alpha\beta}{}^{\gamma} = \begin{cases} 1, & when\, \lambda_\alpha \lambda_\beta = \lambda_\gamma, \\ 0, & otherwise. \end{cases} \tag{2.1}$$

Let $\mathfrak{g}$ be a Lie algebra with basis $\{T_A\}$ and structure constants $C_{AB}{}^{C}$ defined by the commutation relations

$$[T_A, T_B] = C_{AB}{}^{C}\, T_C. \tag{2.2}$$

Denote a basis element of the direct product $S \times \mathfrak{g}$ by $T_{(A,\alpha)} = \lambda_\alpha T_A$ and consider the induced commutator

$$\left[T_{(A,\alpha)}, T_{(B,\beta)}\right] \equiv \lambda_\alpha \lambda_\beta \left[T_A, T_B\right]. \tag{2.3}$$

One can show[8] that the product

$$\mathfrak{g}_S = S \times \mathfrak{g} \tag{2.4}$$

corresponds to the Lie algebra given by

$$\left[T_{(A,\alpha)}, T_{(B,\beta)}\right] = K_{\alpha\beta}{}^{\gamma} C_{AB}{}^{C}\, T_{(C,\gamma)}, \tag{2.5}$$

whose structure constants can be written as

$$C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} = K_{\alpha\beta}{}^{\gamma} C_{AB}{}^{C}. \tag{2.6}$$

For every abelian semigroup S and Lie algebra $\mathfrak{g}$, the algebra $\mathfrak{g}_S$ obtained through product (2.4) is also a Lie algebra, with a Lie bracket given by (2.5). The new, larger Lie algebra thus obtained is called S-expanded algebra, and it is written as $\mathfrak{g}_S = S \times \mathfrak{g}$.

B. Reduced algebras

Let us give the following definition (see Ref. 8) in order to introduce the concept of reduction of Lie algebras:

Definition 2. Let us consider a Lie algebra $\mathfrak{g}$ of the form $\mathfrak{g} = V_0 \oplus V_1$, where V_0 and V_1 are the two subspaces, respectively, given by $V_0 = \{T_{a0}\}$ and $V_1 = \{T_{a1}\}$. When $[V_0, V_1] \subset V_1$, that is to say, when the commutation relations between generators present the following form:

$$[T_{a_0}, T_{b_0}] = C_{a_0 b_0}{}^{c_0} T_{c_0} + C_{a_0 b_0}{}^{c_1} T_{c_1}, \tag{2.7}$$

$$[T_{a_0}, T_{b_1}] = C_{a_0 b_1}{}^{c_1} T_{c_1}, \tag{2.8}$$

$$[T_{a_1}, T_{b_1}] = C_{a_1 b_1}{}^{c_0} T_{c_0} + C_{a_1 b_1}{}^{c_1} T_{c_1}, \tag{2.9}$$

one can show that the structure constants $C_{a_0 b_0}{}^{c_0}$ satisfy the Jacobi identity themselves, and therefore

$$[T_{a_0}, T_{b_0}] = C_{a_0 b_0}{}^{c_0} T_{c_0} \tag{2.10}$$

itself corresponds to a Lie algebra, which is called the reduced algebra of $\mathfrak{g}$ and symbolized as $|V_0|$.

Let us observe that, in general, a reduced algebra does *not* correspond to a subalgebra.

C. 0_S-reduction of S-expanded algebras

The 0_S-reduction of Lie algebras[8] involves the extraction of a smaller algebra from an S-expanded Lie algebra $\mathfrak{g}_S$, when certain conditions are met.

In order to give a brief review of 0_S-reduction, let us consider an abelian semigroup S and the S-expanded (super)algebra $\mathfrak{g}_S = S \times \mathfrak{g}$.

When the semigroup S has a *zero element* $0_S \in S$ (also denoted with the symbol $\lambda_{0_S} \equiv 0_S$ in the literature), this element plays a peculiar role in the S-expanded (super)algebra. In fact, we can split the semigroup S into non-zero elements λ_i, $i = 0, \ldots, N$, and a zero element $\lambda_{N+1} = 0_S = \lambda_{0_S}$. The zero element λ_{0_S} is defined as one for which

$$\lambda_{0_S} \lambda_\alpha = \lambda_\alpha \lambda_{0_S} = \lambda_{0_S}, \tag{2.11}$$

for each $\lambda_\alpha \in S$. Under this assumption, we can write $S = \{\lambda_i\} \cup \{\lambda_{N+1} = \lambda_{0_S}\}$, with $i = 1, \ldots, N$ (the Latin index runs only on the non-zero elements of the semigroup). Then, the two-selector satisfies the relations

$$K_{i,N+1}{}^{j} = K_{N+1,i}{}^{j} = 0, \tag{2.12}$$

$$K_{i,N+1}{}^{N+1} = K_{N+1,i}{}^{N+1} = 1, \tag{2.13}$$

$$K_{N+1,N+1}{}^{j} = 0, \tag{2.14}$$

$$K_{N+1,N+1}{}^{N+1} = 1, \tag{2.15}$$

which means, from the multiplication rules point of view,

$$\lambda_{N+1} \lambda_i = \lambda_{N+1}, \tag{2.16}$$

$$\lambda_{N+1} \lambda_{N+1} = \lambda_{N+1}. \tag{2.17}$$

Therefore, for $\mathfrak{g}_S = S \times \mathfrak{g}$, we can write the commutation relations

$$\left[T_{(A,i)}, T_{(B,j)}\right] = K_{ij}{}^{k} C_{AB}{}^{C} T_{(C,k)} + K_{ij}{}^{N+1} C_{AB}{}^{C} T_{(C,N+1)}, \tag{2.18}$$

$$\left[T_{(A,N+1)}, T_{(B,j)}\right] = C_{AB}{}^{C} T_{(C,N+1)}, \tag{2.19}$$

$$\left[T_{(A,N+1)}, T_{(B,N+1)}\right] = C_{AB}{}^{C} T_{(C,N+1)}. \tag{2.20}$$

If we now compare these commutation relations with (2.7)–(2.9), we clearly see that

$$\left[T_{(A,i)}, T_{(B,j)}\right] = K_{ij}{}^{k} C_{AB}{}^{C} T_{(C,k)} \tag{2.21}$$

are the commutation relations of a reduced Lie algebra generated by $\{T_{(A,i)}\}$, whose structure constants are given by $K_{ij}{}^{k} C_{AB}{}^{C}$.

The reduction procedure, in this particular case, is tantamount to impose the condition

$$T_{A,N+1} = \lambda_{0_S} T_A \equiv 0_S T_A = 0. \tag{2.22}$$

We can notice that, in this case, the reduction abelianizes large sectors of the (super)algebra and that for each j satisfying $K_{ij}{}^{N+1} = 1$ (that is to say, $\lambda_i \lambda_j = \lambda_{N+1}$), we have

$$\left[T_{(A,i)}, T_{(B,j)}\right] = 0. \tag{2.23}$$

The above considerations led the authors of Ref. 8 to the following definition.

Definition 3. Let S be an abelian semigroup with a zero element $\lambda_{0_S} \equiv 0_S \in S$, and let $\mathfrak{g}_S = S \times \mathfrak{g}$ be an S-expanded algebra. Then, the algebra obtained by imposing the condition

$$\lambda_{0_S} T_A \equiv 0_S T_A = 0 \tag{2.24}$$

on $\mathfrak{g}_S$ (or on a subalgebra of it) is called the 0_S-reduced algebra of $\mathfrak{g}_S$ (or of the subalgebra).

Furthermore, when a 0_S-reduced algebra presents a structure which is resonant with respect to the structure of the semigroup involved in the S-expansion process, the procedure takes the name of 0_S-*resonant-reduction*.

D. Resonant subalgebras

Another prescription for getting smaller algebras (subalgebras) from the expanded ones, which read $S \times \mathfrak{g}$, is described in the definition below (see Ref. 8).

Definition 4. Let $\mathfrak{g} = \bigoplus_{p\in I} V_p$ be a decomposition of $\mathfrak{g}$ into subspaces V_p, where I is a set of indices. For each $p, q \in I$, it is always possible to define the subsets $i_{(p,q)} \subset I$ such that

$$\left[V_p, V_q\right] \subset \bigoplus_{r \in i_{(p,q)}} V_r, \tag{2.25}$$

where the subsets $i_{(p,q)}$ store the information on the subspace structure of $\mathfrak{g}$.

Now, let $S = \bigcup_{p\in I} S_p$ be a subset decomposition of the abelian semigroup S such that

$$S_p \cdot S_q \subset \bigcap_{r \in i_{(p,q)}} S_r, \tag{2.26}$$

where the product $S_p \cdot S_q$ is defined as

$$S_p \cdot S_q = \{\lambda_\gamma \mid \lambda_\gamma = \lambda_{\alpha_p}\lambda_{\alpha_q}, \text{ with } \lambda_{\alpha_p} \in S_p, \lambda_{\alpha_q} \in S_q\} \subset S. \tag{2.27}$$

When such subset decomposition $S = \bigcup_{p\in I} S_p$ exists, this decomposition is said to be in resonance with the subspace decomposition of $\mathfrak{g}$, $\mathfrak{g} = \bigoplus_{p\in I} V_p$.

The resonant subset decomposition is essential in order to systematically extract subalgebras from the S-expanded algebra $\mathfrak{g}_S = S \times \mathfrak{g}$, as it was enunciated and proven in Ref. 8 with the following theorem.

Theorem 1. *Let $\mathfrak{g} = \bigcup_{p\in I} V_p$ be a subspace decomposition of $\mathfrak{g}$, with a structure described by Eq. (2.25), and let $S = \bigcup_{p\in I} S_p$ be a resonant subset decomposition of the abelian semigroup S, with the structure given in Eq. (2.26). Define the subspaces of $\mathfrak{g}_S = S \times \mathfrak{g}$ as*

$$W_p = S_p \times V_p, \quad p \in I. \tag{2.28}$$

Then,

$$\mathfrak{g}_R = \bigoplus_{p\in I} W_p \tag{2.29}$$

is a subalgebra of $\mathfrak{g}_S = S \times \mathfrak{g}$, called resonant subalgebra of $\mathfrak{g}_S$.

E. Inönü-Wigner contraction

In what follows, we review the Inönü-Wigner contraction (and, in particular, an instance of its definition, which does fit the definition of standard Inönü-Wigner contraction) and the so-called "generalized Inönü-Wigner contraction" procedures (see Refs. 1, 2, 7, and 37–40 for further details).

Definition 5. Let T_i, $i = 1, 2, \ldots, n$, be a set of basis vectors for a Lie algebra $\mathfrak{g}$. Let a new set of basis vectors $\tilde{T}_i$, $i = 1, 2, \ldots, n$, be related to the T_i's by

$$\tilde{T}_j = U(\varepsilon)_j{}^i T_i, \quad U(\varepsilon = 1)_j{}^i = \delta_j{}^i, \quad det\left[U(\varepsilon = 0)_j{}^i\right] = 0. \tag{2.30}$$

The structure constants of the Lie algebra $\mathfrak{g}$ with respect to the new basis are given by

$$\left[\tilde{T}_i, \tilde{T}_j\right] = C_{ij}{}^k(\varepsilon)\tilde{T}_k. \tag{2.31}$$

When the limit

$$\lim_{\varepsilon \longrightarrow 0} C_{ij}{}^k(\varepsilon) = C'{}_{ij}{}^k \tag{2.32}$$

exists and is well defined, the new structure constants $C'{}_{ij}{}^k$ characterize a Lie algebra that is not isomorphic to the original one. This procedure is called Inönü-Wigner contraction.

The above definition fits the standard Inönü-Wigner contraction, which corresponds to a specific choice of the matrix $U(\varepsilon)$, as well as the generalized Inönü-Wigner contraction, which corresponds to a special case of Definition 5, in which the matrix $U(\varepsilon)$ acts in a peculiar way, as we will see in the following.

We now discuss a particular instance of Definition 5 which does fit the case of *standard Inönü-Wigner contraction*: If we now consider the symmetric coset $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ of simple Lie algebras, where $\mathfrak{h}$ closes a subalgebra and $\mathfrak{p}$ is a complementary subspace, with commutation relations of the form

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \tag{2.33}$$

$$[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \tag{2.34}$$

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}, \tag{2.35}$$

the standard Inönü-Wigner contraction of $\mathfrak{g} \longrightarrow \mathfrak{g}'$ involves the matrix

$$U(\varepsilon) = \begin{pmatrix} I_{dim(\mathfrak{h})} & 0 \\ 0 & \varepsilon I_{dim(\mathfrak{p})} \end{pmatrix}, \tag{2.36}$$

where I is the identity matrix, and $dim(\mathfrak{h})$ and $dim(\mathfrak{p})$ stand for the dimension of $\mathfrak{h}$ and $\mathfrak{p}$, respectively. We can thus write

$$\begin{pmatrix} \mathfrak{h}' \\ \mathfrak{p}' \end{pmatrix} = \begin{pmatrix} I_{dim(\mathfrak{h})} & 0 \\ 0 & \varepsilon I_{dim(\mathfrak{p})} \end{pmatrix} \begin{pmatrix} \mathfrak{h} \\ \mathfrak{p} \end{pmatrix}. \tag{2.37}$$

Therefore, after having performed the limit $\varepsilon \longrightarrow 0$, the contracted algebra $\mathfrak{g}'$ becomes a semidirect product $\mathfrak{g}' = \mathfrak{h}' \ltimes \mathfrak{p}' = \mathfrak{h} \ltimes \mathfrak{p}'$, and we can write the following commutation relations:

$$[\mathfrak{h}', \mathfrak{h}'] \subset \mathfrak{h}', \tag{2.38}$$

$$[\mathfrak{h}', \mathfrak{p}'] \subset \mathfrak{p}', \tag{2.39}$$

$$[\mathfrak{p}', \mathfrak{p}'] = 0, \tag{2.40}$$

from which we can see that $\mathfrak{p}'$ is now an abelian sector.

If we consider non-symmetric cosets (namely, in the case in which $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h} \oplus \mathfrak{p}$), we can prove that the standard Inönü-Wigner contraction still abelianizes this commutator ($[\mathfrak{p}', \mathfrak{p}'] = 0$).

Let us observe that the contracted algebra has the same dimension of the starting one.

The Inönü-Wigner contraction has a lot of applications in Physics, among which the relevant cases of the Galilei algebra as an Inönü-Wigner contraction of the Poincaré algebra and the Poincaré algebra as a contraction of the de Sitter algebra. Both of them involve two universal constants: the velocity of light and the cosmological constant, respectively.

1. Generalized Inönü-Wigner contraction

Any diagonal contraction is equivalent to a generalized Inönü-Wigner contraction (which can also be referred to as "Weimar-Woods generalized contraction")[38–40] with integer parameter powers.

Definition 6. Let $\mathfrak{g}$ be a non-simple algebra which can be written as a sum of $n+1$ subspaces (sets of generators) V_i, $i=0,1,\ldots,n$,

$$\mathfrak{g}=\bigoplus_{i=0}^{n} V_i = V_0 \oplus V_1 \oplus \cdots \oplus V_n \tag{2.41}$$

such that the following Weimar-Woods conditions[38,39] are satisfied:

$$\left[V_p, V_q\right] \subset \bigoplus_{s\leq p+q} V_s, \quad p,q=0,1,\ldots,n. \tag{2.42}$$

The conditions in (2.42) imply that V_0 is a subalgebra of $\mathfrak{g}$. The contraction of $\mathfrak{g}$ can be obtained after having performed a proper rescaling on the generators of each subspace and once a singular limit for the contraction parameter have been considered, namely, by considering

$$V'_i = \varepsilon^{a_i} V_i, \quad i=0,1,\ldots,n, \tag{2.43}$$

with the choice of the powers a's providing a finite limit of the contracted algebra if $\varepsilon \longrightarrow 0$. This procedure is called generalized Inönü-Wigner contraction.

Let us observe that Definition 6 corresponds to a special case of Definition 5, in which the matrix $U(\varepsilon)$ acts as in (2.43). In fact, the generalized Inönü-Wigner contractions are produced by diagonal matrices of the form $U(\varepsilon)_{jj}=\varepsilon^{n_j}$, $n_j \in \mathbb{Z}$. Thus, a contraction $\mathfrak{g} \xrightarrow{U(\varepsilon)_{ij}} \mathfrak{g}'$ is called *generalized Inönü-Wigner contraction*[39] if the matrix $U(\varepsilon)$ has the form

$$U(\varepsilon)_{ij} = \delta_{ij}\varepsilon^{n_j}, \quad n_j \in \mathbb{R}, \quad \varepsilon > 0, \quad i,j=1,2,\ldots,N, \tag{2.44}$$

with respect to a basis of generators $\{T_{a_1}, T_{a_2}, \ldots, T_{a_N}\}$.

F. Ideal subalgebra

In the following, we extend the concept of normal subgroup of a group to the concept of ideal subalgebra of an algebra (see Ref. 41 for further details).

Definition 7. Let us consider a Lie group G and a normal subgroup H of G. Let the Lie algebra $\mathcal{A}$ be the algebra associated with the Lie group G (by exponentiation), and let the subalgebra $\mathcal{I}$ of $\mathcal{A}$ be the subalgebra associated with the normal subgroup H. Then, $\mathcal{I}$ is an ideal (ideal subalgebra) of $\mathcal{A}$, and we can write

$$[\mathcal{A}, \mathcal{I}] \subset \mathcal{I}. \tag{2.45}$$

Let us now consider the homomorphism $\varphi: G \longrightarrow G/H$, where G is a Lie group and H is a normal subgroup of G. Let $\mathcal{A}$ be the Lie algebra associated with the Lie group G.

By definition, $H = ker\varphi$, and G/H is a Lie group. Let $\hat{\mathcal{A}}$ be the Lie algebra associated with the Lie group G/H. Then, φ induces a homomorphism $\hat{\varphi}: \mathcal{A} \longrightarrow \hat{\mathcal{A}}$ between algebras, such that, if $a \in \mathcal{A}$, then $\varphi(e^a) = e^{\hat{\varphi}(a)}$. Since, from Definition 7, $\forall a \in \mathcal{I}$, we have $e^a \in H$, this now implies $\varphi(e^a) = e' \in G/H \iff \hat{\varphi}(a) = 0$. Consequently, $\hat{\varphi}(\mathcal{I}) = \{0\}$.

The algebra $\hat{\mathcal{A}}$ is isomorphic to the coset space $\mathcal{A} \ominus \mathcal{I}$, whose elements are the equivalence classes $[a] = \{a' \in \mathcal{A} : a' - a \in \mathcal{I}\}$.

Let us now consider ρ: $G \longrightarrow Aut(V)$, where V is a vector space. This defines a representation of G/H provided on the vector space V. It has a trivial action on each vector $\Psi \in V$,

$$\rho(H)\Psi = \Psi. \tag{2.46}$$

Then, by defining, $\forall\, [g] \in G/H$,

$$\rho([g]) = \rho(g), \tag{2.47}$$

since

$$\rho(g\cdot h)\Psi=\rho(g)\rho(h)\Psi \tag{2.48}$$
$$=\rho(g)\Psi,\quad \forall\Psi\in V, \tag{2.49}$$

we can say that $\rho(G/H)=\rho(G)$ is the property we need in order to require

$$\rho(\mathcal{I})=0. \tag{2.50}$$

In this way, ρ provides a representation of $\mathcal{A}\ominus\mathcal{I}$ such that,

$$\rho(\mathcal{A})=\rho(\mathcal{A}\ominus\mathcal{I}). \tag{2.51}$$

Thus, if we now write

$$\mathcal{A}\ominus\mathcal{I}=\mathcal{A}_0, \tag{2.52}$$

where $\mathcal{A}_0$ is a coset space, we can say that $\rho(\mathcal{A}_0)$ is homomorphic to $\rho(\mathcal{A})$, and we can finally write

$$\rho(\mathcal{A})\Psi=\rho(\mathcal{A}_0)\Psi,\quad \forall\Psi\in V. \tag{2.53}$$

III. INÖNÜ-WIGNER CONTRACTION AND *S*-EXPANSION

In the following, we show that a standard Inönü-Wigner contraction can be reproduced with a (finite) S-expansion procedure (with resonance and 0_S-reduction). Then, we review the way in which the authors of Ref. 8 used the information on the subspace structure of the original Lie algebra to extract reduced algebras from resonant subalgebras of S-expanded algebras, being able to reproduce, in this way, the generalized Inönü-Wigner contraction.

After doing this, we proceed to the core of the present paper, that is, we describe our prescription for infinite S-expansion (involving an infinite abelian semigroup $S^{(\infty)}$) with ideal subtraction, and we explain how to reproduce a generalized Inönü-Wigner contraction in this context. This method is a new prescription for S-expansion and also an alternative way of seeing the (generalized) Inönü-Wigner contraction.

A. Finite *S*-expansion and Inönü-Wigner contraction

We first of all show the way in which a standard Inönü-Wigner contraction can be reproduced with a (finite) S-expansion procedure. Then, we review, following Ref. 8, the procedure to extract reduced algebras from resonant subalgebras of S-expanded algebras, allowing the S-expansion method to reproduce the case of generalized Inönü-Wigner contraction.

1. Standard Inönü-Wigner contraction as finite S-expansion with resonance and 0_S-reduction

Let us consider a coset $\mathfrak{g}$ with a subalgebra $\mathfrak{h}$ and a complementary subspace $\mathfrak{p}$, namely, $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$, with commutation relations of the form

$$\begin{aligned}&[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h},\\&[\mathfrak{h},\mathfrak{p}]\subset\mathfrak{p},\\&[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{h}\oplus\mathfrak{p}.\end{aligned} \tag{3.1}$$

The Inönü-Wigner contraction of $\mathfrak{g}\longrightarrow\mathfrak{g}'$ involves the following transformation:

$$\begin{pmatrix}\mathfrak{h}'\\ \mathfrak{p}'\end{pmatrix}=\begin{pmatrix}I_{dim(\mathfrak{h})} & 0\\ 0 & \varepsilon I_{dim(\mathfrak{p})}\end{pmatrix}\begin{pmatrix}\mathfrak{h}\\ \mathfrak{p}\end{pmatrix}, \tag{3.2}$$

where I is the identity matrix, and $dim(\mathfrak{h})$ and $dim(\mathfrak{p})$ stand for the dimension of $\mathfrak{h}$ and $\mathfrak{p}$, respectively; ε is the contraction parameter. Then, the commutation relations of $\mathfrak{g}'=\mathfrak{h}'\ltimes\mathfrak{p}'=\mathfrak{h}\ltimes\mathfrak{p}'$ are well defined for all values of ε, including the singular limit $\varepsilon\longrightarrow0$,

$$\begin{aligned} &[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, & & [\mathfrak{h}',\mathfrak{h}'] \subset \mathfrak{h}',\\ &[\mathfrak{h},\mathfrak{p}] \subset \mathfrak{p}, &\xrightarrow{\mathfrak{h}'=\mathfrak{h},\, \mathfrak{p}'=\varepsilon\mathfrak{p},\, \varepsilon\longrightarrow 0} \quad & [\mathfrak{h}',\mathfrak{p}'] \subset \mathfrak{p}',\\ &[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}\oplus\mathfrak{p}, & & [\mathfrak{p}',\mathfrak{p}'] = 0. \end{aligned} \tag{3.3}$$

This Inönü-Wigner contraction (standard Inönü-Wigner contraction) can be reproduced with a finite S-expansion procedure involving 0_S-resonant-reduction and the semigroup $S_E^{(1)} = \{\lambda_0, \lambda_1, \lambda_2\}$ (where the zero element of the semigroup is $\lambda_{0_S} = \lambda_2$) described by the multiplication table

$$\begin{array}{c|ccc} & \lambda_0 & \lambda_1 & \lambda_2 \\ \hline \lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \end{array}. \tag{3.4}$$

This can be done (see Ref. 8) by considering a proper partition over the subspaces of the starting Lie algebra and by multiplying its generators by the elements of the semigroup $S_E^{(1)}$. Then, the resonant-reduced, S-expanded Lie algebra $\mathfrak{g}_{S_{RR}} = \mathfrak{h}' \oplus \mathfrak{p}'$, where $\mathfrak{h}' = \lambda_0 \mathfrak{h}$ and $\mathfrak{p}' = \lambda_1 \mathfrak{p}$, satisfies the following commutation relations:

$$[\lambda_0\mathfrak{h}, \lambda_0\mathfrak{h}] \subset \lambda_0\mathfrak{h}, \tag{3.5}$$

$$[\lambda_0\mathfrak{h}, \lambda_1\mathfrak{p}] \subset \lambda_1\mathfrak{p}, \tag{3.6}$$

$$[\lambda_1\mathfrak{p}, \lambda_1\mathfrak{p}] = 0. \tag{3.7}$$

Here we can see that the role of the zero element $\lambda_2 = \lambda_{0_S}$ is to turn each multiplicand to zero (see Eq. (2.24) in Sec. II). We can easily see that by rewriting $\lambda_0\mathfrak{h}$ and $\lambda_1\mathfrak{p}$ in terms of $\mathfrak{h}'$ and $\mathfrak{p}'$, we arrive to Eq. (3.3). Thus, we can conclude that the Inönü-Wigner contraction can be seen as an S-expansion (involving 0_S-resonant-reduction) performed with the semigroup $S_E^{(1)}$.

2. Reduction of resonant subalgebras and generalized Inönü-Wigner contraction

The authors of Ref. 8 proved that one can extract reduced algebras from the resonant subalgebra of a (finite) S-expanded algebra $S \times \mathfrak{g}$, and they showed that, by following this path, the generalized Inönü-Wigner contraction fits within their scheme.

In particular, Theorem VII.1 of their work provides necessary conditions under which a reduced algebra can be extracted from a resonant subalgebra. It reads as follows:

Theorem 2. *Let* $\mathfrak{g}_R = \bigoplus_{p\in I} S_p \times V_p$ *be a resonant subalgebra of* $\mathfrak{g}_S = S \times \mathfrak{g}$. *Let* $S_p = \hat{S}_p \cup \check{S}_p$ *be a partition of the subset* $S_p \subset S$ *such that*

$$\hat{S}_p \cap \check{S}_p = \emptyset, \tag{3.8}$$

$$\check{S}_p \cdot \hat{S}_q \subset \bigcap_{r \in i_{(p,q)}} \hat{S}_r. \tag{3.9}$$

Conditions (3.8) and (3.9) induce the decomposition $\mathfrak{g}_R = \check{\mathfrak{g}}_R \oplus \hat{\mathfrak{g}}_R$ *on the resonant subalgebra, where*

$$\check{\mathfrak{g}}_R = \bigoplus_{p\in I} \check{S}_p \times V_p, \tag{3.10}$$

$$\hat{\mathfrak{g}}_R = \bigoplus_{p\in I} \hat{S}_p \times V_p. \tag{3.11}$$

When the conditions (3.8) and (3.9) hold, then

$$[\check{\mathfrak{g}}_R, \hat{\mathfrak{g}}_R] \subset \hat{\mathfrak{g}}_R, \tag{3.12}$$

and therefore $|\check{\mathfrak{g}}_R|$ *corresponds to a reduced algebra of* $\mathfrak{g}_R$.

Indeed, $\mathfrak{g}_R$ fits the definition of reduced algebra (2). Using the structure constants for the resonant subalgebra, it is then possible to find the structure constants for the reduced algebra $|\check{\mathfrak{g}}_R|$, as shown in Ref. 8.

Let us observe that when every $S_p \subset S$ of a resonant subalgebra includes the zero element λ_{0_S}, the choice $\hat{S}_p = \{\lambda_{0_S}\}$ automatically satisfies the conditions (3.8) and (3.9). As a consequence of this, the 0_S-reduction previously reviewed can be regarded as a particular case of Theorem 2.

Theorem 2, namely, Theorem VII.1 of Ref. 8, has been then used by the author of the same paper to recover Theorem 3 of Ref. 5 in the context of Lie algebras contractions.

Considering the semigroup $S_E^{(N)} = \{\lambda_\alpha, \alpha = 0, \ldots, N+1\}$, provided with the multiplication rule

$$\lambda_\alpha \lambda_\beta = \lambda_{H_{N+1}(\alpha+\beta)}, \tag{3.13}$$

where H_{N+1} is defined as the function

$$H_n(x) = \begin{cases} x, & \text{when } x < n, \\ n, & \text{when } x \geq n, \end{cases} \tag{3.14}$$

and where λ_{N+1} is the zero element in $S_E^{(N)}$, namely, $\lambda_{N+1} = \lambda_{0_S}$, the authors of Ref. 8 performed S-expansions using $S = S_E$ and found resonant partitions for S_E, constructing resonant subalgebras $\mathfrak{g}_R$'s and then applying a 0_S-reduction to the resonant subalgebras, in different cases. Their aim, in this context, was to recover some results presented for algebra expansions in Ref. 5 within the S-expansion approach.

In particular, they analyzed the case in which a Lie algebra $\mathfrak{g}$ fulfills the Weimar-Woods conditions,[38,39] namely, when one can write the subspace decomposition $\mathfrak{g} = \bigoplus_{p=0}^{n} V_p$ of the Lie algebra $\mathfrak{g}$ for which the following Weimar-Woods conditions hold:

$$\left[V_p, V_q\right] \subset \bigoplus_{p=0}^{H_n(p+q)} V_r. \tag{3.15}$$

In this context, the authors of Ref. 8 considered a subset decomposition of S_E,

$$S_E = \bigcup_{p=0}^{n} S_p, \tag{3.16}$$

where the subsets $S_p \subset S_E$ were defined by

$$S_p = \{\lambda_{\alpha_p},\ \alpha_p = p, \ldots, N+1\}, \tag{3.17}$$

with $N+1 \geq n$. The subset decomposition (3.16) is a resonant one under the semigroup product (2.27) since it satisfies

$$S_p \cdot S_q = S_{H_n(p+q)} \subset \bigcap_{r=0}^{H_n(p+q)} S_r. \tag{3.18}$$

Thus, according to Theorem 1 (which corresponds to Theorem IV.2 of Ref. 8), the direct sum

$$\mathfrak{g}_R = \bigoplus_{p=0}^{n} W_p, \tag{3.19}$$

with

$$W_p = S_p \times V_p, \tag{3.20}$$

is a resonant subalgebra of $\mathfrak{g}$. Then, they considered the following S_p partition, which satisfies (3.8)

$$\check{S}_p = \{\lambda_{\alpha_p},\ \alpha_p = p, \ldots, N_p\}, \tag{3.21}$$

$$\hat{S}_p = \{\lambda_{\alpha_p},\ \alpha_p = N_p + 1, \ldots, N+1\}, \tag{3.22}$$

and they proved that the reduction condition (3.9) on (3.21) and (3.22) is equivalent to the following requirement on the N_p's

$$N_{p+1} = \begin{cases} N_p \text{ or} \\ H_{N+1}(N_p+1) \end{cases}. \tag{3.23}$$

This condition is exactly the one obtained in Theorem 3 of Ref. 5 (requiring that the expansion in the Maurer-Cartan forms closes). In the S-expansion context, the case

$$N_{p+1} = N_p = N + 1 \tag{3.24}$$

for each p corresponds to the resonant subalgebra, and the case

$$N_{p+1} = N_p = N \tag{3.25}$$

to its 0_S-reduction.[8] The generalized Inönü-Wigner contraction corresponds to the case

$$N_p = p \tag{3.26}$$

(see Ref. 5 for details). This, as stated in Ref. 8, means that the generalized Inönü-Wigner contraction does not correspond to a resonant subalgebra but to its reduction. This is an important point because the authors of Ref. 8 have been able to define non-trace invariant tensors for resonant subalgebras and 0_S-reduced algebras but *not* for general reduced algebras.

As we will see in the following, the prescription for infinite S-expansion with ideal subtraction we develop in the present paper leads to reduced algebras. In particular, the subtraction of the infinite ideal subalgebra can be viewed as a 0_S-reduction (since it reproduces the same result) involving an infinite number of semigroup elements that, together with the generators associated, play the role of "generating zeros." In this context, as we will discuss, we are able to write the invariant tensors of the target algebras in terms of those on the original ones.

We can thus conclude that, in the context of finite S-expansion, the path for reproducing a generalized Inönü-Wigner contraction consists in extracting reduced algebras from the resonant subalgebra of an S-expanded algebra.

B. Infinite *S*-expansion with ideal subtraction

As we have previously discussed, in Ref. 8, generalized Inönü-Wigner contractions have been realized as reductions of S-expanded algebras with a finite semigroup.

In the following, we will discuss a new prescription for S-expansion, involving an *infinite* abelian semigroup and the subsequent subtraction of an infinite ideal subalgebra. This scenario also offers an alternative view on the generalized Inönü-Wigner contraction procedure. As we have already mentioned in the Introduction, the subtraction of the infinite ideal subalgebra in this context is crucial since it allows to obtain Lie (super)algebras with a finite number of generators, after having infinitely expanded the original Lie (super)algebras.

Before proceeding to the development of our method, let us spend a few words on the general idea. Thus, let us remind that the generalized Inönü-Wigner contraction (see Ref. 40) can be performed when we consider a non-simple algebra $\mathfrak{g}$ decomposed into $n + 1$ subspaces V_i of generators $T_a^{(i)}$ ($i = 0, 1, \ldots, n$), namely,

$$\mathfrak{g} = V_0 \oplus V_1 \oplus \cdots \oplus V_n, \tag{3.27}$$

where conditions (2.42) (namely the Weimar-Woods conditions[38,39]) are satisfied.

The generalized Inönü-Wigner contraction of (3.27) is obtained by properly rescaling each generator $T_a^{(i)}$ by a power of the contraction parameter ε, namely,

$$T_a^{(i)} \in V_i \longrightarrow \varepsilon^{a_i} T_a^{(i)} \in V_i', \tag{3.28}$$

where the choice of a_i provides finite limits of the contracted algebra when $\varepsilon \longrightarrow 0$.

With this in mind, in our method, we first of all consider an *infinite* S-expansion procedure, namely, an S-expansion which involves an *infinite abelian semigroup* $S^{(\infty)}$, which will be defined in the following.

Then, we will show how to confer the role of "generating zeros" to a particular infinite set of generators, associating them with different elements of the infinite abelian semigroup $S^{(\infty)}$. This will need to pass through *infinite resonant subalgebras*, as we will discuss in a while. We will then be able to subtract an infinite ideal subalgebra from an infinite resonant subalgebra of the infinitely S-expanded algebra. We will explicitly show that the ideal subtraction corresponds to a reduction. This procedure will also be able to reproduce the same result which would have been obtained by having performed a generalized Inönü-Wigner contraction.

We observe that the algebra we end up after the ideal subtraction, in general, is *not* a subalgebra of the starting algebra. It is, instead, a reduced algebra. In particular, the ideal subtraction reproduces the same result of a 0_S-reduction.

In this context, we are taking into account the following operation on a given algebra $\mathcal{A}$,

$$\mathcal{A} \ominus \mathcal{I} = \mathcal{A}_0, \tag{3.29}$$

where $\mathcal{A}_0$ generates a coset space, and where $\mathcal{I}$ is an ideal (ideal subalgebra) of $\mathcal{A}$, namely, a subalgebra of $\mathcal{A}$ that satisfies the property $[\mathcal{A}, \mathcal{I}] \subset \mathcal{I}$ (see Subsection II F for further details).

We now explain in detail our method involving an infinite S-expansion with subsequent ideal subtraction.

1. General formulation of the method of infinite S-expansion with ideal subtraction

As said in Ref. 42, if, in the S-expansion procedure, the finite semigroup is generalized to the case of an infinite semigroup, then the S-expanded algebra will be an *infinite-dimensional algebra*. One can thus see from the fact that $T_{(A,\alpha)} = \lambda_\alpha T_A$ constitutes a base for the S-expanded algebra and from the fact that α takes now the values in an infinite set.

We can thus generate an infinitely S-expanded algebra as a loop-like Lie algebra (see Ref. 42), where the semigroup elements can be represented by the set $(\mathbb{N}, +)$,[57] that presents the same multiplication rules (extended to an infinite set) of the general semigroup $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$, namely, $\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha + \beta \leq N + 1$, and $\lambda_\alpha \lambda_\beta = \lambda_{N+1}$ if $\alpha + \beta > N + 1$.[58]

Definition 8. Let $\{\lambda_\alpha\}_{\alpha=0}^{\infty} = \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_\infty\}$ be an infinite discrete set of elements.[59] *Then, the infinite set $\{\lambda_\alpha\}_{\alpha=0}^{\infty}$ satisfying commutation rules like the ones of the set $(\mathbb{N}, +)$ (that is, of $S_E^{(N)}$), namely,*

$$\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}, \tag{3.30}$$

where

$$\lambda_\alpha \lambda_\infty = \lambda_\infty, \quad \forall \lambda_\alpha \in \{\lambda_\alpha\}_{\alpha=0}^{\infty}, \qquad \lambda_\infty \lambda_\infty = \lambda_\infty, \tag{3.31}$$

is an infinite abelian semigoup.

Hereafter, we will denote such an infinite abelian semigroup by $S^{(\infty)} = \{\lambda_\alpha\}_{\alpha=0}^{\infty}$. Let us notice that since the multiplication rules in (3.31) hold, the element $\lambda_\infty \in S^{(\infty)}$ can be regarded as an "ideal element" of the infinite semigroup $S^{(\infty)}$.

Now, let $\mathfrak{g} = \bigoplus_{p \in I} V_p$ be a subspace decomposition of $\mathfrak{g}$.

With the above assumptions, we perform an infinite S-expansion on $\mathfrak{g}$ using the semigroup $S^{(\infty)}$, and the infinite S-expanded algebra can be rewritten as

$$\begin{aligned} \mathfrak{g}_S^{\infty} &= \{\lambda_\alpha\}_{\alpha=0}^{\infty} \times \mathfrak{g} \\ &= \{\lambda_\alpha\}_{\alpha=0}^{\infty} \times \left[\bigoplus_{p \in I} V_p \right]. \end{aligned} \tag{3.32}$$

Let us observe that the Jacobi identity is fulfilled for the infinite S-expanded algebra since the starting algebra satisfies the Jacobi identity and the semigroup $S^{(\infty)}$ is abelian (and associative by definition): These are the requirements that the starting algebra and the semigroup involved in the procedure must satisfy so that the S-expanded algebra satisfies the Jacobi identity when performing an S-expansion process (see Ref. 8 for further details).

At this point, one can split the infinite semigroup in subsets in such a way to be able to properly extract a resonant subalgebra from the infinitely S-expanded one. It will be then possible, according to the procedure described in Ref. 8, to define partitions on these subsets such that one can isolate an ideal structure from the resonant subalgebra of the infinitely S-expanded algebra reproducing a reduction and ending up, in this way, with a finite algebra. This procedure, as we will see, is also able to reproduce a generalized Inönü-Wigner contraction.

We will now describe the general development of the method, which is based on the following steps:

1. Properly defined subsets S_p of S^∞ such that they satisfy the resonant condition (2.26), in order to be able to extract a resonant subalgebra from $\mathfrak{g}_S^\infty$ (by using Theorem 1).
2. Explicitly show that the ideal subtraction we want to apply satisfies the requirements of a reduction (in the sense of Definition 2). To this aim, one must define a S_p partition such that the conditions (3.8) and (3.9) are fulfilled, in order to extract a reduced algebra from the resonant subalgebra of the infinitely S-expanded one. This reduced algebra will be our target, while the (infinite) ideal subalgebra will be taken apart.

Depending on the subspace decomposition structure of the original algebra $\mathfrak{g}$, the subset decomposition of $S^{(\infty)}$ in subsets S_p will assume different forms in order to satisfy the resonance condition, allowing the extraction of a resonant subalgebra. We will carry on the discussion in general in the following, in order to expose our method, while in Sec. IV, we will study some examples with different subspace decomposition structures, case by case.

To proceed with the extraction of the infinite resonant subalgebra, according to the review we have presented in Sec. II and to the approach presented in Ref. 8, we must define a resonant subset decomposition, under the product (2.27), of the infinite semigroup $S^{(\infty)}$,

$$S^{(\infty)} = \bigcup_{p\in I} S_p, \tag{3.33}$$

Namely, a decomposition such that (2.26) is fulfilled, where S_p's are the infinite subsets.

Once the resonant subset decomposition has been found, the direct sum

$$\mathfrak{g}_R^\infty = \bigoplus_{p\in I} W_p, \tag{3.34}$$

with

$$W_p = S_p \times V_p, \quad p \in I, \tag{3.35}$$

is a *resonant subalgebra* of $\mathfrak{g}_S^\infty$ (where we have used Theorem 1).

In particular, $\mathfrak{g}_R^\infty = \bigoplus_{p\in I} W_p = \bigoplus_{p\in I} S_p \times V_p$ is the direct sum of a finite number of infinite subspaces W_p, which are infinite due to the fact that the subsets S_p's contain an infinite amount of semigroup elements.

Now, according to the procedure described in Ref. 8, we can develop the following theorem.

Theorem 3. *Let $\mathfrak{g}$ be a Lie (super)algebra and let $\mathfrak{g}_S^\infty = S^{(\infty)} \times \mathfrak{g}$ be the infinite S-expanded (super)algebra obtained using the infinite abelian semigroup $S^{(\infty)} = \{\lambda_\alpha,\ \alpha = 0, \dots, \infty\}$.*

Let $\mathfrak{g}_R^\infty$ be an infinite resonant subalgebra of $\mathfrak{g}_S^\infty$ and let $\mathcal{I}$ be an infinite ideal subalgebra of $\mathfrak{g}_R^\infty$. Then, the (super)algebra

$$\check{\mathfrak{g}}_R = \mathfrak{g}_R^\infty \ominus \mathcal{I} \tag{3.36}$$

is a reduced (super)algebra.

Proof. After having performed the infinite S-expansion on $\mathfrak{g}$, obtaining the infinite S-expanded (super)algebra $\mathfrak{g}_S^\infty = S^{(\infty)} \times \mathfrak{g}$, and after having extracted a resonant subalgebra $\mathfrak{g}_R^\infty$ from $\mathfrak{g}_S^\infty$ as described above, we can write an S_p partition $S_p = \hat{S}_p \cup \check{S}_p$, where $\check{S}_p$'s are the finite subsets, while the $\hat{S}_p$'s are infinite ones, satisfying the conditions (3.8) and (3.9). Once such a partition has been found, it induces, according to Theorem 2, the following decomposition on the resonant subalgebra $\mathfrak{g}_R^\infty$,

$$\mathfrak{g}_R^\infty = \check{\mathfrak{g}}_R \oplus \hat{\mathfrak{g}}_R^\infty, \tag{3.37}$$

where

$$\check{\mathfrak{g}}_R = \bigoplus_{p\in I} \check{S}_p \times V_p, \tag{3.38}$$

$$\hat{\mathfrak{g}}_R^\infty = \bigoplus_{p\in I} \hat{S}_p \times V_p. \tag{3.39}$$

Let us observe that $\check{\mathfrak{g}}_R$ is *finite* since it is the direct sum of products between finite subsets and finite subspaces, while $\hat{\mathfrak{g}}_R^\infty$ is *infinite*, due to the fact that the $\hat{S}_p$'s are infinite subsets.

Then, applying Theorem 2, we have

$$\left[\check{\mathfrak{g}}_R, \hat{\mathfrak{g}}_R^{\infty}\right] \subset \hat{\mathfrak{g}}_R^{\infty}, \tag{3.40}$$

and, therefore, $|\check{\mathfrak{g}}_R|$ correspond to a reduced (super)algebra of $\mathfrak{g}_S^{\infty}$.

Furthermore, in the case in which we have

$$\left[\hat{\mathfrak{g}}_R^{\infty}, \hat{\mathfrak{g}}_R^{\infty}\right] \subset \hat{\mathfrak{g}}_R^{\infty}, \tag{3.41}$$

that is to say, when $\hat{\mathfrak{g}}_R^{\infty}$ is an infinite subalgebra of $\mathfrak{g}_R^{\infty}$ (and, consequently, an infinite subalgebra of $\mathfrak{g}_S^{\infty}$), $\hat{\mathfrak{g}}_R^{\infty}$ is, in particular, an *infinite ideal subalgebra* [due to the fact that it also satisfies (3.40)], in the sense described in Sec. II. This allows us to write

$$\check{\mathfrak{g}}_R = \mathfrak{g}_R^{\infty} \ominus \mathcal{I}, \tag{3.42}$$

where we have denoted by $\mathcal{I}$ the infinite ideal subalgebra, $\mathcal{I} \equiv \hat{\mathfrak{g}}_R^{\infty}$, and where $\check{\mathfrak{g}}_R$ corresponds to the reduced algebra that one ends up with at the end of the procedure.

We have thus demonstrated that the subtraction of an infinite ideal subalgebra from an infinite resonant subalgebra of an infinite S-expanded (super)algebra (reached by using the infinite semigroup $S^{(\infty)}$) corresponds to a *reduction*, leading to a *reduced algebra*. □

The ideal subtraction, which is crucial since it allows us to end up with a Lie algebra with a finite number of generators thus satisfies the requirements of a reduction, in the sense given in Sec. II in the context of reduced algebras. As we have already mentioned, a reduced algebra, in general, does *not* correspond to a subalgebra.

In particular, *the ideal subtraction can be viewed as a (generalization of the)* 0_S*-reduction*, in the sense that all the elements of the infinite ideal subalgebra are mapped to zero after the ideal subtraction; this has the same effect given by the zero element λ_{0_S} of a semigroup (see Sec. II), namely,

$$\lambda_{0_S} T_A = 0. \tag{3.43}$$

This is what we meant when we said that we are conferring the role of "generating zeros" to a particular infinite set of generators, that is, the ones belonging to the infinite ideal subalgebra. Thus, the reduced algebra $\check{\mathfrak{g}}_R$ can be viewed, in this sense, as a 0_S-reduced algebra.

Using the structure constants that one can write for the resonant subalgebra, it is then possible to find the structure constants for the (0_S)-reduced algebra, as it was presented in Ref. 8.

As we will see in Sec. IV, the method we have described above is reliable and can be applied to algebras already considered in the literature in the context of expansion and contraction, reproducing the same results shown in the literature. The main point consists in choosing properly the resonant partition of the semigroup $S^{(\infty)}$ and then the S_p partition $S_p = \check{S}_p \cup \hat{S}_p$ (namely, in selecting in a proper way the subsets $\check{S}_p$ and $\hat{S}_p$), in order to be able to extract an infinite ideal subalgebra $\mathcal{I}$ from the infinite resonant subalgebra $\mathfrak{g}_R^{\infty}$.

2. *Formulation of the method for reproducing a generalized Inönü-Wigner contraction*

Let us now apply our method to the case in which the original Lie algebra $\mathfrak{g}$ can be decomposed into $n + 1$ subspaces

$$\mathfrak{g} = V_0 \oplus V_1 \oplus \cdots \oplus V_n \tag{3.44}$$

and satisfies the Weimar-Woods conditions[38,39]

$$\left[V_p, V_q\right] \subset \bigoplus_{s \leq p+q} V_s, \quad p, q = 0, 1, \ldots, n. \tag{3.45}$$

We will now discuss how to properly choose the subset partition of $S^{(\infty)}$ and we apply our method of infinite S-expansion with ideal subtraction in order to show that the generalized Inönü-Wigner contraction fits our scheme. We proceed methodically with the following steps:

1. We perform an infinite S-expansion with the infinite abelian semigroup $S^{(\infty)}$, endowed with the multiplication rule (3.30), on the original algebra $\mathfrak{g}$ satisfying the Weimar-Woods conditions.

2. We properly define subsets S_p of $S^{(\infty)}$ such that they satisfy the resonant condition (2.26) in the case in which the starting algebra $\mathfrak{g}$ satisfies the Weimar-Woods conditions, in order to be able to extract a resonant subalgebra from $\mathfrak{g}_S^\infty$ (by using Theorem 1). The resonant condition (2.26) reads as follows when the Weimar-Woods conditions hold:
$$S_p \cdot S_q \subset \bigcap_{r \leq p+q} S_r. \tag{3.46}$$
3. We define a S_p partition such that the conditions (3.8) and (3.9) are fulfilled, in order to be able to extract a reduced algebra from the resonant subalgebra of the infinitely S-expanded one. This will be done through ideal subtraction. The reduced algebra will be our target, reproducing, in this way, a generalized Inönü-Wigner contraction, while the ideal subalgebra will be taken apart.

In this way, the whole procedure consisting in S-expanding with an infinite semigroup and subtracting the ideal will be able to reproduce a generalized Inönü-Wigner contraction.

We first of all perform the infinite S-expansion on $\mathfrak{g}$, obtaining the infinitely S-expanded algebra
$$\mathfrak{g}_S^\infty = S^{(\infty)} \times \mathfrak{g} = \left(\{\lambda_\alpha\}_0^\infty \times V_0\right) \oplus \left(\{\lambda_\alpha\}_0^\infty \times V_1\right) \oplus \cdots \oplus \left(\{\lambda_\alpha\}_0^\infty \times V_n\right). \tag{3.47}$$
In order to proceed with the extraction of the infinite resonant subalgebra, we must split the semigroup $S^{(\infty)}$ into $n+1$ *infinite* subsets S_p such that when $\mathfrak{g}$ satisfies the Weimar-Woods conditions (that is our case), the condition (3.46) is fulfilled, where $S_p \cdot S_q$ denotes the set of all products of all elements of S_p and all elements of S_q. Thus, we must define such a decomposition for the infinite semigroup $S^{(\infty)}$.

Now, let
$$S^{(\infty)} = \bigcup_{p=0}^{n} S_p \tag{3.48}$$
be a subset decomposition of $S^{(\infty)}$, where the subsets $S_p \subset S^{(\infty)}$ are defined by
$$S_p = \{\lambda_{\alpha_p},\ \alpha_p = p, \ldots, \infty\}, \quad p = 0, \ldots n. \tag{3.49}$$
This can also be visualized, for making it clearer, as
$$\begin{aligned} S^{(\infty)} &= S_0 \cup S_1 \cup S_2 \cup \ldots \cup S_n = \\ &= \{\lambda_0, \lambda_1, \ldots, \lambda_\infty\} \cup \{\lambda_1, \lambda_2, \ldots, \lambda_\infty\} \cup \{\lambda_2, \lambda_3, \ldots, \lambda_\infty\} \cup \ldots \cup \{\lambda_n, \lambda_{n+1}, \ldots, \lambda_\infty\}. \end{aligned} \tag{3.50}$$
The subset decomposition (3.48) is a *resonant* one under the semigroup product (2.27), since it satisfies (3.46). Thus, according to Theorem 1, the direct sum
$$\mathfrak{g}_R^\infty = \bigoplus_{p=0}^{n} W_p, \tag{3.51}$$
with
$$W_p = S_p \times V_p, \tag{3.52}$$
is a *resonant subalgebra* of $\mathfrak{g}_S^\infty$.

Let us now consider the resonant subalgebra $\mathfrak{g}_R^\infty = \bigoplus_{p=0}^n W_p = \bigoplus_{p=0}^n S_p \times V_p$ and write the following S_p partition: $S_p = \hat{S}_p \cup \check{S}_p$, where
$$\check{S}_p = \{\lambda_{\alpha_p},\ \alpha_p = p\} \equiv \{\lambda_p\}, \tag{3.53}$$
$$\hat{S}_p = \{\lambda_{\alpha_p},\ \alpha_p = p+1, \ldots, \infty\}. \tag{3.54}$$
This S_p partition satisfies
$$\hat{S}_p \cap \check{S}_p = \emptyset, \tag{3.55}$$
which is exactly condition (3.8). The second condition which must be fulfilled in order to be able to extract a reduced algebra from the resonant subalgebra $\mathfrak{g}_S^\infty$ when the original algebra $\mathfrak{g}$ satisfies the Weimar-Woods conditions reads

$$\check{S}_p \cdot \hat{S}_q \subset \bigcap_{r \le p+q} \hat{S}_r. \tag{3.56}$$

In the present case, $\check{S}_p$ and $\hat{S}_q$ are, respectively, given by

$$\check{S}_p = \{\lambda_p\}, \tag{3.57}$$

$$\hat{S}_q = \{\lambda_{\alpha_q},\ \alpha_q = q+1, \ldots, \infty\}. \tag{3.58}$$

Thus, condition (3.56) is fulfilled, since, in this case,

$$\bigcap_{r=0}^{p+q} \hat{S}_r = \hat{S}_{p+q}, \tag{3.59}$$

where $\hat{S}_{p+q} = \{\lambda_{p+q+m}, m = 1, \ldots, \infty\}$, and

$$\check{S}_p \cdot \hat{S}_q = \hat{S}_{p+q}, \tag{3.60}$$

where we have taken into account product (3.30). We can thus conclude that the S_p partition we have chosen satisfies the reduction condition, and we can now extract a reduced algebra from the resonant subalgebra $\mathfrak{g}_S^{\infty}$.

Indeed, what we have done by considering this particular S_p partition induce, according to Theorem 2, the following decomposition on the resonant subalgebra:

$$\mathfrak{g}_R^{\infty} = \check{\mathfrak{g}}_R \oplus \hat{\mathfrak{g}}_R^{\infty}, \tag{3.61}$$

where

$$\check{\mathfrak{g}}_R = \bigoplus_{p=0}^{n} \check{S}_p \times V_p, \tag{3.62}$$

$$\hat{\mathfrak{g}}_R^{\infty} = \bigoplus_{p=0}^{n} \hat{S}_p \times V_p. \tag{3.63}$$

Let us observe that, according to what we have previously observed when describing the method in general, $\check{\mathfrak{g}}_R$ is *finite* since it is the direct sum of products between finite subsets and finite subspaces, while $\hat{\mathfrak{g}}_R^{\infty}$ is *infinite*, due to the fact that the $\hat{S}_p$'s are infinite (sub)sets.

We can now write

$$\hat{\mathfrak{g}}_R^{\infty} = \bigoplus_{p=0}^{n} \hat{W}_p, \tag{3.64}$$

with

$$\hat{W}_p = \hat{S}_p \times V_p = S_{p+1} \times V_p, \tag{3.65}$$

where $S_{p+1} = \{\lambda_{p+m},\ m = 1, \ldots, \infty\}$. One can now easily prove that, by construction, we have

$$\left[\hat{\mathfrak{g}}_R^{\infty}, \hat{\mathfrak{g}}_R^{\infty}\right] \subset \hat{\mathfrak{g}}_R^{\infty}, \tag{3.66}$$

that is to say, $\hat{\mathfrak{g}}_R^{\infty}$ is an infinite subalgebra of $\mathfrak{g}_R^{\infty}$ (and, consequently, an infinite subalgebra of $\mathfrak{g}_S^{\infty}$). In particular, it is an *ideal subalgebra* since it also satisfies

$$\left[\check{\mathfrak{g}}_R, \hat{\mathfrak{g}}_R^{\infty}\right] \subset \hat{\mathfrak{g}}_R^{\infty}. \tag{3.67}$$

This allows us to write

$$\check{\mathfrak{g}}_R = \mathfrak{g}_R^{\infty} \ominus \mathcal{I}, \tag{3.68}$$

where we have denoted by $\mathcal{I}$ the infinite ideal subalgebra, $\mathcal{I} \equiv \hat{\mathfrak{g}}_R^{\infty}$, and where, applying Theorem 3, the algebra $\check{\mathfrak{g}}_R$ corresponds to a reduced algebra.

Since, as stated in Ref. 8 and previously reviewed in the present paper, the generalized Inönü-Wigner contraction corresponds to the reduction of a resonant subalgebra of the S-expanded one, we have thus reproduced a generalized Inönü-Wigner contraction by performing our method of infinite S-expansion with subsequent ideal subtraction.

We can summarize our approach saying that after having performed an infinite S-expansion with the semigroup $S^{(\infty)}$ on an original Lie algebra $\mathfrak{g}$, we can reproduce the generalized Inönü-Wigner contraction by first of all extracting a resonant subalgebra $\mathfrak{g}_R^{\infty}$ from the infinitely S-expanded algebra

$\mathfrak{g}_S^\infty = S^{(\infty)} \times \mathfrak{g}$ and by then subtracting an infinite ideal subalgebra from the resonant one. In this way, the procedure reproduces the result of a reduction of a resonant subalgebra of an (infinitely) S-expanded algebra, which, according to Ref. 8, lets the generalized Inönü-Wigner contraction fit the S-expansion scheme. In this way, we have obtained an alternative view of the generalized Inönü-Wigner contraction process.

We will discuss some examples of application of our method in Sec. IV. We now move to the description of the way for finding the invariant tensors of the (super)algebras obtained with our method in terms of those of the original (super)algebras.

C. Invariant tensors of (super)algebras obtained through infinite *S*-expansion with ideal subtraction

In Ref. 8, the authors developed a theorem which describes how to write the components of the invariant tensor of a target algebra obtained through a (finite) S-expansion in terms of those of the initial algebra (precisely, we are referring to Theorem VII.1 of Ref. 8). Then, in Theorem VII.2 of the same paper, they have given an expression for the invariant tensor for a 0_S-reduced algebra.

Taking into account this result, we can now show how to write the invariant tensors for the (super)algebras, which can be obtained by applying our method of infinite S-expansion with ideal subtraction.

Theorem 4. *Let* $\mathfrak{g}$ *be a Lie (super)algebra of basis* $\{T_A\}$ *and let* $\langle T_{A_0} \ldots T_{A_N} \rangle$ be *an invariant tensor for* $\mathfrak{g}$. *Let* $\mathfrak{g}_S^\infty = S^{(\infty)} \times \mathfrak{g}$ *be the infinite S-expanded (super)algebra obtained using the infinite abelian semigroup* $S^{(\infty)} = \{\lambda_\alpha,\ \alpha = 0, \ldots, \infty\}$. *Let* $\mathfrak{g}_R^\infty$ *be an infinite resonant subalgebra of* $\mathfrak{g}_S^\infty$ and let $\mathcal{I}$ *be an infinite ideal subalgebra of* $\mathfrak{g}_R^\infty$. *Then,*

$$\langle T_{A_{p_0}}^{\alpha_{p_0}} \ldots T_{A_{p_N}}^{\alpha_{p_N}} \rangle = \alpha^m \delta_m^{\alpha_{p_0}+\alpha_{p_1}+\alpha_{p_2}+\cdots+\alpha_{p_N}} \langle T_{A_0} \ldots T_{A_N} \rangle, \tag{3.69}$$

where α^m *are the arbitrary constants, corresponds to an invariant tensor for the finite (super)algebra*

$$\check{\mathfrak{g}}_R = \mathfrak{g}_R^\infty \ominus \mathcal{I}, \tag{3.70}$$

having denoted the generators of $\check{\mathfrak{g}}_R$ by $\lambda_{\alpha_{p_i}} T_{A_{p_i}} \equiv T_{A_{p_i}}^{\alpha_{p_i}}$, *with* $i = 0, \ldots, N$, *and where the set* $\{\lambda_{\alpha_p}\}$ *is finite.*

Proof. The proof of Theorem 4 can be developed by applying Theorem 3 of the present paper. Indeed, as stated in Theorem 3, we have that the subtraction of an infinite ideal subalgebra from an infinite resonant subalgebra of an infinitely S-expanded (super)algebra [using the semigroup $S^{(\infty)}$ on the original (super)algebra] corresponds to a reduction. In particular, we have seen that it reproduces the same result of a 0_S-reduction.

In this way, one can write the invariant tensor of the (super)algebra obtained with our method of infinite S-expansion with ideal subtraction by applying Theorem VII.2 of Ref. 8, which, indeed, gives an expression for the invariant tensor for a 0_S-reduced algebra.

Thus, it is straightforward to show that the invariant tensor for the (super)algebra $\check{\mathfrak{g}}_R = \mathfrak{g}_R^\infty \ominus \mathcal{I}$ can be written in the form

$$\langle T_{A_{p_0}}^{\alpha_{p_0}} \ldots T_{A_{p_N}}^{\alpha_{p_N}} \rangle = \alpha^m \delta_m^{\alpha_{p_0}+\alpha_{p_1}+\alpha_{p_2}+\cdots+\alpha_{p_N}} \langle T_{A_0} \ldots T_{A_N} \rangle, \tag{3.71}$$

being α^m the arbitrary constants, where we have denoted the generators of $\check{\mathfrak{g}}_R$ by $\lambda_{\alpha_{p_i}} T_{A_{p_i}} \equiv T_{A_{p_i}}^{\alpha_{p_i}}$, with $i = 0, \ldots, N$, and where the set $\{\lambda_{\alpha_p}\}$ is finite. □

IV. EXAMPLES OF APPLICATION

In the following, we develop some examples of application, in which we replicate the effects of both standard and generalized Inönü-Wigner contractions of some (super)algebras presented in the literature through our infinite S-expansion approach with ideal subtraction.

We also find the invariant tensors of some of the mentioned (super)algebras by applying Theorem 4.

A. Symmetric cosets

As shown in Sec. II, the standard Inönü-Wigner contraction can be applied to symmetric cosets of simple Lie algebras, i.e., to cosets of Lie algebras which can be written as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where the following commutation relations hold:

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subset \mathfrak{h}, \\ [\mathfrak{h}, \mathfrak{p}] &\subset \mathfrak{p}, \\ [\mathfrak{p}, \mathfrak{p}] &\subset \mathfrak{h}. \end{aligned} \tag{4.1}$$

After having performed a standard Inönü-Wigner contraction ($\mathfrak{h}' = \mathfrak{h}'$, $\mathfrak{p}' = \varepsilon\mathfrak{p}$, and $\varepsilon \longrightarrow 0$) on $\mathfrak{g}$, we get the contracted commutation relations

$$\begin{aligned} [\mathfrak{h}', \mathfrak{h}'] &\subset \mathfrak{h}', \\ [\mathfrak{h}', \mathfrak{p}'] &\subset \mathfrak{p}', \\ [\mathfrak{p}', \mathfrak{p}'] &= 0. \end{aligned} \tag{4.2}$$

The standard Inönü-Wigner contraction abelianizes the last commutator ($[\mathfrak{p}', \mathfrak{p}'] = 0$) and thus produces an algebra that is non-isomorphic to the starting one.[60]

With the method presented in this paper, we can reproduce the same result. Indeed, the case of the standard Inönü-Wigner contraction from an algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{h}$ is a subalgebra and $\mathfrak{p}$ is a symmetric coset, to an algebra $\check{\mathfrak{g}}_R = \mathfrak{h}' \ltimes \mathfrak{p}' = \mathfrak{h} \ltimes \mathfrak{p}'$, satisfying (4.2) can be reproduced by performing our prescription of infinite S-expansion with ideal subtraction. Indeed, after having infinitely S-expanded the original algebra using the semigroup $S^{(\infty)}$, we can write

$$\check{\mathfrak{g}}_R = \mathfrak{g}_R^{\infty} \ominus \mathcal{I}, \tag{4.3}$$

where

$$\begin{aligned} \mathfrak{g}_R^{\infty} &= \{\lambda_{2l},\ l = 0, \ldots, \infty\} \times \mathfrak{h} \oplus \{\lambda_{2l+1},\ l = 0, \ldots, \infty\} \times \mathfrak{p}, \\ \mathcal{I} &= \{\lambda_{2m},\ m = 1, \ldots, \infty\} \times \mathfrak{h} \oplus \{\lambda_{2m+1},\ m = 1, \ldots, \infty\} \times \mathfrak{p}, \\ \check{\mathfrak{g}}_R &= \{\lambda_0\} \times \mathfrak{h} \oplus \{\lambda_1\} \times \mathfrak{p} = \mathfrak{h}' \oplus \mathfrak{p}'. \end{aligned} \tag{4.4}$$

1. From the AdS to the Poincaré algebra in D = 3

Let us see the particular feature described above through an explicit, physical example.

The Poincaré algebra $\mathfrak{iso}(D-1, 1)$ can be obtained as a standard Inönü-Wigner contraction of the anti-de Sitter (AdS) algebra $\mathfrak{so}(D-1, 2)$.

We now consider the AdS algebra in three dimensions, whose commutation relations read

$$\begin{aligned} \left[\tilde{J}_a, \tilde{J}_b\right] &= \epsilon_{abc}\tilde{J}_c, \\ \left[\tilde{J}_a, \tilde{P}_b\right] &= \epsilon_{abc}\tilde{P}_c, \\ \left[\tilde{P}_a, \tilde{P}_b\right] &= \epsilon_{abc}\tilde{J}_c. \end{aligned} \tag{4.5}$$

Then, we apply the infinite S-expansion procedure with ideal subtraction described in the present work, considering $V_0 = \{\tilde{J}_a\}$ and $V_1 = \{\tilde{P}_a\}$. After having infinitely S-expanded the original algebra with the semigroup $S^{(\infty)}$, we write

$$\begin{aligned} \mathfrak{g}_R^{\infty} &= \{\lambda_{2l},\ l = 0, \ldots, \infty\} \times V_0 \oplus \{\lambda_{2l+1},\ l = 0, \ldots, \infty\} \times V_1, \\ \mathcal{I} &= \{\lambda_{2m},\ m = 1, \ldots, \infty\} \times V_0 \oplus \{\lambda_{2m+1},\ m = 1, \ldots, \infty\} \times V_1, \\ \check{\mathfrak{g}}_R &= \{\lambda_0\} \times V_0 \oplus \{\lambda_1\} \times V_1. \end{aligned} \tag{4.6}$$

Subsequently, we rename the generators as follows:

$$\lambda_0\tilde{J}_a = J_a, \tag{4.7}$$

$$\lambda_1\tilde{P}_a = P_a. \tag{4.8}$$

In this way, we obtain the Poincaré algebra $\mathfrak{iso}(2, 1)$ in three dimensions, which can be written in terms of the following commutation relations:

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \\
[J_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b, \\
[P_a, P_b] &= 0.
\end{aligned} \tag{4.9}
$$

We have thus reached a Poincaré algebra from an *AdS* algebra by performing an infinite *S*-expansion with ideal subtraction, reproducing, in this way, the effects of a standard Inönü-Wigner contraction.

B. From the *AdS* to the Maxwell-like algebra $\mathcal{M}_5$

The *AdS* algebra $\mathfrak{so}(D-1,2)$ has the set of generators $\left\{\tilde{J}_{ab}, \tilde{P}_a\right\}$, and it is endowed with the following commutation relations:

$$
\begin{aligned}
\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] &= \eta_{bc} \tilde{J}_{ad} - \eta_{ac} \tilde{J}_{bd} - \eta_{bd} \tilde{J}_{ac} + \eta_{ad} \tilde{J}_{bc}, \\
\left[\tilde{J}_{ab}, \tilde{P}_c\right] &= \eta_{bc} \tilde{P}_a - \eta_{ac} \tilde{P}_b, \\
\left[\tilde{P}_a, \tilde{P}_b\right] &= \tilde{J}_{ab}.
\end{aligned} \tag{4.10}
$$

The Maxwell algebras type $\mathcal{M}_m$ (alternatively known as generalized Poincaré algebras, see Ref. 16) can be obtained as a finite *S*-expansion (with resonance and reduction) from the anti-de Sitter algebra $\mathfrak{so}(D-1,2)$, using semigroups of the type $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$, which are endowed with the multiplication rules $\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha + \beta \leq N+1$ and $\lambda_\alpha \lambda_\beta = \lambda_{N+1}$ if $\alpha + \beta > N+1$.

The reduction involved in the procedure takes into account the presence of a zero element in the semigroup. This zero element is defined as $\lambda_{0_S} = \lambda_{N+1}$ and depends on the number N in the semigroup.

We can now reach the Maxwell-like algebra $\mathcal{M}_5$ by performing our method of infinite *S*-expansion with ideal subtraction, starting from the *AdS* algebra.

We follow the notation and the subspaces partition adopted in Ref. 25 and perform an infinite *S*-expansion with the infinite semigroup $S^{(\infty)} = \{\lambda_\alpha\}_{\alpha=0}^{\infty}$, obtaining

$$
\mathfrak{g}_S^{\infty} = \{\lambda_\alpha\}_{\alpha=0}^{\infty} \times \left\{\tilde{J}_{ab}, \tilde{P}_a\right\} = \left(\{\lambda_\alpha\}_{\alpha=0}^{\infty} \times V_0\right) \oplus \left(\{\lambda_\alpha\}_{\alpha=0}^{\infty} \times V_1\right), \tag{4.11}
$$

where $V_0 = \{\tilde{J}_{ab}\}$ and $V_1 = \{\tilde{P}_a\}$.

Now, following our method, we can write (adopting the subspace partition of Ref. 16)

$$
\begin{aligned}
\mathfrak{g}_R^{\infty} &= \{\lambda_{2l},\ l = 0, \ldots, \infty\} \times V_0 \oplus \{\lambda_{2l+1},\ l = 0, \ldots, \infty\} \times V_1, \\
\mathcal{I} &= \{\lambda_{2m},\ m = 2, \ldots, \infty\} \times V_0 \oplus \{\lambda_{2m+1},\ m = 2, \ldots, \infty\} \times V_1, \\
\check{\mathfrak{g}}_R &= \{\lambda_0, \lambda_2\} \times V_0 \oplus \{\lambda_1, \lambda_3\} \times V_1.
\end{aligned} \tag{4.12}
$$

Thus, after having renamed the generators in the following way: $\lambda_0 \tilde{J}_{ab} = J_{ab}$, $\lambda_1 \tilde{P}_a = P_a$, $\lambda_2 \tilde{J}_{ab} = Z_{ab}$, and $\lambda_3 \tilde{P}_a = Z_a$, we finally end up with the Maxwell-like algebra $\mathcal{M}_5$, which is indeed endowed with the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ab} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{4.13}$$
$$[Z_{ab}, J_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{4.14}$$
$$[Z_{ab}, P_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \tag{4.15}$$
$$[J_{ab}, Z_b] = \eta_{bc} Z_a - \eta_{ac} Z_b, \tag{4.16}$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{4.17}$$
$$[P_a, P_b] = Z_{ab}, \tag{4.18}$$
$$[Z_{ab}, Z_{cd}] = [Z_a, Z_b] = 0. \tag{4.19}$$

We have thus reached the Maxwell-like algebra $\mathcal{M}_5$ by performing an infinite *S*-expansion with ideal subtraction on an *AdS* algebra.

1. Invariant tensor of the Maxwell-like algebra in D = 3

In the three-dimensional case, considering Eq. (3.69), the components of the invariant tensor of the Maxwell-like algebra different from zero are the following ones:

$$\langle J_{ab}J_{cd}\rangle = \alpha^0\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right), \qquad \langle J_{ab}P_c\rangle = \alpha^1\epsilon_{abc}, \tag{4.20}$$

$$\langle Z_{ab}J_{cd}\rangle = \alpha^2\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right), \qquad \langle J_{ab}Z_c\rangle = \alpha^3\epsilon_{abc}, \tag{4.21}$$

$$\langle Z_{ab}P_c\rangle = \alpha^3\epsilon_{abc}, \qquad \langle P_aP_b\rangle = \alpha^2\eta_{ab}, \tag{4.22}$$

where α^m, $m = 0, 1, 2, 3$, are the arbitrary constants. One could now construct a Lagrangian for a gravitational theory, as it was done in the literature (see Refs. 19 and 27).

C. Bargmann algebra and Newton-Hooke algebra

In the following, we apply our method involving an infinite S-expansion with subsequent ideal subtraction in order to relate different algebras which have been objects of great interest in the literature (see Refs. 44–51 for further details), namely, the Poincaré and Galilean algebras, the Bargmann algebra, and the Newton-Hooke algebra.

The Bargmann algebra $\mathfrak{b}(D-1,1)$ is the Galilean algebra[45,50] augmented with a central generator M[61] and can be obtained by performing a contraction on $\mathfrak{iso}(D-1,1)\oplus\mathfrak{g}_M$, where $\mathfrak{iso}(D-1,1)$ is the Poincaré algebra in D dimensions, and where $\mathfrak{g}_M$ is a commutative subalgebra[46] spanned by a central generator M.

Some algebras involving the presence of the cosmological constant were deeply studied in the past years. The most relevant symmetry groups with the presence of a cosmological constant Λ are the de Sitter (dS) and the anti-de Sitter (AdS) groups that are related to relativistic symmetries which also involved the velocity of the light in the vacuum, c. In the non-relativistic limit, that is to say, when $c\longrightarrow\infty$ and $\Lambda\longrightarrow 0$, with $-c^2\Lambda$ finite, the (centrally extended) dS and the AdS Lie algebras become centrally extended version of the so-called Newton-Hooke algebra, which describes Galilean physics with a cosmological constant $\lambda = -c^2\Lambda$.[51] The central extensions of the Newton-Hooke Lie algebras can be obtained as contractions of trivial central extensions of the dS and AdS Lie algebras (see Ref. 51).

1. The Bargmann algebra

Let us now consider a central extension (with central generator M) of the Poincaré algebra in $\mathfrak{iso}(D-1,1)$ (this extension of the Poincaré algebra is a commutative algebra $\mathfrak{g}_M$ spanned by the central generator M), namely, $\mathfrak{iso}(D-1,1)\oplus\mathfrak{g}_M$, in order to apply our procedure of infinite S-expansion with ideal subtraction and obtain the Bargmann algebra in three dimensions, $\mathfrak{b}(D-1,1)$. In this way, we will reproduce the result of a contraction with our method.

We thus start by redefining (according to the notation of Ref. 46) the generators of the starting algebra $\mathfrak{iso}(D-1,1)\oplus\mathfrak{g}_M$ as follows:

$$J_{ij} = J_{ij}, \quad P_i = P_i, \quad J_{i0} = G_i, \quad P_0 = H \rightarrow H + M. \tag{4.23}$$

We then write the following partition over subspaces:

$$V_0 = \{J_{ij}, H\}, \tag{4.24}$$

$$V_1 = \{P_i, G_i\}, \tag{4.25}$$

$$V_2 = \{M\}. \tag{4.26}$$

After that, we perform an infinite S-expansion with $S^{(\infty)}$, and, following the procedure described in Subsection III B, we consider the following resonant subset decomposition of the semigroup $S^{(\infty)}$:

$$\begin{aligned} S_0 &= \{\lambda_0\}\cup\{\lambda_1,\ldots,\lambda_\infty\},\\ S_1 &= \{\lambda_1\}\cup\{\lambda_2,\ldots,\lambda_\infty\},\\ S_2 &= \{\lambda_2\}\cup\{\lambda_3,\ldots,\lambda_\infty\}. \end{aligned} \tag{4.27}$$

The resonant subalgebra $\mathfrak{g}_R^\infty = \check{\mathfrak{g}}_R \oplus \hat{\mathfrak{g}}_R^\infty$ results are to be given by the direct sum of the following terms:

$$S_0 \times V_0 = \{\lambda_0\} \times V_0 \oplus \{\lambda_1, \ldots, \lambda_\infty\} \times V_0, \tag{4.28}$$

$$S_1 \times V_1 = \{\lambda_1\} \times V_1 \oplus \{\lambda_2, \ldots, \lambda_\infty\} \times V_1, \tag{4.29}$$

$$S_2 \times V_2 = \{\lambda_2\} \times V_2 \oplus \{\lambda_3, \ldots, \lambda_\infty\} \times V_2. \tag{4.30}$$

Then, the ideal reads

$$\mathcal{I} = (\{\lambda_1, \ldots, \lambda_\infty\} \times V_0) \oplus (\{\lambda_2, \ldots, \lambda_\infty\} \times V_1) \oplus (\{\lambda_3, \ldots, \lambda_\infty\} \times V_2)\,, \tag{4.31}$$

and we can perform the ideal subtraction on the resonant subalgebra $\mathfrak{g}_R^\infty$.

Thus, the target algebra $\check{\mathfrak{g}}_R$, which is, in this case, the Bargmann algebra $\mathfrak{b}(D-1,1)$, is given by $\check{\mathfrak{g}}_R = \mathfrak{g}_R^\infty \ominus \mathcal{I}$.

Indeed, after having properly renamed the generator as follows: $\lambda_0 J_{ij} = \tilde{J}_{ij}$, $\lambda_0 H = \tilde{H}$, $\lambda_1 P_i = \tilde{P}_i$, $\lambda_1 G_i = \tilde{G}_i$, and $\lambda_2 M = \tilde{M}$, we finally get

$$\begin{aligned}
\left[\tilde{J}_{ij}, \tilde{J}_{kl}\right] &= 4\delta_{[i\,[k}\tilde{J}_{l]\,j]},\\
\left[\tilde{J}_{ij}, \tilde{G}_{k}\right] &= -2\delta_{k[i}\tilde{G}_{j]},\\
\left[\tilde{J}_{ij}, \tilde{P}_{k}\right] &= -2\delta_{k[i}\tilde{P}_{j]},\\
\left[\tilde{G}_{i}, \tilde{P}_{j}\right] &= -\delta_{ij}\tilde{M},\\
\left[\tilde{G}_{i}, \tilde{H}\right] &= -\tilde{P}_i,
\end{aligned} \tag{4.32}$$

which are precisely the commutation relations of the Bargmann algebra $\mathfrak{b}(D-1,1)$. Let us also observe that for $M = 0$, this is the Galilean algebra.

2. Centrally extended Newton-Hooke algebra in D = 3

We now perform an infinite S-expansion with ideal subtraction on a central extension (with two central charges) of the AdS algebra in $D = 3$, in order to reach a centrally extended Newton-Hooke algebra in three dimensions (see Ref. 51 for further details on the Newton-Hooke algebras). In this way, we will reproduce the result of a contraction by following our method.

In $D = 3$, we can write the three-dimensional AdS algebra as

$$\begin{aligned}
[J_a, J_b] &= \epsilon_{abc}J^c,\\
[J_a, P_b] &= \epsilon_{abc}P^c,\\
[P_a, P_b] &= \epsilon_{abc}J^c.
\end{aligned} \tag{4.33}$$

We now rename the generators, using the two-dimensional epsilon symbol with non-zero entries $\epsilon_{12} = -\epsilon_{21} = 1$, as follows:

$$\begin{aligned}
\bar{J} &= -J_0,\\
K_i &= -\epsilon_{ij}J_j,\\
\bar{H} &= -P_0.
\end{aligned} \tag{4.34}$$

The generator K_i, $i = 1, 2$, generate boosts in the ith spatial direction. Then, algebra (4.33) reads

$$\begin{aligned}
\left[K_i, K_j\right] &= \epsilon_{ij}\bar{J},\\
\left[K_i, P_j\right] &= \delta_{ij}\bar{H},\\
\left[P_i, P_j\right] &= \epsilon_{ij}\bar{J},\\
\left[K_i, \bar{J}\right] &= \epsilon_{ij}K_j,\\
\left[K_i, \bar{H}\right] &= P_i,\\
\left[\bar{H}, P_i\right] &= K_i,\\
\left[P_i, \bar{J}\right] &= \epsilon_{ij}P_j.
\end{aligned} \tag{4.35}$$

We now consider two central extensions: S and M. Then, according to Ref. 51, we define

$$H = \bar{H} - M, \quad J = \bar{J} - S. \tag{4.36}$$

Namely, $\bar{H} = H + M$ and $\bar{J} = J + S$, and we perform the following subspaces partition:

$$V_0 = \{J, H\}, \tag{4.37}$$

$$V_1 = \{P_i, K_i\}, \tag{4.38}$$

$$V_2 = \{M, S\}. \tag{4.39}$$

After that, applying the procedure developed in Subsection III B, we perform an infinite S-expansion with $S^{(\infty)}$ and we consider the following resonant subset decomposition of the semigroup $S^{(\infty)}$:

$$\begin{aligned} S_0 &= \{\lambda_0\} \cup \{\lambda_1, \ldots, \lambda_\infty\}, \\ S_1 &= \{\lambda_1\} \cup \{\lambda_2, \ldots, \lambda_\infty\}, \\ S_2 &= \{\lambda_2\} \cup \{\lambda_3, \ldots, \lambda_\infty\}. \end{aligned} \tag{4.40}$$

The resonant subalgebra $\mathfrak{g}_R^\infty = \check{\mathfrak{g}}_R \oplus \hat{\mathfrak{g}}_R^\infty$ results are to be given by the direct sum of the following terms:

$$S_0 \times V_0 = \{\lambda_0\} \times V_0 \oplus \{\lambda_1, \ldots, \lambda_\infty\} \times V_0, \tag{4.41}$$

$$S_1 \times V_1 = \{\lambda_1\} \times V_1 \oplus \{\lambda_2, \ldots, \lambda_\infty\} \times V_1, \tag{4.42}$$

$$S_2 \times V_2 = \{\lambda_2\} \times V_2 \oplus \{\lambda_3, \ldots, \lambda_\infty\} \times V_2. \tag{4.43}$$

Then, the ideal reads

$$\mathcal{I} = (\{\lambda_1, \ldots, \lambda_\infty\} \times V_0) \oplus (\{\lambda_2, \ldots, \lambda_\infty\} \times V_1) \oplus (\{\lambda_3, \ldots, \lambda_\infty\} \times V_2), \tag{4.44}$$

and we can perform the ideal subtraction on the resonant subalgebra $\mathfrak{g}_R^\infty$.

The commutators of $\check{\mathfrak{g}}_R = \mathfrak{g}_R^\infty \ominus \mathcal{I}$ can be finally written as follows:

$$\begin{aligned} \left[\tilde{K}_i, \tilde{K}_j\right] &= \epsilon_{ij}\tilde{S}, \\ \left[\tilde{K}_i, \tilde{P}_j\right] &= \delta_{ij}\tilde{M}, \\ \left[\tilde{P}_i, \tilde{P}_j\right] &= \epsilon_{ij}\tilde{S}, \\ \left[\tilde{K}_i, \tilde{J}\right] &= \epsilon_{ij}\tilde{K}_j, \\ \left[\tilde{K}_i, \tilde{H}\right] &= \tilde{P}_i, \\ \left[\tilde{H}, \tilde{P}_i\right] &= \tilde{K}_i, \\ \left[\tilde{P}_i, \tilde{J}\right] &= \epsilon_{ij}\tilde{P}_j, \end{aligned} \tag{4.45}$$

where we have defined $\lambda_0 J = \tilde{J}$, $\lambda_0 H = \tilde{H}$, $\lambda_1 P_i = \tilde{P}_i$, $\lambda_1 K_i = \tilde{K}_i$, $\lambda_2 M = \tilde{M}$, and $\lambda_2 S = \tilde{S}$. These last commutation relations are those of a centrally extended version of the so-called Newton-Hooke algebra in three dimensions (see Ref. 51 and references therein).

We have thus reached a centrally extended version of the Newton-Hooke algebra in $D = 3$ by performing an infinite S-expansion with subsequent ideal subtraction starting from a central extension of the three-dimensional AdS algebra.

D. Non-standard Maxwell superalgebra in D = 3 from the AdS algebra $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$

The Maxwell superalgebra can be obtained as an Inönü-Wigner contraction of the AdS-Lorentz superalgebra, which is an S-expansion of the AdS Lie algebra (see Ref. 27 for further details).

The non-standard Maxwell algebra[52,53] can be recovered from the supersymmetric extension of the AdS-Lorentz algebra by performing a suitable Inönü-Wigner contraction (see Ref. 40).

In the following, we reproduce the non-standard Maxwell superalgebra in three dimensions by performing an infinite S-expansion with $S^{(\infty)}$ on the AdS superalgebra and by subsequently removing an ideal. We also write the components of the invariant tensor of the target superalgebra in terms of those of the AdS superalgebra.

Let us thus consider the AdS superalgebra in three dimensions, $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$, generated by $\tilde{J}_{ab}$, $\tilde{P}_a$, and $\tilde{Q}_\alpha$, which satisfy the following commutation relations:

$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc}, \tag{4.46}$$

$$\left[\tilde{J}_{ab}\tilde{P}_c\right] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b, \tag{4.47}$$

$$\left[\tilde{P}_a, \tilde{P}_b\right] = \tilde{J}_{ab}, \tag{4.48}$$

$$\left[\tilde{P}_a, \tilde{Q}_\alpha\right] = \frac{1}{2}\left(\Gamma_a \tilde{Q}\right)_\alpha, \tag{4.49}$$

$$\left[\tilde{J}_{ab}, \tilde{Q}_\alpha\right] = \frac{1}{2}\left(\Gamma_a \tilde{Q}\right)_\alpha, \tag{4.50}$$

$$\left\{\tilde{Q}_\alpha, \tilde{Q}_\beta\right\} = -\frac{1}{2}\left[\left(\Gamma^{ab}C\right)_{\alpha\beta}\tilde{J}_{ab} - 2(\Gamma^a C)_{\alpha\beta}\tilde{P}_a\right]. \tag{4.51}$$

We perform the following splitting of the AdS superalgebra into two subspaces: $V_0 = \{\tilde{J}_{ab}\}$ and $V_1 = \{\tilde{P}_a, \tilde{Q}_\alpha\}$. This subspace structure satisfies

$$\begin{aligned} &[V_0, V_0] \subset V_0, \\ &[V_0, V_1] \subset V_1, \\ &[V_1, V_1] \subset V_0 \oplus V_1. \end{aligned} \tag{4.52}$$

We then consider the abelian semigroup $S^{(\infty)}$ endowed with product (3.30) and we obtain an infinite-dimensional superalgebra as an infinite S-expansion of $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$, using $S^{(\infty)}$. Thus, a resonant partition of the semigroup $S^{(\infty)} = S_0 \cup S_1$ with respect to product (3.30) is given by

$$S_p = \{\lambda_{2m+p},\ m = 0, \ldots, \infty\}, \quad p = 0, 1. \tag{4.53}$$

Then, following our method, we obtain

$$\begin{aligned} \mathfrak{g}_R^\infty &= \{\lambda_{2l},\ l = 0, \ldots, \infty\} \times V_0 \oplus \{\lambda_{2l+1},\ l = 0, \ldots, \infty\} \times V_1, \\ \mathcal{I} &= \{\lambda_{2m},\ m = 2, \ldots, \infty\} \times V_0 \oplus \{\lambda_{2m+1},\ m = 1, \ldots, \infty\} \times V_1, \\ \check{\mathfrak{g}}_R &= \{\lambda_0, \lambda_2\} \times V_0 \oplus \{\lambda_1\} \times V_1. \end{aligned} \tag{4.54}$$

If we now perform an identification and rename the generators of $\check{\mathfrak{g}}_R$ as follows,

$$J_{ab} = \lambda_0 \tilde{J}_{ab}, \tag{4.55}$$

$$Z_{ab} = \lambda_2 \tilde{J}_{ab}, \tag{4.56}$$

$$P_a = \lambda_1 \tilde{P}_a, \tag{4.57}$$

$$Q_\alpha = \lambda_1 \tilde{Q}_\alpha, \tag{4.58}$$

it is straightforward to show that we end up with a new superalgebra, $[\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)] \ominus \mathcal{I}$, which corresponds to the non-standard Maxwell superalgebra[40] and read

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{4.59}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{4.60}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \tag{4.61}$$

$$[P_a, P_b] = Z_{ab}, \tag{4.62}$$

$$[J_{ab}, Q_\alpha] = \frac{1}{2}(\Gamma_{ab}Q)_\alpha, \tag{4.63}$$

$$[P_a, Q_\alpha] = 0, \tag{4.64}$$

$$[Z_{ab}, Z_{cd}] = 0, \tag{4.65}$$

$$[Z_{ab}, P_c] = 0, \tag{4.66}$$

$$[Z_{ab}, Q_\alpha] = 0, \tag{4.67}$$

$$\left\{Q_\alpha, Q_\beta\right\} = -\frac{1}{2}\left(\Gamma^{ab}C\right)_{\alpha\beta}Z_{ab}. \tag{4.68}$$

We have thus reached the non-standard Maxwell superalgebra[40] by performing an infinite S-expansion with ideal subtraction on the AdS superalgebra.

1. Invariant tensor of the non-standard Maxwell superalgebra in D = 3

We write the components of the invariant tensor of the AdS superalgebra (see Ref. 43) as follows:

$$\langle \tilde{J}_{ab}\tilde{J}_{cd}\rangle = \tilde{\mu}_0\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right), \tag{4.69}$$

$$\langle \tilde{J}_{ab}\tilde{P}_{c}\rangle = \tilde{\mu}_1\epsilon_{abc}, \tag{4.70}$$

$$\langle \tilde{P}_{a}\tilde{P}_{b}\rangle = \tilde{\mu}_0\eta_{ab}, \tag{4.71}$$

$$\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta}\rangle = \left(\tilde{\mu}_0 - \tilde{\mu}_1\right) C_{\alpha\beta}, \tag{4.72}$$

where $\tilde{\mu}_0$ and $\tilde{\mu}_1$ are the arbitrary constants. Using (3.69), the non-zero components of the invariant tensor for the non-standard Maxwell superalgebra can be written as

$$\langle J_{ab}J_{cd}\rangle = \alpha^0\langle \tilde{J}_{ab}\tilde{J}_{cd}\rangle = \alpha^0\tilde{\mu}_0\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right) \equiv \tilde{\alpha}^0\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right), \tag{4.73}$$

$$\langle J_{ab}Z_{cd}\rangle = \alpha^2\langle \tilde{J}_{ab}\tilde{J}_{cd}\rangle = \alpha^2\tilde{\mu}_0\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right) \equiv \tilde{\alpha}^2\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right), \tag{4.74}$$

$$\langle J_{ab}P_{c}\rangle = \alpha^1\langle \tilde{J}_{ab}\tilde{P}_{c}\rangle = \alpha^1\tilde{\mu}_1\epsilon_{abc} \equiv \tilde{\alpha}^1\epsilon_{abc}, \tag{4.75}$$

$$\langle P_{a}P_{b}\rangle = \alpha^2\langle \tilde{P}_{a}\tilde{P}_{b}\rangle = \alpha^2\tilde{\mu}_0\eta_{ab} \equiv \tilde{\alpha}^2\eta_{ab}, \tag{4.76}$$

$$\langle Q_{\alpha}Q_{\beta}\rangle = \alpha^2\tilde{\mu}_0 C_{\alpha\beta} \equiv \tilde{\alpha}^2 C_{\alpha\beta}, \tag{4.77}$$

where α^m, $m = 0, 1, 2$, are the arbitrary constants, and where we have defined

$$\tilde{\alpha}^0 \equiv \alpha^0\tilde{\mu}_0, \tag{4.78}$$

$$\tilde{\alpha}^1 \equiv \alpha^1\tilde{\mu}_1, \tag{4.79}$$

$$\tilde{\alpha}^2 \equiv \alpha^2\tilde{\mu}_0. \tag{4.80}$$

E. PP-wave (super)algebra in $D = 11$ from the $AdS_4 \times S^7$ (super)algebra

The (super-)PP-wave algebra in eleven dimensions can be obtained through a generalized Inönü-Wigner contraction of the $AdS_4 \times S^7$ (super)algebra in $D = 11$ (see Ref. 54).[62]

In the following, we will reach the same result by performing an infinite S-expansion with ideal subtraction. Indeed, as shown in Sec. III, our prescription of infinite S-expansion with ideal subtraction is able to reproduce a generalized Inönü-Wigner contraction.

1. PP-wave algebra from the $AdS_4 \times S^7$ algebra

The $AdS_4 \times S^7$ algebra in $D = 11$ can be written in the following traditional form (see Ref. 54):

$$[P_a, P_b] = 4J_{ab}, \qquad [P_{a'}, P_{b'}] = 4J_{a'b'}, \tag{4.81}$$

$$[J_{ab}, P_c] = 2\eta_{bc}P_a, \qquad [J_{a'b'}, P_{c'}] = 2\eta_{b'c'}P_{a'}, \tag{4.82}$$

$$[J_{ab}, J_{cd}] = 4\eta_{ad}J_{bc}, \qquad [J_{ab}, J_{cd}] = 4\eta_{ad}J_{bc}, \tag{4.83}$$

where the vector index of AdS_4 is $i = 0, 1, 2, 3$ and that of S^7 is $a' = 4, 5, 6, 7, 8, 9, \sharp$.

Let us follow Ref. 54 and define the light cone components of the momenta P's and boost generators P^*'s as

$$P_{\pm} \equiv \frac{1}{\sqrt{2}}\left(P_{\natural} \pm P_0\right), \tag{4.84}$$

$$P_m = \left(P_i, P_{i'}\right), \tag{4.85}$$

$$P^*_m = \left(P^*_i = J_{i0}, P^*_{i'} = J_{i'\natural}\right). \tag{4.86}$$

Thus, we can write the commutation relations of $AdS_4 \times S^7$ as follows:

$$
\begin{aligned}
&[P_i, P_+] = 2\sqrt{2}P_i^*, && [P_{i'}, P_+] = -\frac{1}{\sqrt{2}}P_i^*,\\
&\left[P_i^*, P_+\right] = -\frac{1}{\sqrt{2}}P_i, && \left[P_{i'}^*, P_+\right] = \frac{1}{\sqrt{2}}P_{i'},\\
&[P_i, P_-] = -2\sqrt{2}P_i^*, && [P_{i'}, P_-] = -\frac{1}{\sqrt{2}}P_{i'}^*,\\
&\left[P_i^*, P_-\right] = \frac{1}{\sqrt{2}}P_i, && \left[P_{i'}^*, P_-\right] = \frac{1}{\sqrt{2}}P_{i'},\\
&\left[P_i^*, P_j\right] = -\frac{1}{\sqrt{2}}\eta_{ij}\left(P_- - P_+\right), && \left[P_{i'}^*, P_{j'}\right] = -\frac{1}{\sqrt{2}}\eta_{i'j'}\left(P_- + P_+\right),\\
&\left[P_i, P_j\right] = 4J_{ij}, && \left[P_{i'}, P_{j'}\right] = -J_{i'j'},\\
&\left[P_i^*, P_j^*\right] = J_{ij}, && \left[P_{i'}^*, P_{j'}^*\right] = -J_{i'j'},\\
&\left[J_{ij}, P_k\right] = 2\eta_{jk}P_i, && \left[J_{i'j'}, P_{k'}\right] = 2\eta_{j'k'}P_{i'},\\
&\left[J_{ij}, P_k^*\right] = 2\eta_{jk}P_i^*, && \left[J_{i'j'}, P_{k'}^*\right] = 2\eta_{j'k'}P_{i'}^*,\\
&\left[J_{ij}, J_{kl}\right] = 4\eta_{il}J_{jk}, && \left[J_{i'j'}, J_{k'l'}\right] = 4\eta_{i'l'}J_{j'k'}.
\end{aligned}
\tag{4.87}
$$

Before applying the infinite S-expansion method with subsequent ideal subtraction, we consider the following subspaces partition of the $AdS_4 \times S^7$ algebra:

$$
\begin{aligned}
V_0 &= \left\{P_-, J_{ij}, J_{i'j'}\right\},\\
V_1 &= \left\{P_i, P_{i'}, P_i^*, P_{i'}^*\right\},\\
V_2 &= \{P_+\}.
\end{aligned}
\tag{4.88}
$$

In this scenario, the Weimar-Woods conditions hold. Thus, following the procedure developed in Subsection III B in the case in which the original algebra satisfies the Weimar-Woods conditions, we now perform an infinite S-expansion with the semigroup $S^{(\infty)}$ on the algebra $\mathfrak{g} = AdS_4 \times S^7$, considering the following resonant subset decomposition of the semigroup $S^{(\infty)}$:

$$
\begin{aligned}
S_0 &= \{\lambda_0\} \cup \{\lambda_1, \ldots, \lambda_\infty\},\\
S_1 &= \{\lambda_1\} \cup \{\lambda_2, \ldots, \lambda_\infty\},\\
S_2 &= \{\lambda_2\} \cup \{\lambda_3, \ldots, \lambda_\infty\}.
\end{aligned}
\tag{4.89}
$$

The resonant subalgebra $\mathfrak{g}_R^\infty = \check{\mathfrak{g}}_R \oplus \hat{\mathfrak{g}}_R^\infty$ results are to be given by the direct sum of the following terms:

$$S_0 \times V_0 = \{\lambda_0\} \times V_0 \oplus \{\lambda_1, \ldots, \lambda_\infty\} \times V_0, \tag{4.90}$$

$$S_1 \times V_1 = \{\lambda_1\} \times V_1 \oplus \{\lambda_2, \ldots, \lambda_\infty\} \times V_1, \tag{4.91}$$

$$S_2 \times V_2 = \{\lambda_2\} \times V_2 \oplus \{\lambda_3, \ldots, \lambda_\infty\} \times V_2. \tag{4.92}$$

Then, the ideal reads

$$\mathcal{I} = (\{\lambda_1, \ldots, \lambda_\infty\} \times V_0) \oplus (\{\lambda_2, \ldots, \lambda_\infty\} \times V_1) \oplus (\{\lambda_3, \ldots, \lambda_\infty\} \times V_2), \tag{4.93}$$

and we can perform the ideal subtraction on the resonant subalgebra $\mathfrak{g}_R^\infty$.

Let us notice that the most relevant step consists in writing the commutation relations

$$\left[\lambda_1 P_i^*, \lambda_1 P_j\right] = -\frac{1}{\sqrt{2}}\eta_{ij}\left(\lambda_2 P_- - \lambda_2 P_+\right), \tag{4.94}$$

$$\left[\lambda_1 P_{i'}^*, \lambda_1 P_{j'}\right] = -\frac{1}{\sqrt{2}}\eta_{i'j'}\left(\lambda_2 P_- - \lambda_2 P_+\right), \tag{4.95}$$

where $\lambda_2 P_-$ belongs to the ideal while $\lambda_2 P_+$ does not.

Thus, after the ideal subtraction, we end up with the algebra $\mathfrak{g}_{PP}$ (the PP-wave algebra),

$$\mathfrak{g}_{PP} \equiv \check{\mathfrak{g}}_R = \mathfrak{g}_R^\infty \ominus \mathcal{I}. \tag{4.96}$$

Indeed, if we now rename the generators as follows: $\lambda_0 J_{ij} = \tilde{J}_{ij}$, $\lambda_0 J_{i'j'} = \tilde{J}_{i'j'}$, $\lambda_0 P_- = \tilde{P}_-$, $\lambda_1 P_i = \tilde{P}_i$, $\lambda_1 P_{i'} = \tilde{P}_{i'}$, $\lambda_1 P_i^* = \tilde{P}_i^*$, $\lambda_1 P_{i'}^* = \tilde{P}_{i'}^*$, and $\lambda_2 P_+ = \tilde{P}_+$, we can finally write the PP-wave algebra in terms of the following commutation relations (we adopt the notation of Ref. 54):

$$\begin{aligned}
\left[\tilde{P}_i, \tilde{P}_-\right] &= -2\sqrt{2}\tilde{P}_i^*, & \left[\tilde{P}_{i'}, \tilde{P}_-\right] &= -\frac{1}{\sqrt{2}}\tilde{P}_i^*, \\
\left[\tilde{P}_m^*, \tilde{P}_-\right] &= -\frac{1}{\sqrt{2}}\tilde{P}_m, & \left[\tilde{P}_m^*, \tilde{P}_n\right] &= -\frac{1}{\sqrt{2}}\eta_{mn}\tilde{P}_+, \\
\left[\tilde{J}_{mn}, \tilde{J}_{pq}\right] &= 4\eta_{mq}\tilde{J}_{np}, & \left[\tilde{J}_{mn}, \tilde{P}_p^*\right] &= 2\eta_{np}\tilde{P}_m^*, \\
\left[\tilde{J}_{mn}, \tilde{J}_{pq}\right] &= 4\eta_{mq}\tilde{J}_{np}.
\end{aligned} \tag{4.97}$$

We have thus reached the PP-wave algebra in $D = 11$ with an infinite S-expansion with ideal subtraction, starting from the $AdS_4 \times S^7$ algebra, reproducing, in this way, a generalized Inönü-Wigner contraction.

2. Super-PP-wave algebra from the $AdS_4 \times S^7$ superalgebra

We now extend the previous analysis to the supersymmetric case. We thus consider the addition of fermionic generators to the $AdS_4 \times S^7$ algebra and then perform an infinite S-expansion with consequent ideal subtraction.

The commutators involving the supercharges Q's can be decomposed in the following way:

$$Q = Q_+ + Q_-, \quad Q_\pm = Q_\pm \mathcal{P}_\pm, \tag{4.98}$$

using the light cone projection operators

$$\mathcal{P}_\pm = \frac{1}{2}\Gamma_\pm \Gamma_\mp, \tag{4.99}$$

$$\Gamma_\pm \equiv \frac{1}{\sqrt{2}}\left(\Gamma_\natural \pm \Gamma_0\right), \tag{4.100}$$

where the Γ's are the gamma matrices in eleven dimensions. Now we add $Q_\pm$ into the subspaces partition, before applying the ideal subtraction, and we thus write

$$\begin{aligned}
V_0 &= \{P_-, J_{mn}, Q_-\}, \\
V_1 &= \{P_m, P_m^*, Q_+\}, \\
V_2 &= \{P_+\}.
\end{aligned} \tag{4.101}$$

In this way, proceeding analogously to the previous (bosonic) case, performing an infinite S-expansion involving the infinite abelian semigroup $S^{(\infty)}$ and subsequently subtracting the ideal

$$\mathcal{I} = (\{\lambda_1, \ldots, \lambda_\infty\} \times V_0) \oplus (\{\lambda_2, \ldots, \lambda_\infty\} \times V_1) \oplus (\{\lambda_3, \ldots, \lambda_\infty\} \times V_2), \tag{4.102}$$

we end up with the following commutation relations:

$$[\lambda_0 P_-, \lambda_1 Q_+] = -\frac{3}{2\sqrt{2}}\lambda_1 Q_+ I, \tag{4.103}$$

$$[\lambda_0 P_-, \lambda_0 Q_-] = -\frac{1}{2\sqrt{2}}\lambda_0 Q_- I, \tag{4.104}$$

$$[\lambda_1 P_i, \lambda_0 Q_-] = \frac{1}{\sqrt{2}}\lambda_1 Q_+ \Gamma^- I \Gamma_i, \tag{4.105}$$

$$[\lambda_1 P_{i'}, \lambda_0 Q_-] = \frac{1}{2\sqrt{2}} \lambda_1 Q_+ \Gamma^- I \Gamma_{i'}, \tag{4.106}$$

$$[\lambda_1 P^*_m, \lambda_0 Q_-] = \frac{1}{2\sqrt{2}} \lambda_1 Q_+ \Gamma_m \Gamma^-, \tag{4.107}$$

$$[\lambda_0 J_{mn}, \lambda_0 Q_\pm] = \frac{1}{2} \lambda_0 Q_\pm \Gamma_{mn}, \tag{4.108}$$

$$\{\lambda_1 Q_+, \lambda_1 Q_+\} = -2\mathcal{C}\Gamma^+ \lambda_1 P_+, \tag{4.109}$$

$$\{\lambda_0 Q_-, \lambda_0 Q_-\} = -2\mathcal{C}\Gamma^- \lambda_0 P_- - \sqrt{2}\mathcal{C}\Gamma^- I \Gamma^{ij} \lambda_0 J_{ij} + \frac{1}{\sqrt{2}} \mathcal{C}\Gamma^- I \Gamma^{i'j'} \lambda_0 J_{i'j'}, \tag{4.110}$$

$$\{\lambda_1 Q_+, \lambda_0 Q_-\} = \left(-2\mathcal{C}\Gamma^m \lambda_1 P_m - 4\mathcal{C}I\Gamma^i \lambda_1 P^*_i - 2\mathcal{C}I\Gamma^{i'} \lambda_1 P^*_{i'}\right) \lambda_0 \mathcal{P}_-. \tag{4.111}$$

Then, we only have to properly rename the generators in order to find the algebra

$$\begin{aligned}
\left[\tilde{P}_-, \tilde{Q}_+\right] &= -\frac{3}{2\sqrt{2}} \tilde{Q}_+ I, \\
\left[\tilde{P}_-, \tilde{Q}_-\right] &= -\frac{1}{2\sqrt{2}} \tilde{Q}_- I, \\
\left[\tilde{P}_i, \tilde{Q}_-\right] &= \frac{1}{\sqrt{2}} \tilde{Q}_+ \Gamma^- I \Gamma_i, \\
\left[\tilde{P}_{i'}, \tilde{Q}_-\right] &= \frac{1}{2\sqrt{2}} \tilde{Q}_+ \Gamma^- I \Gamma_{i'}, \\
\left[\tilde{P}^*_m, \tilde{Q}_-\right] &= \frac{1}{2\sqrt{2}} \tilde{Q}_+ \Gamma_m \Gamma^-, \\
\left[\tilde{J}_{mn}, \tilde{Q}_\pm\right] &= \frac{1}{2} \tilde{Q}_\pm \Gamma_{mn}, \\
\left\{\tilde{Q}_+, \tilde{Q}_+\right\} &= -2\mathcal{C}\Gamma^+ \tilde{P}_+, \\
\left\{\tilde{Q}_-, \tilde{Q}_-\right\} &= -2\mathcal{C}\Gamma^- \tilde{P}_- - \sqrt{2}\mathcal{C}\Gamma^- I \Gamma^{ij} \tilde{J}_{ij} + \frac{1}{\sqrt{2}} \mathcal{C}\Gamma^- I \Gamma^{i'j'} \tilde{J}_{i'j'}, \\
\left\{\tilde{Q}_+, \tilde{Q}_-\right\} &= \left(-2\mathcal{C}\Gamma^m \tilde{P}_m - 4\mathcal{C}I\Gamma^i \tilde{P}^*_i - 2\mathcal{C}I\Gamma^{i'} \tilde{P}^*_{i'}\right) \tilde{\mathcal{P}}_-.
\end{aligned} \tag{4.112}$$

These commutation relations correspond to those of the super-PP-wave algebra (we have adopted the same notation of Ref. 54).

We have thus reached the super-PP-wave algebra in $D = 11$ by performing an infinite S-expansion with ideal subtraction, starting from the super-$AdS_4 \times S^7$, reproducing, in this way, a generalized Inönü-Wigner contraction in the case of a supersymmetric algebra.

V. COMMENTS AND POSSIBLE DEVELOPMENTS

The S-expansion method has the peculiarity of being able to reproduce the standard Inönü-Wigner contraction as a special case of the procedure called 0_S-reduction (see Ref. 8).

Furthermore, as shown in Ref. 8, the information on the subspace structure of the original (super)algebra can be used in order to find resonant subalgebras of the S-expanded (super)algebra, and by extracting reduced algebras from the resonant subalgebra, one can reproduce the generalized Inönü-Wigner contraction within this scheme.

In the present work, we have given a new prescription for S-expansion, based on using an *infinite* abelian semigroup $S^{(\infty)}$ and by performing the subtraction of an infinite ideal subalgebra. We have explicitly shown that the subtraction of the infinite ideal subalgebra corresponds to a reduction, leading to a reduced (super)algebra. In particular, it can be viewed as a (generalization of the) 0_S-reduction since it reproduces the same effects. This method can be also interpreted as a different, alternative view of the generalized Inönü-Wigner contraction. Indeed, our "infinite S-expansion" procedure allows one to reproduce the standard as well as the generalized Inönü-Wigner contraction.

The removal of the infinite ideal subalgebra is crucial since it allows to end up with finite dimensional Lie (super)algebras.

This infinite S-expansion procedure represents an extension and generalization of the finite one, allowing a deeper view on the maps linking different algebras.

We have then explained how to write the invariant tensors for the (super)algebras, which can be obtained by applying our method of infinite S-expansion with ideal subtraction. This procedure allows us to develop the dynamics and construct the Lagrangians of physical theories. In particular, in this context, the construction of Chern-Simons forms becomes more accessible, and it would be particularly interesting to develop them in (super)gravity theories in higher dimensions, by following our approach.

This work reproduces the results already presented in the literature, concerning expansions and contractions of Lie (super)algebras, and also gives some new features. Moreover, it gives further connections between the contraction processes and the expansion methods, which was an open question already mentioned in Ref. 55.

On the other hand, this paper can also contribute to understand the nature of the semigroups used in the S-expansion processes since a contraction is usually applied over a physical constant.

In our work, we have restricted our study to the cases involving an infinite semigroup $S^{(\infty)}$ related to the set $(\mathbb{N}, +)$. We leave a possible extension to the set $(\mathbb{Z}, +)$ to future studies. This further analysis would be interesting since it would produce an S-expansion involving an abelian group (with respect to the sum operation) rather than a semigroup.

Another possible development of our work would consists in applying the method developed in the present paper to the case studied in Ref. 56, where the authors showed that interpreting the inverse AdS_3 radius $1/l$ as a Grassmann variable results in a formal map from gravity in AdS_3 to gravity in flat space. The underlying reason for this relies in the Inönü-Wigner contraction. They systematically developed the possibility that Inönü-Wigner contraction could be turned into an algebraic operation through a Grassmann approach.

We argue that this could present a strong chance that a generalization of their algebraic approach should also work in our case as well, and it would thus be interesting to extend our method by exploiting a Grassmann-like approach. Some work is in progress on this topic.

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[57] The loop algebra[42] was constructed by considering the semigroup $(\mathbb{Z}, +)$ (which is an abelian group with the sum operation). In our work, we restrict to $(\mathbb{N}, +)$ and we leave the extension to $(\mathbb{Z}, +)$ to the future.
[58] Different semigroups of the type $S_E^{(N)}$ have been used and discussed in several studies on S-expanded algebras (see Refs. 8, 13, 16, 19, 25, 27, 32, and 43).
[59] We have used the notation λ_∞ just for denoting the fact that we are considering an infinite set, in the sense that it contains an infinite number of elements.
[60] Let us observe that in the case in which $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{h} \oplus \mathfrak{p}$, that is to say, when we are taking into account a non-symmetric coset, the standard Inönü-Wigner contraction still abelianizes this commutator.
[61] Namely, a generator that commutes with all other generators of the algebra. In $D = 3$, three such central generators can be introduced.
[62] This can be viewed as the Penrose limit of the $AdS \times S$ metric.