Properties and Numerical Solution of an Integral Equation to Minimize Airplane Drag

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Dedicated to Ian H. Sloan on the occasion of his 80th birthday.

Abstract In this paper, we consider an (open) airplane wing, not necessarily symmetric, for which the optimal circulation distribution has to be determined. This latter is the solution of a constraint minimization problem, whose (Cauchy singular integral) Euler-Lagrange equation is known. By following an approach different from a more classical one applied in previous papers, we obtain existence and uniqueness results for the solution of this equation in suitable weighted Sobolev type spaces. Then, for the collocation-quadrature method we propose to solve the equation, we prove stability and convergence and derive error estimates. Some numerical examples, which confirm the previous error estimates, are also presented. These results apply, in particular, to the Euler-Lagrange equation and the numerical method used to solve it in the case of a symmetric wing, which were considered in the above mentioned previous papers.

1 Introduction

In [3], the authors have studied the induced drag minimization problem for an open symmetric airplane wing. In particular, by applying a classical variational approach,

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they have derived the associated Euler-Lagrange (integral) equation (ELE) for the unknown wing circulation distribution. In its final form, this equation is a Cauchy singular one, for which existence and uniqueness of its solution have been assumed. For the solution of this equation, the authors have proposed a discrete polynomial collocation method, based on Chebyshev polynomials and a corresponding Gaussian quadrature. Although the convergence of this method has been confirmed by an intensive numerical testing, no error estimates have been obtained.

Later, in [4], by using an alternative (weakly singular) formulation of the above ELE of Symm's type, existence and uniqueness of the optimal circulation has been proved under the assumption that the curve transfinite diameter is different from 1. The authors have however conjectured that this property should hold without this restriction.

In this paper, we consider an open wing (also called lifting curve), not necessarily symmetric, and examine the associated Euler-Lagrange equation. The main physical quantities and formulas, that are needed to describe the minimization problem, are briefly recalled in Section 2. Then, in Section 3, by following an approach different from the more classical one applied in [4], we obtain existence and uniqueness results in suitable weighted Sobolev type spaces, without requiring the above mentioned curve restriction. In Section 3, we derive an error estimate for the collocation-quadrature method we use to solve the ELE. In the case of a symmetric lifting line, the method naturally reduces to that proposed in [3]. Finally, in the last section, to test the efficiency of the proposed method and the error estimate previously obtained for it, we apply the method to four different open curves.

2 The Drag Minimization Problem

Following [3], we consider a wing defined by a single open lifting line ℓ in the cartesian y-z plane. This is represented by a curve ℓ, having parametric representation $\psi(t) = \left[\, \psi_1(t) \;\; \psi_2(t) \,\right]^T$, $|\psi'(t)| \neq 0$, $t \in [-1,1]$. The corresponding arc length abscissa η is then defined by

$$\eta(t) = \int_0^t \left|\psi'(s)\right| \mathrm{d}s, \tag{1}$$

where, here and in the following, $|\cdot|$ denotes the Euclidean norm. This abscissa will run from $\eta(-1) = -b$ to $\eta(1) = a$ for some positive real numbers a and b. Moreover, $\eta(0) = 0$.

For simplicity, it is also assumed that the lifting line ℓ is sufficiently smooth. That is, it is assumed that $\psi_i(t), i = 1,2$, are continuous functions together with their first $m \geq 2$ derivatives on the interval $[-1,1]$ (i.e., $\psi_i \in \mathbf{C}^m[-1,1]$). A point on the lifting line, where the aerodynamic forces are calculated, is denoted by $\mathbf{r} = \left[\, y \; z \,\right]^T \in \ell$, where $\mathbf{r} = \mathbf{r}(\eta) = \left[\, y(\eta) \;\; z(\eta) \,\right]^T = \left[\, \psi_1(t) \;\; \psi_2(t) \,\right]^T$ in correspondence with (1).

Using the arc length abscissa, the expressions of the wing *lift* L and *induced drag* D_{ind} are obtained in terms of the (unknown) *circulation* Γ:

$$L = L(\Gamma) = -\rho_\infty V_\infty \int_{-b}^{a} \tau_y(\eta)\Gamma(\eta)\,\mathrm{d}\eta \tag{2}$$

$$D_{\text{ind}} = D_{\text{ind}}(\Gamma) = -\rho_\infty \int_{-b}^{a} v_n(\eta)\Gamma(\eta)\mathrm{d}\eta. \tag{3}$$

The quantities ρ_∞ and V_∞ are given positive constants which indicate the density and freestream velocity, respectively. Further, $\tau_y(\eta) = y'(\eta)$ is the projection on the y-axis of the unit vector tangent to the lifting line, while v_n is the so-called *normalwash*. This latter has the representation

$$v_n(\eta) = \frac{1}{4\pi} \fint_{-b}^{a} \Gamma'(\xi)\, Y(\eta,\xi)\,\mathrm{d}\xi, \quad -b < \eta < a, \tag{4}$$

where

$$Y(\eta,\xi) = -\frac{\mathrm{d}}{\mathrm{d}\eta} \ln|\mathbf{r}(\xi) - \mathbf{r}(\eta)|. \tag{5}$$

The function $Y(\eta,\xi)$, which is the *kernel* of the associated integral transform, has a singularity of order 1 when $\eta = \xi$, and the integral in (4) is a Cauchy principal value one.

The problem we need to solve is the minimization, in a suitable space, of the functional $D_{\text{ind}}(\Gamma)$, subject to the prescribed lift constraint

$$L(\Gamma) = L_{\text{pres}}. \tag{6}$$

In the next sections we will go back to the interval $[-1,1]$. For this, we use the notations

$$\Gamma_0(t) := \Gamma(\eta(t)), \quad \mathbf{r}_0(t) := \mathbf{r}(\eta(t)) = \psi(t),$$

and

$$Y_0(t,s) := -\frac{\mathrm{d}}{\mathrm{d}t} \ln|\mathbf{r}_0(s) - \mathbf{r}_0(t)| \tag{7}$$

for $t,s \in [-1,1]$, as well as the respective relations

$$\Gamma_0'(t) = \Gamma'(\eta(t))\eta'(t), \quad \psi_1'(t) = y'(\eta(t))\eta'(t) \quad Y_0(t,s) = Y(\eta(t),\eta(s))\eta'(t).$$

Condition (6) then takes the new form

$$\int_{-1}^{1} \psi_1'(t)\Gamma_0(t)\,\mathrm{d}t = \gamma_0 := -\frac{L_{\text{pres}}}{\rho_\infty V_\infty}. \tag{8}$$

Moreover, from (3) and (4) we get

$$D_{\text{ind}} = D_{\text{ind}}(\Gamma_0) = -\frac{\rho_\infty}{4\pi} \int_{-1}^{1} \fint_{-1}^{1} Y_0(t,s)\Gamma_0'(s)\,\mathrm{d}s\,\Gamma_0(t)\,\mathrm{d}t. \tag{9}$$

3 The Euler-Lagrange Equation and its Properties

For a Jacobi weight $\rho(t) := v^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ with $\alpha,\beta > -1$, let us recall the definition of the Sobolev-type space (cf. [1]) $\mathbf{L}^{2,r}_\rho = \mathbf{L}^{2,r}_\rho(-1,1)$, $r \ge 0$. For this, by $\mathbf{L}^2_\rho = \mathbf{L}^{2,0}_\rho$ we denote the real Hilbert space of all (classes of) quadratic summable (w.r.t. the weight $\rho(t)$) functions $f : (-1,1) \longrightarrow \mathbb{R}$ equipped with the inner product

$$\langle f,g\rangle_\rho := \int_{-1}^{1} f(t)g(t)\rho(t)\,\mathrm{d}t$$

and the norm $\|f\|_\rho = \sqrt{\langle f,f\rangle_\rho}$. In case $\alpha = \beta = 0$, i.e., $\rho \equiv 1$, we write $\langle f,g\rangle$ and $\|f\|$ instead of $\langle f,g\rangle_\rho$ and $\|f\|_\rho$, respectively. If $\left\{p^\rho_n : n \in \mathbb{N}_0\right\}$ denotes the system of orthonormal (w.r.t. $\rho(t)$) polynomials $p^\rho_n(t)$ of degree n with positive leading coefficient, then

$$\mathbf{L}^{2,r}_\rho := \left\{ f \in \mathbf{L}^2_\rho : \sum_{n=0}^{\infty}(1+n)^{2r}\left|\langle f,p^\rho_n\rangle_\rho\right|^2 < \infty \right\}.$$

Equipped with the inner product

$$\langle f,g\rangle_{\rho,r} = \sum_{n=0}^{\infty}(1+n)^{2r}\langle f,p^\rho_n\rangle_\rho\langle g,p^\rho_n\rangle_\rho$$

and the norm $\|f\|_{\rho,r} := \sqrt{\langle f,f\rangle_{\rho,r}}$, the set $\mathbf{L}^{2,r}_\rho$ becomes a Hilbert space. Note that, in cases $\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$, the spaces $\mathbf{L}^{2,r}_\rho$ were also introduced in [6, Section 1] with a slightly different notation. Let $\varphi(t) = \sqrt{1-t^2}$ and define

$$\mathbf{V} := \left\{ f = \varphi u : u \in \mathbf{L}^{2,1}_\varphi \right\}$$

together with $\langle f,g\rangle_{\mathbf{V}} := \langle \varphi^{-1}f, \varphi^{-1}g\rangle_{\varphi,1}$ and $\|f\|_{\mathbf{V}} := \left\|\varphi^{-1}f\right\|_{\varphi,1}$.

In the following, we denote by $\mathscr{D}$ the operator of generalized differentiation. An important property of this operator with respect to the $\mathbf{L}^{2,r}_\rho$ spaces is recalled in the next lemma, where we have set $\rho^{(1)}(t) = (1-t)^{1+\alpha}(1+t)^{1+\beta} = \rho(t)(1-t^2)$.

Lemma 1 ([2], Lemma 2.7; cf. also [1], Theorem 2.17). *For $r \ge 0$, the operator $\mathscr{D} : \mathbf{L}^{2,r+1}_\rho \longrightarrow \mathbf{L}^{2,r}_{\rho^{(1)}}$ is continuous.*

Lemma 2. *For $f \in \mathbf{V}$, we have $f \in \mathbf{C}[-1,1]$ with $f(\pm 1) = 0$.*

Proof. Let $f = \varphi g$ with $g \in \mathbf{L}^{2,1}_\varphi$. Due to Lemma 1, $\mathscr{D}g \in \mathbf{L}^2_{\varphi^3}$. Hence, for $0 < t < 1$,

$$|g(t)| = \left| g(0) + \int_0^t (\mathscr{D}g)(s)\,\mathrm{d}s \right| \le |g(0)| + \sqrt{\int_0^t (1-s^2)^{-\frac{3}{2}}\,\mathrm{d}s}\;\|\mathscr{D}g\|_{\varphi^3}$$

and

$$\int_0^t (1-s^2)^{-\frac{3}{2}}\,\mathrm{d}s \le \int_0^t (1-s)^{-\frac{3}{2}}\,\mathrm{d}s = 2\left(\frac{1}{\sqrt{1-t}}-1\right).$$

This implies $f(1) = \lim_{t\to 1-0}\varphi(t)g(t) = 0$. Analogously, one can show that also $f(-1) = 0$ holds for $f \in \mathbf{V}$.

Now, the problem we aim to solve (cf. [3]) is the following:

(P) *Find a function $\Gamma_0 \in \mathbf{V}$, which minimizes the functional* (*cf.* (9))

$$F(\Gamma_0) := -\int_{-1}^{1} ⨍_{-1}^{1} Y_0(t,s)\Gamma_0'(s)\,\mathrm{d}s\,\Gamma_0(t)\,\mathrm{d}t$$

subject to the condition (*cf.* (8)) $\langle \psi_1', \Gamma_0\rangle = \gamma_0$.

If we define the linear operator

$$(\mathscr{A}f)(t) = -\frac{1}{\pi} ⨍_{-1}^{1} Y_0(t,s)f'(s)\,\mathrm{d}s, \quad -1 < s < 1, \tag{10}$$

then the problem can be reformulated as follows:

(P) *Find a function $\Gamma_0 \in \mathbf{V}$, which minimizes the functional $F(\Gamma_0) := \langle \mathscr{A}\Gamma_0, \Gamma_0\rangle$ on $\mathbf{V}$ subject to the condition $\langle \psi_1', \Gamma_0\rangle = \gamma_0$.*

The formulation of this problem is correct, which can be seen from the following lemma.

Lemma 3. *If $\psi_j \in \mathbf{C}^m[-1,1]$ for some integer $m \ge 2$ and $|\psi'(t)| \ne 0$ for $t \in [-1,1]$, then the function $Y_0(t,s)$ has the representation*

$$Y_0(t,s) = \frac{1}{s-t} + K(t,s), \tag{11}$$

where the function $K : [-1,1]^2 \longrightarrow \mathbb{R}$ is continuous together with its partial derivatives $\dfrac{\partial^{j+k}K(t,s)}{\partial t^j \partial s^k}$, $k, j \in \mathbb{N}_0$, $j+k \le m-2$.

Proof. Note that, by definition,

$$Y_0(t,s) = \frac{[\psi_1(s)-\psi_1(t)]\psi_1'(t) + [\psi_2(s)-\psi_2(t)]\psi_2'(t)}{[\psi_1(s)-\psi_1(t)]^2 + [\psi_2(s)-\psi_2(t)]^2}.$$

Hence,

$$K(t,s) = Y_0(t,s) - \frac{1}{s-t} = \frac{\Omega(t,s)}{\Psi(t,s)},$$

where

$$\Omega(t,s) = G_1(t,s)g_1(t,s) + G_2(t,s)g_2(t,s), \quad \Psi(t,s) = [g_1(t,s)]^2 + [g_2(t,s)]^2,$$

$$g_j(t,s) = \frac{\psi_j(s) - \psi_j(t)}{s-t} = \int_0^1 \psi_j'\big(sv + t(1-v)\big)\,\mathrm{d}v$$

$$G_j(t,s) = \frac{\psi_j'(t) - g_j(t,s)}{s-t} = \int_0^1 \psi_j''\big(sv + t(1-v)\big)(1-v)\,\mathrm{d}v,$$

and the assertion of the lemma follows by taking into account $\Psi(t,s) \neq 0$ for all $(t,s) \in [-1,1]^2$.

Lemma 4. *The operator $\mathscr{A} : \mathbf{V} \longrightarrow \mathbf{L}^2_\varphi$ is a linear and bounded one and, consequently, $\langle \mathscr{A} f, f\rangle$ is well defined for all $f \in \mathbf{V}$.*

Proof. Let $U_n = p_n^\varphi$ and $T_n = p_n^{\varphi^{-1}}$. Then, for $f \in \mathbf{V}$,

$$\begin{aligned}\|\mathscr{D} f\|_\varphi^2 &= \sum_{n=0}^\infty \left|\langle \mathscr{D} f, \varphi^{-1} T_n\rangle_\varphi\right|^2 = \sum_{n=0}^\infty |\langle \mathscr{D} f, T_n\rangle|^2 = \sum_{n=1}^\infty |\langle f, nU_{n-1}\rangle|^2 \\ &= \sum_{n=0}^\infty (1+n)^2 \left|\langle \varphi^{-1} f, U_n\rangle_\varphi\right|^2 = \left\|\varphi^{-1} f\right\|_{\varphi,1}^2 = \|f\|_{\mathbf{V}}^2,\end{aligned}$$

i.e., $\mathscr{D} f \in \mathbf{L}^2_\varphi$. By relation (11), the operator $\mathscr{A}$ defined in (10) can be written in the form $\mathscr{A} = -(\mathscr{S} + \mathscr{K})\mathscr{D}$ with

$$(\mathscr{S} f)(t) := \frac{1}{\pi} \fint_{-1}^1 \frac{f(s)\,\mathrm{d}s}{s-t}, \quad (\mathscr{K} f)(t) = \frac{1}{\pi}\int_{-1}^1 K(t,s) f(s)\,\mathrm{d}s, \quad -1 < t < 1.$$

It is well known that the Cauchy singular integral operator $\mathscr{S} : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\varphi$ is bounded ([5, Theorem 4.1]) and that $\mathscr{K} : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\varphi$ is compact. Consequently, for $f = \varphi u \in \mathbf{V}$ we have that $\langle \mathscr{A} f, f\rangle = \langle \mathscr{A} f, u\rangle_\varphi$ is a finite number, since both $\mathscr{A} f$ and u belong to $\mathbf{L}^2_\varphi$.

In the following lemma we give a representation of the operator $\mathscr{A}$ defined in (10), which is crucial for our further investigations. From this representation, it is seen that the operator $\mathscr{A}$ is an example of a hypersingular integral operator in the sense of Hadamard (cf., for example, the representation of Prandtl's integro-differential operator in [2, Section 1], where $\mathbf{r}_0(t) = t$ and $\mathscr{B}$ is equal to the Cauchy singular integral operator).

Lemma 5. *For all $f \in \mathbf{V}$, the relation*

$$\mathscr{A} f = \mathscr{D}\mathscr{B} f \tag{12}$$

holds true, where

$$(\mathscr{B} f)(t) = \frac{1}{\pi}\int_{-1}^1 \ln|\mathbf{r}_0(s) - \mathbf{r}_0(t)|\, f'(s)\,\mathrm{d}s \tag{13}$$

and where $\mathscr{D}$ is the operator of generalized differentiation already used in the proof of Lemma 2.

Proof. Since $\mathscr{D} : \mathbf{V} \longrightarrow \mathbf{L}^2_\varphi$ is an isometrical mapping (cf. the proof of Lemma 4), it suffices to show that $-(\mathscr{S}+\mathscr{K})g = \mathscr{D}\mathscr{B}_0 g$ is valid for all $g \in \mathbf{L}^2_\varphi$, where

$$(\mathscr{B}_0 g)(t) = \frac{1}{\pi}\int_{-1}^{1} \ln|\mathbf{r}_0(s)-\mathbf{r}_0(t)|\, g(s)\,\mathrm{d}s.$$

Since

$$Z_0(t,s) := \ln|\mathbf{r}_0(s)-\mathbf{r}_0(t)| = \ln|s-t| + K_0(t,s) \tag{14}$$

with a function $K_0 : [-1,1]^2 \longrightarrow \mathbb{R}$ which is continuous together with $\dfrac{\partial K_0(t,s)}{\partial t} = -K(t,s)$, the operator $\mathscr{B}_0 : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^{2,1}_{\varphi^{-1}}$ is bounded (see [7, Section 5] and [1, Lemma 4.2]). Moreover, $\mathscr{D} : \mathbf{L}^{2,1}_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_\varphi$ is continuous ([2, Lemma 2.7]), such that on the one hand, the operator $\mathscr{D}\mathscr{B}_0 : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\varphi$ is linear and bounded. On the other hand, the operator $\mathscr{S}+\mathscr{K} : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\varphi$ is also linear and bounded. Thus, it remains to prove that

$$⨍_{-1}^{1} Y_0(t,s)g(s)\,\mathrm{d}s = -\frac{d}{dt}\int_{-1}^{1} Z_0(t,s)g(s)\,\mathrm{d}s, \quad -1<t<1 \tag{15}$$

for all g from a linear and dense subset of $\mathbf{L}^2_\varphi$. For this, let $g : [-1,1] \longrightarrow \mathbb{R}$ be a continuously differentiable function and consider

$$\psi_0(t) := \int_{-1}^{1} Z_0(t,s)g(s)\,\mathrm{d}s = \lim_{\varepsilon\to+0}\psi_\varepsilon(t)$$

with $\psi_\varepsilon(t) := \left(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\right) Z_0(t,s)g(s)\,\mathrm{d}s$. For every $t \in (-1,1)$, it follows

$$\begin{aligned}
\psi_\varepsilon'(t) &= -\left(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\right) Y_0(t,s)g(s)\,\mathrm{d}s + Z_0(t,t-\varepsilon)g(t-\varepsilon) - Z_0(t,t+\varepsilon)g(t+\varepsilon)\\
&= -\left(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\right) Y_0(t,s)g(s)\,\mathrm{d}s + \ln\varepsilon[g(t-\varepsilon)-g(t+\varepsilon)]\\
&\qquad\qquad + K_0(t,t-\varepsilon)g(t-\varepsilon) - K_0(t,t+\varepsilon)g(t+\varepsilon)\\
&\longrightarrow -⨍_{-1}^{1} Y_0(t,s)g(s)\,\mathrm{d}s \quad \text{if} \quad \varepsilon \longrightarrow +0,
\end{aligned}$$

where, as before, the last integral is defined in the Cauchy principal value sense. For every $\delta \in (0,1)$, this convergence is uniform w.r.t. $t \in [-1+\delta, 1-\delta]$. Indeed, for $0<\varepsilon_1<\varepsilon_2<\delta$ and for

$$g_\varepsilon(t) := \left(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1} \right) Y_0(t,s) g(s)\, \mathrm{d}s,$$

we have

$$\begin{aligned} g_{\varepsilon_1}(t) - g_{\varepsilon_2}(t) &= \int_{t-\varepsilon_2}^{t-\varepsilon_1} Y_0(t,s) g(s)\, \mathrm{d}s + \int_{t+\varepsilon_1}^{t+\varepsilon_2} Y_0(t,s) g(s)\, \mathrm{d}s \\ &= \int_{t-\varepsilon_2}^{t-\varepsilon_1} Y_0(t,s)[g(s)-g(t)]\, \mathrm{d}s + \int_{t+\varepsilon_1}^{t+\varepsilon_2} Y_0(t,s)[g(s)-g(t)]\, \mathrm{d}s \\ &\qquad + g(t) \left[\int_{t-\varepsilon_2}^{t-\varepsilon_1} Y_0(t,s) g(s)\, \mathrm{d}s + \int_{t+\varepsilon_1}^{t+\varepsilon_2} Y_0(t,s) g(s)\, \mathrm{d}s \right] \\ &= \left(\int_{t-\varepsilon_2}^{t-\varepsilon_1} + \int_{t+\varepsilon_1}^{t+\varepsilon_2} \right) [1 + (s-t)K(t,s)] \, \frac{g(s)-g(t)}{s-t}\, \mathrm{d}s \\ &\qquad + g(t) \left(\int_{t-\varepsilon_2}^{t-\varepsilon_1} + \int_{t+\varepsilon_1}^{t+\varepsilon_2} \right) K(t,s)\, \mathrm{d}s. \end{aligned}$$

Consequently,

$$|g_{\varepsilon_1}(t) - g_{\varepsilon_2}(t)| \le M(\varepsilon_2 - \varepsilon_1)$$

with $M = 2\,(M_1\,\|g'\|_\infty + M_2\|g\|_\infty)$, $M_1 = 1 + \max\{|(s-t)K(t,s)| : -1 \le s,t \le 1\}$, and $M_2 = \max\{|K(t,s)| : -1 \le s,t \le 1\}$. This uniform convergence implies that $\psi_0(t)$ is differentiable for all $t \in (-1,1)$, where

$$\psi_0'(t) = \frac{d}{dt}\left[\lim_{\varepsilon \to +0} \psi_\varepsilon(t) \right] = \lim_{\varepsilon \to +0} \psi_\varepsilon'(t) = -\,⨍_{-1}^{1} Y_0(t,s) g(s)\, \mathrm{d}s$$

and $\psi_0'(t) = \dfrac{d}{dt} \displaystyle\int_{-1}^{1} Z_0(t,s) g(s)\, \mathrm{d}s$, and (15) is proved.

Lemma 6. *The operator $\mathscr{A} : \mathbf{V} \longrightarrow \mathbf{L}^2_\varphi$ is symmetric and positive, i.e. $\forall\, f,g \in \mathbf{V}$, $\langle \mathscr{A} f, g\rangle = \langle f, \mathscr{A} g\rangle$ and, $\forall\, f \in \mathbf{V} \setminus \{0\}$, $\langle \mathscr{A} f, f\rangle > 0$.*

Proof. Using relation (12), Lemma 2, partial integration, and Fubini's theorem, we get, for all $f,g \in \mathbf{V}$,

$$\langle \mathscr{A} f, g\rangle = -\frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} f'(s) \ln |\mathbf{r}_0(s) - \mathbf{r}_0(t)|\, \mathrm{d}s\, g'(t)\, \mathrm{d}t = \langle f, \mathscr{A} g\rangle. \tag{16}$$

Hence,

$$\langle \mathscr{A} f, f\rangle = \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \ln \frac{1}{|\mathbf{r}_0(s) - \mathbf{r}_0(t)|}\, f'(s) f'(t)\, \mathrm{d}s\, \mathrm{d}t$$

corresponds to the logarithmic energy of the function f', where $\displaystyle\int_{-1}^{1} f'(t)\, \mathrm{d}t = 0$ due to Lemma 2. Consequently (see [8], Section I.1, and in particular Lemma 1.8),

$\langle \mathscr{A} f, f\rangle$ is positive if $f' \neq 0$ a.e. Hence, $\langle \mathscr{A} f, f\rangle = 0$ implies $f'(t) = 0$ for almost all $t \in (-1,1)$ and, due to $f(\pm 1) = 0$, also $f(t) = 0$ for all $t \in [-1,1]$.

For $\gamma \in \mathbb{R}$, define the (affine) manifold $\mathbf{V}_\gamma := \{f \in \mathbf{V} : \langle f, \psi_1'\rangle = \gamma\}$. The following result then holds.

Proposition 1. *The element $\Gamma_0^* \in \mathbf{V}_{\gamma_0}$ is a solution of Problem* (P) *if and only if there is a number $\beta \in \mathbb{R}$ such that*

$$\mathscr{A}\Gamma_0^* = \beta \psi_1'. \tag{17}$$

This solution is unique, if it exists.

Proof. Assume that $\Gamma_0^* \in \mathbf{V}_{\gamma_0}$ and $F(\Gamma_0^*) = \min\{F(\Gamma_0) : \Gamma_0 \in \mathbf{V}_{\gamma_0}\}$. This implies $G'(0) = 0$ for $G(\alpha) = F(\Gamma_0^* + \alpha f)$ and for all $f \in \mathbf{V}_0 \setminus \{0\}$. Since

$$G(\alpha) = F(\Gamma_0^*) + 2\alpha\langle \mathscr{A}\Gamma_0^*, f\rangle + \alpha^2\langle f, f\rangle \tag{18}$$

and

$$G'(\alpha) = 2\langle \mathscr{A}\Gamma_0^*, f\rangle + 2\alpha\langle f, f\rangle,$$

this condition gives $\langle \mathscr{A}\Gamma_0^*, g\rangle_\varphi = 0$ for all $g \in \mathbf{L}_\varphi^{2,1}$ with $\langle g, \psi_1'\rangle_\varphi = 0$, which is equivalent to (17). On the other hand, if $\Gamma_0^* \in \mathbf{V}_{\gamma_0}$ and $\beta \in \mathbb{R}$ fulfil (17) and if $f \in \mathbf{V}_0 \setminus \{0\}$, then we get from (18) for $\alpha = 1$

$$\begin{aligned} F(\Gamma_0^* + f) &= F(\Gamma_0^*) + 2\langle \mathscr{A}\Gamma_0^*, f\rangle + \langle f, f\rangle \\ &= F(\Gamma_0^*) + 2\langle \mathscr{A}\Gamma_0^* - \beta\psi_1', f\rangle + \langle f, f\rangle \\ &= F(\Gamma_0^*) + \langle f, f\rangle > F(\Gamma_0^*), \end{aligned}$$

which shows the uniqueness of the solution (if it exists).

Remark 1. Using relation (12), equation (17) can be written equivalently as

$$\mathscr{B}\Gamma_0^* = \beta\psi_1 + \gamma, \quad \Gamma_0^* \in \mathbf{V}_{\gamma_0},\ \beta, \gamma \in \mathbb{R}. \tag{19}$$

Moreover, by applying partial integration to the integral in (13) and taking into account $f(\pm 1) = 0$ for $f \in \mathbf{V}$ (see Lemma 2), we get

$$\begin{aligned} (\mathscr{B} f)(t) &= \lim_{\varepsilon \to +0} \frac{1}{\pi}\left(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\right) \ln|\mathbf{r}_0(s) - \mathbf{r}_0(t)|\, f'(s)\,\mathrm{d}s \\ &= \lim_{\varepsilon \to +0} \frac{1}{\pi}\left[f(t-\varepsilon)\ln|\mathbf{r}_0(t-\varepsilon) - \mathbf{r}_0(t)| - f(t+\varepsilon)\ln|r_0(t+\varepsilon) - \mathbf{r}_0(t)|\right] \\ &\quad - \lim_{\varepsilon \to +0} \frac{1}{\pi}\left(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\right) f(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\ln|\mathbf{r}_0(s) - \mathbf{r}_0(t)|\,\mathrm{d}s \\ &\overset{(7)}{=} \frac{1}{\pi} \fint_{-1}^{1} Y_0(s,t) f(s)\,\mathrm{d}s \end{aligned}$$

Hence, we obtain the identity

$$\mathscr{B}f = \mathscr{A}_0 f \quad \forall f \in \mathbf{V}, \tag{20}$$

where (cf. (11))

$$(\mathscr{A}_0 f)(t) = \frac{1}{\pi} \fint_{-1}^{1} Y_0(s,t) f(s)\,\mathrm{d}s = -(\mathscr{S} f)(t) + (\mathscr{K}_0 f)(t)$$

with

$$(\mathscr{K}_0 f)(t) = \frac{1}{\pi} \int_{-1}^{1} K(s,t) f(s)\,\mathrm{d}s. \tag{21}$$

Note that equation (17), hence its equivalent representation one obtains from (19) and equality (20), defines the Euler-Lagrange equation for the drag minimization problem.

The following Lemma is a consequence of the well-known relation

$$\mathscr{S} \varphi p_n^{\varphi} = -p_{n+1}^{\varphi^{-1}}, \quad n \in \mathbb{N}_0, \tag{22}$$

Lemma 7. *The operators $\mathscr{S} : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1},0}$ and $\mathscr{S} : \varphi \mathbf{L}^{2,r}_{\varphi} \longrightarrow \mathbf{L}^{2,r}_{\varphi^{-1},0}$, $r > 0$, are invertible, where $\mathbf{L}^2_{\rho,0} = \left\{ f \in \mathbf{L}^2_{\rho} : \langle f, 1 \rangle_{\rho} = 0 \right\}$ and $\mathbf{L}^{2,r}_{\rho,0} = \mathbf{L}^{2,r}_{\rho} \cap \mathbf{L}^2_{\rho,0}$.*

In the following proposition we discuss the solvability of (19).

Proposition 2. *Assume that $\psi_j \in \mathbf{C}^3[-1,1]$. Then,*

(a) *the operator $\mathscr{A}_0 : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ has a trivial null space, i.e.,*

$$N(\mathscr{A}_0) = \left\{ f \in \mathbf{L}^2_{\varphi^{-1}} : \mathscr{A}_0 f = 0 \right\} = \{0\};$$

(b) *if $\psi_1(t)$ is not a constant function, then equation* (19) *possesses a unique solution $(\Gamma_0^*, \beta, \gamma) \in \mathbf{V}_{\gamma_0} \times \mathbb{R}^2$;*

(c) *if $\psi_1(t)$ is not a constant function, then Problem* (P) *is uniquely solvable.*

Proof. Let $f_0 \in \mathbf{L}^2_{\varphi^{-1}}$ and $\mathscr{A}_0 f_0 = 0$. Hence, $\mathscr{S} f_0 = \mathscr{K}_0 f_0 \in \mathbf{C}^1[-1,1] \subset \mathbf{L}^{2,1}_{\varphi^{-1}}$, due to Lemma 3. By Lemma 7, we get $\mathscr{K}_0 f_0 \in \mathbf{L}^{2,1}_{\varphi^{-1},0}$ and, consequently, $f_0 \in \varphi \mathbf{L}^{2,1}_{\varphi} = \mathbf{V}$. On the other hand, due to (12) and (20) (cf. also the proof of Lemma 6), we have

$$0 < \langle \mathscr{A} f, f \rangle = -\langle \mathscr{A}_0 f, \mathscr{D} f \rangle \quad \forall f \in \mathbf{V} \setminus \{0\}.$$

This implies $f_0 = 0$, and (a) is proved.

Since, by Lemma 7, the operator $\mathscr{S} : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ is Fredholm with index -1 and since, due to the continuity of the function $K(s,t)$ (cf. Lemma 3), the operator $\mathscr{K}_0 : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ is compact, also the operator $\mathscr{A}_0 = -\mathscr{S} + \mathscr{K}_0 : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ is Fredholm with index -1. Hence, we conclude that the codimension of the image

$$R(\mathscr{A}_0) = \left\{ \mathscr{A}_0 f : f \in \mathbf{L}^2_{\varphi^{-1}} \right\}$$

is equal to 1. Hence, the intersection $\mathbf{W}_1 := R(\mathscr{A}_0) \cap \{\beta\psi_1 + \gamma : \beta, \gamma \in \mathbb{R}\}$ is at least one-dimensional. If this dimension is equal to 1 and if $\mathbf{W}_1 = \operatorname{span}\{\psi_0\}$, then there is a unique $\Gamma_0 \in \mathbf{L}^2_{\varphi^{-1}}$, such that $\mathscr{A}_0\Gamma_0 = \psi_0$. Again using Lemma 3, we get $\mathscr{S}\Gamma_0 = \mathscr{K}_0\Gamma_0 - \psi_0 \in \mathbf{C}^1[-1,1]$ and, consequently, $\Gamma_0 \in \mathbf{V}$. We show that $\langle \Gamma_0, \psi_1' \rangle \neq 0$. If this is not the case, then, because of Proposition 1 and Remark 1, Problem (P) has only a solution for $\gamma_0 = 0$. But, this (unique) solution is identically zero. This implies $\Gamma_0 = 0$ in contradiction to $\psi_0 \neq 0$. Hence, $\Gamma_0^* = \dfrac{\gamma_0}{\langle \Gamma_0, \psi_1' \rangle}\Gamma_0$ is the solution of (19) with $\beta\psi_1 + \gamma = \dfrac{\gamma_0}{\langle \Gamma_0, \psi_1' \rangle}\psi_0$.

To complete the proof of (b), finally we show that $\dim \mathbf{W}_1 = 2$ is not possible. Indeed, in that case $\mathbf{W}_1 = \operatorname{span}\{\psi_1, \psi_0\}$ with $\psi_0(t) = 1$, and we have (cf. the previous considerations) two linearly independent solutions $\Gamma_0^j \in \mathbf{V}$ of $\mathscr{A}\Gamma_0^j = \psi_j$ with $\langle \Gamma_0^j, \psi_1' \rangle \neq 0$, $j = 0, 1$. Hence, $\Gamma_0^{j,*} = \dfrac{\gamma_0}{\langle \Gamma_0^j, \psi_1' \rangle}\Gamma_0^j$, $j = 1, 2$ are two linearly independent solutions of (19) and, in virtue of Proposition 1, also of Problem (P) in contradiction to the uniqueness of a solution of (P).

Assertion (c) is an immediate consequence of (b), together with Proposition 1 and Remark 1.

In the case of the wing problem examined in [3] and [4], where the line ℓ is symmetric in the $x-z$ plane with respect to the z-axis and where it cannot be a vertical segment, we note that the problem formulation (19) can be simplified significantly. In particular, for it, the following result holds (see also the associated numerical method, described at the end of the next section).

Corollary 1. *If the lifting line ℓ is symmetric w.r.t. the z-axis, i.e., $\psi_1(-t) = -\psi_1(t)$ and $\psi_2(-t) = \psi_2(t)$, and if $\psi_1(t)$ is not constant, then the unique solution $\Gamma_0^* \in \mathbf{V}$ of Problem (P) is an even function. Moreover, in* (19) *we have $\gamma = 0$.*

Proof. In virtue of Proposition 1, Remark 1, and Proposition 2, there exist unique $\beta, \gamma \in \mathbb{R}$ such that $(\Gamma_0^*, \beta, \gamma) \in \mathbf{V}_{\gamma_0} \times \mathbb{R}^2$ is the unique solution of (19). By assumption, $Z_0(-t,-s) = Z_0(t,s)$ (cf. (14)). Set $g(t) = \Gamma_0^*(-t)$. From (19) it follows

$$\begin{aligned}\beta\psi_1(t) - \gamma = -\beta\psi_1(-t) - \gamma &= -\frac{1}{\pi}\int_{-1}^{1} Z_0(-t,s)(\Gamma_0^*)'(s)\,\mathrm{d}s \\ &= -\frac{1}{\pi}\int_{-1}^{1} Z_0(-t,-s)(\Gamma_0^*)'(-s)\,\mathrm{d}s = \frac{1}{\pi}\int_{-1}^{1} Z_0(t,s)g'(s)\,\mathrm{d}s,\end{aligned}$$

which means that $(g, \beta, -\gamma) = (\Gamma_0^*, \beta, \gamma)$.

Remark 2. Note that $\gamma = 0$ implies that the problem defined by (19) can be reformulated as follows: *Find $\Gamma_0^* \in \mathbf{V}$ and $\beta \in \mathbb{R}$ such that*

$$\mathscr{B}\Gamma_0^* = \beta\psi_1, \quad \left\langle \psi_1', \Gamma_0^* \right\rangle = \gamma_0. \tag{23}$$

This is only apparently a system of two equations. Indeed, since we must necessarily have $\beta \neq 0$, by introducing the new unknown $\overline{\Gamma}_0^* = \Gamma_0^*/\beta$ we obtain

$$\mathscr{B}\overline{\Gamma}_0^* = \psi_1, \quad \beta\langle\psi_1', \overline{\Gamma}_0^*\rangle = \gamma_0. \tag{24}$$

which is exactly the decoupled system that has been derived in [3]. Solving the first equation we obtain $\overline{\Gamma}_0^*$, from the second equation we get the value of β, and finally we find the solution Γ_0^*.

4 A Collocation-Quadrature Method

Here, we describe a numerical procedure for the approximate solution of equation (19). For this, we write this equation in the form (cf. (20), (21))

$$\mathscr{A}_0 f = \beta\psi_1 + \gamma, \quad (f, \beta, \gamma) \in \mathbf{V}_{\gamma_0} \times \mathbb{R}^2 \tag{25}$$

with $\mathscr{A}_0 = -\mathscr{S} + \mathscr{K}_0 : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ and

$$(\mathscr{S}f)(t) = \frac{1}{\pi} ⨍_{-1}^{1} \frac{f(s)\,\mathrm{d}s}{s-t} \quad \text{and} \quad (\mathscr{K}_0 f)(t) = \frac{1}{\pi}\int_{-1}^{1} K(s,t)f(s)\,\mathrm{d}s.$$

For any integer $n \geq 1$ we are looking for an approximate solution $(f_n, \beta_n, \gamma_n) \in R(\mathscr{P}_n) \times \mathbb{R}^2$ of (25), where $R(\mathscr{P}_n)$ is the image space of the orthoprojection $\mathscr{P}_n : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ defined by

$$\mathscr{P}_n f = \sum_{k=0}^{n-1} \langle f, U_k \rangle\, \varphi U_k,$$

where $U_k = p_k^{\varphi}$ denotes the normalized second kind Chebyshev polynomial of degree k, by solving the collocation equations

$$-(\mathscr{S}f_n)(t_{jn}) + (\mathscr{K}_n^0 f_n)(t_{jn}) = \beta_n\psi_1(t_{jn}) + \gamma_n, \quad j = 1, \ldots, n+1, \tag{26}$$

together with

$$\frac{\pi}{n+1}\sum_{i=1}^{n} \varphi(s_{in})\psi_1'(s_{in})f_n(s_{in}) = \gamma_0, \tag{27}$$

where $t_{jn} = \cos\dfrac{(2j-1)\pi}{2n+2}$ and $s_{in} = \cos\dfrac{i\pi}{n+1}$ are Chebyshev nodes of first and second kind, respectively, and where

$$(\mathscr{K}_n^0 f_n)(t) = \frac{1}{n+1}\sum_{i=1}^{n} \varphi(s_{in})K(s_{in}, t)f_n(s_{in}). \tag{28}$$

Note that $f_n(t)$ can be written, with the help of the weighted Lagrange interpolation polynomials

$$\widetilde{\ell}_{kn}^{\varphi}(t) = \frac{\varphi(t)\ell_{kn}^{\varphi}(t)}{\varphi(s_{kn})} \quad \text{with} \quad \ell_{kn}^{\varphi}(t) = \frac{U_n(t)}{(t-s_{kn})U_n'(s_{kn})}, \quad k = 1,\ldots,n,$$

in the form

$$f_n(t) = \sum_{k=1}^{n} \xi_{kn}\widetilde{\ell}_{kn}^{\varphi}(t), \quad \xi_{kn} = f_n(s_{kn}). \tag{29}$$

Let $\mathscr{L}_n^j$, $j = 1,2$, denote the interpolation operators which associate to a function $g : (-1,1) \longrightarrow \mathbb{R}$ the polynomials

$$(\mathscr{L}_n^1 g)(t) = \sum_{j=1}^{n+1} \frac{g(t_{jn})T_{n+1}(t)}{(t-t_{jn})T_{n+1}'(t_{jn})} \quad \text{and} \quad (\mathscr{L}_n^2 g)(t) = \sum_{i=1}^{n} \frac{g(s_{in})U_n(t)}{(t-s_{in})U_n'(s_{in})},$$

where $T_n = p_n^{\varphi^{-1}}$. Now, the system (26), (27) can be written as operator equation

$$\mathscr{A}_n f_n = \beta_n \mathscr{L}_n^1 \psi_1 + \gamma_n, \quad f_n \in R(\mathscr{P}_n) \tag{30}$$

together with

$$\langle \mathscr{L}_n^2 \psi_1', f_n \rangle = \gamma_0, \tag{31}$$

where $\mathscr{A}_n = -\mathscr{S}_n + \mathscr{K}_n$, $\mathscr{S}_n = \mathscr{L}_n^1 \mathscr{S} \mathscr{P}_n$, and $\mathscr{K}_n = \mathscr{L}_n^1 \mathscr{K}_n^0 \mathscr{P}_n$. The equivalence of (27) and (31) follows from the algebraic accuracy of the Gaussian rule w.r.t. the Chebyshev nodes of second kind. The assertion of the following lemma is well-known (see [9, Theorem 14.3.1]).

Lemma 8. *For all* $f \in \mathbf{C}[-1,1]$, $\lim_{n\to\infty} \left\| f - \mathscr{L}_n^1 f \right\|_{\varphi^{-1}} = 0$ *and* $\lim_{n\to\infty} \left\| f - \mathscr{L}_n^2 f \right\|_{\varphi} = 0$.

The next lemma provides convergence rates for the interpolating polynomials and will be used in the proof of Proposition 3.

Lemma 9 ([1], Theorem 3.4). *If* $r > \frac{1}{2}$, *then there exists a constant* $c > 0$ *such that, for any real* p, $0 \le p \le r$ *and all* $n \ge 1$,

(a) $\left\| f - \mathscr{L}_n^1 f \right\|_{\varphi^{-1},p} \le c n^{p-r} \|f\|_{\varphi^{-1},r}$ *for all* $f \in \mathbf{L}^{2,r}_{\varphi^{-1}}$,
(b) $\left\| f - \mathscr{L}_n^2 f \right\|_{\varphi,p} \le c n^{p-r} \|f\|_{\varphi,r}$ *for all* $f \in \mathbf{L}^{2,r}_{\varphi}$.

Lemma 10. *Let* $\psi_j \in \mathbf{C}^2[-1,1]$, $j = 1,2$. *Then,*

(a) $\lim_{n\to\infty} \|\mathscr{K}_n - \mathscr{K}_0\|_{\mathbf{L}^2_{\varphi^{-1}} \to \mathbf{L}^2_{\varphi^{-1}}} = 0$,
(b) *there exist constants* $\eta > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\|\mathscr{A}_n f_n\|_{\varphi^{-1}} \ge \eta \, \|f_n\|_{\varphi^{-1}} \quad \forall f_n \in R(\mathscr{P}_n), \, \forall n \ge n_0.$$

Proof. At first, recall that the operator $\mathscr{A}_0 : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ is Fredholm with index -1 (cf. the proof of Proposition 2). By Banach's theorem, the operator

$\mathscr{A}_0 : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \left(R(\mathscr{A}_0), \|.\|_{\varphi^{-1}}\right)$ has a bounded inverse. Hence, there is a constant $\eta_0 > 0$ with

$$\|\mathscr{A}_0 f\|_{\varphi^{-1}} \geq \eta_0 \|f\|_{\varphi^{-1}} \quad \forall\, f \in \mathbf{L}^2_{\varphi^{-1}}. \tag{32}$$

By definition of $\mathscr{K}_n^0$ and in virtue of the algebraic accuracy of the Gaussian rule, for $f_n \in R(\mathscr{P}_n)$ we have

$$(\mathscr{K}_n^0 f_n)(t) = \frac{1}{\pi}\int_{-1}^{1} \mathscr{L}_n^2 \left[K(.,t)\varphi^{-1} f_n\right](s)\varphi(s)\,\mathrm{d}s = \frac{1}{\pi}\int_{-1}^{1} \mathscr{L}_n^2 \left[K(.,t)\right](s) f_n(s)\,\mathrm{d}s,$$

which implies

$$\left\|\left(\mathscr{K}_n^0 - \mathscr{K}_0\right)\mathscr{P}_n f\right\|_\infty \leq \frac{1}{\pi}\sup\left\{\left\|\mathscr{L}_n^2\left[K(.,t)\right] - K(.,t)\right\|_\varphi : -1 \leq t \leq 1\right\}\|f\|_{\varphi^{-1}},$$

where $\|.\|_\infty$ is the norm in $\mathbf{C}[-1,1]$, i.e., $\|f\|_\infty = \max\{|f(t)| : -1 \leq t \leq 1\}$. Since, due to Lemma 8 and the principle of uniform boundedness, the operator sequence $\mathscr{L}_n^1 : \mathbf{C}[-1,1] \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ is uniformly bounded, the last estimate together with Lemma 8 (applied to $\mathscr{L}_n^2$) leads to

$$\lim_{n\to\infty}\left\|\mathscr{K}_n - \mathscr{L}_n^1 \mathscr{K}_0 \mathscr{P}_n\right\|_{\mathbf{L}^2_{\varphi^{-1}} \to \mathbf{L}^2_{\varphi^{-1}}} = 0.$$

Again Lemma 8, the strong convergence of $\mathscr{P}_n = \mathscr{P}_n^* \longrightarrow \mathscr{I}$ (the identity operator), and the compactness of the operator $\mathscr{K}_0 : \mathscr{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{C}[-1,1]$ give us $\lim\limits_{n\to\infty}\left\|\mathscr{L}_n^1 \mathscr{K}_0 \mathscr{P}_n - \mathscr{K}_0\right\|_{\mathbf{L}^2_{\varphi^{-1}} \to \mathbf{L}^2_{\varphi^{-1}}} = 0$, and (a) is proved.

Formula (22) implies the relation $\mathscr{S}_n = \mathscr{S}\mathscr{P}_n$, from which, together with (a), we conclude

$$\|(\mathscr{A}_n - \mathscr{A}_0) f_n\|_{\varphi^{-1}} \leq \alpha_n \|f_n\|_{\varphi^{-1}} \quad \forall\, f_n \in R(\mathscr{P}_n), \tag{33}$$

where $\alpha_n \longrightarrow 0$. Together with (32), this leads to (b).

Proposition 3. *Assume $\psi_j \in \mathbf{C}^3[-1,1]$, $j = 1,2$, $\gamma_0 \neq 0$, and let $\psi_1(t)$ be not constant. Then, for all sufficiently large n (say $n \geq n_0$), there exists a unique solution $(f_n^*, \beta_n^*, \gamma_n^*) \in R(\mathscr{P}_n) \times \mathbb{R}^2$ of* (30), (31). *Moreover,*

$$\lim_{n\to\infty}\sqrt{\|f_n^* - f^*\|^2_{\varphi^{-1}} + |\beta_n^* - \beta^*|^2 + |\gamma_n^* - \gamma^*|^2} = 0, \tag{34}$$

where (f^*, β^*, γ^*) is the unique solution of (25). *If $\psi_j \in \mathbf{C}^m[-1,1]$, $j = 1,2$ for some integer $m > 2$, then*

$$\sqrt{\|f_n^* - f^*\|^2_{\varphi^{-1}} + |\beta_n^* - \beta^*|^2 + |\gamma_n^* - \gamma^*|^2} \leq c\, n^{2-m} \tag{35}$$

with a constant $c > 0$ independent of n.

Proof. Due to the Fredholmness of $\mathscr{A}_0 : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ with index -1 and due to $N(\mathscr{A}_0) = \{0\}$ (Proposition 2, (a)), we have $\mathbf{L}^2_{\varphi^{-1}} = R(\mathscr{A}_0) \oplus \operatorname{span}\{g_0\}$ (direct orthogonal sum w.r.t. $\langle .,. \rangle_{\varphi^{-1}}$) for some $g_0 \in \mathbf{L}^2_{\varphi^{-1}}$ with $\|g_0\|_{\varphi^{-1}} = 1$.

By $\mathbf{H}$ we denote the Hilbert space of all pairs $(f, \delta) \in \mathbf{L}^2_{\varphi^{-1}} \times \mathbb{R}$ equipped with the inner product $\langle (f_1, \delta_1), (f_2, \delta_2) \rangle_{\mathbf{H}} = \langle f_1, f_2 \rangle_{\varphi^{-1}} + \delta_1 \delta_2$. For a continuous function $g_1 \in \mathbf{C}[-1,1]$ with $\langle g_1, g_0 \rangle_{\varphi^{-1}} \neq 0$ and for $n \in \mathbb{N}$, define the linear and bounded operators

$$\mathscr{B}_0 : \mathbf{H} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}, \quad (f, \delta) \mapsto \mathscr{A}_0 f - \delta g_1$$

and

$$\mathscr{B}_n : \mathbf{H} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}, \quad (f, \delta) \mapsto \mathscr{A}_n f - \delta \mathscr{L}^1_n g_1.$$

Let us consider the auxiliary problems

$$\mathscr{B}_0(f, \delta) = g \in \mathbf{C}[-1,1], \quad (f, \delta) \in \mathbf{L}^2_{\varphi^{-1}} \times \mathbb{R} \tag{36}$$

and

$$\mathscr{B}_n(f_n, \delta_n) = \mathscr{L}^1_n g, \quad (f_n, \delta_n) \in R(\mathscr{P}_n) \times \mathbb{R}. \tag{37}$$

An immediate consequence of (33) is the relation

$$\begin{aligned} \|\mathscr{B}_n(f_n, \delta) - \mathscr{B}_0(f_n, \delta)\|_{\varphi^{-1}} &\leq \alpha_n \|f_n\|_{\varphi^{-1}} + |\delta| \left\| \mathscr{L}^1_n g_1 - g_1 \right\|_{\varphi^{-1}} \\ &\leq \beta_n \|(f_n, \delta)\|_{\mathbf{H}} \quad \forall (f_n, \delta) \in R(\mathscr{P}_n) \times \mathbb{R}, \end{aligned} \tag{38}$$

where $\beta_n = \sqrt{\alpha_n^2 + \|\mathscr{L}^1_n g_1 - g_1\|^2_{\varphi^{-1}}} \longrightarrow 0$. Equation (36) is uniquely solvable, since $R(\mathscr{A}_0) = \left\{ f \in \mathbf{L}^2_{\varphi^{-1}} : \langle f, g_0 \rangle_{\varphi^{-1}} = 0 \right\}$ and, consequently, the part $\delta^* \in \mathbb{R}$ of the solution (f^*, δ^*) of (36) is uniquely determined by the condition

$$\langle g + \delta g_1, g_0 \rangle_{\varphi^{-1}} = 0, \quad \text{i.e.,} \quad \delta^* = -\frac{\langle g, g_0 \rangle_{\varphi^{-1}}}{\langle g_1, g_0 \rangle_{\varphi^{-1}}},$$

and $f^* \in \mathbf{L}^2_{\varphi^{-1}}$ is the unique solution (cf. Proposition 1,(a)) of $\mathscr{A}_0 f = g + \delta^* g_1$. Hence, in virtue of Banach's theorem, the operator $\mathscr{B}_0 : \mathbf{H} \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ is boundedly invertible, which implies that there is a constant $\eta_1 > 0$, such that

$$\|\mathscr{B}_0(f, \delta)\|_{\varphi^{-1}} \geq \eta_1 \|(f, \delta)\|_{\mathbf{H}} \quad \forall (f, \delta) \in \mathbf{H}. \tag{39}$$

Putting this together with (38), we can state that there is a number $n_0 \in \mathbb{N}$, such that

$$\|\mathscr{B}_n(f_n, \delta)\|_{\varphi^{-1}} \geq \frac{\eta_1}{2} \|(f_n, \delta)\|_{\mathbf{H}} \quad \forall (f_n, \delta) \in R(\mathscr{P}_n) \times \mathbb{R}, \ \forall n \geq n_0. \tag{40}$$

This implies that, for $n \geq n_0$, the map $\mathscr{B}_n : R(\mathscr{P}_n) \times \mathbb{R} \longrightarrow \operatorname{span}\left\{ T_j : j = 0, 1, \ldots, n \right\}$ is a bijection, such that (37) is uniquely solvable for all $n \geq n_0$. Moreover, if (f^*_n, δ^*_n)

is the solution of (37), then

$$\begin{aligned}
&\|(f_n^*,\delta_n^*)-(\mathscr{P}_nf^*,\delta^*)\|_{\mathbf{H}}\\
&\quad\le\frac{2}{\eta_1}\left\|\mathscr{L}_n^1g-\mathscr{B}_n(\mathscr{P}_nf^*,\delta^*)\right\|_{\varphi^{-1}}\\
&\quad\le\frac{2}{\eta_1}\Big(\left\|\mathscr{L}_n^1g-\mathscr{B}_0(\mathscr{P}_nf^*,\delta^*)\right\|_{\varphi^{-1}}+\|(\mathscr{B}_0-\mathscr{B}_n)(\mathscr{P}_nf^*,\delta^*)\|_{\varphi^{-1}}\Big)\\
&\quad\le\frac{2}{\eta_1}\Big(\left\|\mathscr{L}_n^1g-\mathscr{B}_0(\mathscr{P}_nf^*,\delta^*)\right\|_{\varphi^{-1}}+\beta_n\|(\mathscr{P}_nf^*,\delta^*)\|_{\mathbf{H}}\Big),
\end{aligned}\tag{41}$$

which implies

$$\lim_{n\to\infty}\|(f_n^*,\delta_n^*)-(f^*,\delta^*)\|_{\mathbf{H}}=0.\tag{42}$$

Now, let $(f^*,\beta^*,\gamma^*)\in\mathbf{V}_{\gamma_0}\times\mathbb{R}^2$ be the unique solution of (25) (cf. Proposition 2,(b)). There exist $\beta_1^*,\gamma_1^*\in\mathbb{R}$ such that $\langle g_1,g\rangle_{\varphi^{-1}}=0$ and $\|g_1\|_{\varphi^{-1}}=1$, where $g=\beta^*\psi_1+\gamma^*$ and $g_1=\beta_1^*\psi_1+\gamma_1^*$. Because of

$$\dim\left(R(\mathscr{A}_0)\cap\{\beta\psi_1+\gamma:\beta,\gamma\in\mathbb{R}\}\right)=1$$

(cf. the proof of Proposition 2), we have $g_1\notin R(\mathscr{A}_0)$, i.e., $\langle g_1,g_0\rangle_{\varphi^{-1}}\ne0$. With these notations, $(f^*,0)\in\mathbf{L}^2_{\varphi^{-1}}\times\mathbb{R}$ is the unique solution of (36). Taking into account the previous considerations, we conclude that, for all sufficiently large n, there is a unique $(f_n^1,\delta_n^1)\in R(\mathscr{P}_n)\times\mathbb{R}$ satisfying

$$\mathscr{A}_nf_n^1-\delta_n^1\mathscr{L}_n^1(\beta_1^*\psi_1+\gamma_1^*)=\mathscr{L}_n^1(\beta^*\psi_1+\gamma^*)$$

or equivalently

$$\mathscr{A}_nf_n^1=(\beta^*+\delta_n^1\beta_1^*)\mathscr{L}_n^1\psi_1+\gamma^*+\delta_n^1\gamma_1^*,$$

where, due to (42), $\left\|f_n^1-f^*\right\|_{\varphi^{-1}}\longrightarrow0$ and $\delta_n^1\longrightarrow0$. It follows

$$\langle\mathscr{L}_n^2\psi_1',f_n^1\rangle=\langle\mathscr{L}_n^2\psi_1',\varphi^{-1}f_n^1\rangle_\varphi\longrightarrow\langle\psi_1',\varphi^{-1}f^*\rangle_\varphi=\langle\psi_1',f^*\rangle=\gamma_0.$$

Consequently, for all sufficiently large n, $\langle\mathscr{L}_n^2\psi_1',f_n^1\rangle\ne0$ and $(f_n^*,\beta_n^*,\gamma_n^*)$ with

$$f_n^*=\frac{\gamma_0f_n^1}{\langle\mathscr{L}_n^2\psi_1',f_n^1\rangle},\quad\beta_n^*=\frac{\gamma_0(\beta^*+\delta_n^1\beta_1^*)}{\langle\mathscr{L}_n^2\psi_1',f_n^1\rangle},\quad\gamma_n^*=\frac{\gamma_0(\gamma^*+\delta_n^1\gamma_1^*)}{\langle\mathscr{L}_n^2\psi_1',f_n^1\rangle}$$

is a solution of (30), (31). This solution is unique, since (f_n^1,δ_n^1) was uniquely determined. Furthermore,

$$f_n^*\longrightarrow f^*\text{ in }\mathbf{L}^2_{\varphi^{-1}}\quad\text{and}\quad\beta_n^*\longrightarrow\beta^*,\ \gamma_n^*\longrightarrow\gamma^*,$$

and (34) follows. To prove the error estimate (35), first we recall that $\psi_j\in\mathbf{C}^m[-1,1]$, $j=1,2$ for some $m>2$ implies, due to Lemma 3, the continuity of the partial derivatives $\dfrac{\partial^k K(s,t)}{\partial t^k}$, $k = 1,\ldots,m-2$, for $(s,t) \in [-1,1]^2$. Consequently,

$$-\mathscr{S} f^* = \beta^* \psi_1 + \gamma^* - \mathscr{K}_0 f^* \in \mathbf{C}^{m-2}[-1,1] \subset \mathbf{L}^{2,m-2}_{\varphi^{-1}},$$

i.e., in virtue of Lemma 7, $f^* \in \varphi \mathbf{L}^{2,m-2}_{\varphi}$. Taking into account the uniform boundedness of $\mathscr{L}_n^1 : \mathbf{C}[-1,1] \longrightarrow \mathbf{L}^2_{\varphi^{-1}}$ (see Lemma 8) and Lemma 9, we get, for all $f_n \in R(\mathscr{P}_n)$,

$$\begin{aligned}
&\|(\mathscr{K}_n - \mathscr{K}_0) f_n\|_{\varphi^{-1}} \\
&\quad \le \left\|\mathscr{L}_n^1(\mathscr{K}_n^0 - \mathscr{K}_0) f_n\right\|_{\varphi^{-1}} + \left\|(\mathscr{L}_n^1 \mathscr{K}_0 - \mathscr{K}_0) f_n\right\|_{\varphi^{-1}} \\
&\quad \le \sup\left\{ \left\|\mathscr{L}_n^2 K(.,t) - K(.,t)\right\|_{\varphi} : -1 \le t \le 1 \right\} \|f_n\|_{\varphi^{-1}} + \left\|(\mathscr{L}_n^1 \mathscr{K}_0 - \mathscr{K}_0) f_n\right\|_{\varphi^{-1}} \\
&\quad \le c\, n^{1-m} \left(\sup\left\{ \|K(.,t)\|_{\varphi,m-1} : -1 \le t \le 1 \right\} \|f_n\|_{\varphi^{-1}} + \|\mathscr{K}_0 f_n\|_{\varphi^{-1},m-1} \right) \\
&\quad \le c n^{1-m} \|f_n\|_{\varphi^{-1}},
\end{aligned}$$

where we have also used that $\mathscr{K}_0 : \mathbf{L}^2_{\varphi^{-1}} \longrightarrow \mathbf{C}^{m-2}[-1,1] \subset \mathbf{L}^{2,m-2}_{\varphi^{-1}}$ is bounded (cf. [1, Lemma 4.2]). Hence, in (33) and (38) we have $\alpha_n = \mathscr{O}(n^{2-m})$ and, since $g_1 = \beta_1^* \psi_1 + \gamma^* \in \mathbf{L}^{2,m}_{\varphi^{-1}}$, also $\beta_n = \mathscr{O}(n^{2-m})$. From (41) and $g = \beta^* \psi_1 + \gamma^* \in \mathbf{L}^{2,m}_{\varphi^{-1}}$ we obtain the bound

$$\begin{aligned}
&\left\|(f_n^1, \delta_n^1) - (\mathscr{P}_n f^*, 0)\right\|_{\mathbf{H}} \\
&\quad \le \frac{2}{\eta_1} \left(\left\|\mathscr{L}_n^1 g - g\right\|_{\varphi^{-1}} + \|\mathscr{A}_0\|_{\mathbf{L}^2_{\varphi^{-1}} \to \mathbf{L}^2_{\varphi^{-1}}} \|f^* - \mathscr{P}_n f^*\|_{\varphi^{-1}} + \beta_n \|f^*\|_{\varphi^{-1}} \right) \\
&\quad \le c\, n^{2-m}.
\end{aligned}$$

Now, (35) easily follows.

Remark 3. From the proof of Proposition 3 it is seen that the first assertion including (34) remains true if the assumption $\psi_j \in \mathbf{C}^3[-1,1]$ is replaced by $\psi_j \in \mathbf{C}^2[-1,1]$ together with $\dim N(\mathscr{A}_0) = 0$ (cf. Proposition 2).

5 Implementation Features

Let us discuss some computational aspects. Because of $f_n \in R(\mathscr{P}_n)$ we have, taking into account (22) and $T_{n+1}(t_{jn}) = 0$,

$$
\begin{aligned}
(\mathscr{S} f_n)(t_{jn}) &= \sum_{k=1}^{n} \frac{f_n(s_{kn})}{\varphi(s_{kn})U_n'(s_{kn})} \frac{1}{\pi} \int_{-1}^{1} \frac{\varphi(s)U_n(s)}{(s-s_{kn})(s-t)}\,\mathrm{d}s \\
&= \sum_{k=1}^{n} \frac{f_n(s_{kn})}{\varphi(s_{kn})U_n'(s_{kn})} \frac{1}{\pi} \int_{-1}^{1} \left(\frac{1}{s-s_{kn}} - \frac{1}{s-t_{jn}} \right) \varphi(s)U_n(s)\,\mathrm{d}s\, \frac{1}{s_{kn}-t_{jn}} \\
&= -\sum_{k=1}^{n} \frac{T_{n+1}(s_{kn})}{\varphi(s_{kn})U_n'(s_{kn})} \frac{f_n(s_{kn})}{s_{kn}-t_{jn}} = \sum_{k=1}^{n} \frac{\varphi(s_{kn})}{n+1} \frac{f_n(s_{kn})}{s_{kn}-t_{jn}}.
\end{aligned}
$$

From this, the following expression is obtained:

$$
-(\mathscr{S} f_n)(t_{jn}) + (\mathscr{K}_n^0 f_n)(t_{jn}) = \frac{1}{n+1} \sum_{k=1}^{n} \varphi(s_{kn}) Y_0(s_{kn}, t_{jn}) f_n(s_{kn}), \quad j = 1, \ldots, n+1
$$

(cf. (11), (28), and (26)). Thus, to find the solution (f_n, β_n, γ_n) of (26), (27), we have to solve the algebraic linear system of equations

$$
\mathbb{A}_n \xi_n = \eta_n, \tag{43}
$$

where $\eta_n = \left[\, \eta_{jn} \,\right]_{j=1}^{n+2} = \left[\, 0 \ldots 0 \ \gamma_0 \,\right]^T \in \mathbb{R}^{n+2}$ is given and $\xi_n = \left[\, \xi_{kn} \,\right]_{k=1}^{n+2} = \left[\, f_n(s_{1n}) \ \ldots \ f_n(s_{nn}) \ \beta_n \ \gamma_n \,\right]^T \in \mathbb{R}^{n+2}$ is the vector we are looking for, and where the matrix $\mathbb{A}_n = \left[\, a_{jk} \,\right]_{j,k=1}^{n+2}$ is defined by

$$
\begin{aligned}
& a_{jk} = \frac{\varphi(s_{kn}) Y_0(s_{kn}, t_{jn})}{n+1}, \quad j = 1, \ldots, n+1,\ k = 1, \ldots, n, \\
& a_{j,n+1} = -\psi_1(t_{jn}), \ \ a_{j,n+2} = -1, \quad j = 1, \ldots, n+1, \\
& a_{n+2,k} = \frac{\pi \varphi(s_{kn}) \psi_1'(s_{kn})}{n+1}, \quad k = 1, \ldots, n, \qquad a_{n+2,n+1} = a_{n+2,n+2} = 0.
\end{aligned}
$$

In the case of a symmetric wing, the above numerical method can be significantly simplified. Indeed, it turns out that the ideas used in the proof of Corollary 1, which have led to (24), also work for the discrete system (43). This can be shown as follows.

First, note that in this case we have

$$
Y_0(-t, s) = -Y_0(t, -s) \quad \text{and} \quad Y_0(-t, -s) = -Y_0(t, s). \tag{44}
$$

Then, let $n = 2m$ be sufficiently large, let

$$
\xi_n^* = \left[\, f_n^*(s_{1n}) \ \ldots \ f_n^*(s_{nn}) \ \beta_n^* \ \gamma_n^* \,\right]^T \in \mathbb{R}^{n+2}
$$

be the unique solution of (43), and define $\widetilde{\xi}_n = \left[\, f_n^*(s_{nn}) \ \ldots \ f_n^*(s_{1n}) \ \beta_n^* \ -\gamma_n^* \,\right]^T$. Then, the jth entry of $\mathbb{A}_n \widetilde{\xi}_n$ equals

$$\begin{aligned}
&\left(\mathbb{A}_n\widetilde{\xi}_n\right)_j \\
&= \sum_{k=1}^{n} \frac{\varphi(s_{kn})Y_0(s_{kn},t_{jn})}{n+1} f_n^*(s_{n+1-k,n}) + \beta_n^*\psi_1(t_{jn}) - \gamma_n^* \\
&= -\sum_{k=1}^{n} \frac{\varphi(s_{n+1-k,n})Y_0(s_{n+1-k,n},t_{n+2-j,n})}{n+1} f_n^*(s_{n+1-k,n}) + \beta_n^*\psi_1(t_{n+2-j,n}) + \gamma_n^* = 0
\end{aligned}$$

for $j = 1,\dots,n+1$, and

$$\begin{aligned}
\left(\mathbb{A}_n\widetilde{\xi}_n\right)_{n+2} &= \sum_{k=1}^{n} \frac{\varphi(s_{kn})\psi_1'(s_{kn})}{n+1} f_n^*(s_{n+1-k,n}) \\
&= \sum_{k=1}^{n} \frac{\varphi(s_{n+1-k,n})\psi_1'(s_{n+1-k,n})}{n+1} f_n^*(s_{n+1-k,n}) = \gamma_0
\end{aligned}$$

for $j = n+2$, where we have taken into account (44) and the identities $s_{n+1-k,n} = s_{kn}$ and $t_{n+2-j,n} = t_{jn}$. This means that $\widetilde{\xi}_n$ also solves (43) and, due to the solution uniqueness, we have only to compute the $m+1 = n/2+1$ values $\xi_{kn}^* = f_n^*(s_{kn}) = f_n^*(s_{n+1-k,n})$, $k = 1,\dots,m$, and $\xi_{n+1,n}^* = \beta_n^*$, while $\xi_{n+1,n+1}^* = \gamma_n^* = 0$. The system we have to solve can now be written in the form

$$\sum_{k=1}^{m} b_{jk}\xi_{kn} = \beta_n\psi_1(t_{jn}), \quad j = 1,\dots,m, \qquad \sum_{k=1}^{m} \frac{2\varphi(s_{kn})\psi_1'(s_{kn})}{n+1}\,\xi_{kn} = \gamma_0, \tag{45}$$

where $b_{jk} = a_{jk} + a_{j,n+1-k}$, and where we have used the properties that, for $j = 1,\dots,m$, the $(n+2-j)$th equation in (43) is identical to the jth equation and that the $(m+1)$th equation is automatically fulfilled ($b_{m+1,k} = 0$ and $\psi_1(t_{m+1,n}) = 0$, since $t_{m+1,n} = 0$), in virtue of the assumed symmetries. Of course, (45) is, with $\bar{\xi}_n = \xi_{kn}/\beta_n$, equivalent to

$$\sum_{k=1}^{m} b_{jk}\bar{\xi}_{kn} = \psi(t_{jn}), \quad j = 1,\dots,m, \qquad \beta_n \sum_{k=1}^{m} \frac{2\varphi(s_{kn})\psi_1'(s_{kn})}{n+1}\,\bar{\xi}_{kn} = \gamma_0,$$

since $\beta^* \neq 0$ and $\beta_n^* \longrightarrow \beta^*$ (cf. Remark 2 with (23) and (24)), and thus for all sufficiently large n we have $\beta_n^* \neq 0$. This latter system is precisely the method used in [3], for which we have now proved its convergence and given an error estimate. A similar simplification can be obtained also for $n = 2m+1$.

Finally, we discuss the question if, under the assumptions of Proposition 3, the condition numbers of the matrices $\mathbb{A}_n$ are uniformly bounded or if it is necessary to apply a preconditioning to $\mathbb{A}_n$. Note that, under the assumptions of Proposition 3, the operator sequence $\mathscr{B}_n : R(\mathscr{P}_n)\times\mathbb{R}^2 \longrightarrow \mathbf{P}_n\times\mathbb{R}$ ($\mathbf{P}_n$ being the set of all real algebraic polynomials of degree less than or equal to n) defined by

$$\mathscr{B}_n(f_n,\beta,\gamma) = \left(\mathscr{A}_n f_n - \beta\mathscr{L}_n^1\psi_1 - \gamma, \langle\mathscr{L}_n^2\psi_1', f_n\rangle\right)$$

(cf. (30) and (31)) is a bounded and stable one, i.e., the norms of $\mathscr{B}_n$ and of $\mathscr{B}_n^{-1}$ (which exist for all sufficiently large n) are uniformly bounded (as a consequence of Proposition 3 together with Lemma 10). Hereby, the norms in $\mathbf{H}_n^1 := R(\mathscr{P}_n) \times \mathbb{R}^2$ and $\mathbf{H}_n^2 := \mathbf{P}_n \times \mathbb{R}$ are given by

$$\|(f_n,\beta,\gamma)\|_{\mathbf{H}_n^1} = \sqrt{\|f_n\|_{\varphi^{-1}}^2 + |\beta|^2 + |\gamma|^2} \quad \text{and} \quad \|(p_n,\delta)\|_{\mathbf{H}_n^2} = \sqrt{\|p_n\|_{\varphi^{-1}}^2 + |\delta|^2},$$

respectively. Set $\omega_n = \sqrt{\dfrac{\pi}{n+1}}$ and define the operators

$$\mathscr{E}_n : \mathbf{H}_n^1 \longrightarrow \mathbb{R}^{n+2}, \quad (f_n,\beta,\gamma) \mapsto \big(\omega_n f_n(s_{1n}),\ldots,\omega_n f_n(s_{nn}),\beta,\gamma\big)$$

and

$$\mathscr{F}_n : \mathbf{H}_n^2 \longrightarrow \mathbb{R}^{n+2}, \quad (p_n,\delta) \mapsto \big(\omega_n p_n(t_{1n}),\ldots,\omega_n p_n(t_{n+1,n}),\delta\big),$$

where the space $\mathbb{R}^{n+2}$ is equipped with the usual Euclidean inner product. These operators are unitary ones. To prove this, we recall the representation (29) of $f_n(t)$, in order to see that, for all $(f_n,\beta,\gamma) \in \mathbf{H}_n^1$ and $(\xi_1,\ldots,\xi_{n+2}) \in \mathbb{R}^{n+2}$,

$$\begin{aligned}
\langle \mathscr{E}_n(f_n,\beta,\gamma),(\xi_1,\ldots,\xi_{n+2})\rangle &= \omega_n \sum_{k=1}^n f_n(s_{kn})\xi_k + \beta\xi_{n+1} + \gamma\xi_{n+2} \\
&= \int_{-1}^1 f_n(s) \frac{1}{\omega_n} \sum_{k=1}^n \xi_k \widetilde{\ell}_{kn}^{\varphi}(s)\,\mathrm{d}s + \beta\xi_{n+1} + \gamma\xi_{n+2} \\
&= \langle (f_n,\beta,\gamma), \mathscr{E}_n^{-1}(\xi_1,\ldots,\xi_{n+2})\rangle_{\mathbf{H}_n^1}.
\end{aligned}$$

Analogously, one can show that

$$\langle \mathscr{F}_n(p_n,\delta),(\eta_1,\ldots,\eta_{n+2})\rangle = \langle (p_n,\delta), \mathscr{F}_n^{-1}(\eta_1,\ldots,\eta_{n+2})\rangle_{\mathbf{H}_n^2}$$

holds true for all $(p_n,\delta) \in \mathbf{H}_n^2$ and $(\eta_1,\ldots,\eta_{n+2}) \in \mathbb{R}^{n+2}$. As a consequence we get, that an appropriate matrix $\mathbb{B}_n = \big[\, b_{jk} \,\big]_{j,k=1}^{n+2}$ can be defined by

$$\begin{aligned}
&\mathscr{B}_n(\xi_1,\ldots,\xi_{n+2}) = \mathscr{F}_n \mathscr{B}_n \mathscr{E}_n^{-1}(\xi_1,\ldots,\xi_{n+2}) = \mathscr{E}_n \mathscr{B}_n \left(\omega_n^{-1} \sum_{k=1}^n \xi_k \widetilde{\ell}_{kn}^{\varphi}, \xi_{n+1}, \xi_{n+2} \right) \\
&= \mathscr{E}_n \left(\omega_n^{-1} \sum_{k=1}^n \xi_k \mathscr{A}_n \widetilde{\ell}_{kn}^{\varphi} - \xi_{n+1} \mathscr{L}_n^1 \psi_1 - \xi_{n+2}, \omega_n \sum_{k=1}^n \varphi(s_{kn}) \psi_1'(s_{kn}) \xi_k \right) \\
&= \left(\left[\sum_{k=1}^n \left(\mathscr{A}_n \widetilde{\ell}_{kn}^{\varphi} \right)(t_{jn}) \xi_k - \omega_n \psi_1(t_{jn}) \xi_{n+1} - \omega_n \xi_{n+2} \right]_{j=1}^{n+1}, \omega_n \sum_{k=1}^n \varphi(s_{kn}) \psi_1'(s_{kn}) \xi_k \right) \\
&= \left(\left[\sum_{k=1}^n a_{jk} \xi_k + \omega_n a_{j,n+1} \xi_{n+1} + \omega_n a_{j,n+2} \xi_{n+2} \right]_{j=1}^{n+1}, \sum_{k=1}^n \omega_n^{-1} a_{n+2,k} \xi_k \right),
\end{aligned}$$

i.e., $b_{jk} = a_{jk}$ for $j = 1,\dots,n+1$, $k = 1,\dots,n$, $b_{jk} = \omega_n a_{jk}$ for $j = 1,\dots,n+1$, $k = n+1, n+2$, and $b_{n+2,k} = \omega_n^{-1} a_{n+2,k}$, $k = 1,\dots,n$. This means that

$$\mathbb{B}_n = \mathbb{F}_n \mathbb{A}_n \mathbb{E}_n^{-1} \quad \text{with} \quad \mathbb{E}_n = \operatorname{diag}\left[\, 1 \ \dots \ 1 \ \omega_n^{-1} \ \omega_n^{-1} \,\right], \ \mathbb{F}_n = \operatorname{diag}\left[\, 1 \ \dots \ 1 \ \omega_n^{-1} \,\right],$$

and we can solve the system $\mathbb{B}_n \widetilde{\xi} = \widetilde{\eta}$ instead of $\mathbb{A}_n \xi = \eta$, where $\widetilde{\eta} = \mathbb{F}_n \eta$ and $\widetilde{\xi} = \mathbb{E}_n \xi$.

Therefore, in the following numerical examples we can check the stability of the method by computing the condition number of the matrix $\mathbb{B}_n$ w.r.t. the Euclidean norm, which is equal to the quotient $\frac{s_{\max}(\mathbb{B}_n)}{s_{\min}(\mathbb{B}_n)}$ of its biggest and its smallest singular values. Moreover, the left hand side in (35) can be approximated by the following discretization of it

$$\mathrm{err} = \sqrt{\frac{\pi}{N+1} \sum_{k=1}^{N} \left[f_n^*(s_{kN}) - f_N^*(s_{kN}) \right]^2 + \left| \beta_n^* - \beta_N^* \right|^2 + \left| \gamma_n^* - \gamma_N^* \right|^2} \qquad (46)$$

with $N >> n$.

6 Numerical Examples

To test the numerical method, we have proposed, and the associated convergence estimate (35), we have considered four simple curves. The first one is the following non symmetric part of the unit circle:

$$\psi_1(t) = \cos\left(\frac{\pi}{8}(3t+13)\right), \ \ \psi_2(t) = \sin\left(\frac{\pi}{8}(3t+13)\right), \ \ -1 \le t \le 1.$$

The second one is a symmetric part of the ellipse having semi-axis $a = 1, b = 0.2$ and centered at the point $(0, b)$, given by:

$$\psi_1(t) = a\cos\left((\frac{\pi}{2} + 0.01)t + 3\frac{\pi}{2}\right), \ \ \psi_2(t) = b\sin\left((\frac{\pi}{2} + 0.01)t + 3\frac{\pi}{2}\right), \ \ -1 \le t \le 1.$$

The third one is the non symmetric C^3-continuous curve

$$\psi_1(t) = t, \ -1 \le t \le 1, \quad \psi_2(t) = \begin{cases} \frac{t^4}{4}\ , & -1 \le t \le 0, \\ \frac{t^4}{2}\ , & 0 < t \le 1, \end{cases}$$

while the last one is the (non symmetric and C^2) open curve defined by the following natural (smooth) cubic spline:

$$\psi_1(t) = t, \ -1 \le t \le 1,$$

$$\psi_2(t) = \begin{cases} \frac{a+b}{4}(1+t)^3 - \left(a + \frac{a+b}{4}\right)(1+t) + a \ , \ -1 \le t \le 0, \\ \frac{a+b}{4}(1-t)^3 + \left(b + \frac{a+b}{4}\right)t - \frac{a+b}{4} \quad , \quad 0 < t \le 1, \end{cases}$$

where we have chosen $a = 0.1, b = 0.25$.

In the following tables we report the (global) error, defined by (46), and the errors $|\beta_N^* - \beta_n^*|$ and $|\gamma_N^* - \gamma_n^*|$, we have obtained for some values of n and N. In all examples, we take $\gamma_0 = -1$ (cf. (8)). Moreover, in the last two tables we also present some values $n^r * \text{err}$ for an appropriate r in order to determine the convergence rate, where err is given by (46). We can see that the convergence rate is higher than that forecasted by Proposition 3 (remember that ψ is C^3 in Example 3 and C^2 in Example 4).

Table 1 Example 1: non symmetric circular arc, $N = 256$

n	(46)	β_n^*	γ_n^*	$\lvert\beta_N^* - \beta_n^*\rvert$	$\lvert\gamma_N^* - \gamma_n^*\rvert$	cond($\mathbb{B}_n$)	cond($\mathbb{A}_n$)
4	8.99e-04	-0.6926674	0.1832556	6.68e-06	1.77e-06	2.5770	3.3097
8	3.68e-08	-0.6926607	0.1832538	1.18e-14	9.74e-15	2.5771	5.2093
16	9.98e-14	-0.6926607	0.1832538	1.22e-14	3.97e-15	2.5771	9.1997
256		-0.6926607	0.1832538			2.5771	130.1899

Table 2 Example 2: symmetric ellipse arc, $N = 256$

n	(46)	β_n^*	γ_n^*	$\lvert\beta_N^* - \beta_n^*\rvert$	$\lvert\gamma_N^* - \gamma_n^*\rvert$	cond($\mathbb{B}_n$)	cond($\mathbb{A}_n$)
4	6.05e-03	-0.5984153	0.0000000	1.88e-04	6.40e-16	2.6788	2.8100
8	1.44e-04	-0.5982318	0.0000000	4.92e-06	1.21e-15	2.6919	4.0774
16	7.15e-07	-0.5982269	0.0000000	9.14e-10	1.03e-15	2.6918	7.0137
32	1.25e-10	-0.5982269	0.0000000	2.22e-16	8.98e-16	2.6918	13.0283
256		-0.5982269	0.0000000			2.6918	97.6167

Table 3 Example 3: non symmetric C^3 arc, $N = 256$

n	(46)	β_n^*	γ_n^*	$\lvert\beta_N^* - \beta_n^*\rvert$	$\lvert\gamma_N^* - \gamma_n^*\rvert$	cond($\mathbb{B}_n$)	cond($\mathbb{A}_n$)	$n^4 \cdot$err
4	1.42e-03	-0.5671039	0.0227110	1.69e-04	8.82e-06	2.0341	2.7058	0.3633411
8	2.65e-05	-0.5669336	0.0227055	1.70e-06	3.22e-06	2.0339	4.4381	0.1084367
16	5.27e-07	-0.5669351	0.0227025	1.46e-07	2.87e-07	2.0339	8.0212	0.0345319
32	3.58e-08	-0.5669353	0.0227022	1.06e-08	2.13e-08	2.0339	29.6341	0.0375489
64	2.34e-09	-0.5669353	0.0227022	7.09e-10	1.44e-09	2.0339	15.2228	0.0394176
128	1.41e-10	-0.5669353	0.0227022	4.31e-11	8.75e-11	2.0339	58.4577	0.0403184
256		-0.5669353	0.0227022			2.0339	116.1042	

Table 4 Example 4: non symmetric C^2 arc, $N = 512$

n	(46)	β_n^*	γ_n^*	$\lvert\beta_N^* - \beta_n^*\rvert$	$\lvert\gamma_N^* - \gamma_n^*\rvert$	cond($\mathbb{B}_n$)	cond($\mathbb{A}_n$)	$n^{\frac{5}{2}} \cdot$err
4	2.10e-04	-0.6297931	0.0036251	9.66e-05	3.02e-05	2.1603	2.7822	0.0067258
8	2.32e-05	-0.6297061	0.0036502	9.64e-06	5.09e-06	2.1601	4.5135	0.0042078
16	3.50e-06	-0.6296973	0.0036545	8.24e-07	7.73e-07	2.1601	8.0943	0.0035876
32	6.28e-07	-0.6296965	0.0036552	6.22e-08	1.08e-07	2.1601	15.2931	0.0036382
64	1.16e-07	-0.6296965	0.0036553	4.34e-09	1.43e-08	2.1601	29.6985	0.0037854
128	2.07e-08	-0.6296965	0.0036553	2.90e-10	1.82e-09	2.1601	58.5096	0.0038293
512		-0.6296965	0.0036553			2.1601	231.3716	

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