THE COMMON EVOLUTION OF GEOMETRY AND ARCHITECTURE FROM A GEODETIC POINT OF VIEW

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KEYWORDS: Architecture, Euclidean and non-Euclidean Geometries, Projective Geometry

ABSTRACT:

Throughout history the link between geometry and architecture has been strong and while architects have used mathematics to construct their buildings, geometry has always been the essential tool allowing them to choose spatial shapes which are aesthetically appropriate. Sometimes it is geometry which drives architectural choices, but at other times it is architectural innovation which facilitates the emergence of new ideas in geometry.
Among the best known types of geometry (Euclidean, projective, analytical, Topology, descriptive, fractal,…) those most frequently employed in architectural design are:

- Euclidean Geometry
- Projective Geometry
- The non-Euclidean geometries.

Entire architectural periods are linked to specific types of geometry.
Euclidean geometry, for example, was the basis for architectural styles from Antiquity through to the Romanesque period. Perspective and Projective geometry, for their part, were important from the Gothic period through the Renaissance and into the Baroque and Neo-classical eras, while non-Euclidean geometries characterize modern architecture.

1. INTRODUCTION

For centuries geometry was effectively Euclidean Geometry: it was thought to be the one real geometry, representing space in a realistic way and, thus, no other geometry was believed possible. (Stewart, 2016).
When global navigation began, the natural, roughly spherical geometry of the earth's surface was revealed (Stewart, 2016). On a sphere, geodesics will necessarily meet: in this geometry parallel lines are absent.
However, hyperbolic geometry was developed (in the 19th century) before elliptic geometry, and in the former there are infinite lines parallel to a given line at a given point.
Today there are a variety of non-Euclidean geometries, corresponding to curved surfaces. The general theory of relativity demonstrated that in the vicinity of bodies of great mass such as stars, space-time is not flat but, rather, curved.
There is another type of geometry, called projective geometry, the development of which was based the perspective techniques used by painters and architects. If we are on a Euclidean plane between two parallel lines, we see that these meet on the horizon: the horizon is not part of the plane but is a "line at infinity".
Sometimes it is geometry which drives architectural choices, but at other times it is architectural innovation which facilitates the emergence of new ideas in geometry: in any case, throughout history the link between geometry and architecture has been strong.
Entire architectural periods are linked to specific types of geometry. Euclidean geometry is the basis for architectural styles from Antiquity through to the Romanesque period. Perspective and Projective geometry, for their part, were important from the Gothic period through the Renaissance and into the Baroque and Neo-classical eras, while non-Euclidean geometries characterize modern architecture.

2. FROM ANTIQUITY TO RENAISSANCE

From Antiquity through to the Renaissance, architects defined the proportions and the symmetry of their buildings, and the relationship with the environment by means of geometrical solutions.
Flooding of the Nile symbolized the annual return of chaos, geometry, used to restore the boundaries, was perhaps seen as restoring order on earth: geometry acquired a kind of sacredness.
A golden ratio pyramid is based on a triangle whose three sides represent the mathematical relationship that defines the golden ratio. This triangle is known as a Kepler triangle, where φ is 1.618590347 (Fig. 1).

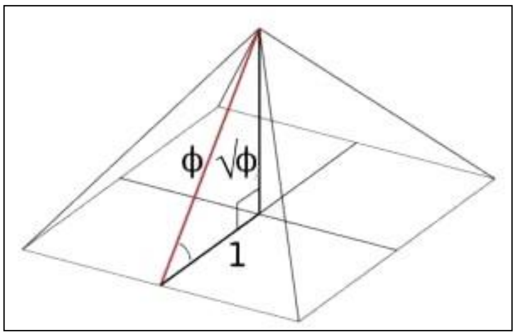

Fig. 1 The Kepler triangle

Recall that we denote by φ (in honor of Fidia) the golden number. If a segment AB is divided into two parts, such as :

AB:AC=AC:CB

the point C divides the segment in the so called golden ratio. The number AB/AC is the golden number. In the case AB was equal to 1, φ is 1.618590347. Indeed:

$$\frac{x}{1} = \frac{1}{x-1} \qquad (1)$$

$$x = \frac{1+\sqrt{5}}{2} \cong 1.618$$

Leonardo Pisano, Fibonacci (1170-1250), discovers a series of particular numbers: the first two terms of the sequence are 1 and 1. All other terms are the sum of the two terms which precede them.
A remarkable feature of these numbers is that the relationship between any Fibonacci number and the one immediately preceding it tends to φ as n tends to infinity.
“ Do the wonderful arrangement of the petals of a rose, the harmonious cycle of certain shells, the breeding of rabbits and Fibonacci's sequence have something in common?.. Behind these very disparate realities there always hides the same irrational number commonly indicated by the Greek letter φ (phi) . A proportion discovered by the Pythagoreans, calculated by Euclid and called by Luca Pacioli divina proportione” (Mario Livio, The Golden Section) .
A logarithmic spiral where the constant relationship between the consecutive beams is equal to φ is called”golden” (Fig.2).

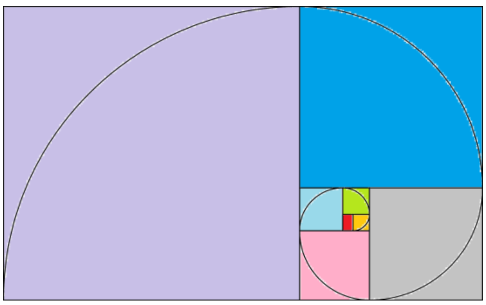

Fig.2 The golden spiral

While some scholars assert that aesthetics did not exist in Ancient Greece, it is well known that the Greeks' architectural works adhered to well-defined standards.
Art was any object whose creation derived from the technical skills or expertise of a craftsman and which recreated an order evoking the order of nature (Κοσμος).
The dimensions of an object of beauty are determined by the relationships between its component parts. The golden ratio was introduced by the Pythagoreans as the ratio between the diagonal and the side of the regular pentagon.
The symbol of the pentagonal star was the sign of the Pythagoreans, for whom it represented love and beauty, health and balance (Corbalan, 2010).
Temples represent the quintessence of Greek architecture, and Euclidean geometry was the canon defining the proportions of the component parts of these buildings. Harmonic ratios derived from the study of this type of geometry, in turn, linked to musical intervals musical intervals. Thus, Greek architecture considers ratios to be bound up with Mathematics, Philosophy and Music.

This tradition is based upon the idea expounded by Pythagoras and by Plato that numerical proportions and symmetries are the pillars holding up the world.
Greek architects also used the *sectio aurea (the golden ratio)* since proportions could not be free, but had, rather, to align with the cosmic order. Golden rectangles are observable on the façade of the Parthenon (Fig. 3). In Parthenon the overall height is the golden section of the width of the front part; then the facade has the size of a golden rectangle. This golden ratio is repeated several times between different elements of the front, for example, between the overall height and the height of which is located the entablature .

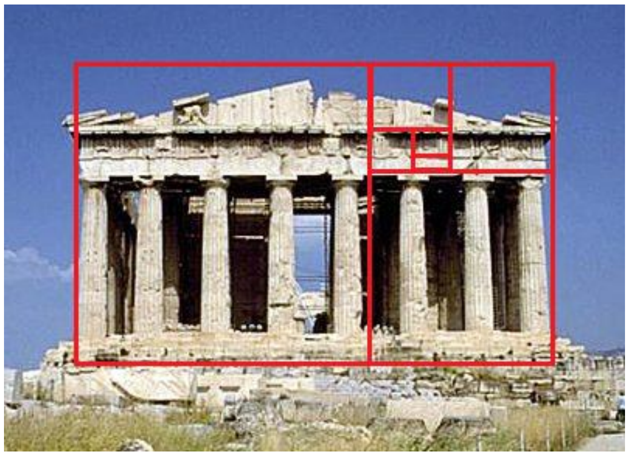

Fig. 3 The Parthenon (Athens, V century b.C.)

In any case, the argument that the architecture of the famous monument was based on the golden section appears unprovable, as it is, moreover, for all the buildings of antiquity.
It might indeed appear that constructions designed using the golden ratio are based on an innate human sense of the aesthetic: Puerta del Sol near La Paz, for example, is based on golden rectangles (Fig. 4)

Fig.4 Puerta del Sol (Tiahunaco, uncertain age)

Thales calculated the height of the Great Pyramid of Cheops, demonstrating that the real height of the pyramid and his pole were in equal proportion to their shadows: this is, at one and the same time, a classical indirect measurement and a basic theorem of Euclidean geometry.
Harmony engendered by proportions and symmetry is also observed and celebrated in Vitruvius’s “*De Architectura*”. Vitruvius describes the three limits of architecture: firmitas, utilitas, venustas (solidity, utility, beauty respectively. Geometry

is involved both in visible and structural characters (venustas and firmitas)

Fig. 5 The Great Pyramid of Cheops (2550 b.C.)

Geometry is involved both in visible and structural characters (venustas and firmitas)
Euclidean geometry informed architectural styles up to the Romanesque period. *Sectio aurea* was of great interest during the Renaissance, from Leonardo da Vinci to Leon Battista Alberti: one charming example of such is the Malatestian Temple at Rimini (L.B. Alberti, 1450-1468) (Fig.6).

Fig. 6 The Malatestian Temple (Rimini, 1503)

The Holy Family by Michelangelo is organized according to the pentagonal star (Fig. 7).
Basic thought of Pythagoreans is that the number is substance of reality: mathematical measurement (μετρου) is adequate for comprehension of the order in the world.
For Pythagoras, numbers exist in the space: so, no contradiction takes palec between mathematics and geometry.
Also, at the end of Plato's research, the science of metrology is formalized as the key of knowledge. This is worth even for man and his behaviour, also related to intercourse with other men and with his own activity, both mental and social.

Fig. 7 The Holy Family (Michelangelo, 1506-1508)

3. PERSPECTIVE AND PROJECTIVE GEOMETRY

As early as 1435, Alberti wrote the first book about the new technique of perspective, the "De Pictura". If some scholars assert that this principle is quite obvious, since the retinal image is in perspective, it is a different matter when discussing vision: Phidias made statues which, in order to be coherently proportioned atop a column, had to be altered in form. Columns closer to the observer seem narrower than those further away, and one famous correction of perspective is to be seen in the inward-slanting columns of the Parthenon.
It is interesting to compare two paintings depicting the Annunciation, with (Fig.9) and without (Fig.8) the use of perspective.

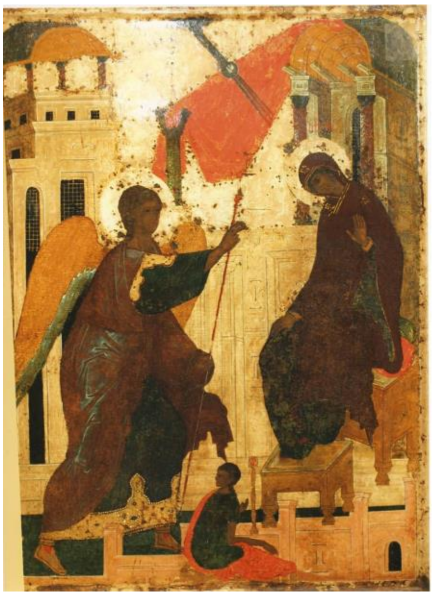

Fig. 8 Annunciation (Andrej Rublev, 1410)

Fig. 9 Annunciation (Leonardo da Vinci, 1472-1475)

In this context, the way to acquire a correct perspective is shown by a comparison between a flat byzantine representation (Fig. 10) and an attempt to draw the perspective in Gothic pictures (Fig. 11).

Fig. 10 The port of Classis (Sant'Apollinare nuovo, Ravenna, VI century)

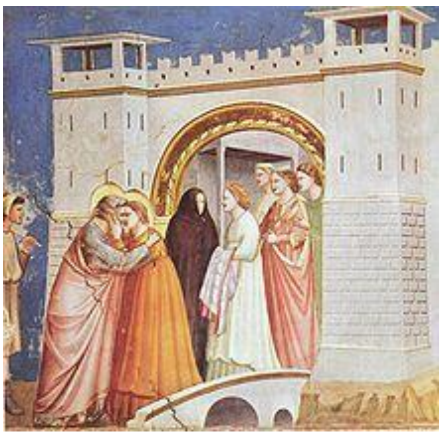

Fig. Joachim and Anne's meeting (Giotto, Scrovegni Chapel, Padua, 1303-1305)

Euclid's optics had already determined that objects create a cone of rays converging in the observer's pupil. Perspective was concerned with the intersection of cone and plane of representation

Perspective conceives of the world from the viewpoint of a "seeing eye", that is an individual who is free to represent the world beyond mere religious dogmas: the relationship between individualism and perspective is an important one. The painting *The ideal city* (Fig. 12), is a precise central projection. The author is unknown, possibly of 15 th century.[1]

The only vanishing point is at the portal of the temple

Fig. 12 The ideal city (Anonymous, 1490)

In architecture, a solid perspective gives the beholder the illusion of a greater depth than is real. One clear example is the apse of Saint Mary by Saint Satyrus Church in Milan by Bramante (Fig. 13).

Although the artist had at his disposal a depth of only 97 cm, he gave the building a monumental impact through the optical illusion provided by an apparent barrel vault.

Fig. 13 Saint Satyrus Church (Milan, 1482)

Perspective was a Renaissance invention, but the basic idea of projection is much older and stemmed from the need to represent the Earth on a plane. Both the Greeks (Hipparchus, Ptolemy) and the Persians (Al Biruni, the astronomer and mathematician who discovered the Earth's radius by indirect measurements, different from those of Eratosthenes, whose work was, nonetheless, known to him), were able to project the globe on a plane and to detect single points using coordinates (latitude and longitude). The Arabs indeed imported into Europe Indian, Arabic, and Persian ideas, not to mention ones from the Greek tradition, in the fields of Mathematics, Cartography and Geography,.

Ptolemy's world map is a polar stereographic conformal projection which was in use until explorations from Marco Polo through to Columbus led first to its modification, and later to its abandonment (Mercator, 1569).

Generally, architects and painters of the 16th century used, particularly for ceiling frescos, a series of vanishing points in order to limit the paradoxical effects of perspective. Some innovators, such as Niceron, Maignan and Andrea del Pozzo, used a unique, central viewpoint which resulted in significant distortion at the edges.

In the convent of Trinità dei Monti, images of saints which are clearly visible from a precise, lateral viewpoint (Fig.14.), dissolve as one comes closer to the center of painting (Fig.15).

In 1632, Niceron wrote "*De perspectiva curiosa*", a treatise on Anamorphosis.

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[1] Some scholars think of Leon Battista Alberti as the author of the painting.

doi:10.5194/isprs-archives-XLII-5-W1-623-2017

Fig.14 St. Francis from Paola - normal view (Maignan, Trinità dei Monti Rome, 1642)

Fig. St. Francis from Paola - anamorphic view (Maignan, Trinità dei Monti Rome, 1642)

The invention of perspective, together with cartographic and anamorphic experimentation, also inspired Projective Geometry. Desargues, a Lyonnais painter, architect and mathematician, broadens perspective, increasing the use of vanishing points (points at infinity). He believed that a geometrical entity could be distorted continuously: when one moves one of the foci of an ellipse to infinity, a parabola is obtained.
Subsequently, Baroque architecture would express ideas of infinite time and space, in step with the philosophical views of Giordano Bruno.
According to Desargues:

- two points identify one and only one straight line
- two straight identify one and only one point

The consequence is that there are no parallel lines and the conical are indistinguishable. (Odifreddi, 2011).
Projective geometry is concerned with the study of the properties of figures, with respect to a series of transformations, defined as projective, obtained by operations of projection and section that can alter the metric properties, but not the projective ones
The Baroque is constantly in search of free and open surfaces, elliptical and in constant transformation. Guarini, in the Chapel of the Holy Shroud (end of XVII century) creates optical effects and continuous geometric transformations (Fig.16); in the San Lorenzo Church (also in Turin), he brings together convex and concave surfaces (Fig. 17).
During his stay in Paris, Guarini studied infinitesimal calculus and the theories of Desargues; many scholars suppose that in the domes of Guarini projective geometry is a theoretical basis.

Fig. 16 Chapel of the Holy Shroud (Guarini, Turin, 1694)

Fig. 17 San Lorenzo Church (Guarini, Turin, 1668-1687)

Borromini shapes undulating architraves, skewed arches and oblique forms (Fig. 18).

Fig. 18 Sant'Ivo alla Sapienza (Borromini, Rome 1642-62)

The colonnade of Piazza San Pietro (by Bernini), which is straight to the eye of the observer at the center of the piazza, appears continuously transformed as the closer up he comes with the columns seemingly oblique (Fig. 19)

Fig. 19 View of St. Peter’s square (Piranesi, 1748)

Since the advent of Computer Vision a few decades ago, projective geometry has been arousing renewed interest. In fact, any understanding of how images are formed depends on an analysis of the process by which a (three-dimensional) scene is projected onto a (two-dimensional) plane.
In a very concise terms, the process can be divided into two distinct parts, one being essentially geometric and while the other radiometric):

- the determination of the image point position;
- the determination of the resulting image point brightness.

Thus, the first phase itself of the process is dependent on concepts and methods of projective geometry. In particular, projective transformations of the plane shape the geometric distortions, which are present when an object is represented in a flat image (captured by a sensor, such as a digital camera, however arranged in 3D space).
Some properties of the object, then, are retained in the passage from object to image (such as collinearity), while others are not (for example, parallelism is not preserved, at least not generally). Therefore, projective geometry models image formation and favours its mathematical representation, adapted to the calculation, i.e. by introducing algebra into geometry, to describe geometric entities in terms of coordinates and other algebraic entities.

4. NON-EUCLIDEAN GEOMETRIES

Perception of parallelism is shaky, perhaps a cultural issue, while perspective vision is perhaps a symbolic form (Panofsky): the time is ripe to abandon the Euclidean space in science and the arts.
It is interesting to compare the paintings of Van Gogh or Cézanne that represent what the artists see, with perspective drawings of the same scene: the effect is of alienation. (Fig. 20).
The curve perception of straight lines depends on the physiology of the eye: the retina is curved. Our eye knows all three geometries Euclidean, spherical and hyperbolic (Odifreddi, 2011).
Topographic work for the Duchy of Hanover caused Gauss to deal with geodetic triangles and theoretical Geodesy, and thus to anticipate hyperbolic and elliptical geometries (Lobačevsky and Riemann).
It is easy to see that all triangles upon a sphere (i.e. the Earth, more or less...) have angles whose global amplitude is over 180°. Also, all geodesics meet at two points only. In this case we can say that curvature is positive. Moreover, in the said geometry, parallels are quite absent.

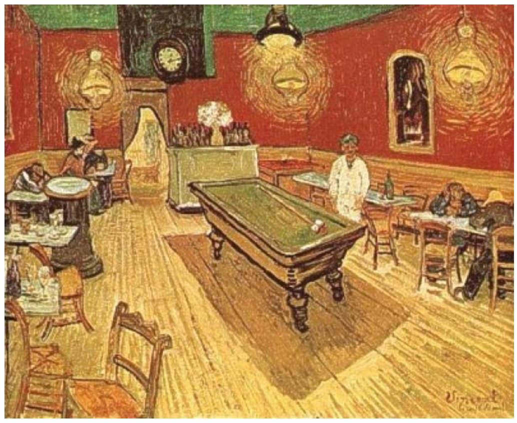

Fig.20 Le café de nuit (Van Gogh, 1888)

On the contrary, for hyperbolic geometry, total amplitude of inner angles for triangles is less than 180° and the curvature is said negative: in this case, the number of parallels at a single point to a given geodesic is unlimited.
Above a spherical surface, when a triangle is enlarged, also the width of inner angles increase (positive curvature); the opposite happens on hyperbolic surfaces (negative curvature). On a flat surface the amplitude doesn’t change. (O’Shea, 2007).
The Earth continues to appear flat in restricted areas, i.e. the sphere can appear, in small scale, Euclidean (Fig. 21)

Fig. 21 The Montréal Biosphère (Fueller,1967)

A forewarning of later artistic revolutions was the dissolution of classical geometric forms led by the Macchiaioli and the early Impressionists (Fig. 22 and Fig. 23) .

Fig. 22 Il caffé Michelangiolo (Ceccioni, 1867)

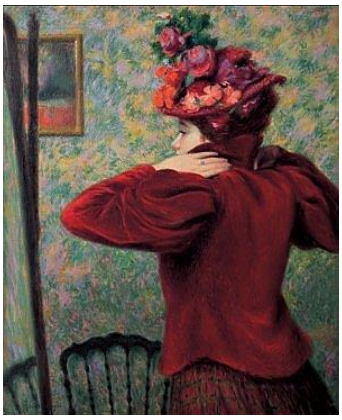

Fig. 23 The red jacket (Zandomenghi, 1895)

Both non-Euclidean Geometry and the Theory of Relativity are at the basis of Cubism. Cubist painting introduces a fourth dimension: time; in point of fact, one picture can have simultaneously different viewpoints, permitting analysis at different times (Fig. 18).

Fig. 24 Portait of Dora Maar (Picasso, 1937)

The idea of watching an object from all viewpoints has to do with going beyond the third dimension, Objects are broken and re-assembled as well as being displayed in a broader context. Non-Euclidean geometries are present in the work of artists like Picasso and Malevich (Fig. 19), and in architectural Constructivism and Deconstructivism.
The theory of relativity concludes that the three geometries Euclidean, hyperbolic and elliptic can be successfully apply in different areas.

Fig. 25 The Woodcutter (Malevich, 1912)

Shukhov, architect, engineer and mathematician, devised, like Gaudi (Fig.26) double-curvature structures, hyperboloids of revolution, based on non-Euclidean geometry.

Fig. 26 Casa Milà (Barcelona, 1906-12)

The hyperboloid tower (Fig. 27) built by Shukhov for the Nizhny Novgorod Exhibition (1896) was the prototype for many industrial buildings, and perhaps it was thanks to this that he had some influence on Russian Constructivism. Indeed, Constructivism encouraged the use of industrial materials and of shapes derived from technical processes.

Fig.27 The Shukhov Tower (1920-1922)

Constructivism created buildings and monuments which broke the rules of compositional unity. Vladimir Tatlin, artist and architect, projected a Tower in honor of the Third International, that was never built (Fig 28). In a period of reaction , Soviet architecture returned to Neo-classicism.
In the nineteen-eighties, some architects declared themselves Deconstructivists, in reference to Derrida and Deconstructionism, on the one hand, and, on the other, to Russian Constructivism - destroyer of Euclidean geometry based upon harmony and proportion - which they sought to infuriate and supplant.

Fig 28 Tower of Third International (Petrograd, 1919-20)

Walls of buildings slope and open out, and the buildings themselves are out of place in their locations: cuts, fragmentations and asymmetry are their main characteristics. One important representation is the Guggenheim Museum in Bilbao, by Frank Gehry (1997) (Fig. 29).

Fig.29 Guggenheim Museum (Bilbao, 1997)

Frank Owen Gehry and Le Corbusier are two symbols in modern architecture. Le Corbusier, in his times, is impressed by the cubistic rationalism (he was a personal friend of some of most important painters of this art movement).
The golden number entered in the architecture of Le Corbusier, as in the *Villa Savoye* (Fig 30.), or in *La Ville radieuse* in Marseille, although in the course of times a noticeable evolution is to be seen in his many works (Zeri, 1994) (Fig.31)
Frank O.Gehry is an eclectic personality; a special feature in his work is avoiding forms and shapes of Euclidean geometry and perspective, with special preference for what appears as disharmonic and asymmetric (Zeri, 1994) (Fig. 32).

Fig. 30 Villa Savoye (Poissy, 1928-31)

Fig. 31 Chapelle Notre dame du Haut (Ronchamp,1950)

Fig. 32 Dancing House (Prague, 1996)

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