On the Hidden Maxwell Superalgebra underlying $D = 4$ Supergravity

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Abstract

In this work, we expand the hidden AdS-Lorentz superalgebra underlying $D = 4$ supergravity, reaching a (hidden) Maxwell superalgebra. The latter can be viewed as an extension involving cosmological constant of the superalgebra underlying $D = 4$ supergravity in flat space. We write the Maurer-Cartan equations in this context and we find some interesting extensions of the parametrization of the 3-form $A^{(3)}$, which appears in the Free Differential Algebra in Minkowski space, in terms of 1-forms. We interestingly find out that the structure of these extensions, and consequently the structure of the corresponding boundary contribution $dA^{(3)}$, strongly depends on the form of the extra fermionic generator appearing in the hidden Maxwell superalgebra.

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1 Introduction

Supergravity theories in various spacetime dimensions $4 \leq D \leq 11$ have a field content that generically includes the metric, the gravitino, a set of 1-form gauge potentials, and $(p+1)$-form gauge potentials of various $p \leq 9$, and they are discussed in the context of Free Differential Algebras (FDAs) [1].

In particular, in the framework of FDAs, the structure of $D = 11$ supergravity, first constructed in [1], was then reconsidered in [2], adopting the superspace geometric approach. In the same paper, the supersymmetric FDA was also investigated in order to see whether the FDA formulation could be interpreted in terms of an ordinary Lie superalgebra in its dual Maurer-Cartan formulation [2]. This was proven to be true, and the existence of a hidden superalgebra underlying the theory was presented for the first time. In fact, in [2], the authors proved that the FDA underlying $D = 11$ supergravity can be traded with a Lie superalgebra which contains, besides the Poincaré superalgebra, also new bosonic 1-forms and a nilpotent fermionic generator Q', necessary for the closure of the superalgebra.

Later, in [3], the authors wonder whether eleven dimensional supergravity can be decontracted into a non-abelian (gauged) model. This problem was reduced to that of finding an algebra whose contraction yields the $D = 11$ algebra of [2]. In the same paper, they also considered the $D = 4$ case, in order to explain their approach through a toy-model.

However, the four dimensional gauged case results of some interest, since its algebraic form (presented in [3]) corresponds to a "hidden AdS-Lorentz-like superalgebra", an extension with an extra nilpotent fermionic generator of the AdS-Lorentz superalgebra presented and largely discussed in [4]. In particular, in [4] where the authors explored the supersymmetry invariance of an extension of minimal $D = 4$ supergravity in the presence of a non-trivial boundary, presenting the explicit construction of the $\mathcal{N} = 1$, $D = 4$ AdS-Lorentz supergravity bulk Lagragian in the rheonomic framework. In particular, they developed a peculiar way to introduce a generalized supersymmetric cosmological term to supergravity. Then, by studying the supersymmetry invariance of the Lagrangian in the presence of a non-trivial boundary, they interestingly found that the supersymmetric extension of a Gauss-Bonnet like term is required in order to restore the supersymmetry invariance of the full Lagrangian.

Recently, in [5], the authors clarified the role of the nilpotent fermionic generator Q' introduced in [2] by looking at the gauge properties of the theory. They found that its presence is necessary, in order that the extra 1-forms of the hidden superalgebra give rise to the correct gauge transformations of the p-forms of the FDA. In particular, in its absence, the extra bosonic 1-forms do *not* enjoy gauge freedom, but generate, together with the supervielbein, new directions of an enlarged superspace, so that the FDA on ordinary superspace is no more reproduced.

On the group theoretical side, in [6], the authors developed the so-called S-expansion procedure, which is based on combining the inner multiplication law of a discrete set S with the structure of a semigroup, with the structure constants of a Lie algebra $\mathfrak{g}$. The new, larger

[1]The FDAs framework is analogous to the Cartan Integrable Systems (CIS) one.

[2]The supergroup structure allows a deeper understanding of the symmetry and topological properties of the theory.

Lie algebra thus obtained is called S-expanded algebra, and it is written as $\mathfrak{g}_S = S \times \mathfrak{g}$.

There are two facets applicable in the S-expansion method, which offer great manipulation on (super)algebras, *i.e. resonance* and *reduction*. The role of *resonance* is that of transferring the structure of the semigroup to the target (super)algebra; Meanwhile, *reduction* plays a peculiar role in cutting the (super)algebra properly, thanks to the existence of a zero element in the set involved in the procedure.

From the physical point of view, several (super)gravity theories have been largely studied using the S-expansion approach, enabling the achievement of several results over recent years (see Ref.s [7–33]). Furthermore, in [34], an analytic method for S-expansion was developed. This method is able to give the multiplication table(s) of the (abelian) set(s) involved in an S-expansion process for reaching a target Lie (super)algebra from a starting one, after having properly chosen the partitions over subspaces of the considered (super)algebras. A complete review of S-expansion can be found in [6] and [34].

Recently, in [35], the authors proposed a new prescription for S-expansion, involving an infinite abelian semigroup $S_E^{(\infty)}$, with subsequent subtraction of a suitable infinite ideal. Their approach is a generalization of the finite S-expansion procedure, and it allows to reproduce a generalized Inönü-Wigner contraction (IW contraction) via infinite S-expansion between two different algebras. Furthermore, the authors of [36] recently presented a generalization of the standard Inönü-Wigner contraction, by rescaling not only the generators of a Lie superalgebra, but also the arbitrary constants appearing in the components of the invariant tensor.

In this work, we obtain a particular hidden Maxwell superalgebra in four dimensions by performing an infinite S-expansion with subsequent ideal subtraction of the hidden AdS-Lorentz superalgebra underlying $D = 4$ supergravity. We then adopt the Maurer-Cartan (and FDA) formalism and we consider the paremetrization of the 3-form $A^{(3)}$, whose field strength is a 4-form $F^{(4)} = dA^{(3)} + \ldots$, modulo fermionic bilinears, in terms of 1-forms, and we show how the (trivial) boundary contribution in four dimensions, $dA^{(3)}$, can be naturally extended by considering particular contributions to the structure of the extra fermionic generator appearing in the hidden Maxwell superalgebra underlying supergravity in four dimensions. This extension involves the cosmological constant. Interestingly, the presence of these terms strictly depends on the form of the extra fermionic generator appearing in the hidden superMaxwell-like extension of $D = 4$ supergravity.

This paper is organized as follows: In Section 2, we perform expansions and contractions of different superalgebras describing and underlying $D = 4$ supergravity, and we also display a map which links different superalgebras in four dimensions. In Section 3, we write some of the superalgebras presented in Section 2 in the Maurer-Cartan formalism and, in particular, we consider a hidden extension, involving cosmological constant, of $D = 4$ supergravity, which corresponds to the (hidden) Maxwell superalgebra. We then write the parametrization of the 3-form $A^{(3)}$ in this context and we show that the (trivial) boundary contribution $dA^{(3)}$ can be naturally extended with the addition of terms involving the cosmological constant. Section 4 contains our outlook and possible future developments. In the Appendix, we give detailed calculations on the infinite S-expansion with ideal subtraction.

2 Expansions and contractions of superalgebras in four dimensions

It is well known that we can construct several theories in four dimensions by choosing different amount of physical (and unphysical) fields, invariant under different superalgebras. One of the simplest case is the Poincaré superalgebra $\overline{osp(1|4)}$, which is abelian in the momenta. On the other side, the (Anti-)de Sitter ($(A)dS$) algebra is characterized by the fact that the translations commute with Lorentz transformations. In Ref. [37], the authors presented a geometric formulation involving the AdS structure group (the AdS one), known as MacDowell-Mansouri action. The generalization of their work consists in considering the supergroup $Osp(\mathcal{N}|4)$.

In Ref. [2], the authors presented a particular superalgebra, now known as *hidden superalgebra*, underlying $D = 11$ supergravity. This hidden superalgebra includes, as a subalgebra, the super-Poincaré algebra, and also involves two extra bosonic generators Z_{ab}, commuting with the generators P_a. Furthermore, an extra nilpotent fermionic generator Q' must be included (in order to satisfy the closure of the superalgebra). It is then possible to consider a *hidden AdS-Lorentz superalgebra*, namely an extension of the AdS superalgebra in which the commutators between the momenta is equal to a Lorentz-like generator, which will be referred as to Z_{ab}. Finally, we should mentioned that the introduction of a second fermionic generator has been considered in the literature; Following this idea, the authors of [25] considered *Maxwell superalgebras* for constructing actions for supergravity theories.

We will now consider (hidden) superalgebras in *four dimensions*. Each of these superalgebras gives rise to the construction of an action for a supergravity theory. The existence of connections between different physical theories motivates to look for connections between the superalgebras underlying these theories. We first consider a "toy model" superalgebra in four dimensions described in [3], namely an *AdS*-Lorentz-like superalgebra with an extra fermionic generator, which is the hidden superalgebra underlying the AdS supergravity theory in $D = 4$. This algebra will be named *hidden AdS-Lorentz superalgebra*. It is generated by the set of generators $\{J_{ab}, P_a, Z_{ab}, Q_\alpha, Q'_\alpha\}$, and can be written as

$$
\begin{aligned}
&[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \\
&[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \\
&[Z_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \\
&[Q_\alpha, Z_{ab}] = -(\gamma_{ab}Q)_\alpha - (\gamma_{ab}Q')_\alpha, \quad [Q'_\alpha, Z_{ab}] = 0, \qquad [J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \\
&[Q_\alpha, P_a] = -i(\gamma_a Q)_\alpha - i(\gamma_a Q')_\alpha, \quad [Q'_\alpha, P_a] = 0, \qquad [P_a, P_b] = -Z_{ab}, \\
&[J_{ab}, Q_\alpha] = -(\gamma_{ab}Q)_\alpha, \quad [J_{ab}, Q'_\alpha] = -(\gamma_{ab}Q')_\alpha, \qquad [Z_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \\
&\{Q_\alpha, Q_\beta\} = -i(\gamma^a C)_{\alpha\beta}P_a - \frac{1}{2}(\gamma^{ab}C)_{\alpha\beta}Z_{ab}, \qquad \{Q_\alpha, Q'_\beta\} = \{Q'_\alpha, Q'_\beta\} = 0,
\end{aligned}
\tag{2.1}
$$

where C stands for the charge conjugation matrix and γ_a and γ_{ab} are Dirac matrices in four dimensions. Let us notice that the Lorentz type algebra generated by $\{J_{ab}, Z_{ab}\}$ is a subalgebra of the above superalgebra. In [4], the authors explored the supersymmetry invariance of an extension of minimal $D = 4$ supergravity in the presence of a non-trivial boundary, and they presented the explicit construction of the $\mathcal{N} = 1$, $D = 4$ AdS-Lorentz supergravity bulk Lagragian in the rheonomic framework. In particular, they developed a peculiar way to introduce a generalized supersymmetric cosmological term in supergravity. The starting superalgebra they considered was a truncation of the hidden AdS-Lorentz one (2.1). In fact, by performing a consistent truncation of the fermionic generator Q'_α in (2.1), we get the AdS-Lorentz superalgebra considered in [4] [3]. In other words, the hidden AdS-Lorentz superalgebra can be consistently viewed as an extension of the AdS-Lorentz algebra described in [4], with the inclusion of an extra fermionic generator Q'_α.

On the other hand, the technique proposed by the authors of [35], which consists in a new prescription for S-expansion, involving an infinite abelian semigroup $S_E^{(\infty)}$, with subsequent subtraction of a suitable infinite ideal [4], allows to obtain a *(hidden) Maxwell superalgebra* [38,39] in four dimensions, generated by the set of generators $\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, Q_\alpha, \Sigma_\alpha\}$ (here and in the following, Σ_α denotes the extra nilpotent fermionic generator appearing in the hidden Maxwell superalgebra), by starting from the hidden AdS-Lorentz superalgebra (2.1). Thus, following the approach described in [35], we can perform a S-expansion with the infinite abelian semigroup $S_E^{(\infty)}$ [5], involving a resonant structure.

For further details on this calculation, see Appendix A. The hidden Maxwell superalgebra thus obtained reads

$$
\begin{aligned}
&[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \\
&[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \\
&\left[J_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \\
&[Z_{ab}, Z_{cd}] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \\
&[Q_\alpha, Z_{ab}] = -(\gamma_{ab}\Sigma)_\alpha, \quad [\Sigma_\alpha, Z_{ab}] = 0, \qquad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \\
&[Q_\alpha, P_a] = -i(\gamma_a \Sigma)_\alpha, \quad [\Sigma_\alpha, P_a] = 0, \qquad [P_a, P_b] = -\tilde{Z}_{ab}, \\
&[J_{ab}, Q_\alpha] = -(\gamma_{ab} Q)_\alpha, \quad [J_{ab}, \Sigma_\alpha] = -(\gamma_{ab}\Sigma)_\alpha, \qquad [Z_{ab}, P_c] = 0, \\
&\{Q_\alpha, Q_\beta\} = -i(\gamma^a C)_{\alpha\beta} P_a - \frac{1}{2}(\gamma^{ab} C)_{\alpha\beta} Z_{ab}, \qquad \{\Sigma_\alpha, \Sigma_\beta\} = 0, \\
&\{Q_\alpha, \Sigma_\beta\} = -2(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}, \\
&\left[\tilde{Z}_{ab}, P_c\right] = \left[Q_\alpha, \tilde{Z}_{ab}\right] = \left[\Sigma_\alpha, \tilde{Z}_{ab}\right] = \left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = 0.
\end{aligned}
\tag{2.2}
$$

It is well known that the Poincaré and the AdS superalgebras are related by Inönü-Wigner contraction, *i.e.* by rescaling and consequently considering a particular limit for the

[3] Let us observe that the authors of [4] adopted the Maurer-Cartan formalism in their work, where the superalgebra generators are properly associated to 1-forms.

[4] Their approach is a generalization of the finite S-expansion procedure, and it allows to reproduce a generalized Inönü-Wigner contraction (IW contraction) with an infinite S-expansion with subsequent ideal subtraction.

[5] The semigroup $S_E^{(\infty)}$ is an extension and generalization of the semigroups of the type $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$, endowed with the following multiplication rules: $\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha + \beta \leq N + 1$, and $\lambda_\alpha \lambda_\beta = \lambda_{N+1}$ if $\alpha + \beta > N + 1$.

$$
\begin{tikzcd}
 & \text{Hidden Maxwell} & \\
AdS \arrow[r, "S\text{-expansion}"] \arrow[d, "\text{IW contraction}"] & \text{Hidden } AdS\text{-Lorentz} \arrow[u, "S_E^{(\infty)} \ominus \mathcal{I}"] \arrow[r, "Q'_\alpha \to 0"] \arrow[d, "\text{IW contraction}"] & AdS\text{-Lorentz} \\
\text{Poincaré} & \text{Hidden Poincaré} \arrow[l, "Q'_\alpha \to 0"'] &
\end{tikzcd}
$$

Figure 1: Map between different superalgebras in four dimensions. Here, $S_E^{(\infty)} \ominus \mathcal{I}$ denotes an infinite S-expansion with subsequent ideal subtraction.

generators. In the same way, by performing an Inönü-Wigner contraction on the hidden AdS-Lorentz algebra (2.1), we obtain the hidden Poincaré superalgebra (introduced and studied in [2,3]). This superalgebra is generated by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha, Q'_\alpha\}$, and can be written as

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \\
[J_{ab}, Z_{cd}] &= \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \\
[J_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b, \quad [Z_{ab}, Z_{cd}] = 0, & [P_a, P_b] &= 0, \\
[Q_\alpha, P_a] &= -\, i (\gamma_a Q')_\alpha, \quad [Q'_\alpha, P_a] = 0, & [J_{ab}, Q_\alpha] &= -\, (\gamma_{ab} Q)_\alpha, \\
[Q_\alpha, Z_{ab}] &= -\, (\gamma_{ab} Q')_\alpha, \quad [Q'_\alpha, Z_{ab}] = 0, & [J_{ab}, Q'_\alpha] &= -\, (\gamma_{ab} Q')_\alpha, \\
\{Q_\alpha, Q_\beta\} &= -\, i (\gamma^a C)_{\alpha\beta} P_a - \frac{1}{2} (\gamma^{ab} C)_{\alpha\beta} Z_{ab}, & \{Q_\alpha, Q'_\beta\} = \{Q'_\alpha, Q'_\beta\} &= 0.
\end{aligned}
\tag{2.3}
$$

Let us observe that, analogously to the case of the hidden AdS-Lorentz and AdS-Lorentz superalgebras in four dimensions, a consistent truncation of the nilpotent fermionic generator Q'_α allows to reproduce the Poincaré superalgebra starting from the hidden Poincaré superalgebra (2.3). In other words, the hidden Poincaré superalgebra in four dimensions is an extension with one extra fermionic generator of the Poincaré superalgebra.

In Figure 1, we have collected and summarized the relationships between the mentioned superalgebras [6].

[6]Let us remind that both the standard and the generalized Inönü-Wigner contractions are reproducible through S-expansion: The standard Inönü-Wigner contraction can be reproduced with a finite S-expansion, while the generalized one can be reproduced through an infinite S-expansions with subsequent ideal subtraction (see Ref. [35] for further details on the latter mentioned procedure).

3 Hidden Maxwell superalgebra in the Maurer-Cartan formalism and parametrization of the 3-form $A^{(3)}$

There are two dual ways of describing a (super)algebra: The first one is provided by the commutation relations between the generators; The second one is instead provided by the so-called Maurer-Cartan equations. These two descriptions are equivalent and dual each other.

The generators T_A's, which form a basis of the tangent space $T(\mathcal{M})$ of a manifold $\mathcal{M}$, satisfy the commutation relations of the (super)algebra and the (super) Jacobi identity. The same information is enclosed in the Maurer-Cartan equations, which read

$$d\sigma^A = -\frac{1}{2} C^A_{\ BC} \sigma^B \wedge \sigma^C, \tag{3.1}$$

where σ^A stands for the forms involved into the Maurer-Cartan equations, and where $C^A_{\ BC}$ are the coupling constants. In the following, for simplicity, we will omit the symbol $\wedge$ denoting the product between 1-forms.

As we can see, the Maurer-Cartan equations are written in terms of the dual forms σ^A's of the generators T_A's, which are related through the expression

$$\sigma^A (T_B) = \delta^A_B, \tag{3.2}$$

up to normalization factors [7].

We now consider the Maurer-Cartan equations associated with the superalgebras in $D = 4$ presented in [3]. In the case of $\overline{osp(4|1)}$ (Poincaré superalgebra), we have

$$\begin{aligned} R^{ab} &= 0, \\ DV^a &= \frac{i}{2} \bar{\Psi} \gamma^a \Psi, \\ D\Psi &= 0, \end{aligned} \tag{3.3}$$

where γ^a, as said before, are the four-dimensional gamma matrices, and where $D = d + \omega$ is the Lorentz covariant exterior derivative. Here we have fixed the normalization of DV^a to $\frac{i}{2}$, according to the usual convention. The closure ($d^2 = 0$) of this superalgebra is trivially satisfied.

In the AdS case, instead, we have that the anticommutator of the generators Q's falls into the Poincaré translations and the Lorentz rotations, generating non-vanishing value of the Lorentz curvature, namely

[7] See the maps between the two formalism presented in [5] for further details.

$$
\begin{aligned}
R^{ab} =& \alpha e^2 V^a V^b + \beta e \bar{\Psi}\gamma^{ab}\Psi, \\
DV^a =& \frac{i}{2}\bar{\Psi}\gamma^a\Psi, \\
D\Psi =& ie\gamma_a\Psi V^a,
\end{aligned}
\tag{3.4}
$$

where $e = 1/2l$ corresponds to the inverse of the AdS radius. Here, α and β are parameters, and we have fixed the normalization of $D\Psi$ to 1. From the closure requirement of the superalgebra ($d^2 = 0$), we get $\beta = \frac{1}{2}\alpha$ and, after having fixed the normalization $\alpha = -1$, we can write $\beta = -\frac{1}{2}$.

We observe that in the limit $e \to 0$ we correctly get the Maurer-Cartan equations in the flat (*i.e.* Minkowski) space, namely equations (3.3).

As shown in [2], with the introduction of a nilpotent fermionic generator Q' we can write the hidden Poincaré (2.3) and the hidden AdS-Lorentz (2.1) superalgebras in terms of the corresponding respective Maurer-Cartan equations. For the hidden Poincaré case (2.3), we have

$$R^{ab} = 0, \tag{3.5}$$

$$DV^a = \frac{i}{2}\bar{\Psi}\gamma^a\Psi, \tag{3.6}$$

$$D\Psi = 0, \tag{3.7}$$

$$DB^{ab} = \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi, \tag{3.8}$$

$$D\eta = \frac{i}{2}\delta\gamma_a\Psi V^a + \frac{1}{2}\epsilon\gamma_{ab}\Psi B^{ab}. \tag{3.9}$$

Here, δ and ϵ are two *arbitrary* parameters. In fact, requiring the closure of the superalgebra, and in particular of $D\eta$, we simply get the identity $0 = 0$, which leads the solution to be given in terms of two free parameters, namely δ and ϵ. In particular, for reaching this result we used the following Fierz identities in four dimensions:

$$\Psi\gamma_a\bar{\Psi}\gamma^a\Psi = 0, \tag{3.10}$$

,

$$\Psi\gamma_{ab}\bar{\Psi}\gamma^{ab}\Psi = 0. \tag{3.11}$$

As we can see, in this superalgebra the Lorentz curvature is zero: $R^{ab} = 0$. However, we have a non-trivial "AdS-like" contribution [8] given by $DB^{ab} = \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi$.

Let us observe that in $D = 4$ we also have a particular subalgebra of the hidden Poincaré one, which can be obtained through an Inöü-Wigner contraction of the hidden AdS-Lorentz superalgebra (2.1). In fact, we do not even need the 1-form B^{ab} to find an underlying group

[8] We call this contribution "AdS-like" since the AdS curvature $\hat{R}^{ab} \equiv R^{ab} + e^2 V^a V^b + \frac{1}{2}e\bar{\Psi}\gamma^{ab}\Psi$ contains a similar term, namely the term involving the gravitino.

for the Cartan Integrable System (CIS) in the four dimensional Minkowski space. This subalgebra reads

$$R^{ab} =0, \tag{3.12}$$
$$DV^a =\frac{i}{2}\bar{\Psi}\gamma^a\Psi, \tag{3.13}$$
$$D\Psi =0, \tag{3.14}$$
$$D\eta =\frac{i}{2}\gamma_a\Psi V^a, \tag{3.15}$$

which endows the CIS with a 3-form $A^{(3)}$ whose parametrization in terms of 1-forms can be simply written as $A^{(3)} = -i\bar{\Psi}\gamma_a\eta V^a$ (see Ref. [3]).

As shown in Section 2, we can write a (hidden) Maxwell superalgebra in four dimensions, by starting from the hidden AdS-Lorentz one (2.1). For completeness, in the following we report the Maurer-Cartan equations associated with the hidden AdS-Lorentz superalgebra (2.1):

$$R^{ab} =0, \tag{3.16}$$
$$DV^a =\frac{i}{2}\bar{\Psi}\gamma^a\Psi - eB^{ab}V_b, \tag{3.17}$$
$$D\Psi =\frac{i}{2}e\gamma_a\Psi V^a + \frac{e}{4}\gamma_{ab}\Psi B^{ab}, \tag{3.18}$$
$$DB^{ab} =\frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi - eB^{ac}B_c^{\ b} + eV^aV^b, \tag{3.19}$$
$$D\eta =\frac{i}{2}\gamma_a\Psi V^a + \frac{1}{4}\gamma_{ab}\Psi B^{ab}, \tag{3.20}$$

where the parameters have been fixed by requiring the closure of the superalgbera and properly fixing the normalization of the 1-form η (see Ref. [3] for further details) [9].

We now write the Maurer-Cartan equations associated with the hidden Maxwell superalgebra in $D = 4$, namely

$$R^{ab} =0, \tag{3.21}$$
$$DV^a =\frac{i}{2}\bar{\Psi}\gamma^a\Psi, \tag{3.22}$$
$$D\Psi =0, \tag{3.23}$$
$$DB^{ab} =\frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi, \tag{3.24}$$
$$D\tilde{B}^{ab} =\alpha e\bar{\Psi}\gamma^{ab}\Phi + \beta eB^{ac}B_c^{\ b} + \gamma eV^aV^b, \tag{3.25}$$
$$D\Phi =\frac{i}{2}\delta\gamma_a\Psi V^a + \frac{1}{2}\varepsilon\gamma_{ab}\Psi B^{ab}, \tag{3.26}$$

[9]The authors of [3] observed that in the hidden AdS-Lorentz superalgebra we can write $D\eta = \frac{1}{e}\Lambda$ and $D\Psi = \Lambda$, where Λ is the 2-form that reads $\Lambda = \frac{i}{2}e\gamma_a\Psi V^a + \frac{1}{4}e\gamma_{ab}\Psi B^{ab}$.

where B^{ab} and $\tilde{B}^{ab}$ are the 1-forms dual to the generators Z_{ab} and $\tilde{Z}_{ab}$, respectively, and where Φ is the spinorial 1-form dual to the extra nilpotent fermionic generator Σ_α appearing in the hidden Maxwell superalgebra.

Once again, we must require the closure $d^2 = 0$ of the superalgebra. In this way, from the first Maurer-Cartan equation we get $\delta\alpha = \gamma$, and $\beta = -2\alpha\epsilon$. We now choose the normalization $\alpha = 1$ and $\delta = 1$. We can thus write $\gamma = 1$ and $\beta = -2\epsilon$, being ϵ a free parameter. We observe that the Lorentz curvature is again zero: $R^{ab} = 0$. In this case, we have two non-trivial "AdS-like" contributions, namely $DB^{ab} = \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi$ and the term $\gamma e V^a V^b$ in $D\tilde{B}^{ab} = \alpha e \bar{\Psi}\gamma^{ab}\Phi + \beta e B^{ac} B_c{}^b + \gamma e V^a V^b$. Then, we can finally write

$$\begin{aligned}
R^{ab} =& 0, \\
DV^a =& \frac{i}{2}\bar{\Psi}\gamma^a\Psi, \\
D\Psi =& 0, \\
DB^{ab} =& \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi, \\
D\tilde{B}^{ab} =& e\bar{\Psi}\gamma^{ab}\Phi + \beta e B^{ac}B_c{}^b + eV^aV^b, \\
D\Phi =& \frac{i}{2}\gamma_a\Psi V^a + \frac{1}{2}\varepsilon\gamma_{ab}\Psi B^{ab},
\end{aligned} \tag{3.27}$$

where $\beta = -2\epsilon$. This superalgebra is the hidden Maxwell superalgebra underlying supergravity in four dimensions.

We observe that setting $\beta = \epsilon = 0$ we get the following subalgebra:

$$\begin{aligned}
R^{ab} =& 0, \\
DV^a =& \frac{i}{2}\bar{\Psi}\gamma^a\Psi, \\
D\Psi =& 0, \\
DB^{ab} =& \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi, \\
D\tilde{B}^{ab} =& e\bar{\Psi}\gamma^{ab}\Phi + eV^aV^b, \\
D\Phi =& \frac{i}{2}\gamma_a\Psi V^a.
\end{aligned} \tag{3.28}$$

In the following, we will write the parametrization of the 3-form $A^{(3)}$ appearing in the CIS in four-dimensional supergravity in terms of 1-forms, both for the hidden Maxwell superalgebra (3.27) and for its subalgebra (3.28). We will then study the particular extensions of the (trivial) boundary contribution $dA^{(3)}$ in four dimensions [10].

[10]Let us observe that the trivial boundary contribution $dA^{(3)}$ can also be considered as a flux contribution in the four dimensional theory.

3.1 Extensions of $dA^{(3)}$ involving the cosmological constant

We start our analysis by considering the hidden Maxwell superalgebra in four dimensions (3.27). Then, we write the parametrization of the 3-form $A^{(3)}$ in terms of 1-forms, both for the hidden Maxwell superalgebra (3.27) and for its subalgebra (3.28), and we study the different extensions of the (trivial) boundary contribution $dA^{(3)}$.

Let us now consider the hidden Maxwell superalgebra valued curvatures, which are defined by

$$R^{ab} \equiv d\omega^{ab} - \omega^a_{\ c}\omega^{cb}\,, \tag{3.29}$$

$$R^a \equiv DV^a - \frac{i}{2}\bar{\Psi}\gamma^a\Psi\,, \tag{3.30}$$

$$F^{ab} \equiv DB^{ab} - \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi\,, \tag{3.31}$$

$$\tilde{F}^{ab} \equiv D\tilde{B}^{ab} - e\bar{\Psi}\gamma^{ab}\Phi - \beta e B^{ac}B_c^{\ b} - eV^aV^b\,, \tag{3.32}$$

$$\rho \equiv D\Psi\,, \tag{3.33}$$

$$\zeta \equiv D\Phi - \frac{i}{2}\gamma_a\Psi V^a - \frac{1}{2}\epsilon\gamma_{ab}\Psi B^{ab}, \tag{3.34}$$

where $D = d + \omega$ is the Lorentz covariant exterior derivative. In the four-dimensional Minkowski space, we can also write

$$F^{(4)} \equiv dA^{(3)} - \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^a V^b\,, \tag{3.35}$$

where the 4-form $F^{(4)}$ is trivially given in terms of a boundary contribution. Our aim is that of writing the deformation to the 4-form $F^{(4)}$ induced by the presence the cosmological constant in the hidden Maxwell superalgebra underlying $D = 4$ supergravity.

We can write the Maurer-Cartan equations in four dimensions for the hidden Maxwell superalgebra, by simply setting the curvatures to zero in the vacuum, namely

$$R^{ab} \equiv d\omega^{ab} - \omega^a_{\ c}\omega^{cb} = 0\,, \tag{3.36}$$

$$R^a \equiv DV^a - \frac{i}{2}\bar{\Psi}\gamma^a\Psi = 0\,, \tag{3.37}$$

$$F^{ab} \equiv DB^{ab} - \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi = 0\,, \tag{3.38}$$

$$\tilde{F}^{ab} \equiv D\tilde{B}^{ab} - e\bar{\Psi}\gamma^{ab}\Phi - \beta e B^{ac}B_c^{\ b} - eV^aV^b = 0\,, \tag{3.39}$$

$$\rho \equiv D\Psi = 0\,, \tag{3.40}$$

$$\zeta \equiv D\Phi - \frac{i}{2}\gamma_a\Psi V^a - \frac{1}{2}\epsilon\gamma_{ab}\Psi B^{ab} = 0\,, \tag{3.41}$$

which simply lead to the expression (3.27).

Now, as done in the $D = 11$ and $D = 7$ supergravity cases in [2] and [5], respectively, we can write the parametrization of the 3-form $A^{(3)}$ in terms of 1-forms. We first of all observe that, since $dA^{(3)}$ is a boundary contribution in four dimensions, we expect a topological form for the parametrization of $A^{(3)}$. We thus start by writing

$$A^{(3)} = \frac{1}{2e}\bar{\Psi}\gamma_{ab}\Psi B^{ab} + \bar{\Psi}\gamma_{ab}\Psi\tilde{B}^{ab} + \beta\tilde{B}_{ab}B^{ac}B_c^{\ b} + \tilde{B}_{ab}V^aV^b - i\bar{\Psi}\gamma_a\Phi V^a - \epsilon\bar{\Psi}\gamma_{ab}\Phi B^{ab}, \quad (3.42)$$

where the topological structure is still *not* manifest. However, we can reorganize and rewrite (3.42) as follows:

$$A^{(3)} = \frac{1}{e}B^{ab}DB_{ab} + \frac{1}{e}\tilde{B}^{ab}D\tilde{B}_{ab} - 2\bar{\Phi}D\Phi, \quad (3.43)$$

where the topological structure is evident: This particular parametrization will give rise to a "topological" structure for the boundary contribution $dA^{(3)}$. Let us observe that by setting $\beta = \epsilon = 0$ in (3.42) we obtain

$$A^{(3)} = \frac{1}{2e}\bar{\Psi}\gamma_{ab}\Psi B^{ab} + \bar{\Psi}\gamma_{ab}\Psi\tilde{B}^{ab} + \tilde{B}_{ab}V^aV^b - i\bar{\Psi}\gamma_a\Phi V^a. \quad (3.44)$$

If we now consider the parametrization (3.43) and compute $dA^{(3)}$, we get the following topological expression:

$$\begin{aligned} dA^{(3)} &= \frac{1}{e}d(B^{ab}DB_{ab}) + \frac{1}{e}d(\tilde{B}^{ab}D\tilde{B}_{ab}) - 2d(\bar{\Phi}D\Phi) = \\ &= \frac{1}{e}DB^{ab}DB_{ab} + \frac{1}{e}D\tilde{B}^{ab}D\tilde{B}_{ab} - 2D\bar{\Phi}D\Phi, \end{aligned} \quad (3.45)$$

which automatically satisfies the closure requirement $d^2 = 0$. If we now substitute the Maurer-Cartan equations (3.27) in the expression (3.45), we get

$$\begin{aligned} dA^{(3)} = &\frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^aV^b + e\bar{\Psi}\gamma_{ab}\Phi\bar{\Psi}\gamma^{ab}\Phi + 2\beta e\bar{\Psi}\gamma_{ab}\Phi B^{ac}B_c^{\ b} + 2e\bar{\Psi}\gamma_{ab}\Phi V^aV^b + \\ &+ 2\beta eB^{ac}B_c^{\ b}V_aV_b - 2i\epsilon\bar{\Psi}\gamma_a\Psi B^{ab}V_b + \epsilon^2\bar{\Psi}\gamma_{ac}\Psi B^{ab}B^c_{\ b}. \end{aligned} \quad (3.46)$$

In the limit $e \to 0$, the expression (3.46) reduces to

$$dA^{(3)} = \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^aV^b - 2i\epsilon\bar{\Psi}\gamma_a\Psi B^{ab}V_b + \epsilon^2\bar{\Psi}\gamma_{ac}\Psi B^{ab}B^c_{\ b}. \quad (3.47)$$

We observe that, interestingly, this solution does *not* reduce to the four-dimensional Minkowski flat space limit when $e \to 0$. However, if we now consider the particular solution $\beta = \epsilon = 0$, which conduces to the subalgebra (3.28) of the hidden Maxwell superalgebra in four dimensions, we clearly see that, interestingly, this particular solution leads to

$$dA^{(3)} = \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^aV^b + e\bar{\Psi}\gamma_{ab}\Phi\bar{\Psi}\gamma^{ab}\Phi + 2e\bar{\Psi}\gamma_{ab}\Phi V^aV^b, \quad (3.48)$$

which exactly reproduces the Minkowski FDA with

$$dA^{(3)} = \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^aV^b \quad (3.49)$$

in the limit $e \to 0$. Thus, the particular subalgebra (3.28) of the Maxwell superalgebra (3.27) underlying supergravity in four dimensions can be written as

$$
\begin{aligned}
R^{ab} &= 0\,, \\
DV^a &= \frac{i}{2}\bar{\Psi}\gamma^a\Psi\,, \\
DB^{ab} &= \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi\,, \\
D\tilde{B}^{ab} &= e\bar{\Psi}\gamma^{ab}\Phi + eV^aV^b\,, \\
D\Psi &= 0\,, \\
D\Phi &= \frac{i}{2}\gamma_a\Psi V^a\,, \\
dA^{(3)} &= \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^aV^b + e\bar{\Psi}\gamma_{ab}\Phi\bar{\Psi}\gamma^{ab}\Phi + 2e\bar{\Psi}\gamma_{ab}\Phi V^aV^b,
\end{aligned}
\tag{3.50}
$$

where, having set $\beta = \epsilon = 0$ in (3.27) and (3.47), we have erased the B^{ab}-contributions in $D\tilde{B}^{ab}$ and $D\Phi$.

The hidden superalgebra (3.50) underlying $D = 4$ supergravity is an extension involving cosmological constant of the hidden superalgebra underlying Poincaré supergravity in four dimensions. In particular, the superalgebra (3.50) is a subalgebra of the hidden Maxwell superalgebra obtained by performing an infinite S-expansion with subsequent ideal subtraction on the hidden AdS-Lorentz superalgebra underlying $D = 4$ supergravity. In the FDAs' framework, the parametrization of the 3-form $A^{(3)}$ appearing in the four-dimensional FDA presents a *topological structure*, which reflects on the (trivial) boundary contribution (or flux contribution) $dA^{(3)}$, as we can see from (3.45) and (3.50). Furthermore, the last expression in (3.50) consistently reproduces the FDA in Minkowski space, and in particular $dA^{(3)} = \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^aV^b$, when $e \to 0$. This new model underlying $D = 4$ supergravity can be considered for the construction of a Lagrangian and for the study of the dynamics of the theory.

For the sake of completeness, we finally observe that the parametrization (3.42) can be also rewritten in the following form:

$$
A^{(3)} = \tilde{B}_{ab}V^aV^b + \beta\tilde{B}_{ab}B^{ac}B_c^{\ b} - i\bar{\Psi}\gamma_a\Phi V^a + \bar{\Psi}\gamma_{ab}\left[\left(\frac{1}{2e}\Psi - \epsilon\Phi\right)B^{ab} + \Phi\tilde{B}^{ab}\right], \tag{3.51}
$$

where we remind that $\beta = -2\epsilon$, which shows us that the parametrization we have considered in the present work is given in terms of 1-forms structures that are pretty similar to the ones appearing in the ("standard") parametrization of $A^{(3)}$ adopted in the Minkowski $D = 11$ case in [2], and later in [3]. For $\beta = \epsilon = 0$, the parametrization (3.51) becomes

$$
A^{(3)} = \tilde{B}_{ab}V^aV^b - i\bar{\Psi}\gamma_a\Phi V^a + \bar{\Psi}\gamma_{ab}\left[\frac{1}{2e}\Psi B^{ab} + \Phi\tilde{B}^{ab}\right]. \tag{3.52}
$$

4 Comments and possible developments

In the present work, we have obtained a particular hidden Maxwell superalgebra underlying supergravity in four dimensions, by performing an infinite S-expansion of the hidden AdS-Lorentz superalgebra underlying the same theory, with subsequent ideal subtraction.

We have then written the hidden Maxwell superalgebra in the Maurer-Cartan formalism, and we have subsequently considered the parametrization of the 3-form $A^{(3)}$ in terms of 1-forms, in order to show the way in which the (trivial) boundary contribution in four dimensions, $dA^{(3)}$, can be naturally extended by considering particular contributions to the structure of the extra fermionic generator appearing in the hidden Maxwell superalgebra. These extensions involve the cosmological constant and, interestingly, their structure strictly depends on the form of the extra fermionic generators appearing in this hidden extension of $D = 4$ supergravity.

It would be interesting to write the Lagrangian in four dimensions considering non-trivial boundary terms, by looking at the new structure of $dA^{(3)}$, and to study other possible extensions of $dA^{(3)}$ depending on the cosmological constant, when considering different (hidden) superalgebras. This study can also be extended to theories in higher dimensions.

Another interesting development of the present work would be the study of Chern-Simons theories in even dimensions, such as the four dimensional case, and Born-Infeld theories, since they are topological theories and they can be affected by the presence of a non-trivial boundary.

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A Infinite S-expansion with ideal subtraction of the hidden AdS-Lorentz superalgebra in four dimensions

In the following, we adopt the technique proposed in [35], namely an infinite S-expansion involving an abelian semigroup $S_E^{(\infty)}$, with subsequent subtraction of a suitable ideal, in order to obtain the hidden Maxwell superalgebra in four dimensions (2.2), generated by the set of generators $\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, Q_\alpha, \Sigma_\alpha\}$, by starting from the hidden AdS-Lorentz superalgebra (2.1), generated by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha, Q'_\alpha\}$.

Thus, we consider the commutation relations of the hidden AdS-Lorentz superalgebra in four dimensions (2.1). We report them here for completeness:

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \\
[J_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \\
[Z_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \\
[Q_\alpha, Z_{ab}] &= -\,(\gamma_{ab}Q)_\alpha - (\gamma_{ab}Q')_\alpha, \quad [Q'_\alpha, Z_{ab}] = 0, \qquad [J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \\
[Q_\alpha, P_a] &= -\,i(\gamma_a Q)_\alpha - i(\gamma_a Q')_\alpha, \quad [Q'_\alpha, P_a] = 0, \qquad [P_a, P_b] = -\,Z_{ab}, \\
[J_{ab}, Q_\alpha] &= -\,(\gamma_{ab}Q)_\alpha, \quad [J_{ab}, Q'_\alpha] = -(\gamma_{ab}Q')_\alpha, \qquad [Z_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \\
\{Q_\alpha, Q_\beta\} &= -\,i(\gamma^a C)_{\alpha\beta}P_a - \frac{1}{2}(\gamma^{ab}C)_{\alpha\beta}Z_{ab}, \qquad \{Q_\alpha, Q'_\beta\} = \{Q'_\alpha, Q'_\beta\} = 0.
\end{aligned}
\tag{A.1}
$$

We consider the subspace partition $V_0 = \{J_{ab}\}$, $V_1 = \{Q_\alpha\}$, $V_2 = \{Z_{ab}\}$, $V_3 = \{Q'_\alpha\}$, $V_4 = \{P_a\}$, and we perform an infinite (resonant) S-expansion with the infinite semigroup $S_E^{(\infty)} = \{\lambda_0, \lambda_1, \lambda_2, \dots, \infty\}$, namely

$$\hat{V}_0 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots, \infty\} \times \{J_{ab}\}, \tag{A.2}$$
$$\hat{V}_1 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots, \infty\} \times \{Q_\alpha\}, \tag{A.3}$$
$$\hat{V}_2 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots, \infty\} \times \{Z_{ab}\}, \tag{A.4}$$
$$\hat{V}_3 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots, \infty\} \times \{Q'_\alpha\}, \tag{A.5}$$
$$\hat{V}_4 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots, \infty\} \times \{P_a\}, \tag{A.6}$$

where $\hat{V}_i,\ i = 0, 1, 2, 3, 4$ are the subspaces of the target superalgebra (here and in the following, we will refer to the quantities related to the target superalgebra as to quantities with the upper $\hat{}$ symbol).

Let us remind that the semigroup $S_E^{(\infty)}$ is an extension and generalization of the semigroups of the type $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$, endowed with the multiplication rules $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha + \beta \leq N + 1$, and $\lambda_\alpha\lambda_\beta = \lambda_{N+1}$ if $\alpha + \beta > N + 1$.

Then, we define

$$\hat{J}_{ab} \equiv \lambda_0 J_{ab}, \tag{A.7}$$
$$\hat{Z}_{ab} \equiv \lambda_2 Z_{ab}, \tag{A.8}$$
$$\hat{\hat{Z}}_{ab} \equiv \lambda_4 Z_{ab}, \tag{A.9}$$
$$\hat{Q}_\alpha \equiv \lambda_1 Q_\alpha, \tag{A.10}$$
$$\hat{\Sigma}_\alpha \equiv \lambda_3 Q'_\alpha, \tag{A.11}$$
$$\hat{P}_a \equiv \lambda_2 P_a, \tag{A.12}$$
$$\tag{A.13}$$

and we perform, by following the procedure described in [35], the subtraction of the infinite ideal given by

$$\mathcal{I} = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus W_4, \tag{A.14}$$

where

$$W_0 = \{\lambda_1, \lambda_2, \lambda_3, \dots, \infty\}, \tag{A.15}$$
$$W_1 = \{\lambda_0, \lambda_2, \lambda_3, \lambda_4 \dots, \infty\}, \tag{A.16}$$
$$W_2 = \{\lambda_0, \lambda_1, \lambda_3, \lambda_5, \lambda_6, \dots, \infty\}, \tag{A.17}$$
$$W_3 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_4, \lambda_5, \dots, \infty\}, \tag{A.18}$$
$$W_4 = \{\lambda_0, \lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \dots, \infty\}. \tag{A.19}$$

We also have to perform the change of basis

$$Q'_\alpha \to Q'_\alpha + Q_\alpha, \tag{A.20}$$

which implies

$$\hat{\Sigma}_\alpha \equiv \lambda_3 Q'_\alpha \to \lambda_3 Q'_\alpha + \lambda_3 Q_\alpha, \tag{A.21}$$

where we remind that $\{\lambda_3 Q_\alpha\}$ belongs to the ideal $\mathcal{I}$.

If we now write the expansion of the commutation relations (A.1), and we rename the target generators by simply removing the upper $\hat{}$ symbol, namely

$$\hat{J}_{ab} \to J_{ab}, \tag{A.22}$$
$$\hat{Z}_{ab} \to Z_{ab}, \tag{A.23}$$
$$\hat{\tilde{Z}}_{ab} \to \tilde{Z}_{ab}, \tag{A.24}$$
$$\hat{P}_a \to P_a, \tag{A.25}$$
$$\hat{Q}_\alpha \to Q_\alpha, \tag{A.26}$$
$$\hat{\Sigma}_\alpha \to \Sigma_\alpha, \tag{A.27}$$

we finally end up with the hidden Maxwell superalgebra in four dimensions (2.2), generated by $\{J_{ab}, Z_{ab}, \tilde{Z}_{ab}, P_a, Q_\alpha, \Sigma_\alpha\}$. For the sake of completeness, we also report its commutation relations in the following:

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \\
[J_{ab}, Z_{cd}] &= \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \\
\left[J_{ab}, \tilde{Z}_{cd}\right] &= \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \\
[Z_{ab}, Z_{cd}] &= \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \\
[Q_\alpha, Z_{ab}] &= -(\gamma_{ab}\Sigma)_\alpha, \quad [\Sigma_\alpha, Z_{ab}] = 0, \qquad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \\
[Q_\alpha, P_a] &= -i(\gamma_a \Sigma)_\alpha, \quad [\Sigma_\alpha, P_a] = 0, \qquad [P_a, P_b] = -\tilde{Z}_{ab}, \\
[J_{ab}, Q_\alpha] &= -(\gamma_{ab} Q)_\alpha, \quad [J_{ab}, \Sigma_\alpha] = -(\gamma_{ab}\Sigma)_\alpha, \qquad [Z_{ab}, P_c] = 0, \\
\{Q_\alpha, Q_\beta\} &= -i(\gamma^a C)_{\alpha\beta} P_a - \frac{1}{2}(\gamma^{ab} C)_{\alpha\beta} Z_{ab}, \qquad \{\Sigma_\alpha, \Sigma_\beta\} = 0, \\
\{Q_\alpha, \Sigma_\beta\} &= -2(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}, \\
\left[\tilde{Z}_{ab}, P_c\right] &= \left[Q_\alpha, \tilde{Z}_{ab}\right] = \left[\Sigma_\alpha, \tilde{Z}_{ab}\right] = \left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = 0.
\end{aligned} \tag{A.28}$$

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