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# Flow simulations in porous media with immersed intersecting fractures

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## Abstract

A novel approach for fully 3D flow simulations in porous media with immersed networks of fractures is presented. The method is based on the discrete fracture model, in which fractures are represented as  $n - 1$ -dimensional objects in an  $n$ -dimensional porous matrix. The problem, written in primal formulation on both the fractures and the porous matrix, is solved resorting to the constrained minimization of a properly designed cost functional that expresses the matching conditions at fracture-fracture and fracture-matrix interfaces. The method, originally conceived for intricate fracture networks in impervious rock matrices, is here extended to fractures in a porous permeable rock matrix. The purpose of the optimization approach is to allow for an easy meshing process, independent of the geometrical complexity of the domain, and for a robust and efficient resolution tool, relying on a strong parallelism. The present work is devoted to the presentation of the new method and of its applicability to flow simulations in poro-fractured domains.

*Keywords:* 3D flows, Darcy flows, matrix-fracture coupled flows, optimization methods for elliptic problems, uncoupled large scale simulations, BEM

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## 1. Introduction

The present work focuses on the simulation of the flow in the underground, which is an active research field as a consequence of its relevance in many practical activities, ranging from environmental applications, to the

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geological storage of pollutants, or to energy production purposes, such as oil and gas production or geothermal applications.

The underground can be regarded as a porous medium, crossed by a network of intersecting fractures. Fractures are regions of the soil typically characterized by a dramatic change in the properties of the porous medium, confined in a layer with one of the spatial dimensions, the thickness, that is orders of magnitude smaller than the other two. On the other hand, fracture main dimensions might also span several orders of magnitude, with the simultaneous presence of small fractures and large fractures or faults. Fractures might have a strong impact on the flow, acting as preferential flow paths, or as barriers: the former is the case of highly conductive fractures, with a permeability much greater than that of the surrounding porous medium, the latter being, instead, the case of almost impervious fractures [1, 2].

The over-mentioned geometrical complexity and multi-scale nature of the subsoil makes the simulation of flows in the underground an extremely challenging task. A major difficulty lies in the representation of the fractures immersed in the porous matrix: due to the very small aperture of fractures with respect to both fracture lengths and domain size, the discretization of the fracture layer would require the introduction of a very large number of elements, thus increasing the computational cost.

The classical approach to flow simulations in fractured media is based on dual or multi-porosity models [3, 4], in which the subsoil is modeled as two (matrix and fractures in the dual porosity model) or multiple homogenized continua with special transfer functions handling the interactions among the continua. Alternative approaches are based instead on discrete fracture models (DFM) that have the advantage of an explicit representation of the fractures, thus improving the accuracy of predictions, mainly when fractures are sparse and with length comparable to that of the simulation domain. DFMs use asymptotic models to describe fractures, in which fractures are seen as  $n - 1$ -dimensional objects in the  $n$ -dimensional matrix. The Darcy law is used in the  $n$ -dimensional domain, whereas an averaged Darcy law across thickness is used in the  $n - 1$ -dimensional fractures. Suitable interface conditions are derived and applied in order to close the problem. This approach is usually denoted by *reduced Darcy model* and is followed for example in [5, 2, 6, 7, 8, 9, 10, 11]. Another possible approach is based, e.g., on hybrid-grid models, where fractures are  $n - 1$ -dimensional only in the geometric grid, whereas an  $n$ -dimensional grid taking into account fracture aperture is then used for computations, as done, among others, in [12, 13].

The limitation in the use of discrete fracture models lie in the difficulty of handling the intricate network of fractures immersed in the porous matrix. In fact, fracture networks for realistic simulations might count up to millions

of fractures mutually intersecting with extremely narrow angles. Further, geometrical models of the underground are built starting from probabilistic data, thus requiring a large number of simulations corresponding to different realizations of the random variables in order to tackle the issue of uncertainty. Numerical methods requiring some sort of conformity of the computational mesh with the network of fractures might therefore have limited applicability in realistic cases, for the impossibility of generating a mesh given the huge number of geometrical constraints. Different approaches have been recently proposed in order to overcome this limitation. Among others, in [7, 14], the authors use extended finite elements (XFEM) [15] with a mixed formulation for the reduced Darcy problem, allowing fractures to be arbitrarily placed with respect to the mesh of the porous medium. The XFEM with special enrichment functions is used also in [16]. In this latter work the authors do not use a reduced Darcy model, as fractures are  $n - 1$ -dimensional interfaces in an  $n$ -dimensional domain only in the geometrical domain, whereas computations are performed still considering a variable fracture aperture. Also the new Virtual Element Method [17, 18, 19, 20, 21, 22] has been successfully used in order to ease the meshing burden for flow simulations in complex networks of fractures [23, 24, 25, 26]. In other works the mesh of the porous medium can not cross fractures, but a non conformity is allowed between the mesh elements on the  $n - 1$ -dimensional fractures and the neighboring elements of the  $n$ -dimensional porous medium, as, for example, in [27, 10, 28] where mimetic finite differences (MFD) [29] are used, or in [30] with mixed finite elements. It is also available a vast literature on finite volumes, two point flux approximation (TPFA) or multi-point flux approximation (MPFA) [31], (see e.g. [32, 6, 33, 13, 8, 34, 35, 36]), also recently studied in the general framework of gradient schemes [11].

This work focuses on the simulation of single phase flows in porous media with an embedded network of fractures, described by means of the discrete fracture model. Fractures are assumed to have a higher permeability than the surrounding porous matrix, thus pressure continuity conditions are prescribed across fractures with discontinuities in the velocity; continuity of pressure and flux conservation is also enforced at fracture intersections. Fully embedded fracture networks, i.e. absence of fractures hitting the domain boundary, are allowed, supposing a no-flow condition on the fracture boundaries lying in the interior of the domain. Assuming, as usual, a power law relating the transmissivity of the fractures to their aperture, the no-flow condition at fracture boundaries can be justified by the fact that the aperture of the fractures goes to zero at the fracture boundary.

The reduced Darcy problem is written in primal formulation for both the fractures and the porous matrix. The problem is formulated in the

correct functional spaces and well posedness is demonstrated, similarly to what proposed in [9]. The novelty of the proposed approach lies in the fact that the resulting system of equations is solved by means of a PDE-constrained optimization problem, thus generalizing the method proposed in [37, 38, 39, 40, 41, 42, 43, 44] for DFNs characterized by an impervious matrix.

The optimization approach naturally leads to the decoupling of the original problem into a set of smaller sub-problems. The computational domain is split into smaller sub-domains not crossing the fractures, and, within the optimization framework, only the resolution of almost independent problems on the small domains need to be solved. Further, as shown in the aforementioned references, the problem on the fracture network is decoupled into smaller problems on the single fractures. A properly designed cost functional measures the error in satisfying the prescribed matching conditions at the interfaces of the new sub-problems, and then the global solution is obtained as the constrained minimum of this functional.

The key advantages of this approach, as already shown for the flow problem in DFNs, lie in the possibility of an extremely easy meshing process, that can be performed independently of the interfaces in the domain. Furthermore, the method has an intrinsic scalable nature, with clear advantages for the handling of problems on large domains with intricate fracture networks.

The purpose of the present manuscript is to introduce the formulation of the new approach, demonstrating its equivalence with the “classical” formulation, and the well posedness of the resulting discrete problem. Numerical results are provided in order to show convergence properties of the method and its applicability and effectiveness, providing both quantitative and qualitative comparisons with reference solutions. The realization of numerical tests on realistic configurations is deferred, instead, to a forthcoming work.

The structure of the work is as follows: in section 2, after an introduction on the relevant notation, the formulation of the problem as a PDE-constrained optimization problem is formulated in a continuous framework. Well posedness of the new formulation is proven. Section 3 reports the discrete formulation of the problem in the optimization framework. A proof of existence and uniqueness of the discrete solution is provided in Section 4. Numerical tests are described in Section 5, whereas some concluding remarks are proposed in Section 6.

## 2. Problem formulation

### 2.1. Notation

The present Section introduces the notation used in the following. As a general introduction, the symbol  $\mathcal{D}$ , possibly with a subscript, is used to denote an open polyhedral subset of  $\mathbb{R}^3$ . Indexes  $k$  denoting rock matrix blocks  $\mathcal{D}_k$  are collected in index sets identified by the symbol  $\mathcal{K}$ . Planar polygons in  $\mathbb{R}^3$  are denoted either by  $\Gamma$ , for matrix block boundaries, or by  $F$ , for fractures, and subscripts are used. The set collecting all the indexes  $\ell$  of faces  $\Gamma_\ell$  is denoted by the letter  $\mathcal{L}$ . Suitable subsets of  $\mathcal{L}$  are marked with subscripts and superscripts as summarized in Table 1. Indexes  $i$  of polygons  $F_i$  are collected in index sets  $\mathcal{I}$ , also accompanied by a subscript restricting the elements contained in the set. Segments in the three dimensional space are denoted by letter  $S$ , for fracture intersections, or by  $\gamma$ , for fracture boundaries, both with subscripts.

Let us consider a polyhedron  $\mathcal{D}$  in the 3-D space, representing a matrix block. Let us further consider a set of intersecting planar polygons  $\bar{F}_i$ ,  $i = 1, \dots, I$  in  $\mathcal{D}$ , representing a discrete fracture network, called  $\Omega$  and given by the union of the fractures  $\bar{F}_i$ . We assume that the fractures in  $\Omega$  intersect each other in segments, called traces, each denoted by  $S_m$ ,  $m = 1, \dots, M$ . The set of all the traces in  $\Omega$  is  $\mathcal{S}$ ,  $\mathcal{S}_i$  is the set of traces in  $\bar{F}_i$  and, under the assumption that each trace is the intersection of exactly two fractures, i.e.  $S_m = \bar{F}_i \cap \bar{F}_j$ , we also define, for each  $m = 1, \dots, M$ , the set  $\mathcal{I}_{S_m} = \{i, j\}$ .

Each fracture  $\bar{F}_i$  lies on a plane  $\pi_i$ , and, for each edge  $\gamma_{F_i}^e$ ,  $e = 1, \dots, n_{i,\gamma}$ , of the boundary of  $\bar{F}_i$  that is in the interior of  $\mathcal{D}$  we introduce a plane  $\pi_i^e$  orthogonal to  $\pi_i$  and such that  $\gamma_{F_i}^e \subset \pi_i^e$ . We denote by  $\Pi_i = \bar{F}_i \cup (\bigcup_{e=1}^{n_{i,\gamma}} \pi_i^e)$  the subset of  $\mathbb{R}^3$  given by the union of fracture  $\bar{F}_i$  and the planes orthogonal to  $\bar{F}_i$  and each containing one of the edges  $\gamma_{F_i}^e$ ,  $e = 1, \dots, n_{i,\gamma}$ . The domain  $\mathcal{D}$  is then split into open sub-domains (polyhedrons)  $\mathcal{D}_k \subset \mathcal{D}$ ,  $k = 1, \dots, n_{\mathcal{D}}$  that do not cross any of the  $\Pi_i$ ,  $i = 1, \dots, I$ , i.e. such that  $\mathcal{D}_k \cap \Pi_i = \emptyset$ ,  $\forall i = 1, \dots, I$ .

The boundary of each block  $\mathcal{D}_k$ ,  $k = 1, \dots, n_{\mathcal{D}}$  is the union of closed planar polygons in the three dimensional space, each denoted by  $\bar{\Gamma}_\ell$ ,  $\ell = 1, \dots, n_\Gamma$ . Let  $\mathcal{L}$  be the set of all the indexes of faces  $\Gamma_\ell$ , and  $\mathcal{L}_k$  the set of face indexes of block  $\mathcal{D}_k$ ,  $k = 1, \dots, n_{\mathcal{D}}$ . Each face  $\bar{\Gamma}_\ell$  is either a boundary face, i.e. a portion of the boundary of  $\mathcal{D}$ ,  $\partial\mathcal{D}$ , or an inner face, shared by exactly two matrix blocks: we denote by  $\mathcal{L}_{\partial\mathcal{D}} \subset \mathcal{L}$  the set of indexes of boundary faces, and by  $\mathcal{L}_{\mathcal{D}} \subset \mathcal{L}$  the set of indexes of inner faces, such that for each  $\ell \in \mathcal{L}_{\mathcal{D}}$ ,  $\bar{\Gamma}_\ell = \bar{\mathcal{D}}_y \cap \bar{\mathcal{D}}_z$ , and we collect these couple of indexes in the index set  $\mathcal{K}_{\Gamma_\ell} = \{y, z\}$ . We have that each fracture  $\bar{F}_i$  in  $\Omega$  can be obtained as the union of some block faces in  $\mathcal{L}$ , and we denote by  $\mathcal{L}_{F_i}$  the set of face

indexes such that  $\bar{F}_i = \bigcup_{\ell \in \mathcal{L}_{F_i}} \bar{\Gamma}_\ell$ . We also define the set of all face indexes that lie on a fracture,  $\mathcal{L}^\star = \bigcup_{i=1}^I \mathcal{L}_{F_i}$  and, for each matrix block, the set of block faces lying on a fracture  $\mathcal{L}_k^\star = \mathcal{L}_k \cap \mathcal{L}^\star$ ; we then call  $\Gamma_{\mathcal{D}_k} = \bigcup_{\ell \in \mathcal{L}_k^\star} \bar{\Gamma}_\ell$  the portion of the boundary of block  $\mathcal{D}_k$  that is part of some fracture. We remark that the set  $\mathcal{L}_{\mathcal{D}}$  differs from the set  $\mathcal{L}^\star$  since in the former there are also indexes of block faces that are not subsets of fractures of the DFN, and this is a consequence of the strategy chosen to split the original matrix block into sub-blocks not crossing the fractures; we then introduce the function  $\varrho : \mathcal{L}^\star \mapsto [1, \dots, I]$  defined such that, for any  $\ell \in \mathcal{L}^\star$ ,  $\bar{\Gamma}_\ell$  is a subset of  $\bar{F}_{\varrho(\ell)}$ . Finally the set of indexes of faces that are not subset of any fracture is denote by  $\mathcal{L}^\diamond$ , i.e.  $\mathcal{L}^\diamond = \mathcal{L}_{\mathcal{D}} \setminus \mathcal{L}^\star$ .

For each trace  $S_m$  in  $\Omega$  we denote by  $\mathcal{L}_{S_m}$  the set of face indexes  $\ell$  such that  $S_m$  is part of the boundary of  $\bar{\Gamma}_\ell$ , i. e. the set of indexes of block faces that are contiguous to fracture intersections:  $\mathcal{L}_{S_m} = \{\ell \in \mathcal{L} : \partial \bar{\Gamma}_\ell \cap S_m \neq \emptyset\}$ .

By  $F_i$  and  $\Gamma_\ell$  we denote the open subsets of  $\mathbb{R}^2$  such that  $\bar{F}_i$  and  $\bar{\Gamma}_\ell$  are the closure of the image of  $F_i$  and  $\Gamma_\ell$ , respectively, through an affine mapping from  $\mathbb{R}^2 \mapsto \mathbb{R}^3$  (see [37] for a possible definition of the mapping).

For any sufficiently regular function  $v$  defined in  $\mathcal{D}$ , let us denote by  $v_k$  the restriction of  $v$  to  $\mathcal{D}_k$  and by  $v_k^\ell$  the trace of  $v_k$  on  $\Gamma_\ell$ . Further, by  $v_{F_i}$  we indicate the trace of  $v$  on  $F_i$  and by  $v_{F_i}^\ell$  the restriction of  $v_{F_i}$  to  $\Gamma_\ell$  for  $\ell \in \mathcal{L}_{F_i}$ .

We assume, for the sake of simplicity, that homogeneous Dirichlet boundary conditions are prescribed on  $\partial \mathcal{D}$  and on the boundaries of the fractures lying on  $\partial \mathcal{D}$ , which we denote by  $\gamma_{F_i, D} = \partial F_i \cap \partial \mathcal{D}$ ,  $\forall i = 1, \dots, I$ , whereas homogeneous Neumann boundary conditions are prescribed on the remaining portion of fracture boundary, denoted by  $\gamma_{F_i, N} = \partial F_i \setminus \gamma_{F_i, D}$ .

Let us set a functional space on the fractures in  $\Omega$ , as

$$H_{i, \gamma}^1(F_i) = \left\{ v \in H^1(F_i) : v|_{\gamma_{F_i, D}} = 0 \right\};$$

and then the functional space on the whole domain  $H_\Omega^1(\mathcal{D})$  is defined as:

$$H_\Omega^1(\mathcal{D}) = \left\{ v \in H_0^1(\mathcal{D}) : v_{F_i} \in H_{i, \gamma}^1(F_i), \ i = 1, \dots, I \right. \\ \left. v_{F_i}|_{S_m} = v_{F_j}|_{S_m}, \ m = 1, \dots, M, \ i, j \in \mathcal{I}_{S_m} \right\}.$$

The space  $H_\Omega^1$  can be easily seen to be a Hilbert space endowed with the scalar product  $(v, w)_{\mathcal{D}, \Omega} = (\nabla v, \nabla w)_{\mathcal{D}} + \sum_{i=1}^I (\nabla_{\pi_i} v_{F_i}, \nabla_{\pi_i} w_{F_i})_{F_i}$ ,  $\forall v, w \in H_\Omega^1$ , being  $\nabla_{\pi_i}$  the gradient on the tangential plane  $\pi_i$ .

The transmissivity of the rock matrix is  $\mathbf{K}^m$  and  $\mathbf{K}_k^m$  is its restriction to  $\mathcal{D}_k$ ;  $\mathbf{K}_{F_i}$  is instead the transmissivity of fracture  $F_i$ .

Table 1: Notation of index sets  $\mathcal{K}$ ,  $\mathcal{L}$ 

| Label                               | Description   |
|-------------------------------------|---|
| $\mathcal{K}$                       | all block indexes (dimension $n_{\mathcal{D}}$ )  |
| $\mathcal{K}_{\Gamma_\ell}$         | block indexes $y, z$ with $y \neq z$ such that $\bar{\Gamma}_\ell = \bar{\mathcal{D}}_y \cap \bar{\mathcal{D}}_z$               |
| $\mathcal{K}_{F_i}$                 | indexes of blocks having one face on $F_i$  |
| $\mathcal{L}$                       | all block-face indexes (dimension $n_{\Gamma}$ )  |
| $\mathcal{L}_k$                     | indexes of block $\mathcal{D}_k$  |
| $\mathcal{L}_{\mathcal{D}}$         | indexes of faces not belonging to the boundary of $\mathcal{D}$   |
| $\mathcal{L}_{\partial\mathcal{D}}$ | indexes of block-faces belonging to the boundary of $\mathcal{D}$   |
| $\mathcal{L}^*$                     | indexes of block-faces lying on a fracture  |
| $\mathcal{L}_{F_i}$                 | indexes of block-faces lying on fracture $F_i$  |
| $\mathcal{L}_k^*$                   | indexes of faces of block $\mathcal{D}_k$ lying on a fracture   |
| $\mathcal{L}^\diamond$              | indexes of block-faces not lying on any fracture ( $\mathcal{L}^\diamond = \mathcal{L}_{\mathcal{D}} \setminus \mathcal{L}^*$ ) |
| $\mathcal{L}_{S_m}$                 | indexes of block-faces contiguous to fracture intersection $S_m$  |

## 2.2. Problem setting

We look for a solution  $H$  of the hydraulic head problem in  $\mathcal{D}$  that satisfies, for a given load  $f$ :

$$-\nabla \cdot (\mathbf{K}_k^m \nabla H_k) = f_k \quad \text{in } \mathcal{D}_k, \quad k = 1, \dots, n_{\mathcal{D}} \quad (1)$$

$$-\nabla_{\pi_i} \cdot (\mathbf{K}_{F_i} \nabla_{\pi_i} H_{F_i}^\ell) = \tilde{Q}^\ell \quad \text{in } \Gamma_\ell, \quad \text{for } i = 1 \dots, I, \quad \forall \ell \in \mathcal{L}_{F_i} \quad (2)$$

$$H = 0 \quad \text{on } \partial\mathcal{D} \quad (3)$$

being  $H_{F_i}^\ell$  the restriction to  $\Gamma_\ell$  of  $H_{F_i}$ , which is, in turn, the trace of  $H$  on  $F_i$ . In (2),  $\tilde{Q}^\ell$  is the unknown load on the face  $\Gamma_\ell$ . Additional conditions need to be imposed at the block faces  $\Gamma_\ell$ , imposing continuity of the solution and balance of fluxes. Recalling that  $\mathcal{L}_{\mathcal{D}}$  is the set of inner face indexes,  $\mathcal{L}^*$  is the set of indexes of faces that are a subset of a fracture, and that, instead  $\mathcal{L}^\diamond$  contains indexes of faces that are not contained in any fracture, matching conditions at block faces are written as:

$$H_y^\ell = H_z^\ell, \quad \forall \ell \in \mathcal{L}_{\mathcal{D}}, \quad y, z \in \mathcal{K}_{\Gamma_\ell}, y \neq z, \quad (4)$$

$$\tilde{Q}_y^\ell + \tilde{Q}_z^\ell = -\tilde{Q}^\ell, \quad \forall \ell \in \mathcal{L}^*, \quad y, z \in \mathcal{K}_{\Gamma_\ell}, y \neq z, \quad (5)$$

$$\tilde{Q}_y^\ell + \tilde{Q}_z^\ell = 0, \quad \forall \ell \in \mathcal{L}^\diamond, \quad y, z \in \mathcal{K}_{\Gamma_\ell}, y \neq z, \quad (6)$$

being  $\tilde{Q}_k^\ell = (\mathbf{K}_k^m \nabla H_k) \cdot \mathbf{n}_k^\ell$  with  $\mathbf{n}_k^\ell$  the unit normal vector to  $\Gamma_\ell$  oriented outward from  $\mathcal{D}_k$ .

Additional conditions need to be enforced at fracture intersections, prescribing continuity of the solution and flux balance at the traces: for  $m =$



$1, \dots, M$ :

$$H_{F_{\varrho(\ell)}|S_m}^\ell - H_{F_{\varrho(r)}|S_m}^r = 0, \quad \text{for } \ell, r \in \mathcal{L}_{S_m}, \ell \neq r \quad (7)$$

$$\sum_{\ell \in \mathcal{L}_{S_m}} \left( \mathbf{K}_{F_{\varrho(\ell)}} \nabla_{\pi_{\varrho(\ell)}} H_{F_{\varrho(\ell)}}^\ell \right) \cdot \mathbf{n}_{\Gamma_\ell}^m = 0, \quad (8)$$

being  $\mathbf{n}_{\Gamma_\ell}^m$  the unit normal vector to trace  $S_m$  with fixed orientation on  $\Gamma_\ell$ . We remark that continuity of the solution at fracture boundaries is implicitly enforced, thanks to the definition of  $H_{F_i} = H|_{F_i}$ ,  $i = 1, \dots, I$  and to equation (4).

In the previous matching conditions, equation (4) prescribes the continuity of the solution across the fracture planes, while equation (5) sets the outflow from the matrix blocks as a source term for the equations on the fracture planes. Equations (6), instead, enforce flux balance conditions across block faces that do not lie on any fracture.

The variational formulation of problem (1)-(8) is given by the following system of equations: Find  $H \in H_\Omega^1(\mathcal{D})$  such that,  $\forall v \in H_\Omega^1(\mathcal{D})$ :

$$(\mathbf{K}^m \nabla H, \nabla v)_\mathcal{D} = (f, v)_\mathcal{D} + \sum_{i=1}^I \sum_{\ell \in \mathcal{L}_{F_i}} \left\langle \tilde{Q}^\ell, v_{F_i}^\ell \right\rangle_{\Gamma_\ell} \quad (9)$$

$$\sum_{i=1}^I (\mathbf{K}_{F_i} \nabla_{\pi_i} H_{F_i}, \nabla_{\pi_i} v_{F_i})_{F_i} = - \sum_{i=1}^I \sum_{\ell \in \mathcal{L}_{F_i}} \left\langle \tilde{Q}^\ell, v_{F_i}^\ell \right\rangle_{\Gamma_\ell}. \quad (10)$$

The previous problem is well posed as its solution  $H \in H_\Omega^1(\mathcal{D})$  is such that

$$H = \min_{w \in H_\Omega^1(\mathcal{D})} \mathcal{E}(w)$$

i.e. it corresponds to the unique minimum of the quadratic functional:

$$\mathcal{E}(w) = \frac{1}{2} \int_\mathcal{D} \left( \left| \sqrt{\mathbf{K}^m} \nabla w \right|^2 - 2fw \right) + \frac{1}{2} \sum_{i=1}^I \int_{F_i} \left| \sqrt{\mathbf{K}_{F_i}} \nabla_{\pi_i} w_{F_i} \right|^2.$$

Another proof of well posedness of the problem in this classical formulation is available for example in [9].

### 2.3. Optimization formulation

The solution of problem (9)-(10) can be seen as the constrained minimum of a properly designed cost functional. To this end, let us introduce the following functional spaces:  $\forall k = 1, \dots, n_\mathcal{D}$ ,

$$V_k = H_\Omega^1(\mathcal{D})|_{\mathcal{D}_k} = \{v \in H^1(\mathcal{D}_k) : v|_{\partial\mathcal{D} \cap \partial\mathcal{D}_k} = 0, \exists w \in H_\Omega^1(\mathcal{D}) : v = w|_{\mathcal{D}_k}\};$$

$V_{F_i} = H_\gamma^1(F_i) \forall i = 1, \dots, I$ ; the space  $\mathcal{H}_\Gamma^\ell = H^1(\Gamma_\ell)$ ,  $\forall \ell = 1, \dots, n_\Gamma$  and its dual  $\mathcal{Q}^\ell = (\mathcal{H}_\Gamma^\ell)'$ ; the space  $\mathcal{H}_S^m = H^{\frac{1}{2}}(S_m)$ ,  $\forall m = 1, \dots, M$  and its dual  $\mathcal{U}^m = (\mathcal{H}_S^m)'$ . Starting from the above definitions, the following product spaces are introduced:

$$V = \prod_{k=1}^{n_D} V_k, \quad V_F = \prod_{i=1}^I V_{F_i}, \quad \mathcal{U}_i = \prod_{S_m \in \mathcal{S}_i} \mathcal{U}^m, \quad \mathcal{U} = \prod_{i=1}^I \mathcal{U}_i, \quad \mathcal{Q} = \prod_{\ell \in \mathcal{L}_D} \mathcal{Q}^\ell.$$

Let us denote by  $U_i^m \in \mathcal{U}^m$  a linear combination of the trace on  $S_m$  of a function  $H_{F_i} \in V_{F_i}$  and the jump of its co-normal derivative across  $S_m$ :

$$U_i^m = \llbracket (\mathbf{K}_{F_i} \nabla_{\pi_i} H_{F_i}) \cdot \mathbf{n}_{F_i}^m \rrbracket + \alpha H_{F_i|S_m}, \quad (11)$$

for a given constant  $\alpha > 0$ , and then we define  $U_i = \prod_{S_m \in \mathcal{S}_i} U_i^m \in \mathcal{U}_i$  and  $U = \prod_{i=1}^I U_i \in \mathcal{U}$ . Further, let us define  $Q_k^\ell \in \mathcal{Q}^\ell$ , for  $\ell \in \mathcal{L}_D$  and  $k \in \mathcal{K}_{\Gamma_\ell}$  as:

$$Q_k^\ell = \tilde{Q}_k^\ell \cdot \mathbf{n}_k^\ell + \beta H_k^\ell, \quad (12)$$

for a given positive constant  $\beta$ . Then it is  $Q^\ell = Q_k^\ell + Q_y^\ell \in \mathcal{Q}^\ell$ ,  $\ell \in \mathcal{L}_D$ ,  $k, y \in \mathcal{K}_{\Gamma_\ell}$ ,  $k \neq y$ , and  $Q = \prod_{\ell \in \mathcal{L}_D} Q^\ell \in \mathcal{Q}$ .

It is now possible to introduce a cost functional  $J$  defined for any function  $H \in V$ ,  $H_F \in V_F$ ,  $U \in \mathcal{U}$ ,  $Q \in \mathcal{Q}$  as:

$$\begin{aligned} J(H, H_F, U, Q) = & \sum_{\ell \in \mathcal{L}^*} \left\| H_k^\ell - H_{F_{\ell(\ell)}}^\ell \right\|_{\mathcal{H}_\Gamma^\ell}^2 + \sum_{\ell \in \mathcal{L}^\diamond} \left\| H_y^\ell - H_z^\ell \right\|_{\mathcal{H}_\Gamma^\ell}^2 \\ & + \sum_{\ell \in \mathcal{L}^\diamond} \left\| Q_y^\ell + Q_z^\ell - \beta (H_y^\ell + H_z^\ell) \right\|_{\mathcal{Q}^\ell}^2 \\ & + \sum_{m=1}^M \left( \left\| H_{F_i|S_m} - H_{F_j|S_m} \right\|_{\mathcal{H}_S^m}^2 + \left\| U_i^m + U_j^m - \alpha (H_{F_i|S_m} + H_{F_j|S_m}) \right\|_{\mathcal{U}^m}^2 \right), \end{aligned} \quad (13)$$

in which, for each  $\ell \in \mathcal{L}_D$ , it is  $y, z \in \mathcal{K}_{\Gamma_\ell}$   $y \neq z$ , whereas for each  $m = 1, \dots, M$ ,  $i, j \in \mathcal{I}_{S_m}$   $i \neq j$ .

The following problems are written in weak formulation on each of the matrix blocks  $\mathcal{D}_k$ : find  $H_k \in V_k$  such that for all  $v_k \in V_k$ :

$$(\mathbf{K}^m \nabla H_k, \nabla v_k)_{\mathcal{D}_k} + \beta \sum_{\ell \in \mathcal{L}_k} (H_k^\ell, v_k^\ell)_{\mathcal{D}_k} = (f_k, v_k)_{\mathcal{D}_k} + \sum_{\ell \in \mathcal{L}_k} \langle Q_k^\ell, v_k^\ell \rangle_{\Gamma_\ell}, \quad (14)$$

and on each of the fracture planes: find  $H_{F_i} \in V_{F_i}$  such that for all  $v_{F_i} \in V_{F_i}$ :

$$\begin{aligned} (\mathbf{K}_{F_i} \nabla H_{F_i}, \nabla v_{F_i})_{F_i} + \alpha \sum_{m=1}^M (H_{F_i|S_m}, v_{F_i|S_m})_{S_m} = \\ \sum_{\ell \in \mathcal{L}_{F_i}} (-(Q^\ell - \beta H^\ell), v_{F_i})_{F_i} + \sum_{m=1}^M \langle U_i^m, v_{F_i} \rangle_{S_m}, \end{aligned} \quad (15)$$

being  $H^\ell = H_y^\ell + H_z^\ell$ , with  $y, z \in \mathcal{K}_{\Gamma_\ell}$ ,  $y \neq z$ . We remark that, if  $\alpha, \beta > 0$ , problems 14-15 are well posed.

The matrix-fracture flow problem, formulated as a PDE-constrained optimization problem, then reads as follows:

$$\begin{aligned} \min J(H, H_F, U, Q) & \quad (16) \\ \text{constrained by} & \quad (14) \ \forall k = 1, \dots, n_{\mathcal{D}} \\ \text{and by} & \quad (15) \ \forall i = 1, \dots, I \end{aligned}$$

in which the variables  $Q$  and  $U$  act as control variables.

**Proposition 1.** *The solution to (16) is equivalent to the solution of equations (9)-(10).*

*Proof.* It is straightforward to see that the solution of (9)-(10) vanishes the functional and also satisfies the constraints (14)-(15). Conversely, let us consider that  $(H^*, H_F^*, U^*, Q^*) \in (V, V_F, \mathcal{U}, \mathcal{Q})$  is the solution of (16), and  $(H, U, Q) \in (H_\Omega^1(\mathcal{D}), \mathcal{U}, \mathcal{Q})$  is the solution of (9)-(10). Let us further observe that  $H_\Omega^1(\mathcal{D}) \subseteq V$  and  $H_\Omega^1(\mathcal{D})|_{F_i} \subseteq V_{F_i}$ , for  $i = 1, \dots, I$ . Given  $H \in H_\Omega^1(\mathcal{D})$ , let  $H_F$  be defined as  $H_F = (H_{F_i})_{i=1, \dots, I}$ . Since  $H, H_F, U, Q$  satisfy the constraints and  $J(H, H_F, U, Q) = 0$ , then necessarily  $J(H^*, H_F^*, U^*, Q^*) = 0$  and hence  $(H^*, H_F^*, U^*, Q^*) \equiv (H, H_F, U, Q)$  for the well posedness of problem (9)-(10).  $\square$

**Remark 2.** *In the above presentation of the problem homogeneous boundary conditions are prescribed on  $\partial\mathcal{D}$ , for simplicity of notation. The presentation can be readily extended to more general boundary conditions, dividing  $\partial\mathcal{D}$  in a Dirichlet part,  $\Gamma_D \subseteq \partial\mathcal{D}$ ,  $\Gamma_D \neq \emptyset$ , and a Neumann part  $\Gamma_N = \partial\mathcal{D} \setminus \Gamma_D$ . The functional space of the solution,  $H_\Omega^1(\mathcal{D})$  is re-defined as:*

$$\begin{aligned} H_\Omega^1(\mathcal{D}) = \{v \in H^1(\mathcal{D}) : v = G_D \text{ on } \Gamma_D, \ v_{F_i} \in H_{i,\gamma}^1(F_i), \ \forall F_i \in \Omega, \\ v_{F_i|S_m} = v_{F_j|S_m}, \ m = 1, \dots, M, \ i, j \in \mathcal{I}_{S_m}\}, \end{aligned}$$

where  $G_D \in H^s(\Gamma_D)$  is the function prescribing the Dirichlet condition on  $\Gamma_D$ . A higher regularity than  $s = \frac{1}{2}$  is required for  $G_D$  in order to give meaning to the functional space for the fractures. A possible choice is  $s = 1$ , which allows for a treatment of boundary conditions in a weak sense, following the approach proposed in [43]. In this case, the functional space on the fractures,  $H_\gamma^1(F_i)$ , would be defined as:

$$H_{i,\gamma}^1(F_i) = \left\{ v \in H^1(F_i) : v|_{\gamma_{F_i,D}} = f_D|_{\gamma_{F_i,D}} \right\},$$

$$f_D \in H^{\frac{1}{2}}(\gamma_{F_i,D}).$$

Table 2: Notation for the number of degrees of freedom (DOFs)

| <i>Label</i>              | <i>Description</i>  |
|---------------------------|---|
| $N_k^{\Gamma_\ell}$       | Number of elements of $h_k^\ell$ and $q_k^\ell$ , collecting DOFs of $H_k^\ell$ and $Q_k^\ell$ on face $\Gamma_\ell$ of block $\mathcal{D}_k$                 |
| $N_k$                     | Number of elements of $h_k$ and $q_k$ , collecting DOFs of $H_k^\ell$ and $Q_k^\ell$ on all the faces of block $\mathcal{D}_k$                                |
| $N_{\mathcal{D}}$         | Number of elements of $h_{\mathcal{D}}$ and $q$ , collecting DOFs of $H_k^\ell$ and $Q_k^\ell$ on all the faces of all blocks                                 |
| $N_{F_i}^{S_m}$           | Number of elements of $u_i^m$ , collecting DOFs of $U_i^m$ on trace $S_m$ of fracture $F_i$   |
| $N^{S_m}$                 | Number of elements of $u^m$ , collecting DOFs of $U_i^m$ and $U_j^m$ on trace $S_m$ of fractures $F_i$ and $F_j$ with $i, j \in \mathcal{I}_{S_m}$ $i \neq j$ |
| $N_{F_i}^{\mathcal{S}_i}$ | Number of elements in $u_i$ , collecting DOFs of $U_i^m$ on all the traces of fracture $F_i$  |
| $N^{\mathcal{S}_i}$       | Number of elements in $u_i^+$ , collecting DOFs of both the functions $U_x^m$ for $x \in \mathcal{I}_{S_m}$ on all the traces of $F_i, S_m \in \mathcal{S}_i$ |
| $N^{\mathcal{S}}$         | Number of elements of $u$ , collecting DOFs of all the functions $U_i^m$ on all the traces of all the fractures   |
| $N_{F_i}$                 | Number of elements of $h_{F_i}$ , collecting DOFs of $H_{F_i}$ on fracture $F_i$  |
| $N_F$                     | Number of elements of $h_F$ , collecting DOFs of $H_{F_i}$ on all the fractures   |

### 3. Discrete formulation

The discrete problem is now written in a quite general framework, without referring to a particular discretization choice. The specialization to a particular discretization method (the Boundary Element Method for the matrix blocks and the Finite Element Method for the fracture planes, in the present case) is provided in Appendix A.

Since the method involves quantities on the interfaces, the discretization of the hydraulic head and of fluxes is introduced only on matrix block faces and on the fractures.

Let us denote by  $\mathbf{x}$  a generic point in the 3D reference system, and let us assume that, on each block face  $\Gamma_\ell$ ,  $\ell \in \mathcal{L}$ , we have that

$$H_k^\ell = \sum_{j=1}^{N_k^{\Gamma_\ell}} h_{k,j}^\ell \phi_{k,j}^\ell(\mathbf{x}), \quad Q_k^\ell = \sum_{j=1}^{N_k^{\Gamma_\ell}} q_{k,j}^\ell \phi_{k,j}^\ell(\mathbf{x}), \quad k \in \mathcal{K}_{\Gamma_\ell},$$

and, for each fracture  $F_i$ ,  $i = 1, \dots, I$  it is:

$$H_{F_i} = \sum_{j=1}^{N_{F_i}} h_{F_i,j} \varphi_{F_i,j}(\mathbf{x}_{F_i}),$$

being  $\mathbf{x}_{F_i}$  a point in the local 2D reference system on  $F_i$ . The notation used for the number of basis functions of the discrete variables is summarized in

Table 2. Let us further denote by  $h_k^\ell$  and  $q_k^\ell$  the vectors obtained collecting column-wise all the coefficients  $h_{k,j}^\ell$  and  $q_{k,j}^\ell$ ,  $j = 1, \dots, N_k^{\Gamma^\ell}$ , respectively, and, similarly let us denote by  $h_{F_i}$  the column vector of coefficients  $h_{F_i,j}$ ,  $j = 1, \dots, N_{F_i}$ . Starting from the above definitions we build the column vectors  $h_k, q_k \in \mathbb{R}^{N_k}$ ,  $k = 1, \dots, n_{\mathcal{D}}$  obtained collecting vectors  $h_k^\ell$  and  $q_k^\ell$ , respectively, for increasing values of  $\ell \in \mathcal{L}_k$ . Finally vectors  $h_{\mathcal{D}}, q \in \mathbb{R}^{N_{\mathcal{D}}}$  and  $h_F \in \mathbb{R}^{N_F}$  are defined as:

$$h_{\mathcal{D}} = \begin{bmatrix} h_1 \\ \vdots \\ h_{n_{\mathcal{D}}} \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ \vdots \\ q_{n_{\mathcal{D}}} \end{bmatrix}, \quad h_F = \begin{bmatrix} h_{F_1} \\ \vdots \\ h_{F_I} \end{bmatrix}. \quad (17)$$

Here and in the following the subscript  $F$  is used to denote the set of all the fractures in the network.

We also assume that on each trace  $S_m$  in  $\Omega$  it is:

$$U_i^m = \sum_{j=1}^{N_{F_i}^{S_m}} u_{i,j}^m \psi_{F_i,j}^m, \quad i \in \mathcal{I}_{S_m},$$

and then we introduce vectors  $u_i^m \in \mathbb{R}^{N_{F_i}^{S_m}}$  obtained collecting column-wise coefficients  $u_{i,j}^m$  for  $j = 1, \dots, N_{F_i}^{S_m}$ . We then build vectors  $u_i \in \mathbb{R}^{N_{F_i}^{S_i}}$ , obtained grouping column-wise vectors  $u_i^m$  for increasing values of index  $m$  such that  $S_m \in \mathcal{S}_i$ , and vectors  $u^m \in \mathbb{R}^{N^{S_m}}$ ,  $N^{S_m} = N_{F_i}^{S_m} + N_{F_j}^{S_m}$  obtained collecting vectors  $u_i^m$  and  $u_j^m$  column-wise for  $i, j \in \mathcal{I}_{S_m}$  with  $i < j$ . Finally, for  $i = 1, \dots, I$ , vectors  $u_i^+ \in \mathbb{R}^{N^{S_i}}$ ,  $N^{S_i} = \sum_{S_m \in \mathcal{S}_i} N^{S_m}$  are obtained collecting column-wise vectors  $u^m$  for increasing values of  $m$  such that  $S_m \in \mathcal{S}_i$ , thus having in  $u_i^+$  coefficients of both the functions  $U_x^m$  for  $x \in \mathcal{I}_{S_m}$  on all the traces  $S_m$  of fracture  $F_i$ . Vector  $u \in \mathbb{R}^{N^S}$  is then obtained grouping column-wise vectors  $u^m$  for increasing values of  $m = 1, \dots, M$ .

The discrete functional  $J(h, h_F, u, q)$  is then obtained replacing the above definitions in the cost functional of equation (13) and substituting the norms in the spaces  $\mathcal{H}_\Gamma^\ell$  and  $\mathcal{Q}^\ell$  with norms in  $L^2(\Gamma_\ell)$ , and the norms in  $\mathcal{H}_S^m$  and  $\mathcal{U}^m$  with norms in  $L^2(S_m)$ . It can be written in compact form as:

$$\begin{aligned} J(h_{\mathcal{D}}, h_F, u, q) = & h_{\mathcal{D}}^T G_{\mathcal{D}, \mathcal{D}}^h h_{\mathcal{D}} + (q - 2\beta h_{\mathcal{D}})^T G^q q + 2h_{\mathcal{D}}^T G_{\mathcal{D}, F}^h h_F + \\ & h_F^T (G_F^h + 2G_{F, F}^h) h_F + u^T G^u u - 2\alpha h_F^T B_F u. \end{aligned} \quad (18)$$

Also the constraint equations can be written in matrix form, introducing matrices  $A_{\mathcal{D}}, \mathcal{B}_F^h$  expressing the action of (discrete) operators on the discrete variable  $h_{\mathcal{D}}$ , matrix  $A_F$  on  $h_F$ , matrices  $\mathcal{B}^q, \mathcal{B}_F^q$  on  $q$  and  $\mathcal{B}^u$  on  $u$ . Vectors  $b_{\mathcal{D}}$

and  $b_F$  collect, instead, source terms and boundary conditions. In particular, concerning the constraint equations on the matrix blocks we formally write:

$$A_{\mathcal{D}}h_{\mathcal{D}} = \mathcal{B}^q q + b_{\mathcal{D}},$$

whereas for equations on the fracture planes:

$$A_F h_F = \mathcal{B}^u u - \mathcal{B}_F^q q + \beta \mathcal{B}_F^h h_{\mathcal{D}} + b_F.$$

The exact expressions of the various terms involved are specified in Appendix A, in the context of a particular discretization choice.

The discrete constrained optimization problem reads as:

$$\min J(h_{\mathcal{D}}, h_F, u, q), \quad (19)$$

$$\text{s.t. : } A_{\mathcal{D}}h_{\mathcal{D}} = \mathcal{B}^q q + b_{\mathcal{D}}, \quad (20)$$

$$A_F h_F = \mathcal{B}^u u - \mathcal{B}_F^q q + \beta \mathcal{B}_F^h h_{\mathcal{D}} + b_F. \quad (21)$$

Let us define  $h = [h_{\mathcal{D}}^T, h_F^T]^T$  and  $w = [q^T, u^T]^T$ , and let us introduce the following matrices:

$$G^h = \begin{bmatrix} G_{\mathcal{D},\mathcal{D}}^h & G_{\mathcal{D},F}^h \\ (G_{\mathcal{D},F}^h)^T & G_F^h + 2G_{F,F}^h \end{bmatrix}, \quad G^w = \begin{bmatrix} G^q & 0 \\ 0 & G^u \end{bmatrix}, \quad B = \begin{bmatrix} \beta G^q & 0 \\ 0 & \alpha B_F \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}^q & 0 \\ \mathcal{B}_F^q & -\mathcal{B}^u \end{bmatrix}, \quad A = \begin{bmatrix} A_{\mathcal{D}} & 0 \\ -\beta \mathcal{B}_F^h & A_F \end{bmatrix}, \quad (22)$$

the optimality conditions for problem (19)-(21) reads as:

$$\begin{bmatrix} G^h & -B & A^T \\ -B^T & G^w & \mathcal{B}^T \\ A & \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} h \\ w \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{b} \end{bmatrix} \quad (23)$$

with  $\tilde{b} = [b_{\mathcal{D}}^T, b_F^T]^T$ .

#### 4. Well posedness of the discrete problem

Let us set  $N_h = N_{\mathcal{D}} + N_F$ ,  $N_w = N_{\mathcal{D}} + N^S$  and  $N = N_h + N_w$ . Let  $G \in \mathbb{R}^{N \times N}$  and  $\mathcal{C} \in \mathbb{R}^{N_h \times N}$  be the matrices defined respectively as follows:

$$G = \begin{bmatrix} G^h & -B \\ -B^T & G^w \end{bmatrix}, \quad \mathcal{C}^T = \begin{bmatrix} A^T \\ \mathcal{B}^T \end{bmatrix}. \quad (24)$$

**Remark 3.** *The following proof of existence and uniqueness of the discrete solution is presented using the BEM for the discretization of the problems on the matrix blocks and finite elements for the discretization of problems on the fractures, following the definitions given in Appendix A. Using similar arguments it is possible to show well posedness of the problem when different discretization choices are used.*

**Lemma 4.** *Let matrix  $A$ , defined as in (22), be full rank, let matrix  $\mathcal{C}$  be defined as in (24) and let  $Z$  be a matrix obtained collecting row-wise column vectors  $z_\xi$ ,  $\xi = 1, \dots, N_w$ , forming a basis of  $\ker(\mathcal{C})$ , then matrix  $Z^T G Z$  is positive definite.*

*Proof.* We start observing that matrix  $A$  is full rank as both matrices  $A_{\mathcal{D}}$  and  $A_F$  are full rank under the assumption that  $\alpha, \beta > 0$ . Then  $\dim(\ker(\mathcal{C})) = N_w$ , since for any vector  $w^* \in \mathbb{R}^{N_w}$  it is possible to find  $h^* \in \mathbb{R}^{N_h}$  such that  $Ah^* = Bw^*$ . To construct a basis of  $\ker(\mathcal{C})$ , let us take  $e_\xi$ , the  $\xi$ -th vector of the canonical basis of  $\mathbb{R}^{N_w}$ , and let us set  $z_\xi = (A^{-1}Be_\xi, e_\xi)$ . According to the index  $\xi$ ,  $e_\xi$  might correspond to: *i*) a non-null function  $Q_k^\ell$  for some  $\ell \in \mathcal{L}^*$  and  $k \in \mathcal{K}_{\Gamma_\ell}$ ; *ii*) a non-null function  $Q_k^\ell$  for some  $\ell \in \mathcal{L}^\diamond$ , and  $k \in \mathcal{K}_{\Gamma_\ell}$ ; *iii*) a non-null function  $U_i^m$  for some  $i = 1, \dots, I$  and  $S_m \in \mathcal{S}_i$ . Let us consider the case *i*) and in particular let us assume that  $Q_{k^*}^{\ell^*} = \phi_{k^*,j}^{\ell^*}$  for certain indexes  $j$ ,  $\ell^* \in \mathcal{L}^*$  and  $k^* \in \mathcal{K}_{\Gamma_{\ell^*}}$ . Let now  $H_{F_{\ell^*}}^{\ell^*}$  be the solution on the fracture on which  $\Gamma_{\ell^*}$  lies and let  $H_{k^{**}}^{\ell^*}$  be the solution on  $\Gamma_{\ell^*}$  of block  $\mathcal{D}_{k^{**}}$ , with  $k^*, k^{**} \in \mathcal{K}_{\Gamma_{\ell^*}}$ ,  $k^* \neq k^{**}$ . Then it is either  $\|H_{k^{**}}^{\ell^*} - H_{F_{\ell^*}}^{\ell^*}\|_{L^2(\Gamma_{\ell^*})} > 0$  or  $\|H_{k^*}^{\ell^*} - H_{F_{\ell^*}}^{\ell^*}\|_{L^2(\Gamma_{\ell^*})} > 0$ , and thus, in both cases,  $z_\xi^T G z_\xi > 0$ . Let us move to case *ii*) and let us now assume that  $Q_{k^\dagger}^{\ell^\dagger} = \phi_{k^\dagger,j}^{\ell^\dagger}$  for certain indexes  $j$ ,  $\ell^\dagger \in \mathcal{L}^\diamond$  and  $k^\dagger \in \mathcal{K}_{\Gamma_{\ell^\dagger}}$ , differing from the previous case for the fact that now  $\ell^\dagger$  corresponds to a face not matching with a fracture. It is possible to say that there is at least one index  $\ell^{\dagger\dagger} \in \mathcal{L}_{k^\dagger}^{\dagger\dagger}$ , such that  $H_{k^{\dagger\dagger}}^{\ell^{\dagger\dagger}}$  is non-zero. If  $\ell^{\dagger\dagger}$  is still an index of a face not matching any fracture, i.e.  $\ell^{\dagger\dagger} \in \mathcal{L}^\diamond$ , there is a neighboring block  $k^{\dagger\dagger} \in \mathcal{K}_{\Gamma_{\ell^{\dagger\dagger}}}$ ,  $k^{\dagger\dagger} \neq k^\dagger$  such that the functional term  $\|H_{k^{\dagger\dagger}}^{\ell^{\dagger\dagger}} - H_{k^\dagger}^{\ell^{\dagger\dagger}}\|_{L^2(\Gamma_{\ell^{\dagger\dagger}})} > 0$ , and thus  $z_\xi^T G z_\xi > 0$ . If, instead,  $\Gamma_{\ell^{\dagger\dagger}}$  lies on a fracture, i.e.  $\ell^{\dagger\dagger} \in \mathcal{L}^*$ , then  $\|H_{k^{\dagger\dagger}}^{\ell^{\dagger\dagger}} - F_{\ell^{\dagger\dagger}}^{\ell^{\dagger\dagger}}\|_{L^2(\Gamma_{\ell^{\dagger\dagger}})} > 0$ , and consequently  $z_\xi^T G z_\xi > 0$  also in this case. Let us now turn the attention to the case *iii*), such that, it is now  $U_i^m = \psi_{F_i,j}^m$  for a certain index  $j$  on a single trace  $S_m$  of fracture  $F_i$ . Substituting into equation (15), and setting, without loss of generality  $\mathbf{K}_{F_i} = 1$ , it is:  $\forall j = 1, \dots, N_{F_i}$

$$0 < (\nabla H_{F_i}, \nabla \varphi_{F_i,j})_{F_i} + \alpha(H_{F_i|_{S_m}}, \varphi_{F_i,j|_{S_m}})_{S_m} = \left\langle \psi_{F_i,j}^m, \varphi_{F_i,j|_{S_m}} \right\rangle_{S_m},$$

and choosing in particular  $H_{F_i}$  as a test function, it results

$$0 < \left\langle \psi_{F_i,j}^m, H_{F_i|_{S_m}} \right\rangle_{S_m} \leq \|\psi_{F_i,j}^m\|_{\mathcal{U}^m} \|H_{F_i|_{S_m}}\|_{\mathcal{H}_S^m} \Rightarrow \|H_{F_i|_{S_m}}\|_{\mathcal{H}_S^m} > 0,$$

from which it is possible to deduce that the functional is again non-zero and then  $z_\xi^T G z_\xi > 0$ . Being  $G$  positive semi-definite it is  $x^T G x \geq 0$  and  $x^T G x = 0$  if and only if  $x \in \ker(G)$ , (see [45]). As a consequence we have that  $z_\xi \notin \ker(G)$  for  $\xi = 1, \dots, N_w$ . The vector space  $\mathcal{Z} = \text{span}\{z_1, \dots, z_{N_w}\}$  is then a subspace of  $\text{Im}(G)$ , and each vector  $y \in \mathcal{Z}$  can be written as  $y = Zv$ , for a vector  $v \in \mathbb{R}^{N_w}$ ,  $v \neq 0$ . Then  $v^T Z^T G Z v > 0$ , which concludes the proof.  $\square$

**Theorem 5.** *Problem (23) has a unique solution  $h^* = [(h_{\mathcal{D}}^*)^T, (h_F^*)^T]^T$ ,  $w^* = [(q^*)^T, (u^*)^T]^T$ ,  $\lambda^*$ , such that  $h_{\mathcal{D}}^*, h_F^*, q^*, u^*$  correspond to the constrained minimum of problem (19)-(21).*

The proof of Theorem 5 follows from Lemma 4 and classical arguments of quadratic programming (see Theorem 16.2 in [46]).

## 5. Numerical Results

This Section is devoted to the presentation of some numerical tests on simple geometrical configurations aimed at showing the viability and convergence properties of the new approach herein proposed. Application of the method to more complex and realistic geometrical settings will be the topic of forthcoming works.

Numerical solutions are obtained on triangular meshes using the BEM on the block faces with element-wise constant basis functions and standard first-order FE on the fractures. Given the simplicity of the considered domains, the meshes on the fractures are conforming to the meshes on the corresponding block faces, even if this is not required by the method. A mesh parameter  $\delta$ , equal to the square root of the maximum element area, is used to characterize the mesh. The numerical solution of the minimization problem (19)-(21) is obtained solving the KKT system (23). For larger problems the use of a conjugate gradient based solver is advisable, in which the solution is reached through the repeated resolution of small local problems, as done in [41], allowing for high a scalability potential.

As a first example, labeled test problem  $T_1$ , let us consider a unit edge-long cubic domain  $\mathcal{D}$ , crossed by a single fracture  $F$  parallel to the horizontal faces of the cube and passing through its barycenter, as shown in Figure 1. A coordinate system centered in  $\mathcal{O}$  is chosen, as shown in the figure, and with reference to it, the face common to the two blocks  $\mathcal{D}_1$  and  $\mathcal{D}_2$  is labeled  $\Gamma_{11}$ ,



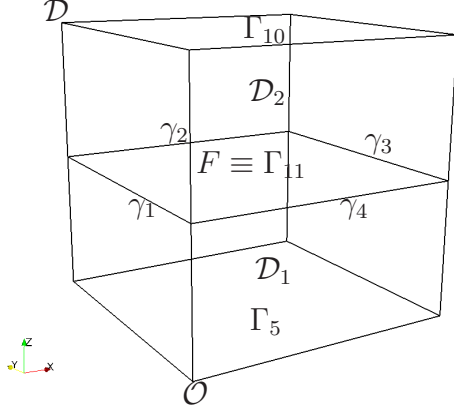


Figure 1: Geometry of test problem  $T_1$  and  $T_2$ .

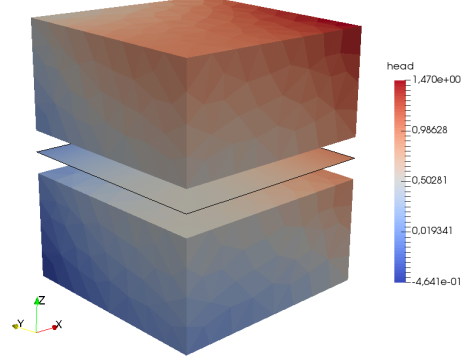


Figure 2: Exploded solution for problem  $T_1$  on grid  $\delta = 0.1$ .

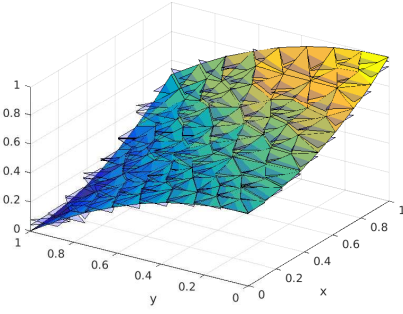


Figure 3: Solution of problem  $T_1$ , mesh  $\delta = 0.1$  on fracture  $F$  (colours proportional to head values) with solution on block  $\mathcal{D}_1$  on face  $\Gamma_{11}$  (light blue patches).

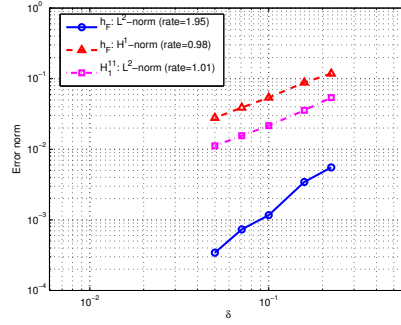


Figure 4: Convergence curves for problem  $T_1$ .

the bottom face is  $\Gamma_5$ , the top face  $\Gamma_{10}$ , whereas the other faces are labeled counter-clockwise starting, for  $\mathcal{D}_1$ , from the face parallel to the  $x - z$  plane and passing through  $\mathcal{O}$ ,  $\Gamma_1$ , and, for  $\mathcal{D}_2$ , from face  $\Gamma_6$ , placed on top of  $\Gamma_1$ . The edges of fracture  $F$  are labeled as in Figure 1, and we have  $\gamma_{F,D} = \gamma_1 \cup \gamma_3$  and  $\gamma_{F,N} = \gamma_2 \cup \gamma_4$ . The following problem is then set in  $\mathcal{D}$ :

$$-\mathbf{K}^m \Delta H_1 = 0, \text{ in } \mathcal{D}_1 \quad (25)$$

$$-\mathbf{K}^m \Delta H_2 = 0, \text{ in } \mathcal{D}_2 \quad (26)$$

$$-\mathbf{K}_F \Delta H_F = -(Q_1^{11} + Q_2^{11}), \text{ in } F \quad (27)$$

$$H_1^{11} = H_F = H_2^{11} \quad (28)$$

with  $\mathbf{K}^m = \mathbf{K}_F = 1$ , and boundary conditions on the block faces:

$$\begin{aligned} H_1^5 &= 1/2(x^2 - y^2) \\ H_2^{10} &= 1/2(x^2 - y^2) + 1 \\ H_1^2 = H_2^7 &= z - 1/2(1 - y^2) \\ H_1^4 = H_2^9 &= z - 1/2y^2 \\ Q_1^1 = Q_2^6 &= 0 \\ Q_1^3 = Q_2^8 &= -1 \end{aligned}$$

and on the fracture boundaries:

$$\begin{aligned} H_{F|\gamma_1} &= 1/2(1 - y^2) \\ H_{F|\gamma_3} &= 1 - 1/2y^2 \\ \nabla H_F \cdot \mathbf{n}_{\gamma_2} &= -1 \\ \nabla H_F \cdot \mathbf{n}_{\gamma_4} &= 0 \end{aligned}$$

where  $\mathbf{n}_{\gamma_i}$  is the outwards unit normal vector to  $\gamma_i$ .

The exact solution to this problem is known and is  $H = 1/2(x^2 - y^2) + z$ . The numerical approximation computed on a mesh with  $\delta = 0.1$  is shown in Figure 2, on an exploded domain, in which it is possible to recognize the two blocks and the fracture plane in between them. Colouring in this figure is proportional to the hydraulic head. The solution on the fracture plane,  $h_F$ , is plotted in Figure 3, and it is represented in grading colours proportional to the value of the hydraulic head. On the same picture the approximated solution  $h_1^{11}$  on face  $\Gamma_{11}$  is also shown in light blue, in order to highlight the very good agreement with  $h_F$ . Error  $\|H|_F - h_F\|$  is computed in

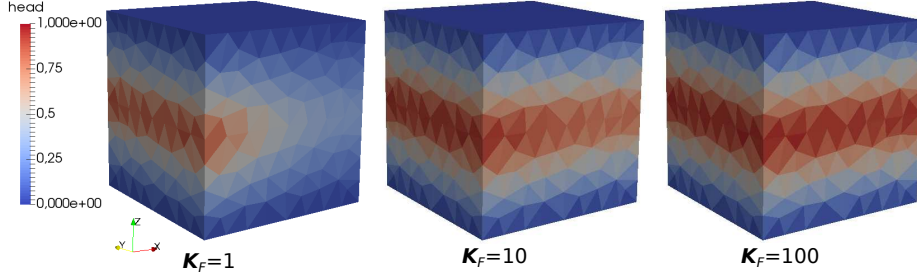


Figure 5: Solution for problem  $T_2$  on a mesh with  $\delta = 0.1$  and three values of  $\mathbf{K}_F$

both the  $L^2(F)$  and  $H^1(F)$ -norms on the fracture for  $h_F$ , whereas the error  $\|H_{|\Gamma_{11}} - h_1^{11}\|$  is computed in the  $L^2(\Gamma_{11})$ -norm for the piece-wise constant function  $h_1^{11}$ . Convergence curves for decreasing values of  $\delta$  ranging between  $0.05 \leq \delta \leq 0.25$  are reported in Figure 4, showing optimal trends for the approximation used.

The second test problem proposed, labeled  $T_2$ , shares the same geometry as  $T_1$ , and the same formulation (25)-(28), with different boundary conditions and coefficients. In particular, in this case homogeneous Dirichlet boundary conditions are imposed on  $\Gamma_5$  and  $\Gamma_{10}$ , whereas homogeneous Neumann boundary conditions are imposed on all the other block faces. A unitary head value is imposed on  $\gamma_{F,D} = \gamma_1$ , and all other fracture edges are insulated. It can be noticed that the method allows for a great flexibility in the imposition of boundary conditions, as shown by the present example, where a Dirichlet boundary condition is prescribed on a fracture edge surrounded by insulated block faces. This reflects realistic configurations, in which, for example the flow in a matrix block is driven by the presence of the fracture. The value of  $\mathbf{K}^m$  is fixed to one, whereas three different values for  $\mathbf{K}_F = (1, 10, 100)$  are used for three different simulations, thus prescribing a maximum difference between the matrix permeability and the fracture permeability of two orders of magnitude. Simulations are all performed on a mesh with  $\delta = 0.1$ . In this setting, then, there is a source of hydraulic head at one edge of the fracture, and a sink is located at the two horizontal faces of the matrix block. The analysis of the results is in this case qualitative: looking at Figure 5, it is possible to see that, at increasing values of  $\mathbf{K}_F$  the hydraulic head reaches higher values more in depth of the matrix block, diffusing through the fracture, and this is an expected behavior. Higher differences of permeability between matrix and fractures could also be handled by the method.

The third proposed example, labeled  $T_3$ , considers a slightly more complex geometry than the previous problems, with two intersecting fractures immersed in an unitary side cube, as shown in Figure 6. With reference to

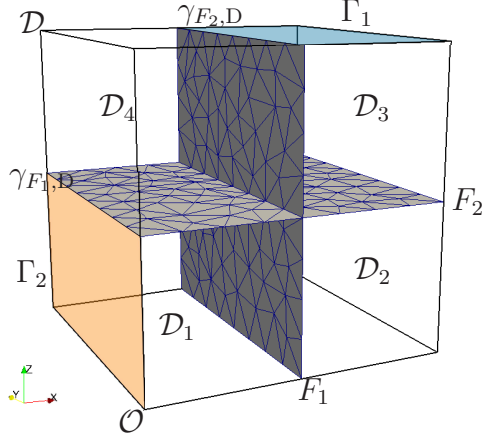


Figure 6: Geometry for test problem  $T_3$

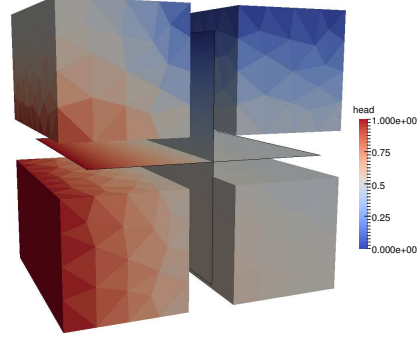


Figure 7: Exploded view of the solution for problem  $T_3$  on a mesh with  $\delta = 0.1$ .

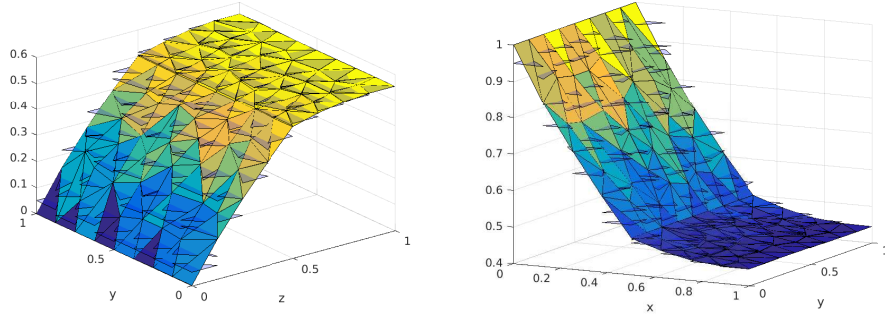


Figure 8: Solution of problem  $T_3$  on the fractures (in grading colours,  $F_1$  left,  $F_2$  right) with solution on the corresponding faces (in light-blue patches,  $\Gamma_{17}$  and  $\Gamma_{18}$ , left,  $\Gamma_{19}$  and  $\Gamma_{20}$ , right).

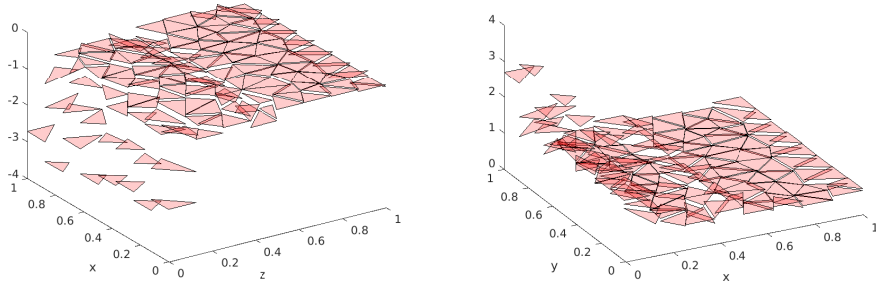


Figure 9: Flux mismatch across fracture  $F_1$  (left) and  $F_2$  (right) for problem  $T_3$ .

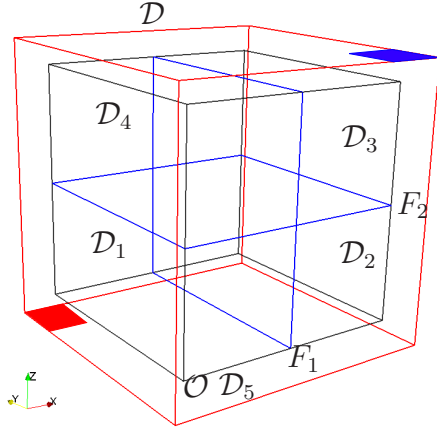


Figure 10: Geometry of problem  $T_4$  along with boundary conditions

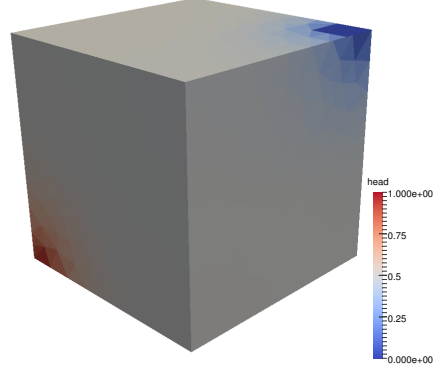


Figure 11: Solution on the external faces of block  $\mathcal{D}_5$  on a mesh with  $\delta = 0.1$

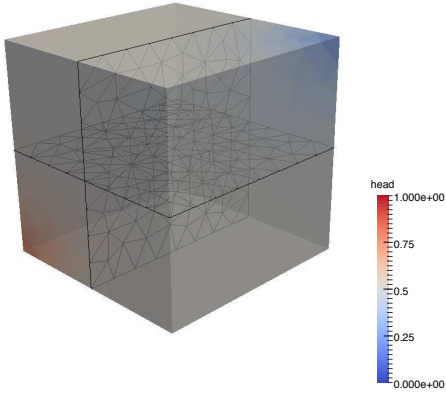


Figure 12: Solution of problem  $T_4$  on the external faces of blocks  $\mathcal{D}_1$ - $\mathcal{D}_4$  on a mesh with  $\delta = 0.1$ , and, in transparency, on the fractures. Mesh with  $\delta = 0.1$ .

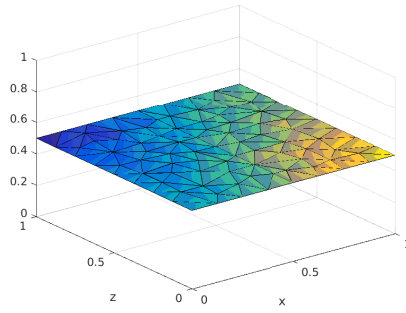


Figure 13: Solution of problem  $T_4$  on fracture  $F_1$ , mesh  $\delta = 0.1$

this figure, in addition to the face labels shown therein,  $\Gamma_{17}$  and  $\Gamma_{18}$  are the lower and upper face on  $F_1$ , respectively, whereas  $\Gamma_{19}$  and  $\Gamma_{20}$  are the left and right face on  $F_2$ , respectively. Problem setting is as in (1)-(2),(4)-(5),(7)-(8), with  $f_k = 0$ ,  $\mathbf{K}_k^m = 1$ ,  $\mathbf{K}_{F_i} = 1$ ,  $i = 1, 2$ , and the following boundary conditions: a unitary head is prescribed on face  $\Gamma_1$  and on the edge of  $F_2$  labeled  $\gamma_{F_2,D}$ , null head on  $\Gamma_2$  and  $\gamma_{F_1,D}$  (see Figure 6 for labels to faces and fracture edges), whereas all other boundary faces and fracture edges are insulated. Parameters  $\alpha$  and  $\beta$  in definitions (11) and (12) are set to one. The obtained solution, on a mesh with parameter  $\delta = 0.1$  is shown in Figure 7, with an exploded view. Looking at Figure 8, where the solution on the fractures is reported along with the solution on the matching faces, it can be seen that there is a good agreement between the two. Also the solution on the fractures has a jump in the first order derivatives along the intersection. Figure 9 shows, instead, the mismatch of the co-normal derivative of the solution on the matrix blocks across the fractures, which corresponds to the forcing terms of the problems on the fractures. Thus it can be concluded that the solution displays the expected behavior.

The last example, problem  $T_4$  shows the behavior of the method in the case of a network of fractures completely immersed in the matrix, i.e. without any fracture hitting the boundary of the computational domain. The domain  $\mathcal{D}$  for this problem is shown in Figure 10, and it consists of two square-shaped intersecting fractures with unit long edge, immersed in a (1.2)-long edge cubical matrix block. The domain is split into five blocks  $\mathcal{D}_k$ ,  $k = 1, \dots, 5$ , and in particular, block  $\mathcal{D}_5$  is a cube with a cubic cavity hosting the remaining blocks  $\mathcal{D}_1$ - $\mathcal{D}_4$ , which, in turn, surround the two fractures  $F_1$  and  $F_2$ . Dirichlet boundary conditions are prescribed on a portion of the top and bottom face of  $\mathcal{D}$  (see the shaded areas in Figure 10), with a prescribed value of head equal to one on the bottom face and equal to zero on the top face. All other edges are insulated. Concerning the fractures, a no flow boundary condition is imposed at fracture borders, as a consequence of the vanishing aperture of the fractures at their borders. The transmissivity of the matrix blocks is equal to one, whereas the transmissivity of fractures is set to  $\mathbf{K}_{F_1} = \mathbf{K}_{F_2} = 10^3$ . The solution on the outer faces of block  $\mathcal{D}_5$  is shown in Figure 11. Figure 12 displays the solution on the faces of blocks  $\mathcal{D}_1$ - $\mathcal{D}_4$ , and in transparency the solution on the two fractures, along with the computing mesh. It can be noticed that the solution on and around the fractures is almost constant, (see also Figure 13) as a consequence of the high value of the transmissivity of the fractures with respect to the transmissivity of the matrix. Therefore, also in this configuration the computed solution reproduces the expected behavior for the solution.

## 6. Conclusions

A new formulation for the simulation of the flow in poro-fractured domains is presented, based on an optimization approach. The three dimensional Darcy law in the porous matrix and the coupled reduced Darcy law on the fractures is solved resorting to the minimization of a properly designed cost functional for the imposition of matching condition at fracture-matrix and fracture-fracture interfaces. The method relaxes the mesh matching constraints at the interfaces, thus paving the way to an easy and reliable meshing process, as already done in the context of discrete fracture network flow simulations. Moreover, the optimization approach allows for the decoupling of the original problem into smaller problems, thus providing a high scalability potential, which is a crucial aspect for computationally demanding simulations. In this work the well posedness of the optimization formulation is stated, both in a continuous and discrete framework. Numerical results on simple three dimensional problems show optimal convergence properties for the proposed method and demonstrate its viability and effectiveness for simulating underground flows in complex geometries.

## Acknowledgments

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## Appendix A.

The following is devoted to a precise definition of the matrices involved in the previous statements. This is done using the Boundary Element Method (BEM) on the matrix blocks and standard first order Finite Elements on the fracture faces. We will assume that the source term  $f$  in the matrices is zero, the extension to the general case being straightforward. For this reason, both vectors  $b_{\mathcal{D}}$  and  $b_F$  only account for boundary conditions. Furthermore, for the sake of simplicity we assume that the matrix trasmissivity is constant on each block, i.e.  $\mathbf{K}_k^m = K_k^m$ ,  $k = 1, \dots, n_{\mathcal{D}}$ .

### *Appendix A.1. Assembling $A_{\mathcal{D}}$ , $\mathcal{B}^q$ and $b$*

The Boundary Element Method (BEM) for the numerical resolution of equation (14) provides the definition of  $h$  on the boundaries of each matrix

block  $\mathcal{D}_k$ . Let us denote by  $\chi_k(\mathbf{x}, \mathbf{x}_0)$  the Green function of the operator  $-K_k^m \Delta(\cdot)$  in 3D, defined as

$$\chi_k(\mathbf{x}, \mathbf{x}_0) = \frac{1}{4K_k^m \pi \|\mathbf{x} - \mathbf{x}_0\|}$$

with  $\|\cdot\|$  the Euclidean norm in  $\mathbb{R}^3$ . Let us fix a block index  $k$ , and let us introduce for the basis functions  $\phi_{k,j}^\ell$ ,  $j = 1, \dots, N_k^{\Gamma_\ell}$ ,  $\ell \in \mathcal{L}_k$  a global numbering:  $\phi_{k,\zeta}$ ,  $\zeta = 1, \dots, N_k$ . We then set:

$$H_k = \sum_{j=1}^{N_k} h_{k,j} \phi_{k,j}(\mathbf{x}), \quad Q_k = \sum_{j=1}^{N_k} q_{k,j} \phi_{k,j}(\mathbf{x}).$$

Furthermore, let us introduce, for each  $k = 1, \dots, n_{\mathcal{D}}$  a set of nodes  $(\mathbf{x}_{k,j})_{j=1, \dots, N_k}$  in  $\mathcal{D}_k$  and, without loss of generality basis functions are such that exists  $\mathbf{x}_{k,j}$  providing  $\phi_{k,i}(\mathbf{x}_{k,j}) = \delta_{i,j}$ ,  $i, j = 1, \dots, N_k$ . For each internal block, i.e. for each  $\mathcal{D}_k$  such that  $\mathcal{L}_k \cap \mathcal{L}_{\partial\mathcal{D}} = \emptyset$ , we have that:

$$\frac{1}{2} h_{k,j} = \int_{\partial\mathcal{D}_k} (Q_k(\mathbf{x}) - \beta H_k(\mathbf{x})) \chi_k(\mathbf{x}, \mathbf{x}_j) - \int_{\partial\mathcal{D}_k} H_k(\mathbf{x}) (K_k^m \nabla \chi_k(\mathbf{x}, \mathbf{x}_j) \cdot \mathbf{n}_{\partial\mathcal{D}_k}).$$

We then collect integrals into matrices  $A_k, \mathcal{B}_k^q \in \mathbb{R}^{N_k \times N_k}$  defined as follows:

$$(A_k)_{j,i} = \int_{\partial\mathcal{D}_k} \beta \phi_{k,i}(\mathbf{x}) \chi_k(\mathbf{x}, \mathbf{x}_j) + \int_{\partial\mathcal{D}_k} \phi_{k,i}(\mathbf{x}) (K_k^m \nabla \chi_k(\mathbf{x}, \mathbf{x}_j) \cdot \mathbf{n}_{\partial\mathcal{D}_k}) + \frac{1}{2} \delta_{ij}, \quad (\text{A.1})$$

$$(\mathcal{B}_k^q)_{j,i} = \int_{\partial\mathcal{D}_k} \phi_{k,i}(\mathbf{x}) \chi_k(\mathbf{x}, \mathbf{x}_j). \quad (\text{A.2})$$

For boundary blocks, i.e. for all  $k = 1, \dots, n_{\mathcal{D}}$  such that  $\mathcal{L}_k \cap \mathcal{L}_{\partial\mathcal{D}} \neq \emptyset$ , without loosing generality, we can assume that boundary faces carrying Dirichlet boundary conditions, correspond to the first face indexes in  $\mathcal{L}_k$ , and consequently the coefficients and the basis functions of  $H_k^\ell$  for  $\ell \in \mathcal{L}_k \cap \mathcal{L}_{\partial\mathcal{D}}$  occupy the first  $N_k^*$  positions in  $h_k$ . We denote by  $H_k^D = \sum_{j=1}^{N_k^*} h_{k,j}^D \phi_{k,j}(\mathbf{x})$  the (known) part of  $H_k$ , corresponding to the Dirichlet faces, being  $h_{k,j}^D$  known coefficients deriving from the discretization of the boundary conditions, collected in the column vector  $h_k^D$ . Instead  $H_k^* = \sum_{j=N_k^*+1}^{N_k} h_{k,j} \phi_{k,j}(\mathbf{x})$  is the remaining (unknown) part of  $H_k$ .

Thus, for boundary blocks, it is: for  $j = 1, \dots, N_k$ ,  $i = 1, \dots, N_k^*$

$$(A_k)_{j,i} = \int_{\partial\mathcal{D}_k} \phi_{k,i}(\mathbf{x}) \chi_k(\mathbf{x}, \mathbf{x}_j),$$



$$(\mathcal{B}_k)_{j,i} = K_k^m \int_{\partial \mathcal{D}_k} \phi_{k,i}(\mathbf{x}) (\nabla \chi_k(\mathbf{x}, \mathbf{x}_j) \cdot \mathbf{n}_{\partial \mathcal{D}_k}) + \frac{1}{2} \delta_{ij},$$

whereas for  $j = 1, \dots, N_k$ ,  $i = N_k^* + 1, \dots, N_k$ ,  $(A_k)_{j,i}$  is as in (A.1), and  $(\mathcal{B}_k^q)_{j,i}$  is as in (A.2).

Finally we have that

$$A_{\mathcal{D}} = \text{diag}(A_k, k = 1, \dots, n_{\mathcal{D}}), \quad \mathcal{B}^q = \text{diag}(\mathcal{B}_k^q, k = 1, \dots, n_{\mathcal{D}}),$$

and

$$b_k = \mathcal{B}_k h_k^{\text{D}}, \quad b_{\mathcal{D}} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ \vdots \\ b_{n_{\mathcal{D}}} \end{bmatrix}.$$

#### Appendix A.2. Assembling $A_F$ , $\mathcal{B}^u$ , $\mathcal{B}_F^q$ and $B_F$

Let us introduce matrices  $A_{F_i} \in \mathbb{R}^{N_{F_i} \times N_{F_i}}$  defined as:

$$(A_{F_i})_{pz} = \int_{F_i} \mathbf{K}_{F_i} \nabla \varphi_{F_i,p} \nabla \varphi_{F_i,z}, \quad i = 1, \dots, I,$$

and we then have  $A_F = \text{diag}(A_{F_i}, i = 1, \dots, I)$ . Furthermore, let us set matrices  $\mathcal{B}_{F_i, F_i, S_m} \in \mathbb{R}^{N_{F_i} \times N_{F_i}^{S_m}}$ , for  $i = 1, \dots, I$  and  $S_m \in \mathcal{S}_i$  as:

$$(\mathcal{B}_{F_i, F_i, S_m})_{p,z} = \int_{S_m} \varphi_{F_i,p} \psi_{F_i,z}^m.$$

The repeated subscript  $F_i$  in  $\mathcal{B}_{F_i, F_i, S_m}$  is used here to distinguish these matrices from matrices  $\mathcal{B}_{F_i, F_j, S_m}$  introduced later in this Appendix. Matrices  $\mathcal{B}_{F_i, F_i, S_m}$  are grouped row-wise for increasing values of  $m$  such that  $S_m \in \mathcal{S}_i$  to form matrices  $\mathcal{B}_{F_i}^u \in \mathbb{R}^{N_{F_i} \times N_{F_i}^{S_i}}$ . Matrices  $\mathcal{R}_{F_i} \in \mathbb{R}^{N^{\mathcal{S}} \times N_{F_i}^{S_i}}$ , instead, represent the action of an operator that extracts from vector  $u$  sub-vectors  $u_i$  for  $i = 1, \dots, I$ , i.e.  $u_i = \mathcal{R}_{F_i} u$ , and then we set:

$$\mathcal{B}^u \in \mathbb{R}^{N_F \times N^{\mathcal{S}}}, \quad \mathcal{B}^u = \begin{bmatrix} \mathcal{B}_{F_1}^u \mathcal{R}_{F_1} \\ \vdots \\ \mathcal{B}_{F_I}^u \mathcal{R}_{F_I} \end{bmatrix}.$$

Let us denote by  $\mathcal{B}_{F_i, k, \Gamma_\ell}^q \in \mathbb{R}^{N_{F_i} \times N_k^{\Gamma_\ell}}$  the matrix defined for  $i = 1, \dots, I$  as:

$$\begin{aligned} (\mathcal{B}_{F_i, k, \Gamma_\ell}^q)_{p,z} &= \int_{\Gamma_\ell} \phi_{k,z}^\ell(\sigma_i(\mathbf{x})) \varphi_{F_i,p}(\mathbf{x}_{F_i}), \quad \ell \in \mathcal{L}_{F_i}, \quad k \in \mathcal{K}_{\Gamma_\ell} \\ (\mathcal{B}_{F_i, k, \Gamma_\ell}^q)_{p,z} &= 0, \quad \ell \notin \mathcal{L}_{F_i}, \quad k \in \mathcal{K}_{\Gamma_\ell} \end{aligned}$$

where  $\sigma_i(\mathbf{x})$  is used to denote a mapping from the 3D reference system restricted to  $\Gamma_\ell$  to the 2D reference system on  $F_i$ , which is actually a rotation and translation. For each  $i = 1, \dots, I$ , let us denote by  $\mathcal{K}_{F_i}$  the set of matrix block indexes having one face on  $F_i$ , i.e.  $\mathcal{K}_{F_i} = \bigcup_{\ell \in \mathcal{L}_{F_i}} \mathcal{K}_{\Gamma_\ell}$ . We now collect, for each  $i = 1, \dots, I$ , matrices  $\mathcal{B}_{F_i, k, \Gamma_\ell}^q$  row-wise to form the new matrices  $\mathcal{B}_{F_i}^q \in \mathbb{R}^{N_{F_i} \times N_{\mathcal{D}}}$  in the following way: for all  $k = 1, \dots, n_{\mathcal{D}}$

$$\mathcal{B}_{F_i, k}^q = \begin{bmatrix} \mathcal{B}_{F_i, k, \Gamma_{\ell_1}}^q & \mathcal{B}_{F_i, k, \Gamma_{\ell_2}}^q & \cdots & \mathcal{B}_{F_i, k, \Gamma_{\ell_{n_k}}}^q \end{bmatrix}, \quad \ell_1 < \ell_2 < \cdots < \ell_{n_k} \in \mathcal{L}_k,$$

$$\mathcal{B}_{F_i}^q = \begin{bmatrix} \mathcal{B}_{F_i, 1}^q & \cdots & \mathcal{B}_{F_i, n_{\mathcal{D}}}^q \end{bmatrix},$$

with  $n_k$  the number of elements of  $\mathcal{L}_k$ .

Finally matrix  $\mathcal{B}_F^q \in \mathbb{R}^{N_F \times N_{\mathcal{D}}}$  is obtained grouping column wise matrices  $\mathcal{B}_{F_i}^q$  for increasing values of  $i = 1, \dots, I$ .

Let us now define, for  $m = 1, \dots, M$  and  $i, j \in \mathcal{I}_{S_m}$ ,  $i \neq j$ , matrices  $\mathcal{B}_{F_i, F_j, S_m} \in \mathbb{R}^{N_{F_i} \times N_{F_j}^{S_m}}$  as follows:

$$(\mathcal{B}_{F_i, F_j, S_m})_{p, z} = \int_{S_m} \varphi_{F_i, p} \psi_{F_j, z}^m,$$

and matrices  $B_{F_i, S_m} \in \mathbb{R}^{N_{F_i} \times N^{S_m}}$  as: for  $i = 1, \dots, I$ ,  $S_m \in \mathcal{S}_i$ ,  $p, z \in \mathcal{I}_{S_m}$  with  $p < z$  (i.e.  $p = i$  or  $z = i$ ):

$$B_{F_i, S_m} = [\mathcal{B}_{F_i, F_p, S_m}, \mathcal{B}_{F_i, F_z, S_m}],$$

which are then grouped row wise to form matrices  $B_{F_i} \in \mathbb{R}^{N_{F_i} \times N^{S_i}}$  for increasing values of  $m$  such that  $S_m \in \mathcal{S}_i$ . Then, let  $R_{F_i} \in \mathbb{R}^{N^{S_i} \times N^{S_i}}$  be a matrix representing the action of an operator that extracts from vector  $u$  sub-vectors  $u_i^+$  for  $i = 1, \dots, I$ , i.e.  $u_i^+ = R_{F_i} u$ . Matrix  $B_F \in \mathbb{R}^{N_F \times N^S}$  is finally given by:

$$B_F = \begin{bmatrix} B_{F_1} R_{F_1} \\ \vdots \\ B_{F_I} R_{F_I} \end{bmatrix}.$$

### Appendix A.3. Assembling matrices $G_{\mathcal{D}, \mathcal{D}}^h$ , $\mathcal{B}_F^h$ and $G_{F, F}^h$

Let us start by introducing matrices  $G_{k, k, \Gamma_\ell} \in \mathbb{R}^{N_k^{\Gamma_\ell} \times N_k^{\Gamma_\ell}}$ ,  $G_{k, y, \Gamma_\ell} \in \mathbb{R}^{N_k^{\Gamma_\ell} \times N_y^{\Gamma_\ell}}$ , and  $G_{k, F_i, \Gamma_\ell} \in \mathbb{R}^{N_k^{\Gamma_\ell} \times N_{F_i}}$ , defined respectively as:

$$(G_{k, k, \Gamma_\ell})_{p, z} = \tilde{\beta} \int_{\Gamma_\ell} \phi_{k, p}^\ell(\mathbf{x}) \phi_{k, z}^\ell(\mathbf{x}), \quad \forall \ell \in \mathcal{L}_{\mathcal{D}}, \quad \tilde{\beta} = \begin{cases} 1 & \ell \in \mathcal{L}^\star, \\ 1 + \beta^2 & \ell \in \mathcal{L}^\diamond, \end{cases}$$

$$\begin{aligned}
G_{k,k,\Gamma_\ell} &= 0, \quad \forall \ell \in \mathcal{L}_{\partial\mathcal{D}}, \quad k \in \mathcal{K}_{\Gamma_\ell}; \\
(G_{k,y,\Gamma_\ell})_{p,z} &= (\beta^2 - 1) \int_{\Gamma_\ell} \phi_{k,p}^\ell(\mathbf{x}) \phi_{y,z}^\ell(\mathbf{x}), \quad \forall \ell \in \mathcal{L}^\diamond, \quad k, y \in \mathcal{K}_{\Gamma_\ell}, \quad k \neq y; \\
G_{k,y,\Gamma_\ell} &= 0, \quad \forall \ell \in \mathcal{L}^\diamond, \quad k \in \mathcal{K}_{\Gamma_\ell}, \quad y \notin \mathcal{K}_{\Gamma_\ell}; \\
G_{k,y,\Gamma_\ell} &= 0, \quad \forall \ell \in \mathcal{L}^\star, \quad k, y \in \mathcal{K}_{\Gamma_\ell}, \quad k \neq y; \\
(G_{k,F_i,\Gamma_\ell})_{p,z} &= \int_{\Gamma_\ell} \phi_{k,p}^\ell(\sigma_i(\mathbf{x})) \varphi_{F_i,z}^\ell(\mathbf{x}_{F_i}), \quad \forall \ell \in \mathcal{L}^\star, \quad k \in \mathcal{K}_{\Gamma_\ell}, \quad i = \varrho(\ell),
\end{aligned}$$

and  $G_{k,F_i,\Gamma_\ell} = 0$ , otherwise.

With these matrices we build for  $k = 1, \dots, n_{\mathcal{D}}$ ,  $\ell_1 < \ell_2 < \dots < \ell_{n_k} \in \mathcal{L}_k$  and  $y \in \mathcal{K}_{\Gamma_{\ell_r}}$ ,  $y \neq k$ :

$$G_{k,k} = \text{diag}(G_{k,k,\Gamma_{\ell_r}}, r = 1, \dots, n_k), \quad G_{k,y} = \text{diag}(G_{k,y,\Gamma_{\ell_r}}, r = 1, \dots, n_k)$$

and then we set  $G_k = G_{k,k} + G_{k,y}$ . Also for  $k = 1, \dots, n_{\mathcal{D}}$  and  $i = 1, \dots, I$  matrices

$$G_{k,F,\Gamma_\ell} = \begin{bmatrix} G_{k,F_1,\Gamma_\ell} & \cdots & G_{k,F_i,\Gamma_\ell} & \cdots & G_{k,F_I,\Gamma_\ell} \end{bmatrix}.$$

Matrix  $G_{\mathcal{D},\mathcal{D}}^h$  is then obtained collecting column-wise matrices  $G_k$ , whereas matrices  $G_{k,F,\Gamma_\ell}$  are grouped column wise for increasing values of  $\ell \in \mathcal{L}_k$  to form matrices

$$G_{k,F} = \begin{bmatrix} G_{k,F,\Gamma_{\ell_1}} \\ \vdots \\ G_{k,F,\Gamma_{\ell_r}} \\ \vdots \\ G_{k,F,\Gamma_{\ell_{n_k}}} \end{bmatrix}, \quad \ell_1 < \ell_2 < \dots < \ell_{n_k} \in \mathcal{L}_k.$$

Finally we have

$$G_{\mathcal{D},F}^h = \begin{bmatrix} G_{1,F} \\ \vdots \\ G_{k,F} \\ \vdots \\ G_{n_{\mathcal{D}},F} \end{bmatrix} \in \mathbb{R}^{N_{\mathcal{D}} \times N_F}, \quad \mathcal{B}_F^h = (G_{\mathcal{D},F}^h)^T.$$

Furthermore, being

$$(G_{F_i,F_i})_{p,z} = \int_{F_i} \varphi_{F_i,p}(\mathbf{x}_{F_i}) \varphi_{F_i,z}(\mathbf{x}_{F_i}), \quad \forall i = 1, \dots, I,$$

matrix  $G_{F,F}^h \in \mathbb{R}^{N_F \times N_F}$  is defined as:

$$G_{F,F}^h = \text{diag}(G_{F_i,F_i}, i = 1 \dots, I).$$

#### Appendix A.4. Assembling matrix $G_F^h$

Let us introduce matrices  $C_{F_i, F_i} \in \mathbb{R}^{N_{F_i} \times N_{F_i}}$  as:

$$(C_{F_i, F_i, S_m})_{p,z} = (1 + \alpha^2) \sum_{S_m \in \mathcal{S}_i} \int_{S_m} \varphi_{F_i, p|_{S_m}} \varphi_{F_i, z|_{S_m}}.$$

For each fracture  $F_i$ , for each fracture  $F_j$ ,  $j = 1, \dots, I$ ,  $j \neq i$  we also have matrices  $C_{F_i, F_j}^{S_m} \in \mathbb{R}^{N_{F_i} \times N_{F_j}}$ , defined as follows:

$$(C_{F_i, F_j, S_m})_{p,z} = \begin{cases} (\alpha^2 - 1) \int_{S_m} \varphi_{F_i, p|_{S_m}} \varphi_{F_j, z|_{S_m}}, & \text{if } \exists m \text{ s.t. } S_m = \bar{F}_i \cap \bar{F}_j \\ 0, & \text{if } \nexists m \text{ s.t. } S_m = \bar{F}_i \cap \bar{F}_j \end{cases}$$

which are then grouped block-wise to form matrix  $C_F^h \in \mathbb{R}^{N_F \times N_F}$  as follows: for  $i = 1, \dots, I$ , for  $j = 1, \dots, I$

$$C_i = \begin{bmatrix} C_{F_i, F_1} & \cdots & C_{F_i, F_j} & \cdots & C_{F_i, F_I} \end{bmatrix}; \quad G_F^h = \begin{bmatrix} C_1 \\ \vdots \\ C_i \\ \vdots \\ C_I \end{bmatrix}.$$

#### Appendix A.5. Assembling matrices $G^u$ and $G^q$

Let us now introduce matrix  $G^u \in \mathbb{R}^{N^S \times N^S}$  defined as follows:

$$G^u = \text{diag}(\mathcal{C}_{S_m}, m = 1 \dots, M),$$

being

$$\mathcal{C}_{S_m} = \begin{bmatrix} \mathcal{C}_{F_i, S_m} & \mathcal{C}_{F_i, F_j, S_m} \\ (\mathcal{C}_{F_i, F_j, S_m})^T & \mathcal{C}_{F_j, S_m}^j \end{bmatrix}, \text{ for } i, j \in \mathcal{I}_{S_m}, i < j,$$

where matrices  $C_{F_i, S_m} \in \mathbb{R}^{N_{F_i}^{S_m}}$ ,  $C_{F_j, S_m} \in \mathbb{R}^{N_{F_j}^{S_m}}$  and  $C_{F_i, F_j, S_m}$  for  $i, j \in \mathcal{I}_{S_m}$  are:

$$(\mathcal{C}_{F_i, S_m})_{p,z} = \int_{S_m} \psi_{F_i, p}^m \psi_{F_i, z}^m, \quad (\mathcal{C}_{F_i, F_j, S_m})_{p,z} = \int_{S_m} \psi_{F_i, p}^m \psi_{F_j, z}^m.$$

Further for  $\ell \in \mathcal{L}_D$  and  $k, y \in \mathcal{K}_{\Gamma_\ell}$ , possibly  $k = y$ , we introduce matrices  $\mathcal{C}_{k, y, \Gamma_\ell} \in \mathbb{R}^{N_k^{\Gamma_\ell} \times N_y^{\Gamma_\ell}}$  as:

$$(\mathcal{C}_{k, y, \Gamma_\ell})_{p,z} = \int_{\Gamma_\ell} \phi_{k, p}^\ell(\mathbf{x}) \phi_{y, z}^\ell(\mathbf{x}), \quad \forall \ell \in \mathcal{L}^\circ, k, y \in \mathcal{K}_{\Gamma_\ell}$$

$\mathcal{C}_{k,y,\Gamma_\ell} = 0, \forall \ell \in \mathcal{L}^\diamond, k \in \mathcal{K}_{\Gamma_\ell}, y \notin \mathcal{K}_{\Gamma_\ell}, \quad \mathcal{C}_{k,y,\Gamma_\ell} = 0, \forall \ell \in \mathcal{L}^\star, k, y \in \mathcal{K}_{\Gamma_\ell}, k \neq y,$   
which are then grouped for  $k = 1, \dots, n_{\mathcal{D}}, \ell_1 < \ell_2 < \dots < \ell_{n_k} \in \mathcal{L}_k$  and  
 $y \in \mathcal{K}_{\Gamma_{\ell_i}}, y \neq k$  to build matrices:

$$\mathcal{C}_{k,k} = \text{diag}(\mathcal{C}_{k,k,\Gamma_{\ell_r}}, r = 1, \dots, n_k), \quad \mathcal{C}_{k,y} = \text{diag}(\mathcal{C}_{k,y,\Gamma_{\ell_r}}, r = 1, \dots, n_k)$$

and then  $G_k^q = \mathcal{C}_{k,k} + \mathcal{C}_{k,y} \in \mathbb{R}^{N_k \times N_{\mathcal{D}}}, k = 1, \dots, n_{\mathcal{D}}$ . Matrix  $G^q \in \mathbb{R}^{N_{\mathcal{D}} \times N_{\mathcal{D}}}$  is  
then obtained grouping matrices  $G_k^q$  column wise for increasing values of  $k$ .

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