Reduced order models for nonlinear dynamic analysis of structures with intermittent contacts

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Abstract

The forced response of structures with complex geometry and intermittent contacts is nonlinear due to contact breathing phenomena that occur during vibration. Therefore, calculation times to predict such responses can be extremely long especially because highly refined finite element models are necessary to properly model geometrically complicated structures. To alleviate this issue, reduced order models can be very beneficial as they can dramatically speed up the analysis process by reducing the size of the system and thus the calculation times.

In this paper a reduced order modeling method for the forced response of structures with intermittent contacts is developed. The proposed approach assumes that the dynamics of the nonlinear system in the frequency range of interest is spatially correlated. The spatial correlation can be dominated by normal modes of the open (or sliding) linear system.

Nonetheless, the boundary conditions of a vibrating structure with intermittent frictionless contacts vary in time (i.e., at any time *t* the contacts are partly open and closed), and the actual extent of the contact area changes over time. Here, this observation is complemented by the assumption that, given the frequency range of the harmonic excitation force, the system dynamics is dominated by one of the modes of either the open or the sliding system, and thus the time evolution of the contact area can be estimated by knowing (a) the normal penetration at the contact node pairs at rest due to pre-stress, and (b) the vector of normal relative displacements at the contact nodes of that dominant mode. As a result, a set of normal modes – referred to as piecewise-linear modes - is computed, by imposing special boundary conditions at the nodes lying on the contact surfaces, and a reduction basis is selected and used to reduce the size of the model of the system.

Two numerical test cases, specifically a cracked plate and two co-axial cylinders are used for validation. Results show that the proposed method allows to accurately compute the nonlinear forced response of structures with intermittent contacts both in case of zero gap and in case of initial pre-stress at the contacts.

Keywords

Nonlinear dynamics; reduced order model; intermittent contact; forced response; harmonic balance method.

1. Introduction

The recent fast increase in computer performance promises to provide the numerical capabilities needed to solve a class of problems that involve nonlinear dynamics of vibrating structures, which is of great interest in a variety of fields. Nevertheless, the solution of nonlinear dynamic equations in case of structures with complex geometries, modeled with highly detailed finite element (FE) models, can still be a formidable task.

Therefore, one of the branches of the scientific research in this field has been focused on the development of reduced order models (ROMs) able to capture the spatial correlations in the vibration and thus reduce the size of the nonlinear system and consequently the amount of computation time necessary for analysis.

A specific class of nonlinear structures consists of structures with localized nonlinearities, where the nonlinearity only involves a relatively small subset of degrees of freedom (DOFs) of the structure, such as is the case of bladed disks with friction dampers (Firrone et al, 2011) or shrouds (Siewert et al. 2010), gears with ring dampers (Zucca et al., 2012), assemblies with bolted joints (Mayer et al, 2007; Schwingshackl et al., 2013), cracked components (Saito et al, 2009; Marinescu et al., 2011), disk-casing rubbing phenomena (Laxalde et al., 2011).

In these cases, a significant reduction of the model size can be obtained using methods based on component mode synthesis (CMS), where the DOFs involved in nonlinearities are included in the ROM (as active DOFs) (Guyan, 1965; Hurty, 1965; Craig and Bampton, 1968). The main drawback of CMS-based models, however, is that the number of nonlinear governing equations is proportional to the number of DOFs involved in nonlinearities, which can be large - for example, many nodes may exist on contacting surfaces, and those nodes are involved in nonlinear no-penetration conditions. To further reduce the size of CMS reduced models for structures with intermittent contacts, an algorithm was developed in (Saito et al, 2010) to select a smaller number of nodes where no-penetration boundary conditions can be enforced (instead of enforcing at all nodes on contacting surfaces).

An alternate way to lower the number of interface DOFs compared to CMS-based models consists of finding a projection basis (i.e., a reduction matrix) whose vectors model the entire structure with no distinction between DOFs involved in the nonlinearities and the other DOFs. This projection matrix can be applied to either full-order models as a single and major reduction or to (already reduced) CMS-based models (as a secondary reduction). Proper orthogonal decomposition (POD) provides such as projection matrix. In POD, a set of snapshots of the nonlinear system response is used to generate a basis of proper orthogonal modes (POMs) which form a reduction matrix (Kerschen et al., 2005). POMs can be interpreted as vectors that represent the spatial coherences in the dynamics of the nonlinear system. POM-based reduction matrices can be effective for model reduction, but their key drawback is that for the POMS to be computed, the solution of the nonlinear system must be computed too.

In Butcher et al. (2007) the local equivalent linear stiffness method (LELSM) is utilized to reduce models with piecewise linear nonlinearities by modifying the rows and columns of the stiffness matrix **K** that correspond to the degrees of freedom that activate the nonlinearity. The LELSM approach is further investigated in AL-Shudeifat et al. (2010), where new Ritz vectors are introduced in the reduction basis, in Butcher et al. (2011), where updated LELSM vectors are formulated, and in AL-Shudeifat et al. (2013) where the approach is applied to switching cracks and damaged boundaries.

In Segalman (2007), systems with local nonlinearities are reduced by means of a reduction basis which includes mode shapes of a reference linear system augmented by a set of additional modes with different boundary conditions at the contacts.

The same approach has been used in Saito and Epureanu (2011) and Zucca and Epureanu (2014), where bi-linear modes (BLMs) have been used to capture the spatial coherences in the dynamics of structures with intermittent contacts by means of two sets of linear modes of the system with special boundary conditions. Only linear modal analyses are necessary to generate BLMs, thus overcoming the computational burden of POD-based methods.

The original BLMs, assumed that, in case of intermittent contacts, transitions between open and closed (i.e., sliding) contacts were instantaneous over the entire contact. In addition, the reduction basis only included vectors from the system with either all open contacts or all sliding contact.

A reduced order modeling method which builds on the concept of BLMs is developed in this paper for structures with intermittent contacts under periodic excitation. The proposed approach assumes that the response of the nonlinear system is periodic also, and that (in the frequency range of interest) the response has a small number of spatial correlations (i.e., a small number of invariant manifolds or nonlinear normal modes dominate the dynamics, so that the physical displacements of the DOFs of the entire system can be expressed as a function of a small subset of generalized coordinates along these manifolds). These spatial correlations can be dominated by normal modes of the open (or sliding) linear system.

As commonly observed experimentally and computationally, a vibrating structure with intermittent frictionless contacts is characterized by boundary conditions that vary in time. That is because the contact surfaces are partly open and/or closed contacts at any instant in time. In addition, the actual extent of the contact area changes over time, during each vibration cycle. Moreover, the static loads applied to the structure define the extent of the contact area at rest. To tackle these complexities, we first assume that in the frequency range of interest the dynamics of the nonlinear system, and therefore the kinematics of the intermittent contact, is dominated by one of the modes (that must be identified in advance) of either the open or the sliding system. This is a strong assumption which limits the applicability of the method. Nonetheless, it is a reasonable assumption in many cases of vibrating structures with complex geometry and relatively simple intermittent contact surfaces.

The dominant mode can be used to approximately track the contact area in time. Specifically, the amplitude of the dominant mode is varied in time and sampled. For each sample, it is examined if either normal penetration exists at the contact node pairs (in case of open system) or the normal contact force is tensile instead of compressive (in case of a sliding system) and a linear system is defined with sliding boundary conditions at the nodes in contact and open boundary conditions at the other nodes.

For each linear system, a set of normal modes – referred to as piecewise-linear modes (PLMs) - is computed. Finally, PLMs are used to construct a reduction basis. Thus, the reduction basis is built in advance, before the actual simulation starts.

In contrast to the original BLMs, the proposed approach does not need transitions (between open and sliding contacts) to be instantaneous over the entire contacts. Instead, the method proposed herein introduces a new set of linear systems, with partially open and partially closed (i.e., sliding) contacts, which are used to extract the PLMs necessary to build the reduction basis.

Two test cases are used to demonstrate the proposed method. Specifically, the forced response analysis of two co-axial cylinders and that of a cracked plate are presented. Results, computed in the frequency domain by means of a harmonic balance method (HBM), show that the reduction method allows to accurately model the nonlinear dynamics of structures with intermittent contacts both for cases of zero gap and for cases of initial pre-stress at the contacts, by significantly reducing the size of the nonlinear system. A surprising and interesting hidden set of solutions are also shown for the nonlinear dynamics of the cracked plate.

2. Methodology

2.1 Governing equations

The governing equations of a vibrating linear structure with linear viscous damping under periodic external excitation can be conveniently written as

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{F}, \tag{1}$$

where overdots represent time derivatives, $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, viscous damping, and stiffness matrices, $\mathbf{q}$ is the vector of generalized DOFs, and $\mathbf{F}$ is the excitation vector. If intermittent contacts occur during vibration, then nonlinear contact forces $\mathbf{F}_{\mathbf{NL}}$ act at the contact areas, and the governing equations become

$$\mathbf{M}\begin{Bmatrix}\ddot{\mathbf{q}}_{\mathbf{NL}}\\ \ddot{\mathbf{q}}_{\mathbf{LN}}\end{Bmatrix}\ddot{\mathbf{q}} + \mathbf{C}\begin{Bmatrix}\dot{\mathbf{q}}_{\mathbf{NL}}\\ \dot{\mathbf{q}}_{\mathbf{LN}}\end{Bmatrix} + \mathbf{K}\begin{Bmatrix}\mathbf{q}_{\mathbf{NL}}\\ \mathbf{q}_{\mathbf{LN}}\end{Bmatrix} = \mathbf{F} + \begin{Bmatrix}\mathbf{F}_{\mathbf{NL}}(\mathbf{q}_{\mathbf{NL}})\\ 0\end{Bmatrix}, \tag{2}$$

where $\mathbf{q}$ is partitioned in $\mathbf{q}_{\mathbf{NL}}$, which contains the DOFs directly involved in the contacts, and $\mathbf{q}_{\mathbf{LN}}$, which is the vector of DOFs not involved in the intermittent contacts.

It is a usual practice in case of a dynamic analysis of structures with sliding or intermittent contacts to create matching meshes on the contact surfaces. If matching meshes do not exist, it is possible (and necessary) either to create a layer of virtual contact pairs (Hohl et al., 2008) or to create virtual nodes on one of the two surfaces perfectly overlapped to the physical nodes on the other (Zucca et al., 2012). The reason is that node-to-node contact elements are used in the analysis and they require contact pairs.

In case of frictionless contacts, like in the cracked component shown Figure 1 (thick contour line shows the visible edges of the crack), the compressive normal contact force N at each contact pair (shown as dots in Figure 1) is defined as

$$\mathrm{N} = \max(k_n\, u_n, 0), \tag{3}$$

where k_n is the normal contact stiffness, and u_n is the normal displacement/penetration between the nodes in contact, with $\mathbf{n}$ being the direction normal to the contact surface.

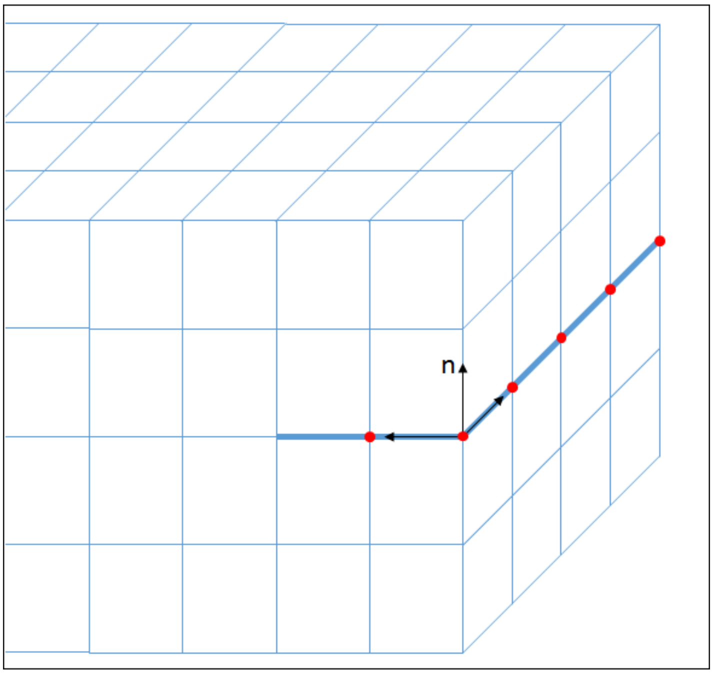

Figure 1 – A cracked component: contact pairs and local axes showing the normal direction

2.2 Conversion from the time domain to the frequency domain

Time domain analysis (TDA) of Eq. (2) may require long calculation times. As an alternative for periodic excitations of frequency ω which lead to periodic responses, harmonic balance method (HBM) can be successfully used to reduce computation time (Cardona et al., 1998). Specifically, the periodic quantities (i.e. displacements and forces) are approximated by a Fourier series in time, truncated at its H^{th} term, and the governing equations can be written in the frequency domain as

$$\left[-(\mathrm{h}\omega)^2\mathbf{M}+i\,\mathrm{h}\omega\mathbf{C}+\mathbf{K}\right]\begin{Bmatrix}\mathbf{q}_{\mathrm{NL}}^{(\mathrm{h})}\\ \mathbf{q}_{\mathrm{L}N}^{(\mathrm{h})}\end{Bmatrix}=\mathbf{F}^{(\mathrm{h})}+\begin{Bmatrix}\mathbf{F}_{\mathrm{NL}}^{(\mathrm{h})}\left(\mathbf{q}_{\mathrm{NL}}^{(\mathrm{h})}\right)\\ 0\end{Bmatrix},\qquad \mathrm{h}=0\ldots\mathrm{H}, \qquad (4)$$

where superscript (h) indicates the h^{th} Fourier coefficient, and $i=\sqrt{-1}$.

The calculation of the Fourier coefficients of the contact force $\mathbf{F}_{\mathrm{NL}}^{(\mathrm{h})}$ is performed during analysis by an alternate frequency time (AFT) method (Cameron and Griffin, 1989), conceptually described in Figure 2. AFT makes use of fast Fourier transforms (FFT) and inverse fast Fourier transforms (IFFT) to convert forces and responses from the frequency domain to the time domain and vice versa. Specifically, the time history of the normal relative displacements of contact pairs is obtained (steps a-b). Then the periodic normal force is computed (step c) by means of Eq. (3), and its Fourier coefficients are assembled in the $\mathbf{F}_{\mathbf{NL}}^{(h)}$ vector (steps d-e).

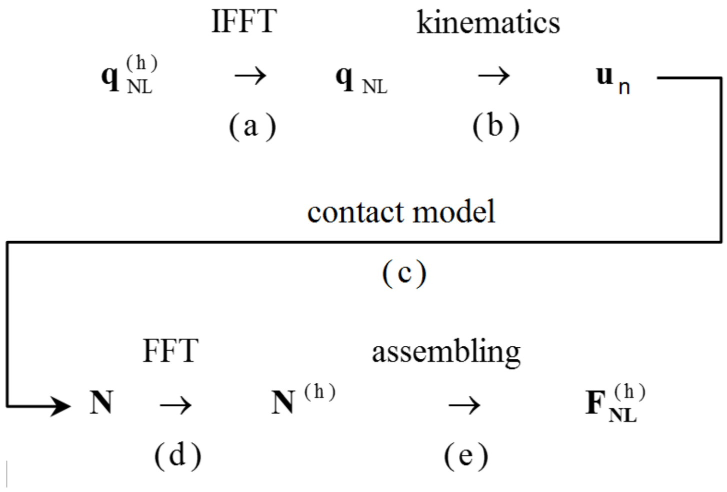

Figure 2 – Conceptual algorithm of the AFT method

Despite the use of HBM and AFT, the nonlinear forced response analysis by means of Eq. (4) can still be a formidable task in terms of memory requirements and computational time especially for highly detailed FE models. That is because the size of the nonlinear system is proportional to the number of DOFs involved in the nonlinear contact, namely $\mathbf{q}_{\mathbf{NL}}$, and to the number of harmonics retained in the analysis. In the next section, a reduced order modeling technique is described to capture the space correlations in the system dynamics, and to exploit these correlations in order to reduce the amount of time and memory necessary for the forced response analysis.

2.3 Reduced order model

Consider a system of nonlinear equations governing the dynamics of a structure with intermittent contacts. A typical method to determine the motion is to time march these equations. One approach to time march these equations is to compute the response at some time t+Δt based on the state of the system at time t and based on the forces acting on the system. During the time interval Δt, the system is considered to not change contact conditions. Hence, the system is a (forced) linear system from time t to t+Δt since the only nonlinearity considered are due to the intermittent contact. Of course, at time t+Δt, contact conditions may change because the contact area varies in time. Thus, one can conclude that the boundary conditions of a vibrating structure with intermittent frictionless contacts vary in time (i.e., at any time *t* the contacts are partly open and closed), and that the actual extent of the contact area changes over time. The dynamics of the system from time t to t+Δt can be transient rather than steady state. Nonetheless, the dynamics are dominated by a set of modes for the corresponding set of boundary conditions. To construct reduced order models which exploit these observations, we note that if the evolution of the contact area can be predicted, then it is possible to extract a set of normal modes able to approximate the space correlation of the system during its forced response. These modes are computed by assuming different extents of the contact areas and are used as a projection basis in a reduced order modeling method for the governing Eq. (4).

2.3.1 Time evolution of the contact area

The proposed method is designed for systems with a periodic response to a periodic excitation, where the opening and closing of intermittent contacts is the source of nonlinearity. Structures with intermittent contacts may by also be pre-stressed (e.g., due to static forces additional to the periodical excitation). Here, we focus on structures where large geometric nonlinearities in the system are not significant. Thus, the linear behavior of the structure dominates the motion. This linear behavior (for a reference contact condition) contains spatial correlations (modes). Hence we can consider the important cases where, in a given frequency range of interest, the kinematics of the contact surface can be approximated by the kinematics of a linear mode of a linear system (referred to as the reference linear system). That mode is referred to as a dominant mode. As a consequence, the time evolution of the contact area during vibration can be approximately predicted by knowing the normal penetration at the contact node pairs at rest due to pre-stress (ΔX_{PS}) and the normal relative displacements ΔV at the contact nodes of the dominant mode of the reference linear system (see next Section for details about the choice of the dominant mode), as shown in Figure 3. Note that this is a surface-surface penetration. For convenience, Figure 3 shows the simplified case of a line contact to explain the concept.

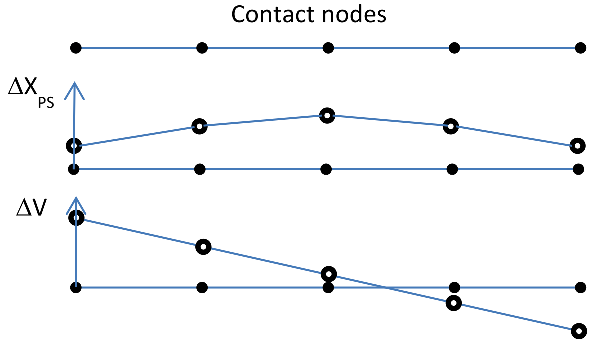

Figure 3 – Example of contact area: contact nodes, static penetration (ΔX_{PS}) due to pre-stress and normal relative displacement (ΔV)

Hence, a nodal penetration can be defined as

$$\Delta X = \Delta X_{PS} + \alpha\ \Delta V \quad (5)$$

at each contact node pair, with $\alpha \in [\alpha_{min}, \alpha_{max}]$, where $\alpha_{min} < 0$ and $\alpha_{max} > 0$ are minimum and maximum values of the factor α. These values are the two extreme values of α for which the contact area does not change anymore if α is further decreased or increased beyond those limits. For each value α, the extent of the contact area is computed, namely, the nodes where the nodal penetration $\Delta X_i = \Delta X_{PS} + \alpha_i\ \Delta V$ is positive are identified (indicated by stars in Figure 4). Any time one of the contact nodes changes its status (from open to closed and vice versa), a sample of the contact area is taken (Figure 4).

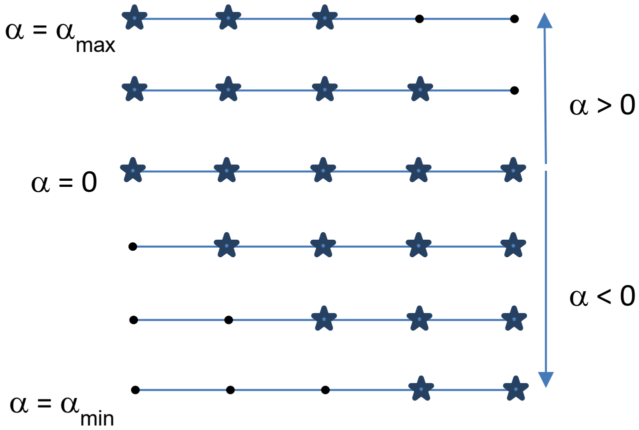

Figure 4 – Samples of the contact area are collected each time one contact node changes status as the parameter α increases from α_{min} to α_{max}

The number of contact samples stored at the end of this process is equal to Z+1, where Z is the number of contact pairs. We assume that such a detailed description of the contact area evolution (node by node) is not necessary to model the evolution of the contact area for the purpose of a nonlinear forced response. Only a few equally spaced contact samples are eventually selected (Figure 5). Of course, a convergence study can be performed to determine the optimal/converged number of contact samples. Also, to improve the ability of the contact samples to represent the contact area evolution during vibration, the set of contact samples is augmented by including a complementary set of contact samples, obtained by inverting the status of each contact pair (open to closed, and closed to open) in the selected contact samples (Figure 5).

The evolution of the contact area during intermittent contacts is more complicated than the one described by means of ΔX. For instance, the contact samples collected by the procedure above neglect the perturbation of the static deflection that occurs during vibration, observed for instance in Zucca and Epureanu (2014). As a consequence, the fine details of the evolution of the contact area during vibration can differ from the one depicted by the contact samples. However, the general effect of the intermittent contact is captured well enough as to produce accurate system-level forced response results. This is similar to the process of node down-sampling applied to lower the computational burden when enforcing no-penetration boundary conditions on intermittent contact surfaces (Akesson and Olhoff, 1988; Butcher and Lu, 2007; Henshell and Ong, 1975; Kammer, 1991; Matta, 1987; Penny et al., 1994; Saito et al., 2010; Zhu and Zhang, 2006).

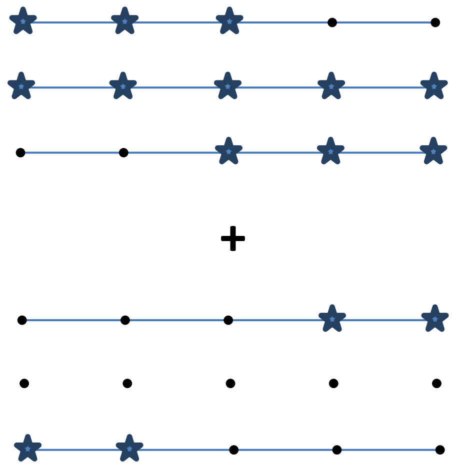

Figure 5 – Selected samples of the contact area (upper set) and the additional set of complementary contact samples (lower set)

In case of zero pre-stress (ΔX_{PS}=0), the procedure based on Eq. (5) predicts that all contact pairs with $\Delta V > 0$ enter simultaneously in contact as α becomes greater/smaller than zero. Actually, a static offset can arise during vibration also for cases of zero pre-stress. The procedure to estimate that static offset is described briefly here (Zucca and Epureanu (2014) provide a full rationale and a detailed description):

- Compute normal relative displacement ΔV of the dominant mode shapes at all node pairs.
- Define a set of static forces $\mathbf{F}_A$ proportional to $|\Delta V|$ and a set of static forces $\mathbf{F}_B = -\mathbf{F}_A$.
- Apply forces $\mathbf{F}_A$ to the contact nodes lying on one surface of the contact and forces $\mathbf{F}_B$ to the contact nodes lying on the other surface of the contact.
- Compute the static offset $\mathbf{X}_A$ and $\mathbf{X}_B$ of contact points on the two contact surfaces due to that set of static loads by solving the corresponding linear static system.
- Define the static offset of the contact surface as $\Delta\mathbf{X}_0 = \mathbf{X}_A - \mathbf{X}_B$.

Equation (5) is then used to model the evolution of the contact area for the zero pre-stress case with $\mathbf{X}_{PS}=\Delta\mathbf{X}_0$. This allows one to track the evolution of the contact area as the amplitude of the vibration, represented by the term α in Eq. (5), changes.

2.3.2 Dominant mode and frequency range selection

The procedure to track the evolution of the contact area described in the previous section implies that the nonlinear system dynamics is dominated by one mode of the reference linear system. That builds on the findings of Saito and Epureanu (2011) on the use of bi-linear modes for model reduction. In that work, the normal modes of the linear systems with sliding or open boundary conditions at the contact DOFs are used as (simpler) dominant modes for the evolution of the contact area.

When a dominant mode exists in the dynamics, it only dominates the response in a given frequency range. Thus, the selection of the dominant mode is accompanied by the selection of the associated frequency range. Two approaches are possible:

1. the dominant mode is chosen first, and then the frequency range where that mode dominates is determined;
2. the frequency range for the forced response is defined, and then it is determined what mode dominates the nonlinear response in that range (if any such mode exists).

The first approach corresponds to the case of an analyst interested in the response of the system near one or more specific resonances (i.e., first bending mode, second torsion mode, etc.), while the second approach corresponds to the case of an analyst interested in the response of the system in a given frequency range of interest.

In this paper, we assume that the first approach is used. Specifically, the FE model of the system with intermittent contact is available, and modal analyses of the linear system with (fully) open and (fully) sliding boundary conditions are performed. Then the analyst selects one of the linear normal modes of either the open or the sliding systems as the mode whose behavior is to be investigated under nonlinear boundary conditions at the contact. That mode is the dominant mode.'

The choice of either the open or the sliding linear system is based on the boundary conditions of the contact at rest. If the number of closed contact pairs is significantly higher than the number of open contact pairs, then the sliding system is best and vice-versa. If the size of the closed area is similar to the size of the open area, or the system is not pre-stressed, then either the open or the sliding system can be chosen.

Then, the frequency range suitable for forced response analysis is estimated because the selected mode may not be dominating outside that frequency range. Two different approaches are used to define that frequency range. One approach is designed for intermittent contacts without pre-stress (A). The other approach is designed for cases with intermittent contacts with pre-stress (B) due to a set of static loads that act on the structure.

A) Case of structures with no pre-stress

In this case, the bi-linear frequency (BLF) approximation (Shaw and Holmes, 1983) is used to estimate the resonance frequency of the nonlinear system. The BLF is obtained using the values of the natural frequencies of two linear systems, by assuming that during vibration the system alternates between these two linear systems, and the nonlinear period is the sum of one half period of the first system and one half period of the second system. Thus, the BLF is defined by following relationship

$$\mathrm{BLF} = \frac{2 f_1 f_2}{f_1 + f_2}, \qquad (6)$$

where f_1 and f_2 are the natural frequencies of the two linear systems.

To define the two linear systems to be used in the BLF approximation, the normal relative displacements ΔV at the contact nodes of the dominant mode shape are used to define two sets of boundary conditions at the contact surfaces. The first linear system is obtained by imposing sliding boundary conditions at nodes with $\Delta V>0$, and open boundary conditions at nodes with $\Delta V<0$. The second linear system is obtained by imposing sliding boundary conditions at nodes with $\Delta V<0$ and open boundary conditions at nodes with $\Delta V>0$. The two sets of modes of these two linear systems are compared to the dominant mode through a standard modal assurance criterion (MAC). The mode pairs with the highest MAC values are selected. The BLF approximation is computed from the frequencies of these two selected mode shapes. Once the BLF is computed, a frequency range around the BLF value is selected for the force response analysis.

B) Case of structures with pre-stress

The BLF approximation was developed for systems without pre-stress, so a different approach is necessary in the case of structures with pre-stress. We observe that at rest the pre-stress makes some node pairs be in contact, and makes other nodes not be in contact. Therefore, a linear system can be defined by using sliding boundary conditions at the node pairs in contact and open boundary conditions at the open node pairs. The modes of this linear system are compared to the dominant mode through the MAC. The mode with the highest MAC value is selected and its frequency is computed. A range around that frequency is selected for the force response analysis.

As a consequence of the procedure described in this section, if multiple dominant modes exist in a frequency range, then the evolution of the contact area is tracked for all the dominant modes, and multiple sets of samples are collected. Thus, in structures characterized by regions of high modal density, like repeated structures (e.g., bladed disks), the method may be not be effective or convenient because the ROM size grows proportionally with the number of dominant modes.

2.3.3 Reduction matrices

Once the samples of the contact area are collected (as shown in Figure 5), several linear modal analyses are performed. Specifically, sliding boundary conditions are applied to the node pairs which are in contact in the j^{th} contact sample (i.e., the contact nodes are coupled along the normal direction by means of a normal stiffness k_n). The corresponding stiffness matrix K_j is obtained. Next, the eigenproblem

$$\mathbf{K}_j \mathbf{V}_j = \lambda_j \mathbf{M} \mathbf{V}_j \tag{7}$$

is solved, where λ_j are the eigenvalues, and $\mathbf{V}_j$ are the mode shapes of the linear system corresponding to the j[th] contact sample.

Modes $\mathbf{V}_j$ are referred to as piecewise-linear modes (PLMs) and are collected in a reduction matrix expressed as

$$\mathbf{P} = \lfloor \mathbf{V}_1 \quad \dots \quad \mathbf{V}_j \quad \dots \quad \mathbf{V}_{N_S} \rfloor, \tag{8}$$

where N_S is the number of contact samples selected to approximate the evolution of the contact area, with $N_S \leq Z+1$. Recalling the analogy with the time marching process, at any moment in time, the nonlinear system is viewed as a system which switches from one linear system to another, to another, etc. according to the changes in the boundary conditions at the contacts over time. Each mode shape $\mathbf{V}_j$ is likely to dominate the dynamics of the j[th] linear system. Of course, not all the PLMs computed from each sample contribute significantly to the response. Hence, a mode selection process is necessary and it is described in the next Section. A final remark about the dominant mode is worthwhile. Since it is only used to approximate the evolution of the contact area and to define the set of linear systems to be used for PLMs generation, it is possible that the dominant mode is not included in the reduction matrix.

The N_S linear systems, defined to compute PLMs, differ from one another just in the boundary conditions applied to the contact pairs. These differences can be small and may result in very similar PLMs. As a consequence, the reduction matrix $\mathbf{P}$ could be ill-conditioned (i.e., some of its columns could be nearly linearly dependent).

To overcome this issue, singular value decomposition (SVD) (Lawson and Hanson, 1995), a well established approach in structural dynamics, has been used to condition the transformation matrix by retaining in the conditioned $\mathbf{P}$ matrix only the eigenvectors corresponding to the highest singular values, and neglecting the others. Specifically, the singular values of the $\mathbf{P}$ matrix are computed, and then scaled by dividing them by the maximum singular value. Then, only vectors whose scaled singular value is higher than 10^{-6} are used to assemble the final reduction matrix $\mathbf{P}$.

Generally, static pre-stress forces are not correlated with the dynamic excitation forces. Hence, the static pre-stressed deformation is not necessarily correlated with the dynamic response. In that case, the projection of the static governing relations in Eq. (4) (for h = 0) onto the subspace spanned by the PLMs in $\mathbf{P}$ will not result in a ROM suitable to represent accurately the static deflection of the structure. To improve the accuracy of the reduction matrix in modeling the static deflection of the system, one can augment the reduction basis by adding the static deflection $\mathbf{q}_{\mathrm{PS}}$ due to the static forces. The resulting reduction matrix for the static equations is

$$\mathbf{P_0} = \begin{bmatrix} \mathbf{P} & \mathbf{q}_{\mathrm{PS}} \end{bmatrix}. \tag{9}$$

In the end, the governing equations of the ROM can be expressed as

$$\begin{gathered} \mathbf{k}_0\, \boldsymbol{\eta}^{(0)} = \mathbf{f}^{(0)} + \mathbf{f}_{\mathbf{NL}}^{(0)}\left(\boldsymbol{\eta}^{(0)}, \boldsymbol{\eta}^{(\mathrm{h})}\right) \\ \left[-(\mathrm{h}\omega)^2 \mathbf{m} + i\,\mathrm{h}\omega\,\mathbf{c} + \mathbf{k}\right]\boldsymbol{\eta}^{(\mathrm{h})} = \mathbf{f}^{(\mathrm{h})} + \mathbf{f}_{\mathbf{NL}}^{(\mathrm{h})}\left(\boldsymbol{\eta}^{(0)}, \boldsymbol{\eta}^{(\mathrm{h})}\right), \end{gathered} \qquad \text{with} \quad \mathrm{h} = 1 \ldots \mathrm{H}, \tag{10}$$

where

$$\begin{aligned} &\mathbf{k}_0 = \mathbf{P}_0^{\,T}\mathbf{K}\,\mathbf{P}_0; \quad \mathbf{q}^{(0)} = \mathbf{P}_0\,\boldsymbol{\eta}^{(0)}; \quad \mathbf{f}^{(0)} = \mathbf{P}_0^{\,T}\mathbf{F}^{(0)}; \quad \mathbf{f}_{\mathbf{NL}}^{\ (0)} = \mathbf{P}_0^{\,T}\left[\mathbf{F}_{\mathbf{NL}}^{\ (0)} \quad 0\right]^T \\ &\mathbf{m} = \mathbf{P}^T\mathbf{M}\,\mathbf{P}; \quad \mathbf{c} = \mathbf{P}^T\mathbf{C}\,\mathbf{P}; \quad \mathbf{k} = \mathbf{P}^T\mathbf{K}\,\mathbf{P}; \\ &\mathbf{q}^{(h)} = \mathbf{P}\,\boldsymbol{\eta}^{(h)}; \quad \mathbf{f}^{(h)} = \mathbf{P}^T\mathbf{F}^{(h)}; \quad \mathbf{f}_{\mathbf{NL}}^{\ (h)} = \mathbf{P}^T\left[\mathbf{F}_{\mathbf{NL}}^{\ (h)} \quad 0\right]^T, \qquad \textit{with} \quad h = 1 \ldots H. \end{aligned} \tag{11}$$

Although two different reduction matrices are used for the static (h = 0) and the dynamic (h > 0) parts of the governing equations, the static and dynamic relations in Eq. (10) are still coupled to each other by the nonlinear forces acting at the contact. The solution of the nonlinear Eq. (10) must be performed by means of an iterative nonlinear solver. In case of intermittent contacts, arc-length continuation methods are needed because of the physical hardening or softening behavior of the response curve (which is known to be related to contact opening and closing).

2.3.4 Piece-wise linear modes selection

The number of PLMs retained in matrix $\mathbf{P}$ determines the size of the ROM. The existence of space correlations (typical of the response of vibrating structures in a specific frequency range of interest) ensures that only a subset of modes $\mathbf{V}_{\mathrm{j}}$ is necessary to properly compute the nonlinear forced response of the system.

Recalling once again the analogy with the time marching process, at any moment in time, the dynamics of the nonlinear system is dominated by the PLMs which to dominate the dynamics of the corresponding linear system at that time. Thus, the selection of the PLMs follows a process which resembles the one used for well known linear systems. Specifically, it is known that only modes whose frequencies are within or close to the frequency range of interest contribute significantly to the dynamics and are therefore retained in the modal reduction.

The approach used to select PLMs to be retained in the reduction matrix **P** is similar in general view, but it is also extended to a multi-harmonic analysis. Specifically (see Figure 6), first a frequency range $[h \cdot f_i \quad h \cdot f_f]$ is defined (horizontal lines in Figure 6) for each harmonic retained in the HBM formulation, where h is the harmonic index, and $[f_i \quad f_f]$ is the frequency range of the excitation. Then all the PLMs whose natural frequencies lie within any of those frequency ranges (circles in Figure 6) are selected and included in the reduction matrix **P**. The rationale for this selection process is that the nonlinear forces $\mathbf{F_{NL}}$ (which act on the contact surface) contain multiple harmonics. Hence, all the PLMs whose natural frequencies are included in the higher order frequency ranges could be excited and could contribute to the nonlinear forced response of the system. Thus, these PLMs are included. With this approach, for each sample, the number of PLMs is fixed and not a parameter that can be changed during a convergence study of the method. Of course, each nonlinear system and each dominant mode may have a different number of PLMs, but once the system and the dominant mode are selected, the number of PLMs is a result of the selection algorithm and should not be changed.

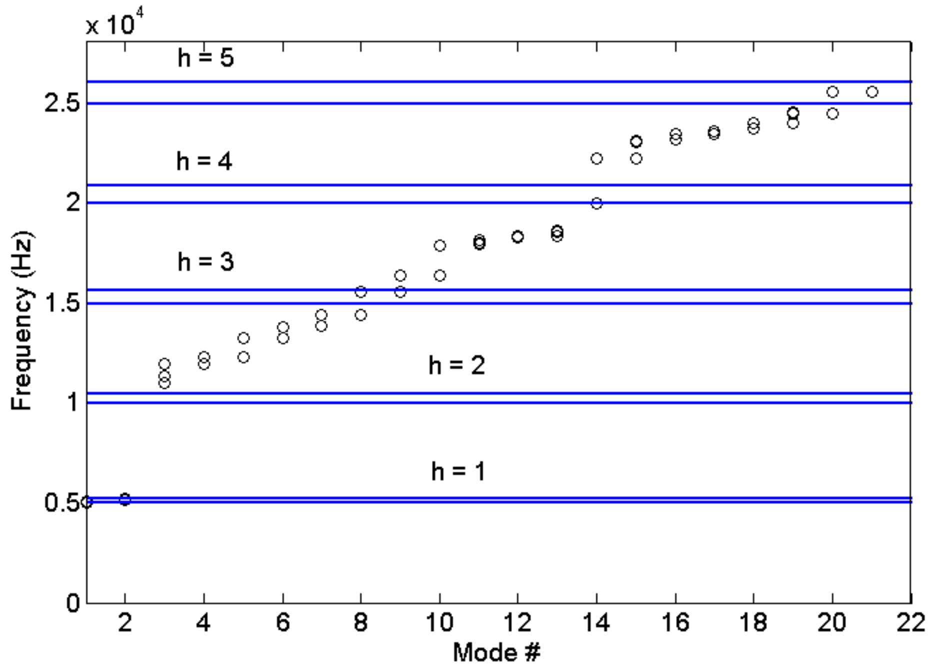

Figure 6 – Graphical example of PLM selection process (5 harmonics are retained in the HBM formulation)

3. Results

The proposed method described in Section 2 is numerically demonstrated here for two relevant test cases. In both cases, the response computed by the proposed ROM is compared to the response obtained by known and commonly used solution methods which can be considered as a reference for evaluating the accuracy of the new reduction technique described herein. Specifically, the response computed by a traditional Craig-Bampton component mode synthesis (CB-CMS) reduced model (Craig and Bampton, 1968) is used for comparisons. This CB-CMS model is a large, converged model, which retains all DOFs involved in the nonlinearity (namely $\mathbf{q}_{NL}$) as master/active DOFs. To reduce the computational time (and without loss of generality), the mass and stiffness matrices ($\mathbf{M}$ and $\mathbf{K}$) of the CB-CMS model are used in Eq. (11) and in the calculation of the reduction matrix $\mathbf{P}$ (instead of full FE matrices). In this way the PLM reduction is demonstrated as a secondary reduction. The PLM reduction can also be a primary reduction when applied to a full FE model. The key goal of the PLMs is to address the main challenge of reducing the number of DOFs in the nonlinear part of the governing equations. To reduce the linear part, there are also other methods which can be used (such as CB-CMS), but those methods do not apply to the nonlinear part.

3.1 Two co-axial cylinders

The first case considered resembles certain combustor assemblies. The structure consists of two co-axial cylinders. Each cylinder is clamped at one edge and partially overlapped (one inserted into the other) at its other end (Figure 7). Each cylinder is additionally constrained to ground at two locations (symmetrical, as shown by triangles in Figure 7). Note that these constraints break the axial symmetry of the assembly. The cylinders are made of steel with $E = 2 \cdot 10^5$ N/mm^2, $\nu = 0.3$, and $\rho = 7{,}800$ kg/m^3. Their wall thickness is 0.6 mm. They are modeled with 15,360 linear brick elements for a total of 23,424 nodes. A total number of 256 contact node pairs exist over the contact surface between them.

A harmonic transverse force is applied to the outer cylinder, while the forced response is computed at point A located on the outer cylinder. The first analysis corresponds to a case with zero pre-stress at the interface at rest, and unitary transverse excitation F. The aim of the analysis is to study the system response around its first resonance, corresponding to its first bending mode. After a convergence study, it was determined that the first 5 harmonics lead to a converged solution. These 5 harmonics are retained in the HBM calculation.

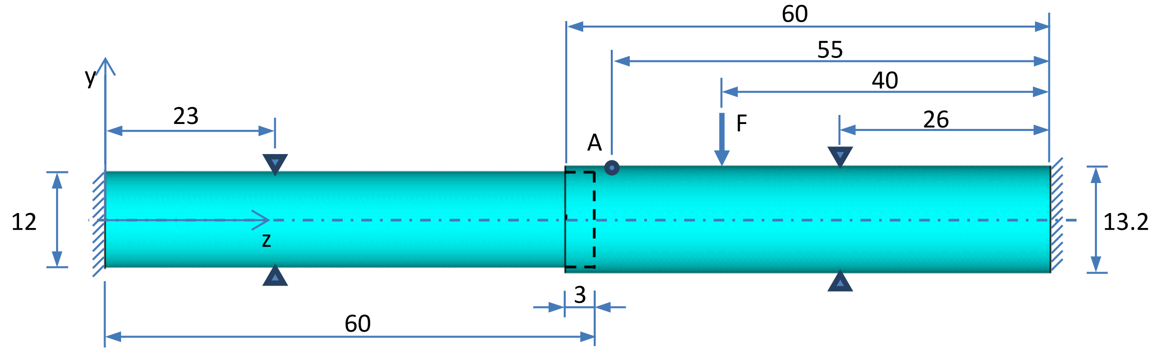

Figure 7 – Geometry of the two co-axial cylinders (dimensions are in mm)

The first step is the selection of the dominant mode to be used in the contact area evolution process. In case of no pre-stress, the procedure A described in Section 2.3.2 is used. The first bending mode of the linear system with sliding boundary conditions is selected as the dominant mode. That is done because we wish to challenge the PLM reduction method and hence we seek a more difficult case to study, namely that of a more complex assembly-level mode. Of course, if open contacts are used for the assembly, then only component-level mode shapes exist (separately for each cylinder, Figure 8).

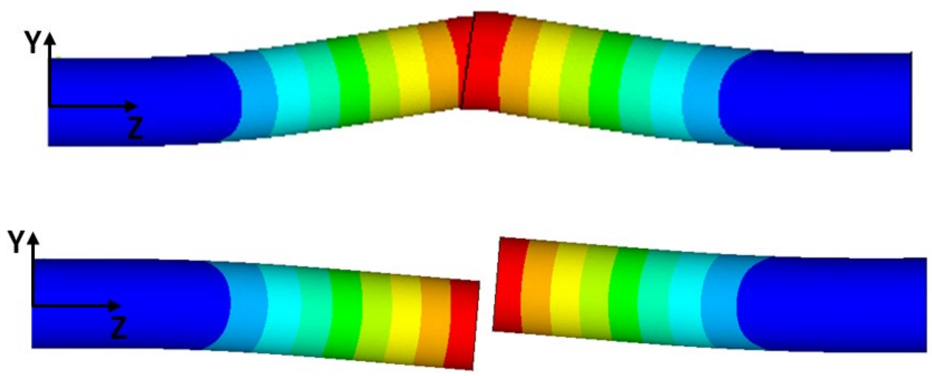

Figure 8 – Linear dominant mode for sliding contacts (up), and component level mode shapes for the open contacts (down)

Next, the $\Delta\mathbf{V}$ vector corresponding to the dominant mode is computed, and the two linear systems are defined (one for each set of boundary conditions). The first 5 modes of each of these two linear systems are computed and compared to the dominant mode using MAC values (Table 1).

Table 1 – MAC between the dominant mode and the modes of the two linear systems used for the BLF calculation

	Mode #1	Mode #2	Mode #3	Mode #4	Mode #5
System #1	9.9E-1	3.0E-18	4.5e-4	5.6e-18	4.1E-20
System #2	9.9E-1	4.5E-18	6.7e-4	6.0e-18	2.6E-20

In both cases, mode #1 has the highest MAC value. Therefore, the natural frequencies of mode #1 of both linear systems are used to compute the BLF, whose obtained value is 5,046 Hz. As shown in Figure 9 (which shows the actual nonlinear forced response of the assembly) the error between the BLF and the actual resonance frequency (5,034.5 Hz) is of about 0.2%, which demonstrates that the BLF gives an excellent scalar to estimate the frequency range of the analysis to be performed with a PLMs-based ROM using the selected dominant mode.

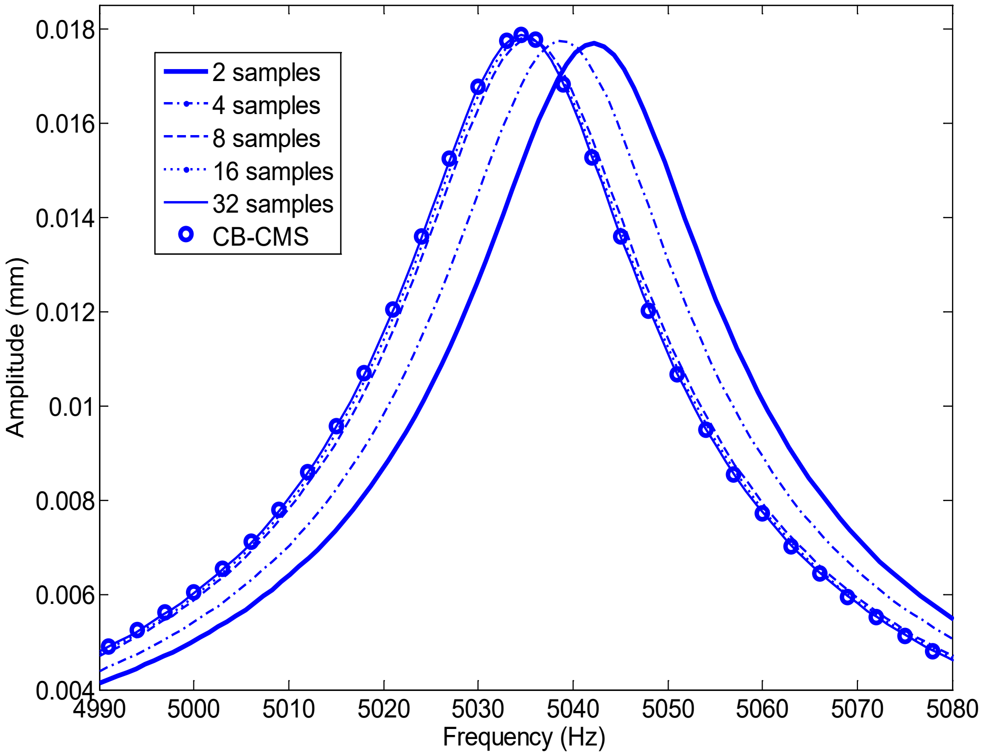

Figure 9 – Two cylinders without pre-stress – Forced response curves

A convergence study of the method is performed by doubling the number of contact samples at each step (Figure 10). The convergence of the nonlinear resonance frequency to its actual value is faster than the convergence of the value of the maximum vibration amplitude. It is noteworthy that with only 4 samples, corresponding to 17 modes, the nonlinear resonance frequency is predicted with 0.1% error, and the maximum vibration amplitude is computed with less than 1% error.

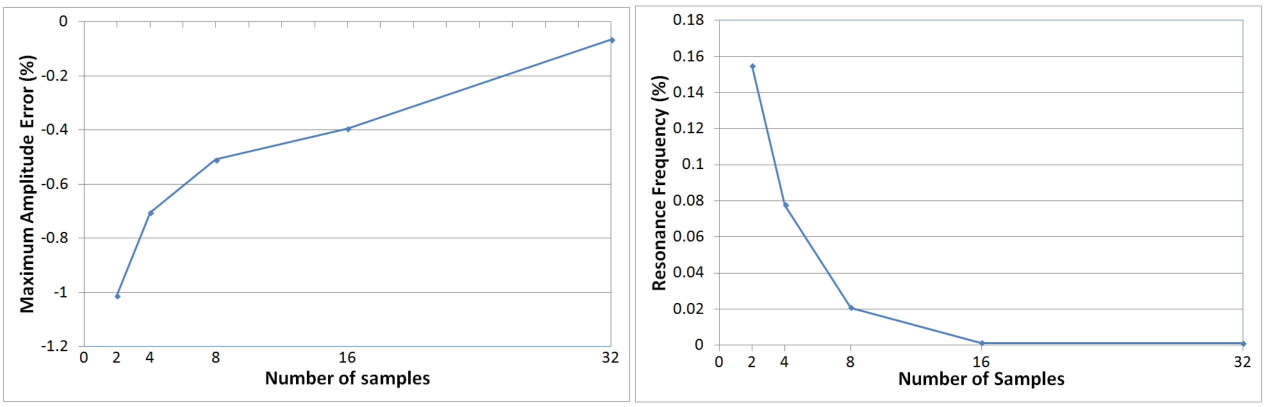

Figure 10 – Convergence study for the two co-axial cylinders with no pre-stress

The main goal of ROMs is to shorten the calculation times of the analysis, while preserving the accuracy of the results with respect to full FE models. Thus, to assess the performance of the proposed method, the average calculation time per iteration was computed for different sets of contact samples. Those times are compared to the average calculation time per iteration necessary when using the CB-CMS model (Figure 11). The ROM obtained with the proposed method proves to be more efficient than the CB-CMS ROM. The comparison would give even more favor to the ROM if the full FE model, instead of the CB-CMS ROM, would have been used. Of course, the CB-CMS reduces the size of the full FE model. The PLM-based ROM reduces that size even further. In both cases, the size of the ROM is significantly lower than the size of the nonlinear (core) of the CB-CMS model, whose size is larger than the number of nonlinear DOFs $\mathbf{q}_{NL}$, and yet it is much smaller that the full-order FE model.

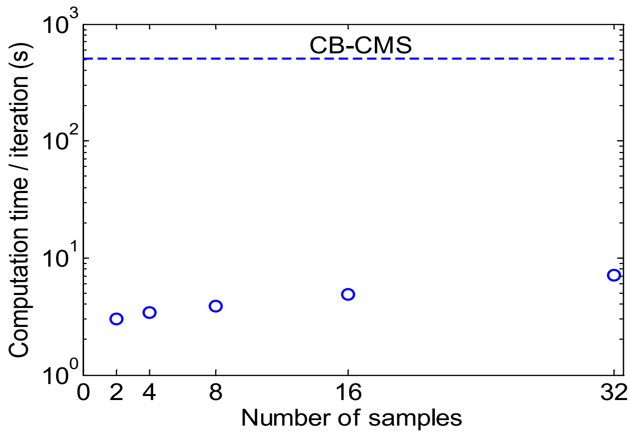

Figure 11 – Comparison of computation times of the proposed method with the CB-CMS method

The second analysis corresponds to a case with pre-stress. In this case, static forces are added to produce pre-stress on the contact area. Specifically, outward unitary radial forces are applied to the contact nodes of the inner cylinder. As a result, a compressive pre-stress is generated on the contact area which is fully closed at rest. When the structure starts vibrating, a partial loss of contact is observed, as pointed out by the softening effect of the forced response (Figure 12). The size of the contact area where contact loss occurs increases with the vibration amplitude.

Also in this case, according to the procedure outlined in Section 2.3.2, the linear system with fully closed contact is used to determine the dominant mode. Like in the first case, we wish to apply the proposed method to a challenging case, where system-level behavior is important. Hence, we focus on the forced response near the first bending vibration. Thus, the first bending mode is selected as the dominant mode. Its natural frequency of 5,158.1 Hz is used to determine the frequency range of the analysis (Figure 10).

The ROM initially obtained by retaining 4 contact samples does not model very accurately the response in the frequency range of interest (Figure 12). These very few contact samples lead to a modest accuracy near the maximum amplitude. Also, the shape of the response curve differs significantly from the shape computed using the CB-CMS model. The reason is that the actual contact area varies with the vibration amplitude due to the pre-stress, and 4 contact samples are not enough to accurately capture the evolution of the contact area.

When the number of contact samples is increased from 4 to 10, a much better match between the CB-CMS model and the ROM is observed. Specifically, an error of just of 0.02% is obtained in the frequency of maximum response while the maximum response amplitude is captured with an error of only 0.05%.

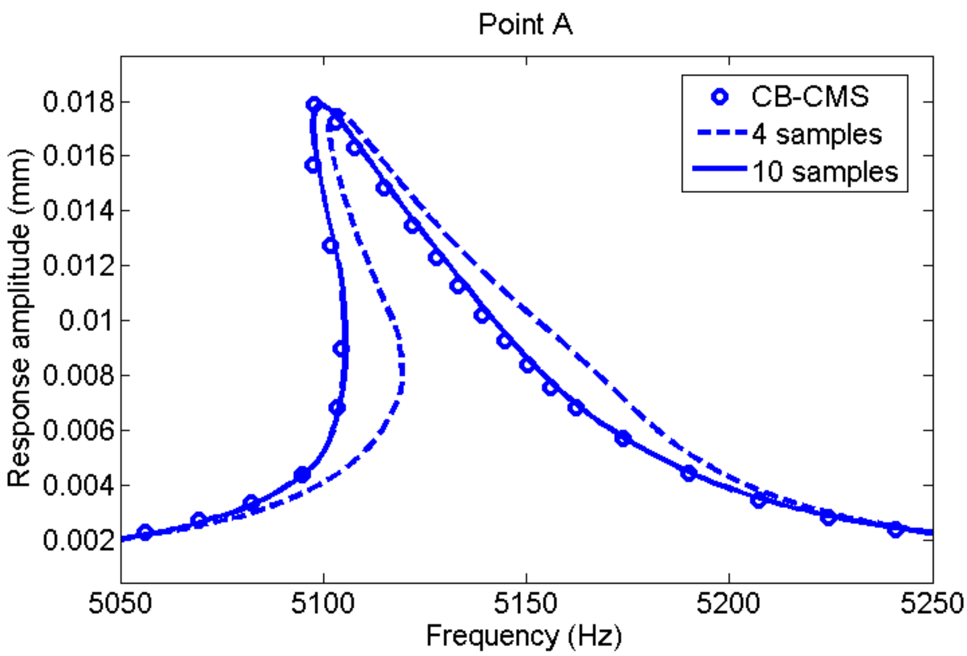

Figure 12 – Two cylinders with pre-stress – Forced response curves

3.2 Cracked plate

The second case analyzes a cracked plate (shown in Figure 13). The plate is made of steel with $E = 2 \cdot 10^5$ N/mm², $\nu = 0.3$, and $\rho = 7{,}800$ kg/m³. The plate was modeled with linear solid elements (a total of 18,630 DOFs), while 90 contact pairs are located on the crack surface.

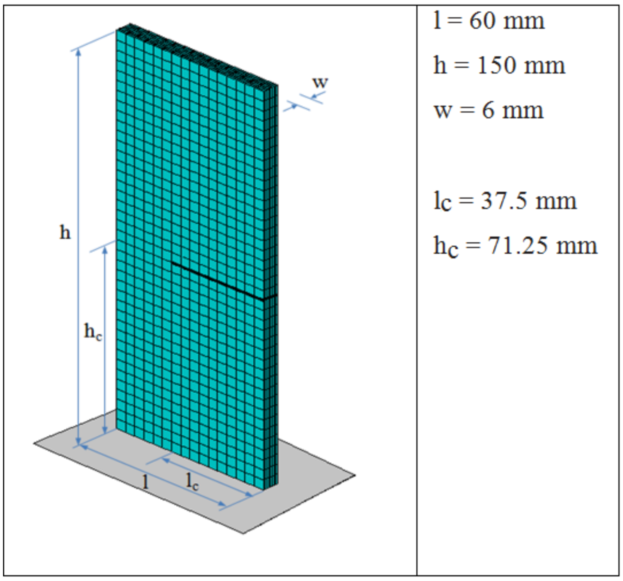

Figure 13 – Geometry of the cracked plate

Two compressive static forces (F_{PS} shown in Figure 14) act on the top of the plate and create pre-stress on the crack surface. At rest the contact occurs only on part of the contact pairs (shown as circles in Figure 14), while the remaining part of the crack is open. A harmonic transverse force of 1N is applied at the plate tip along the y axis to excite the 1st in-plane bending vibration of the structure. Five harmonics (h = 5) are retained in the HBM analysis.

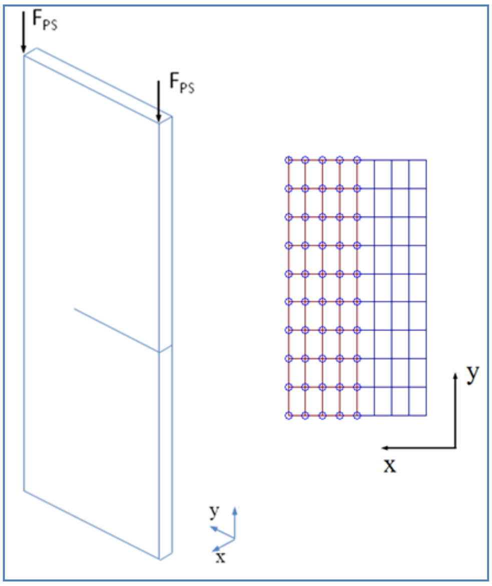

Figure 14 – Static forces and contact area at rest for the cracked plate

The dominant linear mode used to track the evolution of the contact area is the 4th mode of the linear open system, corresponding to the 1st in-plane bending mode of the plate. As detailed in Section 2.3.2, the frequency range of the analysis, where the dynamics is dominated by the selected mode is set around the natural frequency (of 1,861.4 Hz) of the mode with sliding boundary conditions at the node pairs in contact and open boundary conditions at the node pairs not in contact at rest (not all nodes are in contact at rest because of the pre-stress).

The forced response of the structure is characterized by a main response curve and by an isolated branch (Figure 15). The physical intricacies related to the existence of the isolated branch are outside of the scope of this paper. However, it is worth noting that similar intriguing response curves were observed in Detroux et al. (2014) also. The isolated branch was shown there to be connected to the main response curve in the parameters space of the system. Specifically, the isolated branch could be tracked from the main response curve by tracking the turning point in the parameter space using the excitation amplitude as a continuation parameter.

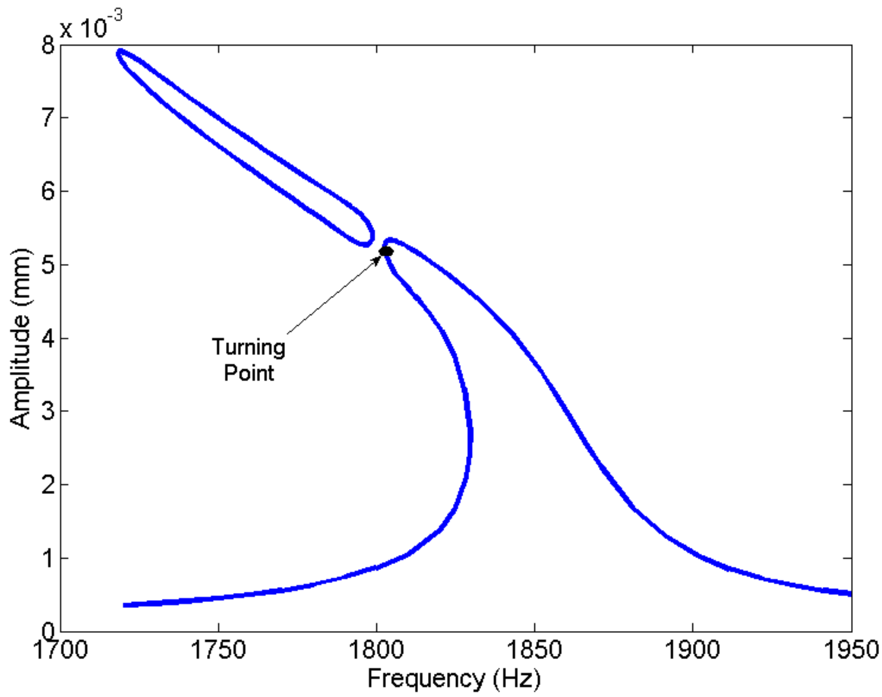

Figure 15 – Cracked plate – Forced response curves for the CB-CMS model

In this case, a set of 6 contact samples, corresponding to a total of 24 PLMs is suitable to model the entire forced response of the system (Figure 16), which includes both the main forced response curve and also the more intriguing isolated branch.

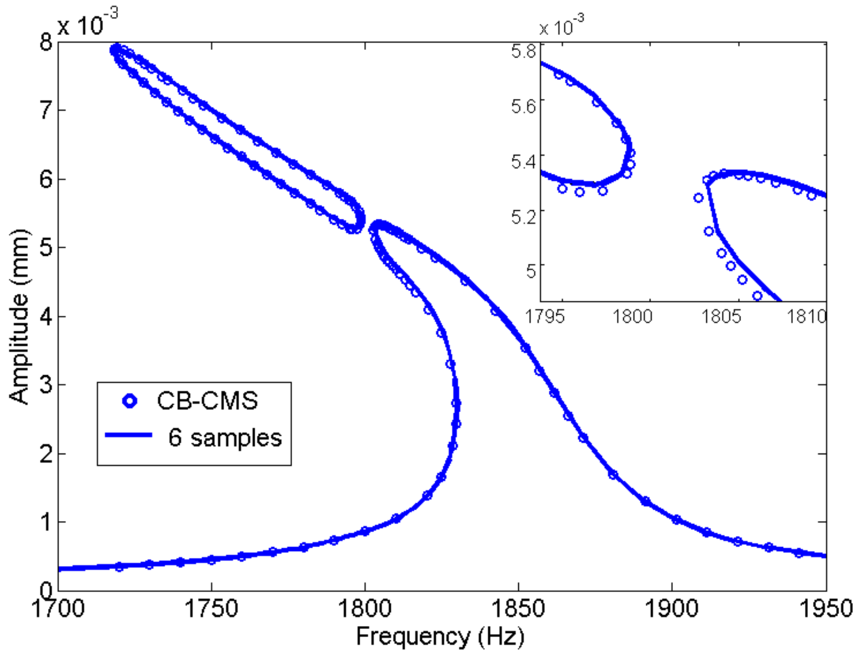

Figure 16 – Cracked plate – Forced response curves

As described in Section 2.3, the reduction matrix $\mathbf{P}_0$ used to project the static governing equations of the system was augmented by including the static deflection of the system at rest $\mathbf{q}_{PS}$. To demonstrate that these static vectors are needed, one can observe the *lack* of spatial correlation between the vector $\mathbf{q}_{PS}$ and the 24 PLMs included in the reduction matrix $\mathbf{P}$ using following residual

$$\mathrm{r} = \frac{\left\| \mathbf{q}_{PS} - \mathbf{P}\,\boldsymbol{\eta} \right\|}{\left\| \mathbf{q}_{PS} \right\|}, \quad (12)$$

where || || represents the L2-norm of a vector, and $\boldsymbol{\eta}$ is computed by least squares solving

$$\mathbf{P}\,\boldsymbol{\eta} = \mathbf{q}_{PS}. \quad (13)$$

In case of perfect correlation, r is zero. When $\mathbf{q}_{PS}$ is orthogonal to $\mathbf{P}$, r is one. For the case under analysis, a value of r = 0.27 is obtained. This demonstrates the expected weak correlation between PLMs and the static deflection.

4. Conclusions

A model reduction technique was discussed for the forced response analysis of structures with intermittent contacts. In this method, the governing equations of the system are projected onto a basis formed of linear normal modes, referred to as piece-wise linear modes, which are computed by imposing special boundary conditions at the contact pairs. A frequency based selection algorithm is used to select the modes to be included in the proposed reduction matrix.

The method assumes that for a given frequency range the dynamics of the system and therefore the kinematics of the contact is dominated by one of the linear modes of either the open or the sliding structure, referred to as the dominant mode.

The harmonic balance method was used to transform the governing equations from the time domain to the frequency domain, and the reduction matrix for the static governing equations (0^{th} harmonic) was augmented by including the static deformation of the system due to pre-stress.

The method was applied to predict the forced response of two co-axial cylinders and of a cracked plate. Different conditions at rest have been investigated; namely zero pre-stress, fully closed contact, and partially closed contact. A fully converged CB-CMS reduced model was used as a reference to check the accuracy of the proposed method.

In all the test cases, the nonlinear forced response was accurately computed with a significant reduction of the size of the nonlinear system, not only with respect to the full-order finite element model, but also with respect to a converged CB-CMS model.

The method was demonstrated for specific test cases, but its application can be extended to structures with more complex geometry, since the physics of the dynamics is expected to be the same.

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References

Åkesson B and Olhoff N (1988) Minimum Stiffness of Optimally Located Supports for Maximum Value of Beam Eigenfrequencies. *Journal of Sound and Vibration* 120 (3): 457–463.

AL-Shudeifat M.A. and Butcher E.A. (2010) Order reduction of forced nonlinear systems using updated LELSM modes with new Ritz vectors. *Nonlinear Dynamics* 62: 821–840.

AL-Shudeifat M.A., Butcher E.A. (2013) On the dynamics of a beam with switching crack and damaged boundaries. *Journal of Vibration and Control* 19: 30–46.

Butcher E.A., AL-Shudeifat M.A. (2011) An efficient mode-based alternative to principal orthogonal modes in the order reduction of structural dynamic systems with grounded nonlinearities. *Mechanical Systems and Signal Processing* 25: 1527–1549.

Butcher E and Lu R (2007) Order Reduction of Structural Dynamic Systems with Static Piecewise Linear Nonlinearities. *Nonlinear Dynamics* 49 (3): 375–399.

Cameron TM and Griffin JH (1989) An Alternating Frequency/Time Domain Method for Calculating the Steady-State Response of Nonlinear Dynamic Systems. *Journal of Applied Mechanics* 56(1): 149-154.

Cardona A, Lerusse A and Geradin M (1998) Fast Fourier nonlinear vibration analysis. *Computational Mechanics* 22: 128–142.

Craig RR and Bampton MCC (1968) Coupling of Substructures for Dynamics Analyses. *AIAA Journal* 6(7): 1313–1319.

Detroux T, Renson L and Kerschen G (2014) The harmonic balance method for advanced analysis and design of nonlinear mechanical systems. In: *Proceedings of the 32nd IMAC, A Conference and Exposition on Structural Dynamics*, Orlando, USA, 3-6 February 2014, pp. 19-34. Cham: Springer International Publishing

Firrone CM, Zucca S and Gola MM (2011) The effect of underplatform dampers on the forced response of bladed disks by a coupled static/dynamic harmonic balance method. *International Journal of Nonlinear Mechanics* 46: 363–375.

Guyan RJ (1965) Reduction of Stiffness and Mass Matrices. *AIAA Journal* 3(2): 380.

Henshell RD and Ong JH (1975) Automatic Masters For Eigenvalue Economization. *Earthquake Engineering and Structural Dynamics* 3(4): 375–383.

Hohl A, Siewert C, Panning L, and Kayser A (2008) Nonlinear Vibration Analysis of Gas Turbine Bladings With Shroud Coupling. In: *ASME Turbo Expo 2008: Power for Land, Sea, and Air – Vol. 5*, Berlin, Germany, 9-13 June 2008, pp. 425–433. New York: ASME.

Hurty WC (1965) Dynamic Analysis of Structural Systems Using Component Modes, *AIAA Journal* 3(4): 678–685.

Kammer DC (1991) Sensor Placement for On-Orbit Modal Identification and Correlation of Large Space Structures. *Journal of Guidance, Control, and Dynamics* 14(2): 251–259.

Kerschen G, Golinval J, Vakakis AF and Bergman LA (2005) The Method of Proper Orthogonal Decomposition for Dynamical Characterization and Order Reduction of Mechanical Systems: An Overview. *Nonlinear Dynamics* 41: 147–169.

Lawson CL and Hanson RJ (1995) *Solving Least Squares Problems*. New York: Society for Industrial and Applied Mathematics.

Laxalde D and Legrand M (2011) Nonlinear modal analysis of mechanical systems with frictionless contact interfaces. *Computational Mechanics* 47: 469–478.

Marinescu O, Epureanu BI and Banu M (2011) Reduced Order Models of Mistuned Cracked Bladed Disks. *ASME Journal of Vibration and Acoustics* 133: 051014.

Matta KW (1987) Selection of Degrees of Freedom for Dynamic Analysis. *Transactions of the ASME, Series J, Journal of Pressure Vessel Technology* 109(1): 65–69.

Mayer MH and Gaul L (2007) Segment-to-segment contact elements for modelling joint interfaces in finite element analysis. *Mechanical Systems and Signal Processing* 21: 724–734.

Penny JET, Friswell MI and Garvey SD (1994) Automatic Choice of Measurement Locations for Dynamic Testing. *AIAA Journal* 32(2): 407–414.

Saito A, Castanier MP, Pierre C and Poudou O (2009) Efficient Nonlinear Vibration Analysis of the Forced Response of Rotating Cracked Blades. *Journal of Computational and Nonlinear Dynamics* 4: 011005.

Saito A, Epureanu BI, Castanier MP and Pierre C (2010) Node Sampling for Nonlinear Vibration Analysis of Structures with Intermittent Contact. *AIAA Journal* 48: 1903–1915.

Saito A and Epureanu BI (2011) Bilinear Modal Representations for Reduced-Order Modeling of Localized Piecewise-Linear Oscillators. *Journal of Sound and Vibration* 330: 3442–3457.

Schwingshackl CW, Di Maio D, Sever I and Green JS (2013)Modeling and Validation of the Nonlinear Dynamic Behavior of Bolted Flange Joints. *Journal of Engineering for Gas Turbines and Power* 135: 122504.

Segalman DJ (2007) Model Reduction of Systems with Localized Nonlinearities. *ASME Journal of Computational and Nonlinear Dynamics* 2: 249-266.

Shaw SW and Holmes PJ (1983) A periodically forced piecewise linear oscillator. *Journal of Sound and Vibration* 90 (1): 129–155.

Siewert C, Panning L, Wallaschek J and Richter C (2010) Multiharmonic Forced Response Analysis of a Turbine Blading Coupled by Nonlinear Contact Forces. *Journal of Engineering for Gas Turbines and Power* 13: 082501.

Zhu J and Zhang W (2006) Maximization of Structural Natural Frequency with Optimal Support Layout. *Structural and Multidisciplinary Optimization* 31 (6): 462–469.

Zucca S, Firrone CM and Facchini M (2012) A Method for the Design of Ring Dampers for Gears in Aeronautical Applications. *Journal of Mechanical Design* 134: 091003.

Zucca S and Epureanu BI (2014) Bi-linear Reduced-Order Models of Structures with Friction Intermittent Contacts, *Nonlinear Dynamics* 77(3): 1055-1067.

Zucca S, Gola MM and Piraccini F (2012) Non-Linear Dynamics of Steam Turbine Blades With Shroud: Numerical Analysis and Experiments. In: *ASME Turbo Expo 2012: Power for Land, Sea, and Air – Vol. 6*, Copenhagen, Denmark, 11-15 June 2012, pp. 665–674. New York: ASME.