Dottorato di Ricerca in Matematica, XI ciclo
1995–1999
Consorzio delle università di Cagliari, Firenze, Modena, Perugia e Siena

Tesi di Dottorato di Ricerca in Matematica

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Discontinuous Ordinary Differential Equations and Stabilization

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Aknowledgements

I am very pleased to have the opportunity here to express my sincere thanks to Professor Bacciotti who has guided me in the study of mathematics for five years, giving me confidence and transmitting his fondness for this discipline.

Particular thanks also to Professor Conti. It has been really a pleasure to attend his course in Control Theory during my first year in Firenze and then to have my thesis read and commented by him.

Contents

Introduction

Mathematical models of physical phenomena are very often based on nonlinear differential equations of the form

$$\dot{x} = f(x).$$

The study of these phenomena is primarily performed looking for qualitative properties of solutions, as asymptotic behaviour and stability of equilibrium positions.
If we suppose that we can act on the system, for example by applying external forces, we have to modify the model by introducing a parameter, called control. One of the main aim of control theory is to find strategies so that solutions of the new system have desired properties.

We consider a nonlinear control system of the form

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$, the parameter $u \in \mathbb{R}^m$ is the control and $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a function that, for the moment, we assume to be at least continuous and such that the origin is an equilibrium point for the system, i.e. $f(0, 0) = 0$.

We are interested in stabilizing the system. This means that we want to find a function $u = k(x)$, called feedback law, such that the implemented system

$$\dot{x} = f(x, k(x)) \tag{2}$$

is asymptotically stable, i.e. the origin is stable in the sense of Lyapunov (see Chapter 3) and its solutions tend to the equilibrium point.

Let us take a step back to linear systems. For these systems stabilizability is a very well studied property. Two facts are particularly relevant: the first is the existence of a linear feedback when the system is stabilizable; the second is that the asymptotic controllability of the system is equivalent to its stabilizability. Roughly speaking, the system is asymptotic controllable to zero if any initial state can be driven to the equilibrium point by means of an open loop control $u = u(t)$ (see Chapter 4 for a more precise definition).

For nonlinear systems the first issue is false, while the second one has been proven true only very recently. More precisely, for nonlinear stabilizable systems, not only a linear feedback does not exist in general, but even the assumption on the feedback to be continuous is restrictive. This can be shown by various examples and by the well known Brockett's condition (see Chapter 4, Section 1).

Alternatively to the only consideration of continuous feedback laws, two approaches can be followed.

In the first one the feedback is taken to be time-varying. This approach began with Sussman and Sontag and with Samson (see [SS1, Sam]) and has produced very good results (see [C1, C2, CR]).

In the second one discontinuous (static) feedback laws are allowed. We focus on this point of view.

We have already mentioned an important result obtained by means of this approach: the proof that asymptotic controllability to zero implies asymptotic stabilizability. This result has been obtained first by Clarke, Ledyaev, Sontag and Subbotin ([CLSS]) and then, in a different way, by Ancona and Bressan ([AB]).

The introduction of discontinuous feedback laws leads to the theoretical problem of defining solutions of systems with discontinuous righthand side. We survey some of the feasible definitions of solution in Chapter 1. In particular we examine Carathéodory, Euler, generalized sampling, Krasovskii and Filippov solutions and we show, with several examples, that the relationships among them are quite weak.

Since in control systems discontinuities essentially come from the implementation of the feedback, either definition of solution can be considered depending on which kinds of discontinuities of the feedback are admitted. For example, in [CLSS], the only assumption on the feedback is local boundedness and (not generalized) sampling solutions are considered, while in [AB], the feedback laws are assumed piecewise smooth and are such that the implemented system have Carathéodory solutions. When different kinds of solutions exist at the same time, the right one to be used should be suggested by the physical model (but this topic is not developed in most papers). We briefly illustrate the effect of considering different kinds of solutions in two examples in Section 2 of Chapter 4.

Still in Chapter 4, Section 3, we analyse discontinuous feedback laws of a particular form which stabilize a large class of systems. More precisely we consider an affine input system of the form

$$\dot{x} = f(x) + G(x)u \tag{3}$$

and the associated unforced system

$$\dot{x} = f(x) \tag{4}$$

If the unforced system is stable, a regular Lyapunov function V is known and some geometric conditions involving the vector fields f and the matrix G are satisfied, then stabilization of the affine system can be achieved by means of the following feedback law (often called damping feedback):

$$u(x) = -\alpha(\nabla V(x)G(x))^T \tag{5}$$

It is important to remark that, if the unforced system is simply Lyapunov stable, converse Lyapunov theorems do not guarantee the existence of a regular Lyapunov function.

According to Yorke ([Y]), Lyapunov stability just implies the existence of a semi-continuous Lyapunov function. On the other hand the existence of more regular Lyapunov functions is equivalent to strengthened concepts of stability (see [AS, BR1]). We consider the case in which a Lipschitz continuous Lyapunov function is known. Note that, except for dimension 1 (see [BR2]), the concept of stability equivalent to the existence of a Lipschitz continuous Lyapunov function is still unknown.

Going back to the damping feedback, it would be desirable to implement it in the affine system even if the Lyapunov function V is only locally Lipschitz continuous. Let us remark that, in this case, thanks to Rademacher's theorem, (5) is almost everywhere defined. Moreover it is measurable and locally essentially bounded. The implemented system's righthand side is then also a locally essentially bounded function. Solutions which fit better in this context are Filippov solutions. We give a stabilization result for them. By slightly modifying the feedback law, we also obtain an analogous result for Krasovskii solutions (Section 3.5, Chapter 4).

In order to get these results we need some tools from nonsmooth analysis and differential inclusions that we collect in Chapter 2 and 3 respectively.

Concerning nonsmooth analysis, though many different generalized derivatives and gradients have been recently introduced and studied, we mainly use Clarke gradient. In fact Clarke gradient is particularly helpful when dealing with Lipschitz continuous functions. We use it, in order to define the set-valued derivative of a Lipschitz continuous function with respect to a differential inclusion. This kind of generalized derivative works particularly well if the function V, besides being Lipschitz continuous, is healthy, in the sense introduced by Valadier in [V]. Note that healthy functions form quite a wide class which contains in particular convex and Clarke-regular functions. In Chapter 3, by means of this set-valued derivative, we give a stability result and a generalization of LaSalle's principle for upper semi-continuous compact and convex valued differential inclusions and Lipschitz continuous and healthy Lyapunov functions.

These tools allow us to widen the class of affine systems considered to systems in which also the drift term f is discontinuous. For these systems, on one hand we prove by means of an example that the application of the damping feedback can turn into a destabilizing action, on the other hand we give some sufficient conditions for the system to be stabilized by it.

Finally, in Chapter 5, we apply the same kind of techniques used for the stabilization of the affine input systems to get a similar result for external stabilization (UBIBS-stabilization).

Chapter 1

Discontinuous Differential Equations

The present chapter is motivated both by the intrinsic mathematical interest in the study of differential equations with discontinuous righthand sides and by the fact that such equations often occur in control theory because of the application of discontinuous feedback laws in control systems. The survey on different possible concepts of solution is then strongly influenced by the requirements of applications in control theory.

We consider both time-dependent and autonomous Cauchy problems:

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases} \tag{1.1}$$

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}^n$.

Let us recall that a *classical solution* of one of the previous Cauchy problems on an interval $I \subset \mathbb{R}$ is an everywhere differentiable function which satisfies (1.1) (or (1.2)) at every $t \in I$.

If the function f is continuous then the Cauchy problem (1.1) is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \tag{1.3}$$

The so called Carathéodory solutions of (1.1) are solutions of (1.3), which can exist even if f is not continuous. The classical conditions on f which guarantee the existence of these solutions are Carathéodory conditions, that we specify in Section 1.1.

If the study of equations whose righthand sides do not satisfy Carathéodory conditions is needed, two alternative ways can be pursued. The first one, consists in looking for the weakest hypothesis on the vector field f which guarantee the existence of Carathéodory solutions (see [AB, BrC, P1, P2]). The second one, followed by most authors who work in control theory, consists in introducing generalized solutions.

Besides Carathéodory solutions, we focus on Euler, generalized sampling, Krasovskii and Filippov solutions.

Among the authors who have introduced other kinds of generalized solutions, let us mention Hermes ([H]), Ambrosio ([A1, A2]) and Sentis ([Se, JS]).

Concerning bibliography, let us stress that a very important article is the one by Hájek ([H]), dated 1979, where Hermes, Krasovskii and Filippov solutions are compared. To the best knowledge of the author there is no recent work in which something similar has been done.

1.1 Carathéodory Solutions

Definition 1 *An absolutely continuous function $\varphi : [t_0, t_0+a] \to \mathbb{R}^n$ is said to be a* Carathéodory solution *of (1.1) if it satisfies (1.3) for all $t \in [t_0, t_0+a]$ or, equivalently, if it satisfies (1.1) for almost every $t \in [t_0, t_0+a]$.*

We denote the set of Carathéodory solutions of (1.1) by $\mathcal{C}$.
We say that there exists a local solution of (1.1) if there exists $\delta > 0$ such that there exists a Carathéodory solution of (1.1) on $[t_0, t_0 + \delta]$.

Definition 2 *Let I be any interval of $\mathbb{R}$, D any subset of $\mathbb{R}^n$, $f : I \times D \to \mathbb{R}$. The function $f : I \times D \to \mathbb{R}^n$ is said to satisfy the* Carathéodory conditions *on $I \times D$ if:*

(i) f is defined and continuous with respect to x for a.e. $t \in I$,

(ii) f is measurable with respect to t for each $x \in D$,

(iii) there exists a nonnegative summable function $m : I \to \mathbb{R}$ such that $\|f(t,x)\| \leq m(t)$ for all $t \in I$.

Theorem 1 *Let f be defined on $R = \{(t,x) :\ t_0 \leq t \leq t_0+a,\ \|x-x_0\| \leq b\}$. If f satisfies the Carathéodory conditions on R, then there exists a Carathéodory solution of (1.1) on $[t_0, t_0+\delta]$, where δ is such that $\int_{t_0}^{t_0+\delta} m(t)dt \leq b$.*

Moreover if there exists a summable function $l : [t_0, t_0+\delta] \to \mathbb{R}$ such that for all $t \in [t_0, t_0+\delta]$ and for all x, y such that $\|x - x_0\| \le b,\ \|y - y_0\| \le b$, one has

$$\|f(t,x) - f(t,y)\| \le l(t)\|x - y\| \tag{1.4}$$

then the solution on $[t_0, t_0+\delta]$ is unique.

In [BrC, AB, P1, P2], the authors consider the autonomous Cauchy problem (1.2) in order to get weaker conditions which guarantee the existence of Carathéodory solutions. Let us introduce these conditions.

Definition 3 *The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be* directionally continuous *if there exists $\delta > 0$ such that, for every $x \in \mathbb{R}^n$, if $f(x) \neq 0$ and $x_n \to x$ with*

$$\left| \frac{x_n - x}{|x_n - x|} - \frac{f(x)}{|f(x)|} \right| < \delta \qquad \forall n \ge 1,$$

then $f(x_n) \to f(x)$.

Directional continuity asks $f(x_n) \to f(x)$ only for sequences converging to x contained inside a cone with vertex at x and opening δ around an axis having the direction of $f(x)$.

Definition 4 *The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be* self tangent *if, for every $x \in \mathbb{R}^n$, there exist two sequences $x_n \to x$ and $t_n > 0$ such that*

$$\lim_{n\to\infty} \frac{x_n - x}{t_n} = \lim_{n\to\infty} f(x_n) = f(x).$$

Let us remark that directional continuity implies self tangency. In [P1, P2] it is proved that directional continuity of f implies the existence of local Carathéodory solution of (1.2). The same is proved in [BrC] under the assumption that f is self tangent and has locally closed graph.
We now define patchy vector fields, which have been introduced in [AB]. They are the superposition of inward-pointing vector fields. They also guarantee the existence of local Carathéodory solutions of (1.2).

Definition 5 *Let $\Omega \subset \mathbb{R}^n$ be an open domain with smooth boundary $\partial\Omega$. A smooth vector field f defined on a neighbourhood of $\overline{\Omega}$ is said to be an* inward pointing vector field *on Ω if at every boundary point $x \in \partial\Omega$ the inner product of f with the outer normal n satisfies $f(x) \cdot n < 0$. The pair (Ω, f) is said to be a* patch.

Definition 6 *$f : \Omega \to \mathbb{R}^n$ is said to be a* patchy vector field *if there exists a family of patches $\{(\Omega_\alpha, f_\alpha) : \alpha \in \mathcal{A}\}$ such that:*
- $\mathcal{A}$ is a totally ordered index set,
- the open sets Ω_α form a locally finite covering of Ω,
- the vector field f can be written in the form

$$f(x) = f_\alpha(x) \quad if \quad x \in \Omega \setminus \bigcup_{\beta > \alpha} \Omega_\beta.$$

1.2 Euler Solutions

In order to define generalized solutions mainly two approaches can be pursued. The first one consists in defining approximate solutions by means of an algorithm and taking as generalized solutions the uniform limits of such approximate solutions. Euler and generalized sampling solutions are constructed in this way. In the second approach one associates a differential inclusion to the differential equation and defines generalized solutions as solutions of the associated differential inclusion. We discuss this approach in Section 1.4.

Definition 7 *An* Euler polygonal ϵ-approximate solution *associated to the Cauchy problem (1.1) and to the partition $\pi = \{t_0, t_1, ..., t_N\}$ of $[t_0, t_0 + a]$, with $t_N = t_0 + a$ and $\mu_\pi = \max\{t_{i+1} - t_i,\ 0 \le i \le N-1\} < \epsilon$, is the piecewise affine function defined by*

$$\begin{cases} \varphi_\pi(t_0) = \overline{x}_0 & t \in [t_0, t_1] \\ \varphi_\pi(t) = \varphi_\pi(t_i) + (e_i + f(t_i, x_i(t_i) + e'_i))(t - t_i) & t \in [t_i, t_{i+1}] \end{cases}$$

where $i = 1, ..., N-1$, $\|x_0 - \overline{x}_0\| < \epsilon$, $e_i, e'_i \in \mathbb{R}^n$ with $|e_i| < \epsilon$, $|e'_i| < \epsilon$.

e_i and e'_i can be seen as respectively inner and outer perturbations.

Let us remark that Euler polygonal ϵ-approximate solutions are absolutely continuous functions.

Definition 8 *A function $\varphi : [t_0, t_0 + a] \to \mathbb{R}^n$ is said to be*

- an Euler solution *of (1.1) if it is the uniform limit as $\epsilon \to 0$ of a sequence of Euler polygonal ϵ-approximate solutions with $\overline{x}_0 = x_0$, $e_i \equiv e'_i \equiv 0$ for all i;*

- an Euler externally disturbed solution *of (1.1) if it is the uniform limit as $\epsilon \to 0$ of a sequence of Euler polygonal ϵ-approximate solutions with $e'_i \equiv 0$ for all i;*

- an Euler disturbed solution *of (1.1) if it is the uniform limit as $\epsilon \to 0$ of a sequence of Euler polygonal ϵ-approximate solutions.*

We denote the set of Euler solutions of (1.1) by $\mathcal{E}$, the set of Euler externally disturbed solutions by $\mathcal{E}_E$ and the set of Euler disturbed solutions with $\mathcal{E}_D$. Note that Euler disturbed solutions are sometimes addressed as weak generalized solutions (see [P]).
Obviously we have that $\mathcal{E} \subseteq \mathcal{E}_E \subseteq \mathcal{E}_D$. We mainly focus on Euler solutions. Let us remark that Euler solutions are interesting from a mathemathical point of view, while they don't seem to have a physical meaning. Then let us point out their mathematical interest. One possible proof of Peano existence theorem of classical solutions of (1.1) is based on the construction of a sequence of Euler ϵ-approximate solutions. By means of Ascoli and Arzelà theorem, this sequence is proved to admit a subsequence convergent to a continuous function, which is a solution of the Cauchy problem (see [Sa], page 36). If f is not continuous, the limit function does not necessarly verifies the equation, but it is taken as a (Euler) solution by definition. Going on this way, we prove the following theorem.

Theorem 2 *If f is bounded on the set $R = \{(t,x) : \ t_0 \leq t \leq t_0 + a, \ \|x - x_0\| \leq b\}$, then a local Euler solution of (1.1) exists. Moreover it is absolutely continuous.*

Proof Let M be such that $\|f(t,x)\| \leq M$ for all $(t,x) \in R$, $\delta = \min\left\{a, \frac{b}{M}\right\}$. Let π_j be a sequence of partitions of $[t_0, t_0+\delta]$ with $\mu_{\pi_j} < \epsilon_j \to 0$, and φ_j the corresponding Euler polygonal ϵ_j-approximate solution. Let us first remark that $(t, \varphi_j(t)) \in R$ for every j and for every $t \in [t_0, t_0+\delta]$. In fact $\|\varphi_j(t) - x_0\| \leq M\delta \leq b$. From this inequality it also follows that the set of continuous functions $\{\varphi_j\}$ is equi-bounded. Let us show that it is also equi-continuous. For all $\epsilon > 0$ we have that, for all $t_1, t_2 \in [t_0, t_0+\delta]$, if $|t_1 - t_2| < \frac{\epsilon}{M}$, then $\|\varphi_j(t_1) - \varphi_j(t_2)\| \leq \delta M < \epsilon$ for all j. By the Ascoli and Arzelà theorem it follows that there exists a subsequence of φ_j uniformly converging to a continuous function φ defined on $[t_0, t_0+\delta]$. Such a function is an Euler solution of (1.1) by definition.

Let us now show that φ is absolutely continuous. We have already remarked that the sequence $\{\varphi_j\}$ is equibounded, then for each $t \in [t_0, t_0+\delta]$ the set $\{\varphi_j(t)\}$ is reletively compact. Moreover $\|\dot{\varphi}_j(t)\| \leq M$ for all $t \in [t_0, t_0+\delta]$. By Theorem 4 page 13 in [AC], it follows that there exists a subsequence of $\{\varphi_j\}$ uniformly converging to an absolutely continuous function. Since we already know that the whole sequence $\{\varphi_j\}$ uniformly converges to φ, it follows that φ is absolutely continuous. $\square$

Let us now compare Euler and Carathéodory solutions.

The following examples respectively show that $\mathcal{C} \not\subseteq \mathcal{E}$ and $\mathcal{E} \not\subseteq \mathcal{C}$.

Example 1 ([Cl2]) Let us consider the Cauchy problem (1.2) with $f(x) = \frac{3}{2}x^{\frac{1}{3}}$, $t_0 = 0$, $a = 1$ and $x_0 = 0$. Carathéodory solutions are $t^{\frac{3}{2}}, t^{-\frac{3}{2}}$ and 0, while the only Euler solution is 0.

Example 2 Let us consider the Cauchy problem (1.2) with

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}\backslash\mathbf{Q} \\ -1 & \text{if } x \in \mathbf{Q} \end{cases}$$

$t_0 = 0$, $a = 1$, $x_0 = 0$ and the sequence of partitions of the interval $[0, 1]$ given by $\pi_N = \{0, \frac{1}{N}, \frac{2}{N}, ..., 1\}$. The corresponding Euler solution is $-t$, while the unique Carathéodory solution is t.

Note that in the previous example the Euler solution $-t$ does not satisfy (1.1) at a.e. $t \in [0, 1]$, but, at least, it satisfies it on a dense subset of $[0, 1]$. Exercise 1.6 (b) in [CLSW2] is an example of an Euler solution of a Cauchy problem of the form (1.2) on an interval I that does not satisfy the equation at any $t \in I$.

In Theorem 4.1.7 in [CLSW2], page 183, it is proved that, under the assumptions that f is continuous with respect to (t, x), $\mathcal{E} \subseteq \mathcal{C}$. Example 1 shows that $\mathcal{C} \not\subseteq \mathcal{E}$ even if f is continuous.

In order to get $\mathcal{C} \equiv \mathcal{E}$, the easiest possibility is to ask f to be continuous with respect to (t, x) and Lipschitz continuous with respect to x, in the sense of condition (1.4). In fact, in this case, there exist both an Euler and a Carathéodory solution, which is unique. Since the Euler solution is also a Carathéodory solution, the two must coincide. In [AB] it is proved that if the system is autonomous and f is patchy, then $\mathcal{C} \equiv \mathcal{E}_E$. From this fact it follows in particular that, if f is patchy, $\mathcal{E} \subseteq \mathcal{C}$.

1.3 Generalized Sampling Solutions

Sampling solutions have been introduced by Krasovskii and Subbotin (see [KS]) in the contest of differential games, and then used in [CLSS] in order to prove that asymptotic controllability implies feedback stabilization. Here we consider generalized sampling solutions which are uniform limits of (not generalized) sampling solutions. Roughly speaking, generalized sampling solutions are obtained as limits of solutions of a sequence of systems in which the control is piecewise constant. The aim of the present section is to see to what extent they have sense in the general contest of discontinuous differential equations. We introduce them for systems of the form

$$\begin{cases} \dot{x}(t) = f(t, x(t), k(t, x(t))) \\ x(t_0) = x_0 \end{cases} \tag{1.5}$$

where $k : \mathbb{R}^{n+1} \to \mathbb{R}^m$ is, in general, a discontinuous function, not necessarly to be thought as a control.

Definition 9 *An* ϵ-trajectory *associated to the Cauchy problem (1.5) and to the partition* $\pi = \{t_0, t_1, ..., t_N\}$ *of* $[t_0, t_0 + a]$ *with* $t_N = t_0 + a$ *and* $\mu_\pi < \epsilon$, *is a function obtained by iteratively solving the following integral equations*

$$\begin{array}{ll} \varphi_\pi(t) = x_0 + \int_{t_0}^t f(\tau, \varphi_\pi(\tau), k(t_0, x_0))d\tau & t \in [t_0, t_1] \\ \varphi_\pi(t) = \varphi_\pi(t_i) + \int_{t_i}^t f(\tau, \varphi_\pi(\tau), k(t_i, \varphi_\pi(t_i))d\tau & t \in [t_i, t_{i+1}],\ i = 1, ..., N-1 \end{array}$$

Let us remark that ϵ-trajectories do not necessarly exist, nor are unique. Nevertheless, if an ϵ-trajectory exists, then it is absolutely continuous.

Definition 10 *A function* $\varphi : [t_0, t_0 + a] \to \mathbb{R}^n$ *is said to be a* generalized sampling solution *of (1.1) if it is the uniform limit of a sequence of* ϵ*-trajectories as* $\epsilon \to 0$.

It is important to emphasize that, in the definition of generalized sampling solution on $[t_0, t_0 + a]$, it is implicit that a sequence of ϵ-trajectories of (1.5) on $[t_0, t_0 + a]$ does exist.

We denote the set of generalized sampling solutions of (1.5) by $\mathcal{S}$. As for Euler solutions it would be possible to define also externally disturbed generalized sampling solutions and disturbed generalized sampling solutions.

Let us now state a local existence theorem for generalized sampling solutions.

Theorem 3 *Let* f *be defined on the set* $Q = \{(t, x, u) :\ t_0 \le t \le t_0 + a,\ \|x - x_0\| \le b,\ u \in \mathbb{R}^m\}$. *If* f *is such that*

(i) for all fixed $u \in \mathbb{R}^m$, f *is measurable in* t *for all* x *and continuous in* x *for a.e.* t

(ii) there exists a positive summable function $m : [t_0, t_0 + a] \to \mathbb{R}$ *such that* $\|f(t, x, u)\| \le m(t)$ *for all* $(t, x, u) \in Q$

then there exists at least one local generalized sampling solution of (1.5). Moreover it is absolutely continuous.

Proof Let $\{\pi_j\}$ be a sequence of partitions of $[t_0, t_0 + a]$ with $\mu_{\pi_j} < \epsilon_j \to 0$. By Theorem 1, there exists $\delta > 0$ such that for each partition π_j a corresponding ϵ_j-trajectory φ_j exists (it is sufficient that $\int_{t_0}^{t_0+\delta} m(t)dt \le b$).

Let us remark that for every $t \in [t_0, t_0 + \delta]$ and for all u, $(t, \varphi_j(t), u) \in Q$. In fact $\|\varphi_j(t) - x_0\| \le \| \int_{t_0}^{t} m(\tau)d\tau\| \le \int_{t_0}^{t_0+\delta} m(\tau)d\tau \le b$. From this inequality it also follows that the sequence of functions $\{\varphi_j\}$ is equi-bounded.

Let us show that it is equi-continuous. Let $\epsilon > 0$ be arbitrarily fixed and let $\gamma > 0$ be such that for any $t_1, t_2 \in [t_0, t_0 + a]$, if $|t_1 - t_2| < \gamma$ then $\int_{t_1}^{t_2} m(\tau)d\tau < \epsilon$ (such a γ exists because of the absolute continuity of Lebesgue integral). Let us then consider $t_1, t_2 \in [t_0, t_0 + a]$ such that $|t_1 - t_2| < \gamma$. We get that $\|\varphi_j(t_1) - \varphi_j(t_2)\| \le \int_{t_1}^{t_2} m(\tau)d\tau < \epsilon$.

By the Ascoli and Arzelà theorem it follows that the sequence $\{\varphi_j\}$ admits a subsequence uniformly converging to a continuous function φ, that is a generalized sampling solution of (1.5) by definition. As in Theorem 2, the absolute continuity of φ follows by Theorem 4 page 13 in [AC]. $\square$

Remark 1 Analogous theorems could be stated if f verifies some conditions which guarantee the existence of ϵ-trajectories for any sequence of partitions of an interval $[t_0, t_0 + \delta] \subseteq I$ for some δ and for every fixed u.

Remark 2 If we assume that the feedback law k is locally bounded and $M > 0$ is such that $\|k(t, x) - k(t_0, x_0)\| \le M$ for all $t \in [t_0, t_0 + a]$ and for all x such that $\|x - x_0\| \le b$, then hypothesis (ii) in the previous theorem can be weakened to the following:

(iibis) there exists a summable function $m : [t_0, t_0 + a] \to \mathbb{R}$ such that $\|f(t, x, u)\| \le m(t)$ for all $(t, x, u) \in Q$ such that $\|u - k(t_0, x_0)\| \le M$

As for Euler solutions, the existence a.e. of the derivative does not imply that a generalized sampling solution satisfies (1.5) a.e.. We can reinterpret Example 2 in terms of generalized sampling solutions by posing $f(x) = k(x)$. We get that $-t$ is a generalized sampling solution, but not a Carathéodory solution, then $\mathcal{S} \nsubseteq \mathcal{C}$. Analogously, by reinterpreting Example 1, we get $\mathcal{C} \nsubseteq \mathcal{S}$. This can be also seen by means of the example in [AB].

It is then natural to look for conditions which guarantee $\mathcal{S} \subseteq \mathcal{C}$. The following theorem is analogous to Theorem 4.1.7 in [CLSW2], page 183, for Euler solutions.

Theorem 4 *If f is continuous with respect to (t, x, u) on Q, there exists a positive and summable function $m : [t_0, t_0 + a]$ such that $\|f(t, x, u)\| \le m(t)$ for all $(t, x, u) \in Q$ and k is continuous with respect to (t, x), then every* *generalized sampling solution of (1.5) on* $[t_0, t_0 + a]$ *is also a Carathéodory solution.*

Proof Let φ be a generalized sampling solution of (1.5), $\{\varphi_j\}$ a sequence of ϵ_j-trajectories corresponding to the sequence $\{\pi_j\}$ of partitions of the interval $[t_0, t_0 + a]$ with $\mu_{\pi_j} < \epsilon_j \to 0$, such that $\varphi_j \to \varphi$ uniformly.

Let us posit $k_j(t) = k(t_i, \varphi_j(t_i)), \quad t \in [t_i, t_{i+1}]$. Since k is continuous in (t, x) and φ_j is continuous in t for all j, we have that $k_j(t) \to k(t, \varphi(t))$ and also $f(t, \varphi_j(t), k_j(t)) \to f(t, \varphi(t), k(t, \varphi(t)))$ for every t. Moreover $\|f(t, \varphi_j(t), k_j(t))\| \le m(t)$, then $\varphi_j(t) \to x_0 + \int_{t_0}^t f(\tau, \varphi(\tau), k(\tau, \varphi(\tau)))d\tau$ for all t.

On the other hand we know that $\varphi_j \to \varphi$ uniformly, then
$\varphi(t) = x_0 + \int_{t_0}^t f(\tau, \varphi(\tau), k(\tau, \varphi(\tau)))d\tau$ for all t, i.e. φ is a Carathéodory solution of (1.5). $\square$

Let us end this section by comparing generalized sampling solutions with Euler solutions. In general, Euler solutions can be seen as a particular case of generalized sampling solutions, when the equation in the Cauchy problem is given by $\dot{x} = k(t, x)$. Nevertheless these two kinds of generalized solutions are not really tightly connected. The example in [AB], if reinterpreted in terms of equation (1.5), shows that $\mathcal{E} \not\subseteq \mathcal{S}$. Moreover the following example shows that $\mathcal{S} \not\subseteq \mathcal{E}$.

Example 3 Let us consider the Cauchy problem (1.5) with $f(t, x, k(t, x)) = \frac{3}{2}k(x)x^{\frac{1}{3}}$, where k is defined by

$$k(x) = \begin{cases} 1 & \text{if } x \in \{0\} \cup \{(\mathbb{R}\backslash\mathbf{Q}) \cap [0, 1]\} \\ 0 & \text{if } x \in \mathbf{Q} \cap (0, 1] \end{cases}$$

$t_0 = 0, \ a = \pi$ and $x_0 = 0$. The unique Euler solution is 0.

On the other hand, by considering the sequence of partitions
$\pi_j = \left\{0, \frac{\pi}{j}, \frac{2\pi}{j}, ..., (j-1)\frac{\pi}{j}, \pi\right\}$, we get that, besides 0, also $t^{\frac{3}{2}}$ and $t^{-\frac{3}{2}}$ are generalized sampling solutions.

1.4 Krasovskii and Filippov Solutions

The idea behind the concepts of Krasovskii and Filippov solutions is that the value of a solution at a certain point should be determined by the behaviour of its derivative in the nearby points. Moreover the definition of Filippov solution suggests that possible misbehaviour of the derivative on null measure sets could be ignored.

More precisely, if we denote by $\overline{\mathrm{co}}$ the convex closure and by μ the usual Lebesgue measure in $\mathbb{R}^n$, we have the following definitions.

Definition 11 *An absolutely continuous function $\varphi : [t_0, t_0 + a] \to \mathbb{R}^n$ is said to be*

- a Krasovskii solution *of (1.1) if it is a solution of the differential inclusion*

$$\dot{x} \in Kf(t,x) = \bigcap_{\delta > 0} \overline{\mathrm{co}} f(t, B(x, \delta)) \tag{1.6}$$

i.e. φ satisfies (1.6) for a.e. $t \in [t_0, t_0 + a]$,

- a Filippov solution *to (1.1) if it is a solution of the differential inclusion*

$$\dot{x} \in Ff(t,x) = \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} \overline{\mathrm{co}} f(t, B(x, \delta) \setminus N) \tag{1.7}$$

i.e. φ satisfies (1.7) for a.e. $t \in [t_0, t_0 + a]$.

We denote the sets of Krasovskii and Filippov solutions to (1.1) respectively by $\mathcal{K}$ and $\mathcal{F}$.

Let us consider the set-valued functions $Kf, Ff : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. If f is locally bounded, then Kf and Ff are both upper semicontinuous and have nonempty, compact and convex values. The same is still true for Ff if f is just locally essentially bounded. From this remark and a classical existence theorem for differential inclusion (see [AC], page 97), the local existence theorem for Krasovskii and Filippov solutions follows.

Theorem 5 *If $f : [t_0, t_0 + a] \times \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded (locally essentially bounded), then a local Krasovskii (Filippov) solution of (1.1) exists.*

Obviously, if f is locally bounded, $\mathcal{F} \subseteq \mathcal{K}$. An interesting condition in order to get $\mathcal{K} = \mathcal{F}$ is given in [H] (Lemma 2.8). As in the mentioned paper, we report it for autonomous systems, but it can be generalized to nonautonomous ones.

Proposition 1 *If there exists a disjoint decomposition $\mathbb{R}^n = \bigcup \Omega_i$, with $\Omega_i \subset \overline{\mathrm{Int}\Omega_i}$ and continuous functions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $f = f_i$ on Ω_i, then each Krasovskii solution of (1.2) is a Filippov solution, i.e. $\mathcal{K} \subseteq \mathcal{F}$.*

If f is continuous with respect to x, then $Kf(t,x) = Ff(t,x) = \{f(t,x)\}$, so that $\mathcal{K} \equiv \mathcal{F} \equiv \mathcal{C}$. Let us remark that this does not always occur when Carathéodory solutions exist. The example in [AB] shows that $\mathcal{K} \not\subseteq \mathcal{C}$ and

$\mathcal{F} \not\subseteq \mathcal{C}$ (note that in this example, thanks to Proposition 1, $\mathcal{K} = \mathcal{F}$). As far as the opposite inclusions are concerned, we have that $\mathcal{C} \subseteq \mathcal{K}$, while $\mathcal{C} \not\subseteq \mathcal{F}$. The inclusion $\mathcal{C} \subseteq \mathcal{K}$ is due to the fact that $f(t,x) \in f(t, B(x,\delta))$ for every (t,x) and $\delta > 0$ and then $f(t,x) \in Kf(t,x)$. On the other hand $\mathcal{C} \not\subseteq \mathcal{F}$ is shown by the following example.

Example 4 Let us consider the Cauchy problem (1.2) with

$$f(x_1, x_2) = \begin{cases} (0,0) & \text{if } (x_1,x_2) = (0,0) \\ (1,0) & \text{if } (x_1,x_2) \neq (0,0) \end{cases}$$

$t_0 = 0$, $(x_{1_0}, x_{2_0}) = (0,0)$. The function $x_1(t) \equiv 0,\ x_2(t) \equiv 0$ is a Carathéodory solution of the Cauchy problem (1.5). On the other hand, since $F(x_1,x_2) = (1,0)$ for each $(x_1,x_2) \in \mathbb{R}^2$, the unique Filippov solution is $x_1(t) = t,\ x_2(t) = 0$.

Let us now compare Krasovskii and Filippov solutions with Euler solutions. The example in [AB] shows that $\mathcal{F} \not\subseteq \mathcal{E}$. In the same example $\mathcal{K} = \mathcal{F}$, then also $\mathcal{K} \not\subseteq \mathcal{E}$. The opposite inclusions don't hold too. This can be seen by means ofthe following example.

Example 5 Let us consider the Cauchy problem (1.1) with

$$f(t,x) = \begin{cases} 1 & \text{if } t \in \mathbb{R}\backslash\mathbf{Q} \\ -1 & \text{if } t \in \mathbf{Q} \end{cases}$$

$t_0 = 0$, $a = 1$, $x_0 = 0$ and the sequence of partitions of the interval $[0,1]$ given by $\pi_N = \{0, \frac{1}{N}, \frac{2}{N}, ..., 1\}$. The corresponding Euler solution is $-t$, while the unique both Krasovskii and Filippov solution is t.

Nevertheless, as mentioned in [Br], if the system is autonomous it may be proved that $\mathcal{E}_D \equiv \mathcal{K}$, then also $\mathcal{E} \subseteq \mathcal{K}$. On the other hand, even for the autonomous case, $\mathcal{E} \not\subseteq \mathcal{F}$, as it can be seen by means of Example 2: $-t$ is an Euler solution, while the only Filippov solution is t.

We now compare Krasovskii and Filippov solutions with generalized sampling solutions. The example in [AB], if reinterpreted in terms of the Cauchy problem (1.5), shows that $\mathcal{K} = \mathcal{F} \not\subseteq \mathcal{S}$. Moreover, if in Example 5 we pose $f(t,x) = k(t,x)$, we also get that $\mathcal{S} \not\subseteq \mathcal{K}$ and $\mathcal{S} \not\subseteq \mathcal{F}$. Finally, in the autonomous case, we have that $\mathcal{S} \subseteq \mathcal{K}$ (see [LS], Lemma 2.9).

Chapter 2

Elements in nonsmooth analysis

In our study of stability and stabilization we need to deal with nonsmooth Lyapunov functions. The central problem is then to have conditions which guarantee the decrease of functions which, in general, are not differentiable. Such conditions can be based on different kinds of generalized derivatives and gradients. We only introduce generalized derivatives and gradients that are needed in the following chapters.

2.1 Functions of one variable

Let I be any interval of $\mathbb{R}$ and $V : I \to \mathbb{R}$. We recall some basic results (see [MS], page 207).

Proposition 2 *Let V be absolutely continuous on each compact subinterval of I. V is non-increasing on I if and only if $\dot{V}(t) \leq 0$ for a.e. $t \in I$.*

If V is just continuous its decrease can be characterized by means of Dini derivatives. Let us then recall some definitions. We denote $r(h,t) = \frac{V(t+h)-V(t)}{h}$.

upper right Dini derivative: $\overline{D^+}V(t) = \limsup_{h\downarrow 0} r(h,t)$

upper left Dini derivative: $\overline{D^-}V(t) = \limsup_{h\uparrow 0} r(h,t)$

lower right Dini derivative: $\underline{D^+}V(t) = \liminf_{h\downarrow 0} r(h,t)$

lower left Dini derivative: $\underline{D^-}V(t) = \liminf_{h\uparrow 0} r(h,t)$.

Proposition 3 *Let V be continuous. Then the following statements are equivalent:*

(i) V *is non-increasing on* I

(ii) $\overline{D^+}V(t) \leq 0$ *for all* $t \in I$

(iii) $\overline{D^-}V(t) \leq 0$ *for all* $t \in I$

(iv) $\underline{D^+}V(t) \leq 0$ *for all* $t \in I$

(v) $\underline{D^-}V(t) \leq 0$ *for all* $t \in I$.

Also in the case V is just either lower semi-continuous or upper semi-continuous, criteria for the decrease of V based on Dini derivatives can be formulated.

2.2 Functions of several variables: generalized directional derivatives and gradients

A fundamental notion associated to functions of several variables is that of directional derivative. Many different definitions can be given for nonsmooth functions and, in connection with them, also different notions of generalized gradients come out.

Let $V : \mathbb{R}^n \to \mathbb{R}$ and $R(h, x, v) = \frac{V(x+hv)-V(x)}{h}$. Dini generalized directional derivatives are

upper right Dini directional derivative: $\overline{D^+}V(x, v) = \limsup_{h\downarrow 0} R(h, x, v)$

upper left Dini directional derivative: $\overline{D^-}V(x, v) = \limsup_{h\uparrow 0} R(h, x, v)$

lower right Dini directional derivative: $\underline{D^+}V(x, v) = \liminf_{h\downarrow 0} R(h, x, v)$

lower left Dini directional derivative: $\underline{D^-}V(x, v) = \liminf_{h\uparrow 0} R(h, x, v)$.

Other generalized directional derivatives are *contingent directional derivatives.* The upper right contingent derivative is defined as

$$\overline{D^+}_K V(x, v) = \limsup_{h\downarrow 0,\ w\to v} R(h, x, w)$$

The other contingent derivatives can be defined analogously. If V is locally Lipschitz continuous contingent derivatives and Dini derivatives coincide.

We mainly focus on Clarke generalized derivatives and gradient (see [Cl1]):

upper Clarke directional derivative: $\overline{D}_C V(x,v) = \limsup_{h\to 0\ y\to x} R(h,y,v)$

lower Clarke directional derivative: $\underline{D}_C V(x,v) = \liminf_{h\to 0\ y\to x} R(h,y,v)$

From the definitions it follows that

$$\underline{D}_C V(x,v) \le \underline{D^+} V(x,v) \le \overline{D^+} V(x,v) \le \overline{D}_C V(x,v)$$

Dini, contingent and also Clarke directional derivatives are positively homogeneous with respect to v. Moreover Clarke directional derivative is subadditive (and hence convex) as a function of v.

Clarke generalized gradient is defined by means of Clarke directional derivatives as

$$\partial_C V(x) = \{p \in \mathbb{R}^n :\ \underline{D}_C V(x,v) \le p\cdot v \le \overline{D}_C V(x,v)\ \forall v \in \mathbb{R}^n\}$$

In a similar way, by means of other generalized directional derivatives, other generalized gradients can be defined.

For each x, the set $\partial_C V(x)$ is convex and closed.

The connection between Clarke generalized derivatives and gradient can also be seen by means of the following equalities:

$$\overline{D}_C V(x,v) = \sup\{p\cdot v,\ p \in \partial_C V(x)\} \qquad \underline{D}_C V(x,v) = \inf\{p\cdot v,\ p \in \partial_C V(x)\}$$

From these it follows that $\underline{D}_C V(x,v) = -\overline{D}_C(x,-v)$.

Very important properties of Clarke generalized directional derivatives and gradient arise if V is locally Lipschitz continuous. Let us recall that in this case, by the well known Rademacher theorem, the gradient ∇V of V exists a.e.. Moreover for every $v \in \mathbb{R}^n$ $\nabla V(\cdot)\cdot v$ is a measurable function (see [EG], page 83). Let us then denote by N the set of zero measure where the gradient of V does not exist and let S be any subset of $\mathbb{R}^n$ of zero measure. We have that

$$\partial_C V(x) = \mathrm{co}\{\lim_{i\to+\infty} \nabla V(x_i) \,:\, x_i \to x,\ \ x_i \notin S\ x_i \notin N\} \tag{2.1}$$

Still if V is Lipschitz continuous, its gradient is bounded, then $\partial_C V(t,x)$ is not just closed but also bounded, and then compact. Thanks to this characterization of $\partial_C V(x)$ it can be proved that, if V is $\mathcal{C}^1$, then $\partial_C V(x) = \{\nabla V(x)\}$. Still in the case V is locally Lipschitz continuous, if U_x is a compact neighbourhood of x and L_x is the Lipschitz constant of V on U_x, we have that

$$-L_x\|v\| \leq \underline{D}_C V(y,v) \leq \overline{D}_C V(y,v) \leq L_x\|v\| \quad \forall y \in U_x \ \forall v \in \mathbb{R}^n.$$

Concluding this paragraph let us recall how Clarke upper directional derivative is used in the definition of C-regular functions.

Definition 12 *$V : \mathbb{R}^n \to \mathbb{R}$ is said to be* C-regular *at x if for every $v \in \mathbb{R}^n$*

(i) there exists the usual right directional derivative
$D^+V(x,v) = \lim_{h\downarrow 0} R(h,x,v)$

(ii) $\overline{D}_C V(x,v) = D^+V(x,v)$.

V is said to be C-regular if it is regular at each $x \in \mathbb{R}^n$.

Let us remark that a convex function is not only Lipschitz continuous, but it is also regular.

2.3 A chain rule

We now restrict our attention to functions which are the composition of a locally Lipschitz continuous funtion $V : \mathbb{R}^n \to \mathbb{R}$ and an absolute continuous function $\psi : \mathbb{R} \to \mathbb{R}^n$. First of all, let us remark that, in these hypothesis, $V \circ \psi : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function and then its derivative exists a.e.. In [MV] the authors prove a chain rule, that we now state in the particular case that is of interest for us.

Proposition 4 *If $V : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and $\psi : \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous, then for a.e. t there exists $p_0 \in \partial_C V(\psi(t))$ such that $\frac{d}{dt}V(\psi(t)) = p_0 \cdot \dot{\psi}(t)$.*

In this context it becomes very interesting the notion of healthy function introduced in [V], and that we slightly modify for functions V having an explicit dependence on time.

Definition 13 *We say that a locally Lipschitz function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is* healthy *if for every absolute continuous function $\varphi : \mathbb{R} \to \mathbb{R}^n$ and for a.e. t the set $\partial_C V(t,\varphi(t))$ is a subset of an affine subspace orthogonal to $(1,\dot{\varphi}(t))$.*

Let us remark that C-regular functions (and then also convex functions) are healthy. The interest in healthy functions is motivated by the following proposition, that can be seen as a chain rule for healthy functions and is easily proved by means of Proposition 4.

Proposition 5 *If $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is healthy and $\psi : \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous, then the set $\{p \cdot (1, \dot{\varphi}(t)),\ p \in \partial_C V(t, \varphi(t))\}$ is reduced to the singleton $\left\{\frac{d}{dt} V(t, \varphi(t))\right\}$ for a.e. t.*

Proof Let $\psi : \mathbb{R} \to \mathbb{R}^{n+1}$ be defined by $\psi(t) = (t, \varphi(t))$. ψ is absolutely continuous and the same is true for $V \circ \psi(t) = V(t, \varphi(t)$. Then $\frac{d}{dt} V \circ \psi(t) = \frac{d}{dt} V(t, \varphi(t))$ exists a.e.. By Proposition 4 we have that for a.e. t there exists $p_0 \in \partial_C V(t, \varphi(t))$ such that $\frac{d}{dt} V \circ \psi(t) = p_0 \cdot \dot{\psi}(t) = p_0 \cdot (1, \dot{\varphi}(t))$ and, by the definition of healthy function, it follows that $\frac{d}{dt} V(t, \varphi(t)) = p \cdot (1, \dot{\varphi}(t))$ for all $p \in \partial_C V(t, \varphi(t))$. $\square$

2.4 Monotonicity along solutions of differential inclusions

In this section we still consider functions which are the composition of a healthy function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ and an absolutely continuous function $\varphi : \mathbb{R} \to \mathbb{R}^n$, but in the particular case φ is a solution of a differential inclusion of the form

$$\dot{x} \in F(t, x) \tag{2.2}$$

with the initial condition $x(t_0) = x_0$, where $F : \mathbb{R}^{n+1} \to 2^{\mathbb{R}^n} \backslash \emptyset$ is an upper semi-continuous set-valued function with compact and convex values.

Definition 14 *A function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is said to be decreasing along F if for each solution $\varphi(t)$ of (2.2) on I and all $t_1, t_2 \in I$ one has*

$$t_1 \leq t_2 \ \Rightarrow V(t_2, \varphi(t_2)) \leq V(t_1, \varphi(t_1)) \tag{2.3}$$

Note that a function V with the property (2.3) is said to be a Lyapunov function for (2.2). We don't emphazise the point of view of Lyapunov functions now because we will focus on them in Chapter 3, in connection with the problem of stability of differential inclusions.

Let us remark that the definition of monotonicity we have given is "strong", in the sense that it refers to all solutions of (2.2). One could analogously define “weak” monotonicity, by refering only to some solutions of (2.2).

In the case V is C^1 the classical condition which guarantees its decrease along F is that for all t and all x

$$D^+V((t,x),(1,v)) = \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x)\cdot v \leq 0 \quad \forall v \in F(t,x)$$

It is from this condition that one gets inspiration for the nonsmooth case.

If V locally Lipschitz continuous, a condition based on Dini derivatives which implies (2.3) is that for a.e.t and all x (see [B3]):

$$\underline{D^+}V((t,x)(1,v)) \leq 0 \qquad \forall v \in F(t,x) \tag{2.4}$$

Note that (2.3) does not imply (2.4). This can be seen by means of Example 1 in Chapter 3.
Since $\underline{D^+}V((t,x)(1,v)) \leq \overline{D}_C V((t,x)(1,v)) = \max\{p\cdot(1,v),\ p \in \partial_C V(t,x)\}$ for all (t,x) and for all $v \in F(t,x)$, also the condition

$$\text{for a.e. } t\ \forall x\ \forall v \in F(t,x)\ \ \max\{p\cdot(1,v), p \in \partial_C V(t,x)\} \leq 0 \tag{2.5}$$

guarantees the monotonicity of V along F. The advantage of this last condition is that, thanks to the characterization of Clarke generalized gradient for locally Lipschitz continuous functions, it can be relatively easily computed. On the other hand it is not very sharp, except for the case $F(t,x) = \{F(x)\}$, where F is continuous. In this case we have the following proposition (see [BCM]).

Proposition 6 *Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}\backslash\emptyset$ be continuous and let $V : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous. V decreases along all solutions of*

$$\dot{x} \in F(x) \tag{2.6}$$

if and only if

$$\forall x \in \mathbb{R}^n\ \forall v \in F(x)\ \ \max\{p\cdot v,\ p \in \partial_C V(x)\} \leq 0$$

Proof The proof that if for all $x \in \mathbb{R}^n$ and for all $v \in F(x)$ one has $\max\{p\cdot v,\ p \in \partial_C V(x)\} \leq 0$ then V decreases along all solutions of (2.6) trivially follows from the previous discussion.

Let us prove that if V decreases along all solutions of (2.6) then for all x and for all $p \in \partial_C V(x)$ one has $p\cdot v \leq 0$. Let us suppose by contradiction that there exists a $x_0 \in \mathbb{R}^n$, $v_0 \in F(x_0)$ and $p_0 \in \partial_C V(x_0)$ such that $p_0 \cdot v_0 > 0$. By (2.1) there exist

a) $\lambda_1, ..., \lambda_m > 0$ such that $\sum_{i=1}^m \lambda_i = 1$

b) $p^{(1)}, ..., p^{(m)} \in \mathbb{R}^n$ and $\{x_k^{(1)}\} \subset \mathbb{R}^n, ..., \{x_k^{(m)}\} \subset \mathbb{R}^n$ such that for all $i \in \{1, ..., m\}$ and for all k there exists $\nabla V(x_k^{(i)})$,

$p^{(i)} = \lim_{k\to+\infty} \nabla V(x_k^{(i)})$ and $\lim_{k\to+\infty} x_k^{(i)} = x_0$

such that $p_0 = \lambda_1 p^{(1)} + ... + \lambda_m p^{(m)}$. Since $p_0 \cdot v_0 = \lambda_1 p^{(1)} \cdot v_0 + ... + \lambda_m p^{(m)} \cdot v_0 > 0$, there exists $j \in \{1, ..., m\}$ such that $p^{(j)} \cdot v_0 > 0$. Let $\{x_k^{(j)}\}$ be a sequence as in b).

Let us fix $\epsilon < \min\left\{1, \frac{p^{(j)} \cdot v_0}{2(\|v_0\| + \|p^{(j)}\| + 1)}\right\}$.

Since $p^{(j)} = \lim_{k\to+\infty} \nabla V(x_k^{(j)})$, there exists $\overline{k}$ such that for all $k > \overline{k}$ there exists $w_k \in B(0,1)$ such that $\nabla V(x_k^{(j)}) = p^{(j)} + \epsilon w_k$, where $B(0,1)$ is the unit ball in $\mathbb{R}^n$ centered at the origin.

Moreover $x_k^{(j)} \to x_0$ and F is continuous, then there exists $\tilde{k}$ such that for all $k > \tilde{k}$ there exist $v_k \in F(x_k^{(j)})$ and $z_k \in B(0,1)$ such that $v_k = v_0 + \epsilon z_k$.

Then for all $k > \max\{\overline{k}, \tilde{k}\}$ there exist $v_k \in F(x_k^{(j)})$ and $w_k, z_k \in B(0,1)$ such that $\nabla V(x_k^{(j)}) \cdot v_k = (p^{(j)} + \epsilon w_k) \cdot (v_0 + \epsilon z_k) = p^{(j)} \cdot v_0 - \epsilon |w_k \cdot v_0 + z_k \cdot p^{(j)} + \epsilon w_k \cdot z_k| \geq p^{(j)} \cdot v_0 - \epsilon(\|v_0\| + \|p^{(j)}\| + 1) > \frac{p^{(j)} \cdot v_0}{2} > 0$.

Let us fix $K > \max\{\overline{k}, \tilde{k}\}$ and let us consider the solution $\varphi(t)$ of (1.2) with the initial conditions $\varphi(t_0) = x_K^{(j)}$ and $\dot{\varphi}(t_0) = v_K^{(j)}$. The existence of such a solution is guaranteed by Theorem 2.3 in [C].

Since V decreases along solutions of (2.6), then $\frac{d}{dt} V(\varphi(t)) \leq 0$ a.e., i.e. on the set where $V \circ \varphi$ is differentiable. In particular we have that $\frac{d}{dt} V(\varphi(t_0)) \leq 0$. On the other hand $\frac{d}{dt} V(\varphi(t_0)) = \nabla V(\varphi(t_0)) \cdot v_K = \nabla V(x_K^{(j)}) \cdot v_K > 0$, that is a contradiction. $\square$

In the general case, in order to get a condition sharper than (2.5), we need to define the *set-valued derivative of V with respect to (2.2)*:

$$\overline{\dot{V}}^{(2.2)}(t,x) = \{a \in \mathbb{R} : \exists v \in F(t,x) \text{ such that } p \cdot (1,v) = a \quad \forall p \in \partial_C V(t,x)\}$$

Remark 3 By Proposition 5, if $\varphi(t)$ is any solution of (2.2) we have that $\frac{d}{dt} V(t, \varphi(t)) \in \overline{\dot{V}}^{(2.2)}(t, \varphi(t))$ for a.e.t.

Lemma 1 *Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be a Lipschitz continuous function. For each fixed $(t,x) \in \mathbb{R}^{n+1}$ the set $\overline{\dot{V}}^{(2.2)}(t,x)$ is a closed and bounded interval, possibly empty.*

Moreover, if V is differentiable, then
$\dot{\overline{V}}^{(2.2)}(t,x) = \left\{ \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot v,\ v \in F(t,x) \right\}$.

Proof We first prove that $\dot{\overline{V}}^{(2.2)}(t,x)$ is closed. Let $\{a_n\} \subset \dot{\overline{V}}^{(2.2)}(t,x)$, $a_n \to a$. For each n there exists $v_n \in F(t,x)$ such that $p \cdot (1, v_n) = a_n$ for all $p \in \partial_C V(t,x)$. Since $F(t,x)$ is compact there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ converging to $v \in F(t,x)$. We get that $a_{n_j} = p \cdot (1, v_{n_j}) \to p \cdot v$. By the uniqueness of the limit we have $p \cdot (1, v) = a \in \dot{\overline{V}}^{(2.2)}(t,x)$.

Let us now prove that $\dot{\overline{V}}^{(2.2)}(t,x)$ is bounded. Let $U_{t,x}$ be a compact neighbourhood of (t,x), $L_{t,x}$ be the Lipschitz constant of V on $U_{t,x}$ and $M_{t,x}$ be such that $\|(1,v)\| \leq M_{t,x}$ for all $v \in F(t,x)$. Let $a \in \dot{\overline{V}}^{(2.2)}(t,x)$. Since there exists $v \in F(t,x)$ such that $a = p \cdot (1,v)$ for all $p \in \partial_C V(t,x)$, we get that $|a| \leq \|p\| \|(1,v)\| \leq L_{t,x} M_{t,x}$.

Let us show that $\dot{\overline{V}}^{(2.2)}(t,x)$ is convex. Let $a_1, a_2 \in \dot{\overline{V}}^{(2.2)}(t,x)$. There exist $v_1, v_2 \in F(t,x)$ such that $a_1 = p \cdot (1, v_1)$ and $a_2 = p \cdot (1, v_2)$ for all $p \in \partial_C V(t,x)$. Let $\tau \in [0,1]$ and let us consider $v = \tau v_1 + (1-\tau) v_2$. Since $F(t,x)$ is convex, $v \in F(t,x)$. For all $p \in \partial_C V(t,x)$ we have that $p \cdot v = \tau a_1 + (1-\tau) a_2$, so that $\tau a_1 + (1-\tau) a_2 \in \dot{\overline{V}}^{(2.2)}(t,x)$.

Finally $\dot{\overline{V}}^{(2.2)}(t,x) = \left\{ \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot v,\ v \in F(t,x) \right\}$ is an immediate consequence of the fact that, if V is differentiable, then
$\partial_C V(t,x) = \left\{ \left(\frac{\partial V}{\partial t}(t,x), \nabla V(t,x) \right) \right\}$. $\square$

Proposition 7 *If $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is locally Lipschitz continuous then for all $(t,x) \in \mathbb{R}^{n+1}$*

$$\max \dot{\overline{V}}^{(2.2)}(t,x) \leq \max_{v \in F(t,x)} \underline{D^+} V((t,x)(1,v)) \tag{2.7}$$

Proof Let $\overline{a} = \max \dot{\overline{V}}^{(2.2)}(t,x)$. There exists $\overline{v} \in F(t,x)$ such that $p \cdot \overline{v} = \overline{a}$ for all $p \in \partial_C V(t,x)$, so that

$$\overline{a} = \underline{D}_C V((t,x),(1,v)) \leq \underline{D^+} V((t,x),(1,v)) \leq \max_{v \in F(t,x)} \underline{D^+} V((t,x),(1,v)). \square$$

Proposition 8 *If $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is a healthy function and $\max \dot{\overline{V}}^{(2.2)}(t,x) \leq 0$ for a.e. t and all x, then V decreases along F.*

Proof Let $\varphi(t)$ be any solution of (2.2) and let us consider the absolute continuous function $V(t,\varphi(t))$. By Proposition 5 the set $\{p\cdot(1,\dot\varphi(t)),\ p\in\partial_C V(t,\varphi(t))\}$ reduces to the singleton $\left\{\frac{d}{dt}V(t,\varphi(t))\right\}$ for a.e. t.

Since $\dot\varphi(t)\in F(t,\varphi(t))$ for a.e.t, we have that $\left\{\frac{d}{dt}V(t,\varphi(t))\right\}\subseteq\{p\cdot(1,\dot\varphi(t)),\ p\in\partial_C V(t,\varphi(t))\}\subseteq\{a\in\mathbb{R}:\ \exists v\in F(t,\varphi(t))\text{ such that } p\cdot v=a\ \forall p\in F(t,x)\}=\overline{\dot V}^{(2.2)}(t,x)$. Then, if $\max\overline{\dot V}^{(2.2)}(t,x)\le 0$ for a.e.t and for all x, we get that $\frac{d}{dt}V(t,\varphi(t))\le 0$ for a.e. t. Finally Proposition 2, implies that V decreases along $\varphi(t)$. $\square$

Remark 4 Let $\varphi(t)$ be any solution of (2.2). $\overline{\dot V}^{(2.2)}(t,\varphi(t))=\emptyset$ can occur only on a zero measure set. In fact for a.e. t there exists $\frac{d}{dt}V(t,\varphi(t))\in\overline{\dot V}^{(2.2)}(t,\varphi(t))$ so that $\overline{\dot V}^{(2.2)}(t,\varphi(t))\neq\emptyset$ for a.e. t. We can then extend the conclusion of Proposition 8 to the case $\overline{\dot V}^{(2.2)}(t,x)=\emptyset$ by posing
$\max\overline{\dot V}^{(2.2)}(t,x)=-\infty$ if $\overline{\dot V}^{(2.2)}(t,x)=\emptyset$.

Remark 5 An analogous version of Proposition 8 was given in [SP] for C-regular functions. The set-valued derivative used in that paper is slightly different and, in general, it is a set larger than $\overline{\dot V}^{(2.2)}$. We show this by means of Example 6 in Chapter 3.

Remark 6 The converse of inequality (2.7) does not hold (see Example 6 in next chapter). This means that, if V is healthy, the stability criterion based on $\overline{\dot V}^{(2.2)}$ works better than the criterion based on Dini lower right derivative.

Remark 7 It is important to emphasize that, if instead of the differential inclusion (2.2) we consider the autonomous differential inclusion (2.6) and Lyapunov functions $V:\mathbb{R}^n\to\mathbb{R}$ not depending on time, we obtain results perfectly analogous to those described in this section (see [BC]).

Chapter 3

Stability of differential inclusions

The problem of stability of differential inclusions is of primary interest for us. In fact, as we have seen in Chapter 1, discontinuous differential equations are often interpreted in terms of differential inclusions, so that the study of stability of discontinuous differential equations can coincide with the study of stability of differential inclusions. Since in general a differential inclusion has not a unique solution, the stability property is usually said to be strong or weak according to the fact that it refers to all or just some of its solutions. We only consider strong stability, so we omit to mention the adjective strong in the following. But let us define stability precisely.

Definition 15 *The differential inclusion (2.2) is said to be* stable *at* $x = 0$ *if for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for each initial condition* (t_0, x_0) *and each solution* $\varphi(t)$ *of (2.2) such that* $\varphi(t_0) = x_0$*,*

$$\|x_0\| < \delta \Rightarrow \|\varphi(t)\| < \epsilon \quad \forall t \geq t_0.$$

More precisely this concept of stability is uniform, in the sense that δ does not depend on t_0.

Note that if the differential inclusion (2.2) is stable at $x = 0$, then $x = 0$ is an equilibrium point for it, i.e. $0 \in F(t, 0)$ for all $t \geq t_0$. In fact, if $x_0 = 0$, then $\|x_0\| < \delta$ for all δ and , if $\varphi(t)$ is a solution of (2.2) with $\varphi(t_0) = 0$, then for all ϵ one has $\|\varphi(t)\| < \epsilon$ for all $t \geq t_0$, i.e. $\varphi(t) \equiv 0$. Since $\varphi(t) \equiv 0$ is a solution of (2.2) and $\dot{\varphi}(t) \equiv 0$, we get that $0 \in F(t, 0)$ for all t. In this way, we have also proved that $\varphi(t) \equiv 0$ is the unique solution of (2.2) such that $\varphi(t_0) = 0$.

3.1 Lyapunov's direct method

Lyapunov's direct method (also called Lyapunov's second method), originated in order to study stability of differential equations, but it can be also successfully applied to differential inclusions. It makes it possible to investigate the stability of (2.2) without knowing the explicit form of its solutions, but just using the differential inclusion itself. The method is based on the knowledge of Lyapunov functions, which can be seen as a generalization of the concept of energy.

Definition 16 *$V : \mathbb{R}^n \to \mathbb{R}$ is said to be a* Lyapunov function *for (2.2)$\varphi(t)$ if for each solution of (2.2) on $I \subseteq \mathbb{R}$ and for all $t_1, t_2 \in I$*
condition (2.3) holds.

Let us emphasize that we always consider differential inclusions of the form (2.2) where F is an upper semi-continuous set-valued map with non-empty, compact and convex values. In these hypothesis the existence of at least one solution of (2.2) is ensured (see, e.g., [AC], page 37).

A well known version of first Lyapunov theorem for differential inclusions is the following. Because of its importance we also prove it.

Theorem 6 *If there exist a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ for (2.2) and two continuous, strictly increasing functions $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

(i) $a(0) = b(0) = 0$ *and* $a(r) > 0$ *for* $r > 0$

(ii) $a(\|x\|) \le V(t, x) \le b(\|x\|)$ *for all* (t, x)

then (2.2) is stable at $x = 0$.

Remark 8 Note that hypothesis (i) and (ii) imply that

1) $V(t, x) \ge 0$ for all (t, x)

2) $V(t, 0) = 0$ for all t

3) $V(t, x)$ is continuous with respect to x at $(t, 0)$ for all t.

Proof We want to prove that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all t_0 and all solutions $\varphi(t)$ of (2.2) with $\varphi(t_0) = x_0$, $\|x_0\| < \delta$ implies $\|\varphi(t)\| < \epsilon$ for all $t \ge t_0$.

Let $\epsilon > 0$ and t_0 be given, and let us consider $a(\epsilon)$. By the continuity of b there exists $\delta > 0$ such that if $\|x_0\| \le \delta$ then $V(t_0, x_0) \le b(\|x_0\|) < a(\epsilon)$.

Since V is a Lyapunov function, for every solution $\varphi(t)$ of (2.2) with $\varphi(t_0) = x_0$

$$V(t, \varphi(t)) \leq V(t_0, x_0) < a(\epsilon) \quad \forall t \geq t_0 \tag{3.1}$$

From this inequality it follows that $\|\varphi(t)\| < \epsilon$ for all $t \geq t_0$.
In fact otherwise there would exist $\bar{t}$ such that $\|\varphi(\bar{t})\| \geq \epsilon$ and, by (i) and (ii), $V(\bar{t}, \varphi(\bar{t})) \geq a(\|\varphi(\bar{t})\|) \geq a(\epsilon)$, that is a contradiction to (3.1). $\square$

In order to apply Lyapunov's second method and prove the stability of a differential inclusion, the fundamental tool is to find a Lyapunov function and, in particular, to verify condition (2.3), without knowing the explicit form of its solutions. We are then led back to the problem of monotonicity along solutions of a differential inclusion that we discussed in Section 2.4.

The following result is a corollary of the previous theorem and Proposition 8 in Chapter 2.

Corollary 1 *If there exists a function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\max \overline{\dot{V}}^{(2.2)}(t, x) \leq 0$ for a.e.t and for all x and two continuous strictly increasing functions $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ such that hypothesis (i) and (ii) of Theorem 6 hold, then the differential inclusion (2.2) is stable at $x = 0$.*

Remark 9 If instead of (2.2) we consider the autonomous differential inclusion (2.6), it makes more sense to consider Lyapunov functions not depending on t. Hypothesis (i) and (ii) in Theorem 6 can then be changed into the following:

$V : \mathbb{R}^n \to \mathbb{R}$ is positive definite and continuous at $x = 0$.

The conclusion of the previous corollary still holds with $\overline{\dot{V}}^{(2.2)}$ replaced by $\overline{\dot{V}}^{(2.6)}(x) = \{a \in \mathbb{R} : \exists v \in F(x) \text{ s.t. } p \cdot v = a \ \forall p \in \partial_C V(x)\}$ (see [BC]).

Remark 10 Nonsmooth Lyapunov functions and generalized derivatives have been previously used in the literature on stability mainly in connection with the problem of asymptotic stability and stabilization: see for instance [SP, SS2, CLSS, R2, FK].

Example 6 We consider a system of the form (1.2) in $\mathbb{R}^2$ where $f(x_1, x_2) = (-\operatorname{sgn} x_2, \operatorname{sgn} x_1)^T$ (Fig.1). According to the Filippov's approach, this leads to the differential inclusion (2.6), where

$$F(x_1, x_2) = Ff(x_1, x_2) =$$

$$= \begin{cases} \{-\operatorname{sgn} x_2\} \times \{\operatorname{sgn} x_1\} & \text{at } (x_1, x_2),\ x_1 \neq 0 \text{ and } x_2 \neq 0 \\ [-1, 1] \times \{\operatorname{sgn} x_1\} & \text{at } (x_1, 0),\ x_1 \neq 0 \\ \{-\operatorname{sgn} x_2\} \times [-1, 1] & \text{at } (0, x_2),\ x_2 \neq 0 \\ \overline{\text{co}}\{(1,1), (-1,1), (-1,-1), (1,-1)\} & \text{at } (0,0) \end{cases}$$

Let us now consider $V(x_1, x_2) = |x_1| + |x_2|$. We have

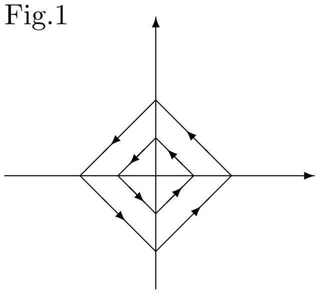
$$\partial_C V(x_1, x_2) = \begin{cases} \{\operatorname{sgn} x_1\} \times \{\operatorname{sgn} x_2\} & \text{at } (x_1, x_2),\ x_1 \neq 0 \text{ and } x_2 \neq 0 \\ \{\operatorname{sgn} x_1\} \times [-1, 1] & \text{at } (x_1, 0),\ x_1 \neq 0 \\ [-1, 1] \times \{\operatorname{sgn} x_2\} & \text{at } (0, x_2),\ x_2 \neq 0 \\ \overline{\text{co}}\{(1,1), (-1,1), (-1,-1), (1,-1)\} & \text{at } (0,0) \end{cases}$$

so that

$$\dot{\overline{V}}^{(2.6)}(x_1, x_2) = \begin{cases} \{0\} & \text{at } (x_1, x_2),\ x_1 \neq 0 \text{ and } x_2 \neq 0 \\ \emptyset & \text{at } (x_1, 0),\ x_1 \neq 0 \\ \emptyset & \text{at } (0, x_2),\ x_2 \neq 0 \\ \{0\} & \text{at } (0,0) \end{cases}$$

Since for all $(x_1, x_2) \in \mathbb{R}^2$ one has $\max \dot{\overline{V}}^{(2.6)}(x_1, x_2) \leq 0$, by Corollary 1, the system is stable at $x = 0$. Let us remark that $\max \dot{\overline{V}}^{(2.6)}(0, x_2) = -\infty < \underline{D^+}V((0, x_2), (-1, 1)) = 2 \leq \max_{v \in F(0, x_2)} \underline{D^+}V((0, x_2), v)$. This means that a test based on Dini derivative is inconclusive. Moreover $\dot{\overline{V}}^{(2.6)}(0, x_2) = \emptyset$ is a strict subset of the set $\dot{\tilde{V}}^{(2.6)}(0, x_2) = \{0\}$ considered in [SP].

Fig.1



Remark 11 The previous example clearly shows that, in general, there is no hope to find smooth Lyapunov functions for discontinuous equations.

Let us assume by contradiction that a smooth Lyapunov function V does exist. V is constant on cycles and any cycle is a level set for V. Then V has nonsmooth level sets, and this contradicts the fact that V is smooth.

Example 7 *Nonsmooth harmonic oscillator.* Let us consider the scalar differential equation

$$\ddot{x} = -\mathrm{sgn} x \tag{3.2}$$

We can associate to this equation a system of the form (1.2) in $\mathbb{R}^2$ where $f(x_1, x_2) = (x_2, \ -\mathrm{sgn}\, x_1)^T$. The Filippov multivalued map associated to the system is

$$F(x_1, x_2) = Ff(x_1, x_2) == \begin{cases} \{x_2\} \times \{-\mathrm{sgn}\, x_1\} & \text{at } (x_1, x_2), \ x_1 \neq 0 \\ \{x_2\} \times [-1, 1] & \text{at } (0, x_2) \end{cases}$$

Let us now consider $V(x_1, x_2) = |x_1| + \frac{x_2^2}{2}$. We have

$$\partial_C V(x_1, x_2) = \begin{cases} \{\mathrm{sgn}\, x_1\} \times \{x_2\} & \text{at } (x_1, x_2), \ x_1 \neq 0 \\ [-1, 1] \times \{x_2\} & \text{at } (0, x_2) \end{cases}$$

so that

$$\overline{\dot{V}}^{(2.6)}(x_1, x_2) = \begin{cases} \{0\} & \text{at } (x_1, x_2), \ x_1 \neq 0 \\ \{0\} & \text{at } (x_1, x_2), \ x_1 = 0, \, x_2 = 0 \\ \emptyset & \text{at } (0, x_2), \ x_2 \neq 0 \end{cases}$$

Since for all $(x_1, x_2) \in \mathbb{R}^2$ one has $\max \overline{\dot{V}}^{(2.6)}(x_1, x_2) \leq 0$, by Corollary 1, the system is stable at $x = 0$.

Example 8 *Gradient vector fields.* It is well known that if $V : \mathbb{R}^n \to \mathbb{R}$ is a positive definite smooth function then the equation

$$\dot{x} = -\nabla V(x)$$

has an asymptotically stable equlibrium at the origin. For a locally Lipschitz, healthy positive definite function V, a natural substitute of the previous equation is the differential inclusion

$$\dot{x} \in -\partial_C V(x).$$

Let $a \in \overline{\dot{V}}^{(2.6)}(x)$, where $F(x) = -\partial_C V(x)$. Then there exists $v \in -\partial_C V(x)$ such that $p \cdot v = a$ for each $p \in \partial_C V(x)$. In particular the equality must be true for $p = v$. But then $a = -|v|^2 \leq 0$. According to Corollary 1, we conclude that any differential inclusion of the form $\dot{x} \in -\partial_C V(x)$ is stable at the origin.

3.2 Inverse Lyapunov's theorems

We devote just a short paragraph to the problem of inverting Theorem 6. The cases of a continuous or smooth, time dependent or autonomous single valued righthandside of (2.2) have been widely treated in the literature: see [BS, K, Ku, Y1, KV, AS] and [BR1] for an overview on the problem. It is important to emphasize that really the most important issue in these papers is not the existence of a Lyapunov function, that is relatively easily obtained, but its regularity. Actually, regularity of Lyapunov functions plays an important role in the applications, for example in connection with the problem of asymptotic stabilization of a control system, as we will see in Chapter 4. We have already remarked that, in general, for discontinuous systems, there is no hope to find a smooth Lyapunov function. In [KV] the authors give an example of a system of the form (1.1) with f continuous such that a continuous Lyapunov function does not exist. Moreover they prove that the existence of a continuous Lyapunov function becomes a necessary and sufficient condition if the notion of stability is conveniently strengthened (see also [AS] for the autonomous case).

From our point of view, it is interesting to know a notion of stability equivalent to the existence of a Lipschitz continuous (or regular or healthy) Lyapunov function, both for autonomous and time dependent systems. For time dependent systems the problem has been solved by Kurzweil and Vrkoč ([KV]) in the case f is continuous, by means of the notion of robust stability. For autonomous systems the problem has been very recently solved in the scalar case (see [BR2]), but it is still open in $\mathbb{R}^n$.

Going back to discontinuous equations and differential inclusions, let us mention just two results. The first one can be found in [D], page 205. It essentially is the inverse of Theorem 6.

In the second one ([BR1]), the authors generalize the mentioned result in [KV] to differential inclusions of the form (2.2).

3.3 Asymptotic Stability and Invariance Principle

As shown in the previous section, if a Lyapunov function is known, one can get some conclusions about stability, but nothing can be said in general about asymptotic stability, whose definition is the following.

Definition 17 *The differential inclusion (2.2) is said to be asymptotically stable at $x = 0$ if* *(i)* *(2.2) is stable at* $x = 0$

(ii) *there exists* $\eta > 0$ *such that if* $\|x_0\| \le \eta$ *then for all* t_0 *and for all solutions* $\varphi(t)$ *of (2.2) with initial condition* $\varphi(t_0) = x_0$ *one has* $\lim_{t\to\infty} \varphi(t) = 0$.

Note that the given concept of asymptotic stability is strong, in the sense that it refers to all solutions of (2.2), and uniform, because η does not depend on t_0.

Both for autonomous and time dependent differential equations with continuous right handside there are classical results which give asymptotic stability. They are based on the knowledge of a smooth Lyapunov functions whose derivative with respect to the system is negative definite. In the autonomous case some asymptotic stability results can be achieved by means of LaSalle principle (also called invariance principle) even if the derivative of the Lyapunov function with respect to the system is not known to be negative definite. Beside its application in the study of asymptotic stability, LaSalle principle is interesting by itself, because it provides information on the behaviour of solutions. Even if a "plain" version of LaSalle principle for time-dependent systems does not exist, many results in this direction can be mentioned: [RHL, A, AP] and references therein.

Going back to differential inclusions and nonsmooth Lyapunov functions, we now limit ourselves to consider the autonomous case. We give a version of the invariance principle based on the notion of set-valued derivative with respect to (2.6) and compare it with similar early results.

The following definitions (see [F2], page 129) are useful to formulate and prove such an invariance theorem.

Definition 18 *A point* $q \in \mathbb{R}^n$ *is said to be a* limit point *for a solution* $\varphi(t)$ *of (2.6) if there exists a sequence* $\{t_i\}$, $t_i \to +\infty$ *as* $i \to +\infty$, *such that* $\varphi(t_i) \to q$ *as* $i \to +\infty$.

The set of the limit points of $\varphi(t)$ *is said to be the* limit set *of* $\varphi(t)$ *and is denoted by* $\Omega(\varphi)$.

Definition 19 *A set* Ω *is said to be a* weakly invariant set *for (2.6) if through each point* $x_0 \in \Omega$ *there exists a maximal solution of (2.6) lying in* Ω.

We recall that under the assumption that F is an upper semi-continuous multivalued map with compact, convex values, if $\varphi(t)$ is a solution of the autonomous differential inclusion (2.6) and $\Omega(\varphi)$ is its limit set, then $\Omega(\varphi)$

is weakly invariant and if $\varphi(t), t \in \mathbb{R}_+$, lies in a bounded domain, then $\Omega(\varphi)$ is nonempty, bounded, connected and $\mathrm{dist}(\varphi(t), \Omega(\varphi)) \to 0$ as $t \to +\infty$ (see [F2], page 129).

Theorem 7 *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous and healthy Lyapunov function for (2.6). Let us assume that for some $l > 0$, the connected component L_l of the level set $\{x \in \mathbb{R}^n : V(x) \le l\}$ such that $0 \in L_l$ is bounded. Let $x_0 \in L_l$ and $\varphi(t)$ be any solution of (2.6) such that $\varphi(t_0) = x_0$. Let*

$$Z_V^{(2.6)} = \{x \in \mathbb{R}^n : 0 \in \dot{\overline{V}}^{(2.6)}(x)\}$$

and let M be the largest weakly invariant subset of $\overline{Z_V^{(2.6)}} \cap L_l$.

Then $\mathrm{dist}(\varphi(t), M) \to 0$ *as $t \to +\infty$.*

Proof Let $\Omega(\varphi)$ be the limit set of $\varphi(t)$. Let us remark that $\varphi(t)$ is bounded. In fact otherwise there would exist $t_1 > 0$ such that $\varphi(t_1) \notin L_l$ and, since $\varphi(t)$ is continuous, $\varphi(t_1)$ is not in any other connected component of $\{x \in \mathbb{R}^n : V(x) \le l\}$. Then $V(\varphi(t_1)) > l \ge V(x_0)$, that is impossible since $V \circ \varphi$ is decreasing.

Let us prove that $\Omega(\varphi) \subseteq \overline{Z_V^{(2.6)}} \cap L_l$. Because of the definition of L_l, $\Omega(\varphi) \subseteq L_l$.

We now prove that $\Omega(\varphi) \subseteq \overline{Z_V^{(2.6)}}$.

Let us remark that V is constant on $\Omega(\varphi)$. Indeed, since $V \circ \varphi$ is decreasing and bounded from below, there exists $\lim_{t\to+\infty} V(\varphi(t)) = c \ge 0$. Let $y \in \Omega(\varphi)$. There exists a sequence $\{t_n\}, t_n \to +\infty$, such that $\lim_{n\to+\infty} \varphi(t_n) = y$ and, by the continuity of V, $V(y) = c$.

Let $y \in \Omega(\varphi)$ and $\psi(t)$ be a solution of (2.2) lying in $\Omega(\varphi)$ such that $\psi(0) = y$. Since $V(\psi(t)) = c$ for all t, we have $\frac{d}{dt}V(\psi(t)) = 0$ for all t. Therefore $0 \in \dot{\overline{V}}^{(2.6)}(\psi(t))$ almost everywhere, namely $\psi(t) \in Z_V^{(2.6)}$ almost everywhere.

Let $\{t_i\}$, $t_i \to 0$, be a sequence such that $\psi(t_i) \in Z_V^{(2.6)}$ for all i. Since ψ is continuous $\lim_{i\to+\infty} \psi(t_i) = \psi(0) = y \in \overline{Z_V^{(2.6)}}$.

From the fact that $\Omega(\varphi)$ is weakly invariant it follows that $\Omega(\varphi) \subseteq M$ and from the fact that $\mathrm{dist}(\varphi(t), \Omega(\varphi)) \to 0$ as $t \to +\infty$ it follows that $\mathrm{dist}(\varphi(t), M) \to 0$ as $t \to +\infty$. □

Remark 12 Early versions of the invariance principle for differential inclusions can be found in [SP] and [R2]. Although the result presented here has been largely inspired by both of them, certain differences should be pointed out. First of all, we emphasize that Theorem 7 is more general than Theorem 3.2 of [SP] since no assumption about uniqueness of solutions is required. As far as Ryan's invariance principle is concerned, essentially two remarks have to be done. On one hand Ryan's result refers to merely locally Lipschitz continuous Lyapunov functions, while we deal with locally Lipschitz continuous and also healthy Lyapunov functions. On the other hand our identification of the "bad" set $Z_V^{(2.6)}$ is sharper than Ryan's one. Finally, Example 10 shows a case in which Theorem 7 can be used in order to compute the limit set, while Ryan's invariance principle doesn't help.

Remark 13 Example 6 of the previous Section shows that, in the conclusion of Theorem 7, we cannot avoid to take, in general, the closure of $Z_V^{(2.6)}$. Indeed, in Example 6 each trajectory is a closed path that coincides with its limit set and crosses the coordinates axis.

Remark 14 As a consequence of the invariance principle we get asymptotic stability in the case that a Lyapunov function for (2.6) is known and the set $Z_V^{(2.6)}$ reduces to the origin.

Example 9 *Smooth oscillator with nonsmooth friction and uncertain coefficients.* Let us consider a differential inclusion of the form (2.6) in $\mathbb{R}^2$, where

$$F(x_1,x_2)=\begin{cases}[-2x_2-1,-x_2-1]\times\{x_1\} & \text{at } (x_1,x_2), x_1>0 \text{ and } x_2>0\\ \{-x_2-\operatorname{sgn} x_1\}\times\{x_1\} & \text{at } (x_1,x_2)\in\mathbb{R}^2\backslash(\{(0,x_2)\}\\ & \cup\{(x_1,x_2), x_1>0,\ x_2>0\})\\ [-2x_2-1,-x_2+1]\times\{0\} & \text{at } (0,x_2), x_2>0\\ [-x_2-1,-x_2+1]\times\{0\} & \text{at } (0,x_2), x_2<0\\ [-1,1]\times\{0\} & \text{at } (0,0).\end{cases}$$

Let us now consider the smooth function $V(x_1,x_2)=\frac{x_1^2+x_2^2}{2}$. In this case

$$\overset{\cdot}{\overline{V}}^{(2.6)}(x)=\begin{cases}\{[-1,0]x_1x_2-x_1\} & \text{at } (x_1,x_2), x_1>0 \text{ and } x_2>0\\ \{-|x_1|\} & \text{at } (x_1,x_2)\in\mathbb{R}^2\backslash(\{(0,x_2)\mathbb{R}\}\\ & \cup\{(x_1,x_2), x_1>0 \text{ and } x_2>0\})\\ \{0\} & \text{at } (0,x_2), x_2\neq 0\\ \{0\} & \text{at } (0,0),\end{cases}$$

then $Z_V^{(2.6)}=\{(0,x_2), x_2\in\mathbb{R}\}$.

Let us now determine the largest weakly invariant subset M of $Z_V^{(2.6)}$.

Let us remark that , if $|x_2| \le 1$, then $(0,0) \in F(x_1,x_2)$, hence the segment $\overline{P_1P_2}$, where $P_1 = (0,1)$ and $P_2 = (0,-1)$, is a weakly invariant subset of $Z_V^{(2.6)}$.

Moreover, if $|x_2| > 1$, all the vectors $v \in F(0,x_2)$ point in the same direction, hence each trajectory, starting in $(0,x_2)$, with $|x_2| > 1$, leaves the x_2-axis.

We conclude that $M = \overline{P_1P_2}$, i.e. all trajectories of the differential inclusion (2.6) tend to the segment $\overline{P_1P_2}$ as $t \to +\infty$. In fact each solution is attracted by a single point of the segment $\overline{P_1P_2}$. This follows by the proof of Theorem 7. Indeed each solution is attracted by the set $Z_V^{(2.6)} \cap L_l \cap V^{-1}(c)$ for some c.

Example 10 *Nonsmooth harmonic oscillator with nonsmooth friction* (Fig. 2). Let us consider a system of the form (1.2) in $\mathbb{R}^2$ where $f(x_1,x_2) = (-\operatorname{sgn} x_2 - \frac{1}{2}\operatorname{sgn} x_1, \operatorname{sgn} x_1)^T$. Filippov solutions of (1.2) are solutions of the differential inclusion (2.6), where

$$F(x_1,x_2) = Ff(x_1,x_2) =$$

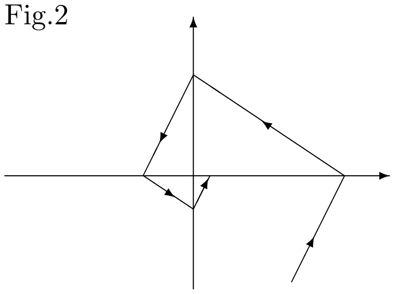
$$= \begin{cases} \{-\text{sgn}x_2 - \frac{1}{2}\text{sgn}x_1\} \times \{\text{sgn}x_1\} & \text{at } (x_1,x_2),\ x_1 \neq 0,\ x_2 \neq 0 \\ \overline{\text{co}}\left\{\left(-\frac{3}{2},1\right),\left(\frac{1}{2},1\right)\right\} & \text{at } (x_1,0),\ x_1 > 0 \\ \overline{\text{co}}\left\{\left(-\frac{1}{2},-1\right),\left(\frac{3}{2},-1\right)\right\} & \text{at } (x_1,0),\ x_1 < 0 \\ \overline{\text{co}}\left\{\left(-\frac{3}{2},1\right),\left(-\frac{1}{2},-1\right)\right\} & \text{at } (0,x_2),\ x_2 > 0 \\ \overline{\text{co}}\left\{\left(\frac{3}{2},-1\right),\left(\frac{1}{2},1\right)\right\} & \text{at } (0,x_2),\ x_2 < 0 \\ \overline{\text{co}}\left\{\left(-\frac{1}{2},-1\right),\left(\frac{1}{2},1\right)\left(-\frac{3}{2},1\right),\left(\frac{3}{2},-1\right)\right\} & \text{at } (0,0) \end{cases}$$

Let us now consider $V(x_1,x_2) = |x_1| + |x_2|$. In this case

$$\dot{\overline{V}}^{(2.6)}(x_1,x_2) = \begin{cases} \{-\frac{1}{2}\} & \text{at } (x_1,x_2),\ x_1 \neq 0 \text{ and } x_2 \neq 0 \\ \emptyset & \text{at } (x_1,0),\ x_1 \neq 0 \\ \emptyset & \text{at } (0,x_2),\ x_2 \neq 0 \\ \{0\} & \text{at } (0,0) \end{cases}$$

then V is a Lyapunov function for the system, that is stable at $x = 0$. Moreover $Z_V^{(2.6)} = \{(0,0)\}$, hence the solutions tend to $(0,0)$ as $t \to +\infty$ (see Fig.2). Let us remark that in this example Ryan's invariance principle doesn't help if we want to compute the limit set of the differential inclusion. In fact, if $x_2 > 0$, we have that $\max\{V^o((0,x_2),v),\ v \in F(0,x_2)\} = \frac{5}{2} > 0$.

Fig.2



Chapter 4

Asymptotic stabilization of control systems

We now turn our attention to control systems of the form

$$\begin{cases} \dot{x} = f(x,u) \\ x(t_0) = x_0 \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is locally essentially bounded and continuous with respect to u and $f(0,0) = 0$. The parameter u is said to be the *control*.

Definition 20 *System (4.1) is said to be (locally)* asymptotically stabilizable *at $x = 0$ if there exists $\delta > 0$ and a measurable function $u : \mathbb{R}^n \to \mathbb{R}^m$ (called* feedback law*) such that for any initial condition x_0 such that $\|x_0\| < \delta$ the system*

$$\begin{cases} \dot{x} = f(x,u(x)) \\ x(t_0) = x_0 \end{cases} \tag{4.2}$$

is asymptotically stable at $x = 0$.

Note that, from now on, if a system is discontinuous, solutions are intended in some generalized sense (see Chapter 1).

4.1 Asymptotic stabilizability and asymptotic controllability

The problem of asymptotic stabilizability is historically tied to the problem of asymptotic controllability to zero. In fact, for linear systems, these two concepts are equivalent.

Definition 21 *System (4.1) is said to be (locally)* asymptotically controllable to zero *if*

1) there exists $\eta > 0$ such that for all x_0 with $\|x_0\| < \eta$ there exists a control $u : \mathbb{R} \to \mathbb{R}^m$ such that for every solution $\varphi(t)$ of the system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(t_0) = x_0 \end{cases} \tag{4.3}$$

$\varphi(t) \to 0$ as $t \to +\infty$

2) for all $\epsilon > 0$ there exists $\delta > 0$, $(\delta \leq \eta)$, such that for all x_0 with $\|x_0\| < \delta$ there exists a control u as in 1) such that for every solution $\varphi(t)$ of (4.3) one has $\|\varphi(t)\| < \epsilon$ for all $t \geq t_0$.

It is evident that an asymptotically stabilizable system is also asymptotically controllable, but the converse is not obvious at all.

In an important paper Sussmann ([S]) shows an analytic system which is globally asymptotically controllable but not globally asymptotically stabilizable by means of a continuous (static) feedback law. More examples of controllable systems which can not be stabilized by means of continuous static feedback laws can be given by means of the following Brockett's condition ([Bro]). It is a topological necessary condition for a nonlinear smooth control system to be asymptotically stabilizable by means of a continuous (static) feedback law.

Theorem 8 *If f is locally Lipschitz continuous and the control system (4.1) can be (locally) asymptotically stabilized by means of a continuous (static) feedback law, then the image of any neighbourhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ is a neighbourhood of $0 \in \mathbb{R}^n$.*

From this result it arises the problem of stabilizing systems which don't admit continuous stabilizing feedback laws. Mainly two alternative ways can be pursued. The first one consists in introducing continuous time-varying feedback laws (see [C2] for an overview on this point of view), while the second one makes use of discontinuous feedback laws. We devote our attention to this second point of view.

With the introduction of discontinuous feedback laws two problems arise: one must choose which kind of discontinuities allow and then, according to that choice, an appropriate definition of solution.

Let us very briefly examine from this point of view some important papers.

In [CR], the authors consider an affine input system of the form

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{4.4}$$

where $f, g_1, ..., g_m$ are continuous vector fields of $\mathbb{R}^n$ and G is the matrix whose columns are $g_1, ..., g_m$. The feedback laws are taken to be such that $u \in L^{\infty}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ and

$$\text{esssup}\,\{\|u(x)\|,\ \|x\| < \epsilon\} \to 0 \text{ as } \epsilon \to 0 \tag{4.5}$$

This last condition can be seen as a sort of continuity of the feedback at the origin. Admitting this kind of feedback the most natural concept of solution is that of Filippov. In this context Coron and Rosier prove that the existence of a discontinuous feedback law implies the existence of a continuous one. Note that if (4.5) is not satisfied, Coron and Rosier's result does not hold anymore. This can be seen by means of the following example (see [R1]).

Example 11 Let us consider the scalar differential equation

$$\dot{x} = x + u|x| \tag{4.6}$$

where $x, u \in \mathbb{R}$. The feedback law $u(x) = -2\text{sgn}x$ asymptotically stabilizes it but it doesn't satisfy (4.5). Let us prove that there doesn't exist a continuous asymptotically stabilizing feedback by contradiction. Assume that $\tilde{u}(x)$ is a continuous asymptotically stabilizing feedback. Note that (1.2) is asymptotically stable at $x = 0$ if and only if for all $x \in \mathbb{R}\backslash\{0\}$ one has $xf(x) < 0$. This implies that
- if $x > 0$ then $xf(x) = x^2(1 + \tilde{u}(x)) < 0$ and $\tilde{u}(x) < -1$
- if $x > 0$ then $xf(x) = x^2(1 - \tilde{u}(x)) < 0$ and $\tilde{u}(x) > 1$
that is a contradiction to the continuity of $\tilde{u}$. Finally note that in this example Filippov solutions of the implemented system are simply classical solutions.

In [R1] essentially affine systems are considered and feedback laws are assumed to be upper semi-continuous multivalued maps with compact and convex values. Ryan proves that, if solutions are intended in the Filippov's or Krasovskii' s sense, then Brockett's topological necessary condition still holds.

The previous two papers suggest that Filippov and Krasovskii solutions are not the most adequate in order to prove that, for general nonlinear systems, asymptotic controllability implies asymptotic stabilizability. Actually this problem has been recently solved by means of different kinds of feedbacks and solutions.

In [CLSS] the authors solve the problem by considering locally bounded feedback laws and (not generalized) sampling solutions. A technique analogous to that used in [CLSS] is used by Rifford ([Ri]) for Euler solutions.

Finally, a totally different approach has been used by Ancona and Bressan ([AB]). They introduce a new class of piecewise smooth feedback laws, called patchy feedback, and consider Carathéodory solutions.

4.2 Discontinuous feedbacks: two examples

The previous paragraph should have motivated the use of discontinuous feedback laws, but, actually, there is still the problem of constructing them. In many papers (see, for example, [BD, CS, FM]) concrete strategies in order to stabilize class of systems which do not satisfy Brockett's condition are suggested. Moreover the fact that they work well is also proved by means of numerical experiments. Nevertheless it is not always clear in which sense solutions have to be considered. In particular, sometimes they are taken in the Filippov's sense, while some other times they seem to be thought in the Carathéodory's sense. We now try to illustrate the problem of choosing a good definition of solution by means of two examples. The first one is the classical example of the nonholonomic integrator ([Bro]). The feedback law we consider has been suggested by Bloch and Drakunov ([BD]).

Example 12 *Nonholonomic integrator.* Let us consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = v \\ \dot{x}_3 = x_1 v - x_2 u \end{cases} \tag{4.7}$$

This system does not satisfy Brockett's condition, then a continuous stabilizing feedback does not exist. Let $(u_0, v_0) \in \mathbb{R}^2 \backslash \{(0,0)\}$ be a fixed vector, α, β be positive constants and $\mathcal{P} = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{\beta}{2\alpha}(x_1^2 + x_2^2) < |x_3|\Big\}$.

We consider the feedback law

$$\begin{pmatrix} u(x_1,x_2,x_3) \\ v(x_1,x_2,x_3) \end{pmatrix} = \begin{cases} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} & \text{if } (x_1,x_2,x_3) \in \mathcal{P} \\ \begin{pmatrix} -\alpha x_1 + \beta x_2 \mathrm{sgn} x_3 \\ -\alpha x_2 - \beta x_1 \mathrm{sgn} x_3 \end{pmatrix} & \text{if } (x_1,x_2,x_3) \in \mathbb{R}^3 \backslash \mathcal{P} \end{cases} \tag{4.8}$$

Note that on the surfaces $x_3 = 0$ and $\partial\mathcal{P}$ the feedback is discontinuous. Let us denote $k_1(x_1,x_2,x_3) = (u_0, v_0, v_0x_1 - u_0x_2)^T$ and $k_2(x_1,x_2,x_3) = (-\alpha x_1 + \beta x_2 \mathrm{sgn} x_3, -\alpha x_2 - \beta x_1 \mathrm{sgn} x_3, -\beta x_1^2 \mathrm{sgn} x_3 - \beta x_2^2 \mathrm{sgn} x_3)^T$, i.e. k_1 and k_2 are the values of the implemented system respectively on $\mathcal{P}$ and on $\mathbb{R}^3 \backslash \mathcal{P}$. Let us remark that on the set $S = \partial\mathcal{P} \cap \Big\{(x_1,x_2,x_3) \in \mathbb{R}^3 :$ $(\alpha u_0 + \beta v_0)x_2 = (\alpha v_0 - \beta u_0)x_1,\ \mathrm{sgn} x_1 = \mathrm{sgn}(\alpha u_0 + \beta v_0)\}$ the vectors k_1 and k_2 are parallel and have opposite directions, then the righthand side of the implemented system is not patchy (in the sense of [AB]).

We briefly examine the behaviour of some of the different kinds of solutions of the implemented system that can be considered. All of Carathéodory and Euler solutions actually tend to the origin, but the same is not true for for Krasovskii and Filippov solutions. In fact the points of S are equilibrium points for the associated differential inclusions.

Note that, in this example, the value given to the feedback on the discontinuity surfaces is essential. In particular if, in a natural way, we define either $(u(x_1,x_2,0), v(x_1,x_2,0))^T = (-\alpha x_1 + \beta x_2, -\alpha x_2 - \beta x_1)^T$ or $(u(x_1,x_2,0), v(x_1,x_2,0))^T = (-\alpha x_1 - \beta x_2, -\alpha x_2 + \beta x_1)^T$ Carathéodory solutions of the implemented system do not exist for arbitrary initial conditions anymore.

In the following example the system considered is not stabilizable by means of a continuous feedback law even if Brockett's condition is satisfied (for a proof see [Ar] and also [S2]). We consider the stabilizing feedback law suggested in [CS].

Example 13 Let us consider the system

$$\begin{cases} \dot{x}_1 = (x_1^2 - x_2^2)u \\ \dot{x}_2 = 2x_1x_2u \end{cases} \tag{4.9}$$

The trajectories of the system when $u = 1$ are shown in Fig.3.

Let us introduce the arc length of the circles passing through the points (x_1, x_2) and $(0,0)$ and with the center on the $x_2 - axis$:

$$a(x_1,x_2) = \begin{cases} x_1 & \text{if } x_2 = 0 \\ \frac{x_1^2+x_2^2}{x_2} \arctan \frac{x_2}{x_1} & \text{if } x_2 \neq 0 \end{cases} \tag{4.10}$$

We define the feedback law $u(x_1, x_2) = -ka(x_1, x_2)$, where k is a positive constant. The x_2-axis is a discontinuity line for the feedback, in fact, for $x_2 \neq 0$, we have that $\lim_{x_1 \to 0^+} u(x_1, x_2) = k\frac{\pi}{2}x_2$ and $\lim_{x_1 \to 0^-} u(x_1, x_2) = -k\frac{\pi}{2}x_2$. From this fact it immediatelly follows that the points of the x_2-axis are equilibrium points for the associated Krasovskii and Filippov differential inclusions, and then not all of Krasovskii and Filippov solutions of the implemented system converge to the origin.

Note that if we posit either $u(0, x_2) = k\frac{\pi}{2}x_2$ or $u(0, x_2) = -k\frac{\pi}{2}x_2$, the feedback is not patchy. Nevertheless Carathéodory solutions do exist: the trajectories of the system are either the right or the left half circles (see Fig. 4). The same happens if, instead of Carathéodory solutions, we consider Euler solutions.

Intuitively, when initial conditions are taken on the x_2-axis, it would be more desirable to have both the right and the left half circles as trajectories of the system. It is possible to get them by considering Euler externally disturbed solutions.

Fig.3

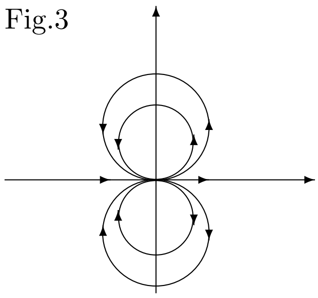
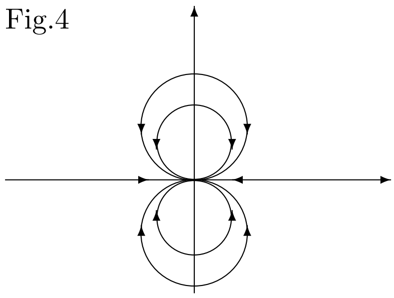


Fig.4



4.3 Discontinuous damping feedback

In this paragraph we see an example of discontinuous feedback laws which stabilize a wide class of systems.

We study stabilization of autonomous systems affine in the control by means of discontinuous damping feedbacks. Let us first go back to smooth systems for a while.

In one of the first papers devoted to nonlinear feedback stabilization, Jurdjevic and Quinn used the idea introduced in [J] that the stability properties of the affine system (4.4) can be enhanced by setting

$$u = u(x) = -\alpha(\nabla V(x)G(x))^T \tag{4.11}$$

where V is a Lyapunov function for the unforced system (1.2), the row vector $\nabla V(x)$ denotes its gradient and α is a positive real parameter (see [JQ]; see also [B1] for subsequent developments and improvements). More precisely, assume that

(A) the origin is Lyapunov stable for (1.2) and a positive definite Lyapunov function $V \in C^1$ such that $\dot{V}$ is negative semi-definite is known;

(B) an additional condition, involving Lie brackets of the vector fields $f, g_1, ..., g_m$, holds.

Then, Jurdjevic and Quinn proved that (4.4) can be asymptotically stabilized by means of the feedback law (4.11).

Although it has been largely and successfully exploited in the literature both from a practical and a theoretical point of view, a weakness of the method related to assumption (A) should be pointed out. Indeed, we have already remarked that, even for smooth f, Lyapunov stability does not imply in general the existence of a (not even) continuous Lyapunov function.

When it is known that the unforced system is stable but the existence of a C^1 Lyapunov function cannot be guaranteed, two alternative ways can be pursued:

1) to introduce time dependent Lyapunov functions. In this case the Jurdjevic and Quinn method can be extended (see [MT]) but it gives rise, of course, to a time dependent feedback;

2) to replace the (classical) gradient in (4.11) by some type of generalized gradient. This in general leads to discontinuous feedback, so that we have to choose in which sense solutions of the discontinuous differential equation involved have to be interpreted.

We devote our attention to the second point of view.

4.3.1 Filippov solutions of the closed loop system

The main assumption we make in the following is that we know a Lipschitz continuous and healthy Lyapunov function for the unforced system (1.2). In general, this implies that the origin is a stable equilibrium point for system (1.2). Moreover, if G is continuous, the feedback law (4.11) is defined a.e. and it is locally essentially bounded and measurable.

In fact, if L_x is the Lipschitz constant of V in a compact neighbourhood U_x of x,

$$\|u(x)\| \leq \alpha \|\nabla V(x)\| \|G(x)\| \leq \alpha L_x \|G(x)\| \text{ a.e. in } U_x,$$

that is bounded in U_x because G is continuous. As already remarked, for every $v \in \mathbb{R}^n$, $\nabla V(\cdot) \cdot v$ is measurable; hence u is also measurable. On the other hand note that, in general, u does not satisfy (4.5).

From this fact it follows that the right hand-side of the equation

$$\dot{x} = f(x) - \alpha G(x)(\nabla V(x) G(x))^T \tag{4.12}$$

is also locally essentially bounded and measurable on $\mathbb{R}^n$.

Among the various solutions introduced in Chapter 1, the most adeguate to this context seem then to be Filippov solutions. We make the following assumptions:

(f0) $f \in L^\infty_{loc}(\mathbb{R}^n; \mathbb{R}^n)$, $0 \in Ff(0)$;

(G0) $G \in C(\mathbb{R}^n; \mathbb{R}^{n \times m})$;

(V0) $V : \mathbb{R}^n \to \mathbb{R}$ is a positive definite, locally Lipschitz continuous and healthy function.

In particular, note that in general we don't assume f to be continuous.

In the smooth case, the proof that (4.11) stabilizes system (4.4) is divided into two steps. First one proves that the stability property of system (4.4) is not affected by the application of the feedback (4.11). After that, by means of LaSalle's principle, it is proved that solutions actually tend to the origin. In the particular case the function f is continuous the first step still holds, as the following proposition shows.

Proposition 9 *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $V : \mathbb{R}^n \to R$ is a Lyapunov function for the unforced system (1.2) and (V0) and (G0) hold, then system (4.2) is stable at $x = 0$.*

Before proving the proposition let us recall some results that simplify the calculation of Filippov's multivalued maps (see [PS]).

Proposition 10 *(i) If $f \in C(\mathbb{R}^n; \mathbb{R}^n)$ then $Ff(x) = \{f(x)\}$ for all $x \in \mathbb{R}^n$.*

(ii) *If* $f, g \in L^\infty_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ *then* $F(f+g)(x) \subseteq Ff(x) + Fg(x)$ *for all* $x \in \mathbb{R}^n$.

Moreover if $f \in C(\mathbb{R}^n;\mathbb{R}^n)$ *then* $F(f+g)(x) = f(x) + Fg(x)$ *for all* $x \in \mathbb{R}^n$.

(iii) *If* $G \in C(\mathbb{R}^n;\mathbb{R}^{n\times m})$, $u \in L^\infty_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ *then* $F(Gu)(x) = G(x)Fu(x)$ *for all* $x \in \mathbb{R}^n$.

(iv) *If* $V : \mathbb{R}^n \to \mathbb{R}$ *is locally Lipschitz continuous, then* $F(\nabla V)(x) = \partial_C V(x)$ *for all* $x \in \mathbb{R}^n$.

Proof of Proposition 9 Let us prove that $\max \overline{\dot{V}}^{(4.12)} \leq 0$. Indeed from this, by Corollary 1 in Chapter 3, it follows the thesis. Let $a \in \overline{\dot{V}}^{(4.12)}(x)$. By Proposition 10, we have that $F(f - \alpha G(\nabla V G)^T)(x) = f(x) - \alpha G(x)(\partial_C V(x) G(x))^T$, then there exists $q \in \partial_C V(x)$ such that $p \cdot v = p \cdot f(x) - \alpha(pG(x)) \cdot (q \cdot G(x))$ for all $p \in \partial_C V(x)$. In particular, for $p = q$ we get that $a = q \cdot f(x) - \alpha \|qG(x)\|^2$. Since by Proposition 6 in Chapter 2 $p \cdot f(x) \leq 0$ for all $p \in \partial_c V(x)$, we get that $a \leq 0$. $\Box$

If f is not continuous the Proposition 9 fails to be true. This is proved by means of the following example.

Example 14 Let us consider a single-input system of the form (4.4) in $\mathbb{R}^2$, where

$$f(x_1,x_2) = \begin{cases} (\mathrm{sgn} x_1,\ -2)^T & \text{at } (x_1,x_2),\ x_2 \geq 0 \\ (0,0)^T & \text{at } (x_1,x_2),\ x_2 < 0 \end{cases}$$

$G(x_1,x_2) = (0,1)^T$, and the function $V(x_1,x_2) = |x_1| + |x_2|$. By computing $\overline{\dot{V}}^{(1.2)}(x_1,x_2)$, it is easily proved that system (3) is stable at $x = 0$ (see Fig. 5).

Let us now consider system (4.12).

$$F(f - \alpha G(\nabla V G)^T)(x_1,x_2) =$$

$$= \begin{cases} \{\mathrm{sgn} x_1\} \times \{-2-\alpha\} & \text{at } (x_1,x_2),\ x_1 \neq 0,\ x_2 > 0 \\ \{0\} \times \{\alpha\} & \text{at } (x_1,x_2),\ x_2 < 0 \\ [-1,1] \times \{-2-\alpha\} & \text{at } (0,x_2),\ x_2 > 0 \\ \overline{\mathrm{co}}\{(\mathrm{sgn} x_1, -2-\alpha)^T, (0,\alpha)^T\} & \text{at } (x_1,0),\ x_1 \neq 0 \\ \overline{\mathrm{co}}\{(1,-2-\alpha)^T, (0,\alpha)^T, (-1,-2-\alpha)^T\} & \text{at } (0,0) \end{cases}$$

Let us remark that for all $\alpha > 0$ and for all the points $(x_1, 0)$ with $x_1 \neq 0$, there exists a trajectory starting from $(x_1, 0)$ which lies on the x_1-axis and goes to infinity. This is obtained by considering the vector $\left(\frac{\alpha}{2(1+\alpha)}, 0\right) \in F(f - \alpha G(\nabla V G))(x_1, 0)$ if $x_1 > 0$, and the vector $\left(-\frac{\alpha}{2(1+\alpha)}, 0\right) \in F(f - \alpha G(\nabla V G))(x_1, 0)$ if $x_1 < 0$ (see Fig.6 in the case $\alpha = 1$).

Fig.5

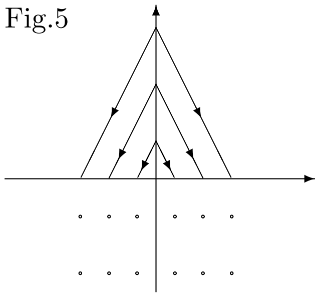
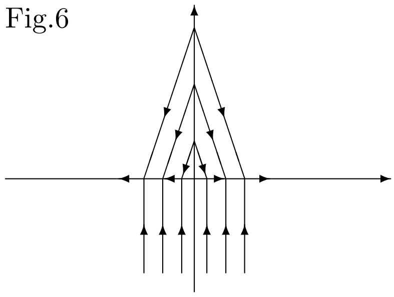


Fig.6



As the previous example shows, in the nonsmooth case, in order to guarentee the conservation of stability for the closed loop system, we need to add some extra assumptions. Actually, we do not present a unique condition, but we list some alternative conditions which, combined together in a convenient way, allow us to get not only the stability of system (4.12), but also the stabilizability of system (4.4). Note that in these conditions the variable x is not yet quantified. Since the role of x will depend on the circumstances, it is convenient to specify it later. The possible conditions are the following:

(f1) $\max \overline{\dot{V}}^{(1.2)}(x) \leq 0$;

(f2) for all $v \in Ff(x)$ there exists $p \in \partial_C V(x)$ such that $p \cdot v \leq 0$;

(f3) for all $v \in Ff(x)$ and for all $p \in \partial_C V(x)$, $p \cdot v \leq 0$;

(G1) there exists $c \in \mathbb{R}$ such that for all $p, q \in \partial_C V(x)$, $(pG(x)) \cdot (qG(x)) = c^2$ (c may depend on x);

(G2) either $(pG(x)) \cdot (qG(x)) > 0$ for all $p, q \in \partial_C V(x)$, or $(pG(x)) \cdot (qG(x)) = 0$ for all $p, q \in \partial_C V(x)$;

(G3) $(pG(x)) \cdot (qG(x)) \geq 0$ for all $p, q \in \partial_C V(x)$;

(fG1) there exists $\alpha > 0$ such that for all $v \in Ff(x)$ and for all $q \in \partial_C V(x)$ there exist $p_1, p_2 \in \partial_C V(x)$ such that

$$(p_1 - p_2) \cdot (v - \alpha G(x)(qG(x))^T) \neq 0.$$

By definition of $\dot{\overline{V}}^{(1.2)}(x)$, (f1) can be restated by saying that if there exists $v \in Ff(x)$ such that for all $p \in \partial_C V(x)$ one has $p \cdot v = a$, then $a \leq 0$. Conditions (f1),(f2) and (f3) can then be seen as geometric conditions on mutual positions of the sets $Ff(x)$ and $\partial_C V(x)$. Moreover we have that (f3) $\Rightarrow$ (f2) $\Rightarrow$ (f1) and all of these conditions imply that V is a Lyapunov function for the unforced system. Note that if f is continuous, (f3) is equivalent to the fact that V is a Lyapunov function for the unforced system (see Proposition 6, Chapter 2).

In order to interpretate conditions (G1), (G2) and (G3), let us consider the set $H(x) = \{pG(x),\ p \in \partial_C V(x)\}$. (G1) implies that $H(x)$ reduces to a single vector, while (G2) and (G3) are conditions on the size of $H(x)$. For these conditions it holds that (G1) $\Rightarrow$ (G2) $\Rightarrow$ (G3).

The meaning of condition (fG1) is explained by the following lemma.

Lemma 2 *Assume that conditions* (f0), (G0) *and* (V0) *hold for some* $x \in \mathbb{R}^n$. *There exists* $\alpha = \alpha(x) > 0$ *such that condition* (fG1) *holds if and only if* $\dot{\overline{V}}^{(4.12)}(x) = \emptyset$.

Proof We prove the statement by contradiction.

Let us suppose that for all $\alpha > 0$ one has $\dot{\overline{V}}^{(4.12)}(x) \neq \emptyset$. Then there exist $a \in \mathbb{R}$, $w \in F(f - \alpha G(\nabla V G)^T)(x)$ such that, for all $p \in \partial_C V(x)$, $p \cdot w = a$. By (ii), (iii) and (iv) in Proposition 10 it follows that there exist $v \in Ff(x)$ and $q \in \partial_C V(x)$ such that for all $p \in \partial_C V(x)$, $p \cdot (v - \alpha G(x)G(x)^T q) = a$. Let $p_1, p_2 \in \partial_C V(x)$. We have $p_1 \cdot (v - \alpha G(x)G(x)^T q) = p_2 \cdot (v - \alpha G(x)G(x)^T q) = a$, hence $(p_1 - p_2) \cdot (v - \alpha G(x)G(x)^T q) = 0$, which is a contradiction to (fG1).

The viceversa is easily proved by contradiction. $\square$

4.3.2 Conservation of stability

From the previous discussion it follows that, in order to prove a stabilization result for system (4.4) by means of the feedback law (4.11), the first step is to give some sufficient conditions for system (4.12) being stable. We do that in the following lemma.

Lemma 3 *Let us assume that* (f0), (G0), (V0) *hold and* (f1) *holds for all* $x \in \mathbb{R}^n \backslash N$*, where*

$$N = \{x \in \mathbb{R}^n \ \ \textit{such that } V \textit{ is not differentiable at } x\}.$$

Let us suppose further that for each $x \in N$ *one of the following combinations of conditions holds: (i)* (f1) *and* (G1)*, (ii)* (fG1) *for some* α *independent of* x*, (iii)* (f2) *and* (G3)*, (iv)* (f3)*.*

Then for each $x \in \mathbb{R}^n$*,* $\max \overset{\cdot}{\overline{V}}^{(4.12)}(x) \leq 0$.

Moreover, if the use of (fG1) *can be avoided, the choice of* α *can be arbitrary.*

Proof Let $a \in \overset{\cdot}{\overline{V}}^{(4.12)}(x)$. Then there exists $w \in F(f - \alpha G(\nabla V G)^T)(x)$ such that, for all $p \in \partial_C V(x)$, $p \cdot w = a$. From (ii), (iii) and (iv) in Proposition 10 it follows that there exist $v \in Ff(x)$ and $q \in \partial_C V(x)$ such that $w = v - \alpha G(x)(qG(x))^T$. In the following we will use this representation for w without mentioning it explicitly.

We distinguish five cases: (o) for $x \in \mathbb{R}^n \backslash N$ and (i), (ii), (iii), (iv) for $x \in N$.

(o) In this case $\partial_C V(x) = \{\nabla V(x)\}$, then $a = \nabla V(x) \cdot w = \nabla V(x) \cdot (v - \alpha G(x)(\nabla V(x) G(x))^T)$ and $\nabla V(x) \cdot v = a + \alpha \|(\nabla V(x) G(x))^T\|^2 = b$, where $b \in \overset{\cdot}{\overline{V}}^{(1.2)}(x)$. Since by assumption $\max \overset{\cdot}{\overline{V}}^{(1.2)}(x) \leq 0$, we also have that $b \leq 0$, hence $a = b - \alpha \|(\nabla V(x) G(x))^T\|^2 \leq 0$.

(i) In this case $a = p \cdot w = p \cdot v - \alpha (pG(x))(qG(x))^T = p \cdot v - \alpha c^2$ for each $p \in \partial_C V(x)$. Hence the proof that $\max \overset{\cdot}{\overline{V}}^{(4.12)}(x) \leq 0$ is analogous the one in (o).

(ii) From assumption (fG1) and Lemma 2 it follows that $\overset{\cdot}{\overline{V}}^{(4.12)}(x) = \emptyset$ for suitable choice of α.

(iii) Since (f2) implies (f1), clearly it is sufficient to prove that for all $w \in F(f + Gu)(x)$ there exists $p \in \partial_C V(x)$ such that $p \cdot w \leq 0$. Let $p \in \partial_C V(x)$ such that $p \cdot v \leq 0$ (such a p exists because of (f2)). By (G3) we get $a = p \cdot w = p \cdot v - \alpha (pG(x))^T \cdot (qG(x))^T \leq 0$, as required.

(iv) For all $p \in \partial_C V(x)$, $a = p \cdot w = p \cdot v - \alpha (pG(x))^T \cdot (qG(x))^T$. In particular, for $p = q$ we get $a = q \cdot w = q \cdot v - \alpha \|(qG(x))^T\|^2$ that is non-positive because of (f3). $\square$

From the previous lemma and Corollary 1 it follows that system (4.12) is stable at $x = 0$.

4.3.3 Improvement of stability

In order to study asymptotic stabilization of system (4.4) let us introduce the sets

$$Z_V^{(4.12)} = \{x \in R^n : 0 \in \overline{\dot{V}}^{(4.12)}(x)\}$$

and

$$Z_V^{(1.2)} = \{x \in R^n : 0 \in \overline{\dot{V}}^{(1.2)}(x)\}.$$

Let us recall that, if the connected component L_l of the level set $\{x \in \mathbb{R}^n : V(x) \leq l\}$ such that $0 \in L_l$ is bounded, by the invariance theorem stated in Chapter 3, the solutions of systems (1.2) and (4.12), with initial condition $x_0 \in L_l$, respectively tend to $\overline{Z_V^{(1.2)}} \cap L_l$ and $\overline{Z_V^{(4.12)}} \cap L_l$.

Lemma 4 *Let us assume that* (f0), (G0), (V0) *hold and that* (f1) *holds for all* $x \in \mathbb{R}^n \backslash N$. *Let us suppose that for each* $x \in N$ *one of the following pairs of conditions holds: (i)* (f1) *and* (G1), *(ii)* (f1) *and* (fG1) *for some* α *independent of* x, *(iii)* (f2) *and* (G2), *(iv)* (f3) *and* (G3).

Then $Z_V^{(4.12)} \subseteq Z_V^{(1.2)}$.

Proof $x \in Z_V^{(4.12)}$ means that there exists $w \in F(f - \alpha G(\nabla V G)^T)(x)$ such that, for all $p \in \partial_C V(x)$, $p \cdot w = 0$. Using the decomposition of w already mentioned in the proof of Lemma 3, we get that there exist $v \in Ff(x)$ and $q \in \partial_C V(x)$ such that $p \cdot w = p \cdot v - \alpha(pG(x))^T \cdot (qG(x))^T = 0$, i.e. $p \cdot v = \alpha(pG(x))^T \cdot (qG(x))^T$. Again, we distinguish five cases: (o) $x \in \mathbb{R}^n \backslash N$, (i), (ii), (iii), (iv).

(o) In this case $\partial_C V(x) = \{\nabla V(x)\}$ then $\nabla V(x) \cdot v = \alpha \|(\nabla V(x) G(x))^T\|^2 = b \geq 0$. On the other hand, since $b \in \overline{\dot{V}}^{(1.2)}(x)$ and $\max \overline{\dot{V}}^{(1.2)}(x) \leq 0$, $b \leq 0$, hence $b = 0$, i.e. there exists $v \in Ff(x)$ such that $\nabla V(x) \cdot v = 0$ and $x \in Z_V^{(1.2)}$.

(i) The proof is analogous to the one in (o).

(ii) By Lemma 2, $\overline{\dot{V}}^{(4.12)}(x) = \emptyset$, so that $0 \notin \overline{\dot{V}}^{(4.12)}(x)$ and $x \notin Z_V^{(4.12)}$.

(iii) $p \cdot v = \alpha(pG(x))^T \cdot (qG(x))^T$ implies that x is such that for all $p, q \in \partial_C V(x)$, $(pG(x))^T \cdot (qG(x))^T = 0$, otherwise for all $p \in \partial_C V(x)$ one has $p \cdot w > 0$, which contradicts (f2). We conclude that, for all $p \in \partial_C V(x)$, $p \cdot v = 0$, i.e. $x \in Z_V^{(4.12)}$.

(iv) $p \cdot v = \alpha(pG(x))^T \cdot (qG(x))^T \geq 0$ because of (G3). On the other hand, by condition (f3), for all $p \in \partial_C V(x)$ we have $p \cdot v \leq 0$, hence, for all $p \in \partial_C V(x)$, $p \cdot v = 0$, i.e. $x \in Z_V^{(3)}$. $\square$

We can finally summarize the results of the present section in the following theorem.

Theorem 9 *Let us assume that* (f0), (G0) *and* (V0) *hold and* (f1) *holds for all* $x \in \mathbb{R}^n \backslash N$. *If* N *can be decomposed as a union* $N = N_{11} \cup N_{12} \cup N_2 \cup N_3$ *such that*

(i) for all $x \in N_{11} \cup N_{12}$ (f1) *holds; for all* $x \in N_{11} \backslash \{0\}$, (G1) *holds and for all* $x \in N_{12} \backslash \{0\}$, (fG1) *holds with* α *independent of* x*;*

(ii) for all $x \in N_2$, (f2) *holds and for all* $x \in N_2 \backslash \{0\}$, (G2) *holds;*

(iii) for all $x \in N_3$, (f3) *holds and for all* $x \in N_3 \backslash \{0\}$, (G3) *holds.*

Then, there exists $\alpha > 0$ *such that*

(A) (4.12) *is stable at* $x = 0$,

(B) $Z_V^{(4.12)} \subseteq Z_V^{(1.2)}$.

Moreover let us assume that

(V1) *there exists* $l > 0$ *such that the connected component* L_l *of the level set* $\{x \in \mathbb{R}^n : V(x) \leq l\}$ *such that* $0 \in L_l$ *is bounded,*

(fG2) *the largest weakly invariant subset of* $\overline{Z_V^{(4.12)}} \cap L_l$ *is* $\{0\}$.

Then

(C) (4.4) *is asymptotically stabilizable by means of the feedback law* (4.11).

Finally, if the use of (fG1) *can be avoided, the choice of* α *is arbitrary.*

Corollary 2 *Let us assume that* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and that, if there exists* $p \in \partial_C V(x)$ *such that* $\|pG(x)\| = 0$, *then* $x = 0$. *Then system (4.4) is asymptotically stabilizable by means of the feedback law (4.11).*

Remark 15 If $V \in C^1$ then $N = \emptyset$, hence we only need to check condition (f1) in order to get the stability of (4.12), and conditions (V1) and (fG2) to get the asymptotic stabilization of (4.4), i.e. we have a classical-like stabilization theorem that can be applied in the case the only assumptions on f are measurability and local boundedness.

4.3.4 Examples

In the present subsection we illustrate the various situations described in Theorem 9 by means of some examples.

Example 15 Let us consider a system of the form (4.4) in $\mathbb{R}^2$, where $f(x_1, x_2) = (-\mathrm{sgn} x_2, \, \mathrm{sgn} x_1)^T$ and $G(x_1, x_2) = (x_1, x_2)^T$, and the function $V(x_1, x_2) = |x_1| + |x_2|$.

As shown in Example 6, for all $(x_1, x_2) \in \mathbb{R}^2$ we have $\max \overline{\dot{V}}^{(1.2)}(x_1, x_2) \le 0$. $N = \{(x_1, 0), x_1 \in \mathbb{R}\} \cup \{(0, x_2), x_2 \in \mathbb{R}\}$. Let us consider $p = (p_1, p_2)$ and $q = (q_1, q_2) \in \partial_C V(x_1, x_2)$

$$(pG(x_1, x_2)) \cdot (qG(x_1, x_2)) = \begin{cases} x_1^2 & (x_1, 0), \ x_1 \neq 0 \\ x_2^2 & (0, x_2), \ x_2 \neq 0 \end{cases}$$

i.e. condition (G1) is verified for all $(x_1, x_2) \in N$. Then, by (A) in Theorem 9, for all $\alpha > 0$ system (4.4) with the feedback (4.11) is stable at $x = 0$.

Moreover let us consider the set

$$\overline{\overline{\dot{V}}}^{(4.12)}(x_1, x_2) = \{a \in \mathbb{R} : \ \exists v \in Ff(x_1, x_2) \ \exists q \in \partial_C V(x_1, x_2) \text{ such that}$$
$$\forall p \in \partial_C V(x_1, x_2), \quad p \cdot (v - \alpha G(x_1, x_2)((qG(x_1, x_2))^T) = a\}$$

$$\overline{\overline{\dot{V}}}^{(4.12)}(x_1, x_2) = \begin{cases} -\alpha(|x_1| + |x_2|)^2 & \text{at } (x_1, x_2), \ x_1 \neq 0 \text{ and } x_2 \neq 0 \\ \emptyset & \text{at } (x_1, 0), \ x_1 \neq 0 \\ \emptyset & \text{at } (0, x_2), \ x_2 \neq 0 \\ \{0\} & \text{at } (0, 0). \end{cases}$$

From Proposition 10 it follows that for all $(x_1, x_2) \in \mathbb{R}^2$, $\overline{\dot{V}}^{(4.12)}(x_1, x_2) \subseteq \overline{\overline{\dot{V}}}^{(4.12)}(x_1, x_2)$, hence $Z_V^{(4.12)} = \{(0, 0)\}$ and, by (C) in Theorem 9, the system is asymptotically stabilizable by means of the feedback law (4.11).

Example 16 Let us consider a system of the form (4.4) in $\mathbb{R}^2$, where $f(x_1, x_2) = (-\mathrm{sgn} x_2, \, \mathrm{sgn} x_1)^T$ and $G(x_1, x_2) = (1, 0)^T$, and the function $V(x_1, x_2) = |x_1| + |x_2|$.

As shown in Example 6, for all $(x_1, x_2) \in \mathbb{R}^2$ one has $\max \overline{\dot{V}}^{(1.2)}(x_1, x_2) \le 0$. Also in this case $N = \{(x_1, 0), x_1 \in \mathbb{R}\} \cup \{(0, x_2), x_2 \in \mathbb{R}\}$. Condition (G1) is verified on $\{(x_1, 0), x_1 \in \mathbb{R}\}$ but not on $\{(0, x_2), x_2 \in \mathbb{R}\}$. Nevertheless, for $\alpha \in (0, 1)$, condition (fG1) is verified on $\{(0, x_2), x_2 \in \mathbb{R}\}$, in fact

$$\{v - \alpha G(0, x_2)(qG(0, x_2))^T, \text{ for } v \in Ff(0, x_2), \ q \in \partial_C V(0, x_2), x_2 \neq 0\} =$$

$$= \begin{cases} [-1-\alpha, -1+\alpha] \times [-1,1] & \text{at } (0,x_2), x_2 > 0 \\ [1-\alpha, 1+\alpha] \times [-1,1] & \text{at } (0,x_2), x_2 < 0 \end{cases}$$

and

$$\{(p_1 - p_2);\ p_1, p_2 \in \partial_C V(0,x_2), x_2 \neq 0\} = ([-2,2],0)^T.$$

By (A) in Theorem 9, it follows that system (4.12) is stable at $x = 0$ with $\alpha \in (0,1)$. Moreover computations analogous to those of the previous example show that $Z_V^{(4.12)} = \{(0,0)\}$. Hence, by (C) in Theorem 9, the system is asymptotically stabilizable by means of the feedback law (4.11) with a fixed $\alpha \in (0,1)$. Example 10 is actually a particular case of the present example, with $\alpha = \frac{1}{2}$. Fig.1 and Fig.2 show the behaviour of the system before and after the application of the feedback.

Remark 16 By direct computation, it is possible to see that the closed loop system considered in the previous example is actually stable for all $\alpha > 0$. However, for $\alpha > 1$, no one of the alternative conditions of Theorem 9 can be applied. This shows that Theorem 9 does not cover all the possible cases.

Example 17 Let us consider a system of the form (4.12) in $\mathbb{R}^2$, where $f(x_1,x_2) = (-\mathrm{sgn}x_2, \mathrm{sgn}x_1)^T$ and $G(x_1,x_2) = \left(x_1 + \frac{1}{2}x_2, x_2 + \frac{1}{2}x_1\right)^T$, and the function $V(x_1,x_2) = |x_1| + |x_2|$.

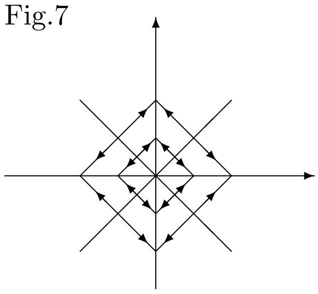
As shown in Example 6, for all $(x_1,x_2) \in \mathbb{R}^2$ we have $\max \overline{\dot{V}}^{(1.2)}(x_1,x_2) \leq 0$, so that, also in this case, $N = \{(x_1,0), x_1 \in \mathbb{R}\} \cup \{(0,x_2), x_2 \in \mathbb{R}\}$. On N condition (f2) is satisfied. Moreover for all $(x_1,x_2) \in N$ and for all $p, q \in \partial_C V(x_1,x_2)$ we have $(pG(x)) \cdot (qG(x)) > 0$, i.e. condition (G2) is satisfied in N. By (A) in Theorem 9, it follows that system (4.12) is stable at $x = 0$ for all $\alpha > 0$. Moreover $Z_V^{(4.12)} = \{(0,0)\}$, hence, by (C) in Theorem 9, the system is asymptotically stabilizable by means of the feedback law (4.11) for all $\alpha > 0$.

Example 18 Let us consider a system of the form (1.2) in $\mathbb{R}^2$, where

$$f(x_1,x_2) = \begin{cases} (x_2, -x_2)^T & \text{at } (x_1,x_2),\ 0 \leq x_2 \leq x_1 \\ (-x_1, x_1)^T & \text{at } (x_1,x_2),\ 0 \leq x_1 \leq x_2 \\ (-x_1, -x_1)^T & \text{at } (x_1,x_2),\ 0 \leq -x_1 \leq x_2 \\ (-x_2, -x_2)^T & \text{at } (x_1,x_2),\ 0 \leq x_2 \leq -x_1 \\ (x_2, -x_2)^T & \text{at } (x_1,x_2),\ x_1 \leq x_2 \leq 0 \\ (-x_1, x_1)^T & \text{at } (x_1,x_2),\ x_2 \leq x_1 \leq 0 \\ (-x_1, -x_1)^T & \text{at } (x_1,x_2),\ x_2 \leq -x_1 \leq 0 \\ (-x_2, -x_2)^T & \text{at } (x_1,x_2),\ -x_1 \leq x_2 \leq 0 \end{cases}$$

and $G(x_1,x_2) = (x_1+x_2, x_1+x_2)^T$, and the function $V(x_1,x_2) = |x_1|+|x_2|$. By computing $Ff(x_1,x_2)$, it is easy to see that (f3) is verified, then (1.2) is stable at $x=0$ (see Fig.7). Since condition (G3) is satisfied on N (note that (G2) is not satisfied on N), then not only system (4.12) is stable at $x=0$, but also $Z_V^{(4.12)} \subseteq Z_V^{(1.2)}$. Actually in this case it can be shown that the feedback law (4.11) does not stabilize system (4.12) asymptotically.

Fig.7



4.3.5 Krasovskii solutions of the closed loop system

In the present section we briefly investigate the effect of the Jurdjevic and Quinn's feedback on the affine system in the case its solutions are intended in the Krasovskii's sense. Note that, thanks to the definition of Filippov solutions, in the previous paragraphs it was not important to specify explicitly the values taken by the righthand side of (4.12) on the subset N of $\mathbb{R}^n$ where $\nabla V(x)$ does not exist. Now, in order to consider Krasovskii solutions, we can't anymore ignore the values of the righthand side of (4.12) on zero measure sets, then we define the feedback law in a slightly different way.

Let

$$\tilde{\nabla} V(x) = \begin{cases} \nabla V(x) & \text{if } x \in \mathbb{R}^n \backslash N \\ \tilde{p} & \text{where } \tilde{p} \text{ is any fixed vector in } \partial_C V(x), \text{ if } x \in N \end{cases} \tag{4.13}$$

We define

$$u(x) = -\alpha(\tilde{\nabla} V(x) G(x))^T \tag{4.14}$$

The righthand side of (4.12) is locally bounded and it makes sense to consider its Krasovskii solutions.

The essential tool in order to deal with Krasovskii solutions and make explicit computations is the analogous of Proposition 10.

Proposition 11 *(i) If $f \in C(\mathbb{R}^n;\mathbb{R}^n)$ then $Kf(x) = \{f(x)\}$ for all $x \in \mathbb{R}^n$.*

(ii) If f, g are locally bounded then $K(f+g)(x) \subseteq Kf(x)+Kg(x)$ for all $x \in \mathbb{R}^n$.

Moreover if $f \in C(\mathbb{R}^n;\mathbb{R}^n)$ then $K(f+g)(x) = f(x)+Kg(x)$ for all $x \in \mathbb{R}^n$.

(iii) If $G \in C(\mathbb{R}^n;\mathbb{R}^{n\times m})$ and u is locally bounded then $K(Gu)(x) = G(x)Ku(x)$ for all $x \in \mathbb{R}^n$.

(iv) If $V : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous, then $K(\tilde{\nabla}V)(x) = \partial_C V(x)$ for all $x \in \mathbb{R}^n$.

The proof of (i), (ii) and (iii) is perfectly analogous to that of Paden and Sastry (see [PS]), while the proof of (iv) needs some extra remarks.

Proof of (iv) Let us first remark that from the definition of the multivalued map Kf it follows that, in general, $Kf(x) = \mathrm{co}\{\lim_i f(x_i),\, x_i \to x\}$.

Let us prove that $\partial_C V(x) \subseteq K(\tilde{\nabla}V(x))$. Let $p \in \partial_C V(x)$. Because of (2.1) and there exist m sequences $\{x_i^{(k)}\}$, $x_i^{(k)} \to x$ as $i \to \infty$, $x_i^{(k)} \notin N$, $k = 1, ..., m$ and m scalars $\lambda_k > 0$ with $\sum_{k=1}^m \lambda_k = 1$ such that $p = \sum_{k=1}^m \lambda_k \lim_i \nabla V(x_i^{(k)})$. Since $\nabla V(x) = \tilde{\nabla}V(x)$ on N, we have that $p = \sum_{k=1}^m \lambda_k \lim_i \tilde{\nabla}V(x_i^{(k)})$, and then $p \in K(\tilde{\nabla}V(x))$.

Let us now prove that $K(\tilde{\nabla}V(x)) \subseteq \partial_C V(x)$. $p \in K(\tilde{\nabla}V(x))$ can be written as $p = \sum_{k=1}^m \lambda_k p_k$, where, for all $k = 1, ..., m$, $p_k = \lim_i \tilde{\nabla}V(x_i^{(k)})$, $\lambda_k > 0$, $\sum_{k=1}^m \lambda_k = 1$ and $x_i^{(k)} \to x$ as $i \to \infty$.

Let us emphazise that $\tilde{\nabla}V(x_i^{(k)}) \in \partial_C V(x_i^{(k)})$ for all k, i. Since $\partial_C V$, as a multivalued map from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}\backslash\emptyset$, is upper semi-continuous then its graph is closed and $p_k \in \partial_C V(x)$ for all k.

From the convexity of $\partial_C V(x)$ it finally follows that $p \in \partial_C V(x)$. $\square$

Thanks to the previous proposition one can get stabilizations results perfectly analogous to those obtained in the context of Filippov solutions, simply by replacing Filippov's multivalued maps with Krasovskii's ones. In particular, also when Krasovskii solutions are considered, the damping feedback may destabilize the system. This can be still proven by means of Example 14 and slightly different computations.

Chapter 5

External Stabilization

In the present chapter we apply the technique we have used for the stabilization of discontinuous systems affine in the control to the problem of external stabilization of the same kind of systems.

5.1 UBIBS Stability

Let us consider a time dependent nonlinear system of the form

$$\begin{cases} \dot{x} = f(t, x, u) \\ x(t_0) = x_0 \end{cases} \tag{5.1}$$

where $f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ is locally essentially bounded and measurable with respect to (t, x) and continuous with respect to u. As in Chapter 4, solutions of (5.1) are intended in the Filippov's sense.

We are interested in intrinsic stability properties which take into account the presence of the control in the system. Many different concepts have been recently introduced: ISS (input-to-state stability), iISS (integral input-to-state stability), IOS (input-to-output stability), OSS (output-to-state stability), BIBO (bounded-input bounded-output) stability, UBIBS (uniform bounded input bounded state) stability (see [S2] and [BM] for an overview on these problems).

We focus on UBIBS stability.

Definition 22 *System* (5.1) *is said to be* UBIBS stable *if for each $R > 0$ there exists $S > 0$ such that for each $(t_0, x_0) \in \mathbb{R}^{n+1}$, and each input $u \in L^\infty_{loc}(\mathbb{R}; \mathbb{R}^m)$, if $\varphi(t)$ indicates any solution of (5.1), one has*

$$\|x_0\| < R,\ \|u\|_\infty < R \ \Rightarrow\ \|\varphi(t)\| < S \quad \forall t \geq t_0.$$

UBIBS stability is related to Lagrange stability, that, roughly speaking, can be seen as (Lyapunov) stability "in the large".

If we posit $u = 0$ in system (5.1) we get the unforced system

$$\begin{cases} \dot{x} = f(t, x, 0) \\ x(t_0) = x_0 \end{cases} \tag{5.2}$$

Definition 23 *System* (5.2) *is said to be* (uniformly) Lagrange stable *if for each $R > 0$ there exists $S > 0$ such that for each $(t_0, x_0) \in \mathbb{R}^{n+1}$, if $\varphi(t)$ indicates any solution of (5.2), one has*

$$\|x_0\| < R \;\Rightarrow\; \|\varphi(t)\| < S \quad \forall t \geq t_0.$$

It is clear that if (5.1) is UBIBS stable then (5.2) is Lagrange stable, while the converse is not true.

Lagrange stability has been characterized in terms of (smooth) Lyapunov-like functions in [Y2] in the case of a system with continuous righthand side and then this result has beem generalized in [BR1] to the case of discontinuous systems.

Definition 24 $V : \mathbb{R}^{n+1} \to \mathbb{R}$ *is a* Lyapunov-like function *for*

$$\dot{x} = f(t, x) \tag{5.3}$$

if there exists $S > 0$ such that for each solution $\varphi(t)$ of (5.3) and each pair of points t_1, t_2 such that $\|\varphi(t)\| \geq S$ for all $t \in [t_1, t_2]$, condition (2.3) holds.

Lyapunov-like functions differ from Lyapunov functions for the fact that they have to be defined "in the large", while Lyapunov functions could have been defined only on a neighbourhood of the origin.
In [VL, BR1], sufficient conditions for UBIBS stability in terms of (smooth) control Lyapunov-like functions have been given.

Definition 25 $V : \mathbb{R}^{n+1} \to \mathbb{R}$ *is a* control Lyapunov-like function *for*

$$\dot{x} = f(t, x, u) \tag{5.4}$$

if for all $R > 0$ there exists $S > 0$ such that for each control such that $\|u\|_\infty < R$, and each solution $\varphi(t)$ of

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{5.5}$$

one has that for each pair of points t_1, t_2 such that $\|\varphi(t)\| \geq S$ for all $t \in [t_1, t_2]$, condition (2.3) holds.

As for Lyapunov functions, in order to verify if a given function V is actually a control Lyapunov-like function, it is important to have sufficient conditions for V to decrease along trajectories of (5.5) that don't involve explicitly neither the control function, nor the solutions of the system.

We give a result analogous to Theorem 1 in [VL] and Theorem 6.2 in [BR1]. It differs from both for the fact that it involves control Lyapunov-like functions which are not of class C^1, but just locally Lipschitz continuous and healthy.

Lemma 5 *If there exists a control Lyapunov-like function V for (5.1), such that*

(V0t) *there exist $L > 0$ and two continuous, strictly increasing, positive functions $a, b : \mathbb{R} \to \mathbb{R}$ such that $\lim_{r\to+\infty} a(r) = +\infty$ and for all $t \geq t_0$ and for all x*

$$\|x\| > L \quad \Rightarrow a(\|x\|) \leq V(t,x) \leq b(\|x\|)$$

then system (5.1) is UBIBS stable.

Proof We prove the statement by contradiction, by assuming that there exists $\overline{R}$ such that for all $S > 0$ there exist $\overline{x_0}$ and $\overline{u} : [0, +\infty) \to \mathbb{R}^m$ such that $\|\overline{x_0}\| < \overline{R}$, $\|\overline{u}\|_\infty < \overline{R}$ and there exists a solution $\overline{\varphi}(t)$ of (5.5) with $u = \overline{u}$, and $\overline{t} > 0$ such that $\|\overline{\varphi}(\overline{t})\| \geq S$.

Because of (V0t), there exists $S_M > 0$ such that if $\|x\| > S_M$, then $V(t,x) > M = b(\overline{R}) \geq \max\{V(t,x),\ \|x\| = \overline{R},\ t \geq 0\}$ for all $t \geq t_0$.

Let us consider $S > \max\{\overline{R}, S_M\}$. By hypothesis there exist $\overline{x_0}$ and $\overline{u} : [0, +\infty) \to \mathbb{R}^m$ such that $\|\overline{x_0}\| < \overline{R}$, $\|\overline{u}\|_\infty < \overline{R}$ and there exists a solution $\overline{\varphi}(t)$ of (5.5) with $u = \overline{u}$, and $\overline{t} > 0$ such that $\|\overline{\varphi}(\overline{t})\| \geq S$. Then there also exist $t_1, t_2 > 0$ such that $\overline{t} \in [t_1, t_2]$, $\|\overline{\varphi}(t_1)\| = \overline{R}$, $\|\overline{\varphi}(t)\| \geq \overline{R}$ for all $t \in [t_1, t_2]$ and $\|\overline{\varphi}(t_2)\| \geq S$. Then

$$V(t_2, \overline{\varphi}(t_2)) > M \geq V(t_1, \overline{\varphi}(t_1)). \tag{5.6}$$

that contradicts (2.3). $\square$

Before stating next lemma, let us introduce some set-valued derivatives.

$$\begin{aligned}\overline{\dot{V}}^{(5.1)}(t,x,u) = \{a \in \mathbb{R} :\ & \exists v \in Ff(t,x,u) \text{ such that} \\ & \forall p \in \partial_C V(t,x)\ \ p \cdot (1,v) = a\}.\end{aligned}$$

Analogously, if $t > 0$, $x \in \mathbb{R}^n$ and a measurable and locally essentially bounded $u : \mathbb{R} \to \mathbb{R}^m$ are fixed, we set

$$\overline{\dot{V}}^{(5.5)}_{u(\cdot)}(t,x) = \{a \in \mathbb{R} : \exists v \in Ff(t,x,u(t)) \text{ such that}$$

$$\forall p \in \partial_C V(t,x) \quad p \cdot (1,v) = a\}$$

and, if $t > 0$ and $x \in \mathbb{R}^n$ are fixed, we define

$$\overline{\dot{V}}^{(5.2)}(t,x) = \{a \in \mathbb{R} : \exists v \in Ff(t,x,0) \text{ such that}$$

$$\forall p \in \partial_C V(t,x) \;\; p \cdot (1,v) = a\}.$$

Note that if $\overline{\varphi}(t)$ is any solution of (5.5) with $u(t) = \overline{u}(t)$ we have $\overline{\dot{V}}^{(5.5)}_{\overline{u}(\cdot)}(t,\overline{\varphi}(t)) \subseteq \overline{\dot{V}}^{(5.1)}(t,\overline{\varphi}(t),\overline{u}(t))$.

Lemma 6 *Let* $V : \mathbb{R}^{n+1} \to \mathbb{R}$ *locally Lipschitz continuous and healthy. If* (V0t) *holds and*

(fut) *for all* $R > 0$ *there exists* $S > \max\{L,R\}$ *such that for all* $x \in \mathbb{R}^n$ *and for all* $u \in \mathbb{R}^m$ *the following holds:*

$$\|x\| > S,\ \|u\| < R \ \Rightarrow\ \max \overline{\dot{V}}^{(5.1)}(t,x,u) \leq 0 \ \textit{for a.e. } t \geq 0$$

then V *is a control Lyapunov-like function for (5.1).*

Proof Let $R > 0$ be fixed and let us choose S corresponding to R as in (fut). Let us also fix $\overline{u}$ such that $\|\overline{u}\|_\infty < R$. Let $\overline{\varphi}(t)$ be any solution of (5.5) with $u = \overline{u}$ and let t_1, t_2 be such that for all $t \in [t_1,t_2]$ one has $\|\overline{\varphi}(t)\| \geq S$. By Remark 3, $\frac{d}{dt}V(t,\overline{\varphi}(t)) \in \overline{\dot{V}}^{(5.1)}_{\overline{u}(\cdot)}(t,\overline{\varphi}(t))$ a.e.. Moreover, as remarked before stating the lemma, $\overline{\dot{V}}^{(5.5)}_{\overline{u}(\cdot)}(t,\overline{\varphi}(t)) \subseteq \overline{\dot{V}}^{(5.1)}(t,\overline{\varphi}(t),\overline{u}(t))$. Since $\|\overline{u}(t)\| < R$ a.e. and $\|\overline{\varphi}(t)\| > S$ for all $t \in [t_1,t_2]$, by virtue of (fut) we have $\frac{d}{dt}V(t,\overline{\varphi}(t)) \leq 0$ for a.e. $t \in [t_1,t_2]$. We get that $V \circ \overline{\varphi}$ decreases in $[t_1,t_2]$. $\square$

The following theorem is now an obvious consequence of the two previous lemmas.

Theorem 10 *Let* $V : \mathbb{R}^{n+1} \to \mathbb{R}$ *be locally Lipschitz continuous and healthy and such that* (V0t) *and* (fut) *hold. Then system (5.1) is UBIBS stable.*

Remark 17 In order to get a sufficient condition for system (5.2) to be uniformly Lagrange stable, one can state Theorem 10 in the case $u = 0$. In this case the control Lyapunov-like function V simply becomes a Lyapunov-like function.

Remark 18 If system (5.1) is autonomous it is possible to state a theorem analogous to Theorem 10 for a control Lyapunov-like function V not depending on time.

5.2 UBIBS Stabilizability

We now turn our attention to the external stabilizabity property associated to UBIBS stability.

Definition 26 *System* (5.1) *is said to be* UBIBS stabilizable *if there exists a function* $k \in L^\infty_{loc}(\mathbb{R}^{n+1}; \mathbb{R}^m)$ *such that the closed loop system*

$$\dot{x} = f(t, x, k(t, x) + v) \tag{5.7}$$

(with v as input) is UBIBS stable.

We study UBIBS stabilization for systems of the form

$$\dot{x} = f(t, x) + G(t, x)u = f(t, x) + \sum_{i=1}^{m} u_i g_i(t, x) \tag{5.8}$$

where $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is measurable and locally essentially bounded, $g_1, ..., g_m \in C(\mathbb{R}^{n+1}; \mathbb{R}^n)$ for all $i \in \{1, ..., m\}$ and G is the matrix whose columns are $g_1, ..., g_m$.

We are interested in finding conditions which guarantee UBIBS stabilizabilty of system (5.8) when the unforced system (1.1) is known to be Lagrange stable. Our result essentially recalls Theorem 6.2 in [BR1] and Theorem 5 in [Ro], with the difference that the control Lyapunov-like function involved is not smooth.

We do not give a unique condition for system (5.8) to be UBIBS stabilizable, but some alternative conditions which, combined together, give the external stabilizability of the system. Before stating the theorem we list these conditions. Note that the variable x is not yet quantified. Since its role depends on different situations, it is convenient to specify it later.

(f1t) $\max \overline{\dot{V}}^{(1.1)}(t, x) \le 0$;

(f2t) for all $z \in Ff(t,x)$ there exists $\overline{p} \in \partial_C V(t,x)$ such that $\overline{p} \cdot (1,z) \leq 0$;

(f3t) for all $z \in Ff(t,x)$ and for all $p \in \partial_C V(t,x)$, $p \cdot (1,z) \leq 0$;

(G1t) for each $i \in \{1,...,m\}$ there exists $c^i_{t,x} \in \mathbb{R}$ such that for all $p \in \partial_C V(t,x)$, $p \cdot (1, g_i(t,x)) = c^i_{t,x}$;

(G2t) for each $i \in \{1,...,m\}$ only one of the following mutually exclusive conditions holds:

- for all $p \in \partial_C V(t,x)$ $\quad p \cdot (1, g_i(t,x)) > 0$,
- for all $p \in \partial_C V(t,x)$ $\quad p \cdot (1, g_i(t,x)) < 0$,
- for all $p \in \partial_C V(t,x)$ $\quad p \cdot (1, g_i(t,x)) = 0$;

(G3t) there exists $\bar{i} \in \{1,...,m\}$ such that for each $i \in \{1,...,m\} \setminus \{\bar{i}\}$ only one of the following mutually exclusive conditions holds:

- for all $p \in \partial_C V(t,x)$ $\quad p \cdot (1, g_i(t,x)) > 0$,
- for all $p \in \partial_C V(t,x)$ $\quad p \cdot (1, g_i(t,x)) < 0$,
- for all $p \in \partial_C V(t,x)$ $\quad p \cdot (1, g_i(t,x)) = 0$;

Let us remark that (f3t) $\Rightarrow$ (f2t) $\Rightarrow$ (f1t) and (G1t) $\Rightarrow$ (G2t) $\Rightarrow$ (G3t).

Theorem 11 *Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be locally Lipschitz continuous, healthy and such that there exists $L > 0$ such that* (V0t) *holds.*

If for all $x \in \mathbb{R}^n$ with $\|x\| > L$ one of the following couples of conditions holds for a.e. $t \geq 0$:
(i) (f1t) *and* (G1t), *(ii)* (f2t) *and* (G2t), *(iii)* (f3t) *and* (G3t),
then system (5.8) *is UBIBS stabilizable.*

Let us make some remarks.
If for all $x \in \mathbb{R}^n$ with $\|x\| > L$ assumption (f1t) (or (f2t) or (f3t)) holds for a.e. $t \geq t_0$, then, by Theorem 10, system (1.1) is uniformly Lagrange stable. Actually in [BR1] the authors introduce the concept of robust uniform Lagrange stability and prove that it is equivalent to the existence of a locally Lipschitz continuous Lyapunov-like function. Then assumption (f1t) (or (f2t) or (f3t)) implies more than uniform Lagrange stability of system (1.1). In [Ro], the author has also proved that, under mild additional assumptions on f, robust Lagrange stability implies the existence of a C^∞ Lyapunov-like function, but the proof of this result is not actually constructive. Then we

could still have to deal with nonsmooth Lyapunov-like functions even if we know that there exist smooth ones.

Moreover Theorem 11 can be restated for autonomous systems with the function V not depending on time. In this case the feedback law is autonomous and it is possible to deal with a situation in which the results in [Ro] do not help.

Finally let us remark that if f is locally Lipschitz continuous, then, by [Y2] (page 105), the Lagrange stability of system (1.1) implies the existence of a time-dependent Lyapunov-like function of class C^∞. In this case, in order to get UBIBS stabilizability of system (5.8), the regularity assumption on G can be weakened to $G \in L^\infty_{loc}(\mathbb{R}^{n+1};\mathbb{R}^m)$ (as in [BB]).

5.2.1 Proof of Theorem 11

We first state and prove a lemma.

Lemma 7 *Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that there exists $L > 0$ such that* (V0t) *and* (V1) *hold. If $(\bar{t},\overline{x})$, with $\|\overline{x}\| > L$, is such that, for all $p \in \partial_C V(\bar{t},\overline{x})$ $\ p\cdot(1,g_i(\bar{t},\overline{x})) > 0$, then there exists $\delta_{\overline{x}} > 0$ such that, for all $x \in B(\overline{x},\delta_{\overline{x}})$, for all $p \in \partial_C V(\bar{t},x)$, $\ p\cdot(1,g_i(\bar{t},x)) > 0$.*

Analagously if $(\bar{t},\overline{x})$, with $\|\overline{x}\| > L$, is such that for all $p \in \partial_C V(\bar{t},\overline{x})$, $p\cdot(1,g_i(\bar{t},\overline{x})) < 0$, then there exists $\delta_{\overline{x}} > 0$ such that, for all $x \in B(\overline{x},\delta_{\overline{x}})$, for all $p \in \partial_C V(\bar{t},x)$, $\ p\cdot(1,g_i(\bar{t},x)) < 0$.

Proof Let $\gamma > 0$ be such that $\|\overline{x}\| > L+\gamma$, and let $L_{\overline{x}} > 0$ be the Lipschitz constant of V in the set $\{\bar{t}\}\times B(\overline{x},\gamma)$. For all $(\bar{t},x) \in \{\bar{t}\}\times B(\overline{x},\gamma)$ and for all $p \in \partial_C V(\bar{t},x)$ $\ \|p\| \le L_{\overline{x}}$ (see [Cl1], page 27).

Since g_i is continuous there exist η and M such that $\|(1,g_i(\bar{t},x))\| \le M$ in $\{\bar{t}\}\times B(\overline{x},\eta)$.

Let $d = \min\{p\cdot(1,g_i(\bar{t},\overline{x})),\ p \in \partial_C V(\bar{t},\overline{x})\}$. By assumption $d > 0$.

Let us consider $\epsilon < \frac{d}{2(L_{\overline{x}}+M)}$.

By the continuity of g_i, there exists δ_i such that, if $\|x-\overline{x}\| < \delta_i$, then $\|(1,g_i(\bar{t},x)) - (1,g_i(\bar{t},\overline{x}))\| < \epsilon$.

By the upper semi-continuity of $\partial_C V$ (see [Cl1], page 29), there exists $\delta_V > 0$ such that, if $\|x-\overline{x}\| < \delta_V$, then $\partial_C V(\bar{t},x) \subseteq \partial_C V(\bar{t},\overline{x}) + \epsilon B(0,1)$, i.e. for all $p \in \partial_C V(\bar{t},x)$ there exists $\overline{p} \in \partial_C V(\bar{t},\overline{x})$ such that $\|p-\overline{p}\| < \epsilon$.

Let $\delta_{\overline{x}} = \min\{\gamma,\eta,\delta_i,\delta_V\}$, x be such that $\|x-\overline{x}\| < \delta_{\overline{x}}$ and $p \in \partial_C V(\bar{t},x)$, $\overline{p} \in \partial_C V(\bar{t},\overline{x})$ be such that $\|p-\overline{p}\| < \epsilon$.

It is easy to see that $|p\cdot(1,g_i(\bar{t},x)) - \overline{p}\cdot(1,g_i(\bar{t},\overline{x}))| < \frac{d}{2}$, hence $p\cdot(1,g_i(\bar{t},x)) > \overline{p}\cdot(1,g_i(\bar{t},\overline{x})) - \frac{d}{2} = \frac{d}{2} > 0$.

The second part of the lemma can be proved in a perfectly analogous way. $\square$

Proof of Theorem 11 For each $x \in \mathbb{R}^n$, let N_x be the zero-measure subset of $\mathbb{R}^+$ in which no one of the couples of conditions $(i), (ii)$ and (iii) holds. Let $k : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $k(x) = (k_1(t,x), ..., k_m(t,x))$, be defined by

$$k_i(t,x) = \begin{cases} -\|x\| & \text{if } \forall p \in \partial_C V(t,x) \;\; p \cdot (1, g_i(t,x)) > 0 \\ 0 & \text{if } \forall p \in \partial_C V(t,x) \;\; p \cdot g_i(t,x) = 0, \\ & \text{or (f3) and (G3) hold and } i = \bar{i}, \text{ or } t \in N_x \\ \|x\| & \text{if } \forall p \in \partial_C V(t,x) \;\; p \cdot (1, g_i(t,x)) < 0. \end{cases}$$

It is clear that $k : \mathbb{R} \to \mathbb{R}^m$ is locally essentially bounded.

By Theorem 10 it sufficient to prove that for all $R > 0$ there exists $\rho > L, R$ such that for all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ the following holds:

$$\|x\| > \rho, \;\; \|v\| < R \;\Rightarrow\; \max \overline{\dot{V}}^{(5.7)}(t,x) \leq 0 \text{ for all } t \in \mathbb{R}^+ \backslash N_x$$

where $\overline{\dot{V}}^{(5.8)}(t,x) = \{a \in \mathbb{R} : \exists w \in F(f(t,x) + G(t,x)k(t,x) + G(t,x)v)$ such that$\forall p \in \partial_C V(t,x) \;\; p \cdot (1,w) = a\}$.

Let x be fixed and $t \in \mathbb{R}^+ \backslash N_x$. Let $a \in \overline{\dot{V}}^{(5.8)}(t,x)$, $w \in F(f(t,x) + G(t,x)k(t,x) + G(t,x)v)$ be such that for all $p \in \partial_C V(t,x) \;\; p \cdot w = a$.

By Proposition 10 we have that
$F(f(t,x) + G(t,x)(k(t,x) + v)(x) \subseteq Ff(t,x) + \sum_{i=1}^m g_i(t,x) F(k_i(t,x) + v_i)$, then there exists $z \in Ff(t,x)$, $z_i \in F(k_i(t,x) + v_i)$, $i \in \{1, ..., m\}$, such that $w = z + \sum_{i=1}^m g_i(t,x) z_i$.

Let us show that $a \leq 0$. We distinguish the three cases $(i), (ii), (iii)$.
(i) $b = p \cdot (1,z) = a - \sum_{i=1}^m c^i_{t,x} z_i$ does not depend on p, then $b \in \overline{\dot{V}}^{(1.1)}(t,x)$ and, by (f1t), $b \leq 0$.

Let us now show that for each $i \in \{1, ..., m\}$ $\;c^i_{t,x} z_i \leq 0$. If i is such that $c^i_{t,x} = 0$, obviously $c^i_{t,x} z_i \leq 0$. If i is such that $c^i_{t,x} > 0$ then, by Lemma 7, there exists δ_x such that $k_i(t,y) = -\|y\|$ in $\{\bar{t}\} \times B(x, \delta_x)$, then k_i is continuous at x with respect to y. This implies that $F(k_i(t,x) + v_i) = -\|x\| + v_i$, i.e. $z_i = -\|x\| + v_i$ and $c^i_{t,x} z_i \leq 0$, provided that $\|v\| > \rho \geq \max\{L, R\}$.

The case in which i is such that $c^i_{t,x} < 0$ can be treated analogously. We finally get that $a = b + \sum_{i=1}^m c^i_{t,x} z_i \leq 0$.
(ii) By (f2t) there exists $\overline{p} \in \partial_C V(t,x)$ such that $\;\overline{p} \cdot (1,z) \leq 0$. $\;a = \overline{p} \cdot (1,z) + \sum_{i=1}^m \overline{p} \cdot (1, g_i(t,x)) z_i$. The fact that for each $i \in \{1, ..., m\}$ we

have $\overline{p} \cdot (1, g_i(t,x))z_i \leq 0$ can be proved as in (i) we have proved that for each $i \in \{1, ..., m\}$ $c^i_{t,x} z_i \leq 0$. We finally get that $a \leq 0$.
(iii) Let us remark that if (G2t) is not verified, i.e. we are not in the case (ii), there exists $\overline{p} \in \partial_C V(t,x)$ corresponding to $\overline{i}$ such that $\overline{p} \cdot (1, g_{\overline{i}}(t,x)) = 0$. Indeed, because of the convexity of $\partial_C V(t,x)$, for all $v \in \mathbb{R}^n$, if there exist $p_1, p_2 \in \partial_C V(t,x)$ such that $p_1 \cdot v > 0$ and $p_2 \cdot v < 0$, then there also exists $p_3 \in \partial_C V(t,x)$ such that $p_3 \cdot v = 0$.

Let $\overline{p} \in \partial_C V(t,x)$ be such that $\overline{p} \cdot (1, g_{\overline{i}}(t,x)) = 0$. For all $p \in \partial_C V(t,x)$ $a = p \cdot (1, w)$. In particular we have $a = \overline{p} \cdot (1, w) = \overline{p} \cdot (1, z) + \sum_{i \neq \overline{i}} \overline{p} \cdot (1, g_i(t,x))z_i + \overline{p} \cdot (1, g_{\overline{i}}(t,x))z_{\overline{i}}$. By (f3t), $\overline{p} \cdot (1, z) \leq 0$. If $i \neq \overline{i}$ the proof that $\overline{p} \cdot (1, g_i(t,x))z_i \leq 0$ is the same as in (ii). If $i = \overline{i}$, because of the choice of $\overline{p}$, $\overline{p} \cdot (1, g_{\overline{i}}(t,x)) = 0$. Also in this case we can then conclude that $a \leq 0$.
$\square$

Bibliography

[A] D. Aeyels, *Asymptotic stability of nonautonomous systems by Lyapunov's direct method*, Systems and Control Letters, 25, 1995, 273-280

[AP] D. Aeyels and J. Peuteman, *A New Asymptotic Stability Criterion for Nonlinear Time-Variant Differential Equations*, IEEE Trans. on Aut. Control, Vol.43, No.7, July 1993, 968-971

[A1] L. Ambrosio, *Relaxation of autonomous functionals with discontinuous integrands*, Ann. Univ. Ferrara, Nuova Serie, Sez. VII, 34, 1988, 21-47

[A2] L. Ambrosio, *A lower closure theorem for autonomous orientor fields*, Proc. R. Soc. Edinb., Sect. A110, No. 3/4, 1988, 249-254

[AB] F. Ancona and A. Bressan, *Patchy Vector Fields and Asymptotic Stabilization*, COCV, vol. 4, 1999, 419-444

[Ar] Z. Artstein, *Stabilization with relaxed controls*, Nonlinear Anal., 7, 1983, 1163-1173

[AC] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1994

[AS] J. Auslander and P. Seibert, *Prolongations and Stability in Dynamical Systems*, Ann. Inst. Fourier (Grenoble), 14, 1964, 237-268

[B1] A. Bacciotti, *Local Stabilizability Theory of Nonlinear Systems*, World Scientific, 1992

[B2] A. Bacciotti, *External Stabilizability of Nonlinear Systems with some applications*, International Journal of Robust and Nonlinear Control, 8, 1998, 1-10

[B3] A. Bacciotti, *Monotonicity and Generalized Derivatives*, Rapporto interno del Dipartimento di Matematica del Politecnico di Torino, 1998

[BB] A. Bacciotti and G. Beccari, *External Stabilizability by Discontinuous Feedback*, Proceedings of the second Portuguese Conference on Automatic Control, 1996

[BC] A. Bacciotti and F. Ceragioli, *Stability and Stabilization of Discontinuous Systems and Nonsmooth Lyapunov Functions*, COCV, vol. 4, 1999, 361-376

[BCM] A. Bacciotti, F. Ceragioli and L. Mazzi, *Differential Inclusions and Monotonicity Conditions for Nonsmooth Lyapunov Functions*, Rapporto interno del Dipartimento di Matematica del Politecnico di Torino, 1999

[BM] A. Bacciotti and L. Mazzi, *A necessary and sufficient condition for bounded-input bounded-state stability of nonlinear systems*, to appear on SIAM J. Contr. and Optim.

[BR1] A. Bacciotti and L. Rosier, *Liapunov and Lagrange Stability: Inverse Theorems for Discontinuous Systems*, Mathematics of Control, Signals and Systems, 11, 1998, 101-128

[BR2] A. Bacciotti and L. Rosier, *On the converse of first Liapunov Theorem: the regularity issue*, Rapporto interno del Dipartimento di Matematica del Politecnico di Torino, 1999

[BS] N.P. Bhatia and G.P. Szëgo, *Stability Theory of Dynamical Systems*, Springer Verlag, 1970

[BD] A. Bloch and S. Drakunov *Stabilization and tracking in the nonholonomic integrator via sliding modes*, Syst. and Control Lett., 29, 1996, 91-99

[Br] A. Bressan, *Singularities of Stabilizing Feedbacks*, to appear in Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino

[BrC] A. Bressan and G. Colombo, *Existence and Continuous Dependence for Discontinuous O.D.E.'s*, Bollettino U.M.I. (7) 4-B, 1990, 295-311

[Bro] R.W. Brockett, *Asymptotic stability and feedback stabilization*, in Differential Geometric Control Theory, R.W. Brockett, R.S. Millmann and J.H. Sussmann Eds., Birkhauser, Boston, 1983, 181-191

[CS] O.J. Canudas de Wit and O.J. Sørdalen, *Examples of Piecewise Smooth Stabilization of Driftless NL Systems with less inputs than States*, Proc. Symp. on Nonlinear Control System Design, Bordeaux, France (IFAC, 1992), 57-61

[C] P.I. Chugunov, *Regular Solutions of Differential Inclusions*, Diff. Urav., 17, 1981, 660-668, translation in Diff. Eq., 17, 1981, 449-455

[Cl1] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley and Sons, New York, 1983

[Cl2] F.H. Clarke, CIRM course on "Mathematical Control Theory", Levico Terme, June 1998

[CLSS] F.H. Clarke, YU.S. Ledyaev, E.D. Sontag and A.I. Subbotin, *Asymptotic Controllability Implies Feedback Stabilization*, IEEE Transactions on Automatic Control, vol.XX, No.Y, Month 199

[CLSW1] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern and P.R. Wolenski, *Qualitative Properties of Trajectories of Control Systems: A Survey*, J. Dynam. Control Systems, 1, 1995, 1-47

[CLSW2] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern and P.R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, 1998

[C1] J.M. Coron, *Global asymptotic stabilization for controllable systems without drift*, Math. Control Signals Systems, 5, 1992, 295-312

[C2] J.M. Coron, *Stabilization of Control Systems*, in Nonlinear Analysis, Differential Equations and Control, F.H. Clarke and R.J. Stern editors, NATO Sciences Series, Series C: Mathematical and Physical Sciences, vol. 528, Kluwer Academic Publishers, 1999

[CR] J.M. Coron and L. Rosier, *A Relation between Continuous Time-Varying and Discontinuous Feedback Stabilization*, Journal of Mathematical Systems, Estimation and Control, Vol.4, No.1, 1994, 67-84

[D] K. Deimling, *Multivalued Differential Equations*, De Gruyter, 1992

[EG] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC, 1992

[F1] A.F. Filippov, *Differential Equations with Discontinuous Right-hand Sides*, Mat. Sb., 5, 1960, 99-127

[F2] A.F. Filippov, *Differential Equations with Discontinuous Right-hand Sides*, Kluwer Academic Publishers, 1988

[FM] M. Fliess and F. Messager, *Methods of nonlinear discontinuous stabilization*, Proceedings of IIASA Workshop held in Sopron, Hungary, June 1989

[FK] R.A. Freeman and P.V. Kokotovic, *Backstepping Design with Nonsmooth Nonlinearities*, IFAC NOLCOS 1995, Tahoe City, California, 483-488

[J] D.H. Jacobson, *Extensions of Linear-Quadratic Control, Optimization and Matrix Theory*, Academic Press, New York, 1977

[JS] R. Janin and R. Sentis, *Existence of an Optimal Feedback in a Control Problem*, Nonlinear Anal., 5, 1981, no.1, 27-45

[JQ] V. Jurdjevic and J.P. Quinn, *Controllability and Stability*, Journal of Differential Equations, 28, 1978, 381-389

[H] O. Hájek, *Discontinuous Differential Equations I, II*, Journal of Differential Equations, 32, 1979, 149-170, 171-185

[K] N.N. Krasovskii, *The Converse of the Theorem of K.P. Persidskij on Uniform Stability*, Prikladnaya Matematika i Mekhanika, 19, 1955, 273-278 (in Russian)

[KS] N.N. Krasovskii and A.I. Subbotin, *Game-Theoretical Control Problems*, Springer-Verlag, New York, 1988

[Ku] J. Kurzweil, *On the Invertibility of the First Theorem of Lyapunov Concerning the Stability of Motion*, Czechoslovak Mathematical Journal, 80, 1955, 382-398 (in Russian)

[KV] J. Kurzweil and I. Vrkoč, *The Converse Theorems of Lyapunov and Persidskij Concerning the Stability of Motion*, Czechoslovak Mathematical Journal, 80, 1955, 382-398 (in Russian)

[LS] Y. Ledyaev and E.D. Sontag, *A Lyapunov Characterization of Robust Stabilization*, J. Nonlin. Anal., 37, 1999, 813-814

[MT] L. Mazzi and V. Tabasso, *On Stabilization of Time-Dependent Affine Control Systems*, Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino, Vol.54, No.1, 1996, 53-66

[MS] E.J. McShane, *Integration*, Princeton University Press, 1947

[MV] J.J. Moreau and M. Valadier, *A Chain Rule Involving Vector Functions of Bounded Variation*, J. of Funct. Anal., 74, 1987, 333-345

[PS] B. Paden and S. Sastry, *A Calculus for Computing Filippov's Differential Inclusion with Application to the Variable Structure Control of Robot Manipulators*, IEEE Transaction on Circuits and Systems, Vol. Cas-34, No. 1, January 1997, 73-81

[P] C. Prieur, *Uniting local and global controllers*, Preprints of the first Nonlinear Control Network (NCN) Pedagogical School, Athens, September 1999

[P1] A. Pucci, *Sistemi di equazioni differenziali con secondo membro discontinuo rispetto all'incognita*, Rend. Ist. Mat. Univ. Trieste, 3, 1971, 75-80

[P2] A. Pucci, *Traiettorie di campi di vettori discontinui*, Rend. Ist. Mat. Univ. Trieste, 8, 1976, 84-93

[Ri] L. Rifford, *Stabilisation des systèmes globalement asymptotiquement commandables*, Preprint

[Ro] L. Rosier, *Smooth Lyapunov Functions for Lagrange Stable Systems*, to appear on Set-valued analysis

[RHL] N. Rouche, P. Habets and M. Laloy, *Stability Theory by Liapunov's Direct Method*, Springer-Verlag, 1977

[R1] E.P. Ryan, *On Brockett's condition for smooth stabilizability and its necessity in a context of nonsmooth feedback*, SIAM J. Control and Optim., 32, 1994, 1597-1604

[R2] E.P. Ryan, *An Integral Invariance Principle for Differential Inclusions with Applications in Adaptive Control*, SIAM J. Control and Optim., Vol.36, May 1998, No.3, 960-980

[Sam] C. Samson, *Velocity and torque feedback control of nonholonomic cart*, in Advanced Robot Control (C. Canudas de Wit ed.), Lectures Notes in Control and Inform. Sci., 162, Springer-Verlag, 1991, 125-151

[Sa] G. Sansone, *Equazioni differenziali nel campo reale*, Zanichelli, Bologna, 1941

[Se] R. Sentis, *Equations differentielles à second membre mesurable*, Boll. Unione Matematica Italiana, 5, 15-B, 1978, 724-742

[SP] D. Shevitz and B. Paden, *Lyapunov Stability Theory of Nonsmooth Systems*, IEEE Transaction on Automatic Control, Vol. 39, No. 9, September 1994, 1910-1914

[S1] E.D. Sontag, *A Lyapunov-like Characterization of Asymptotic Controllability*, SIAM J. Control and Opt., 21, 1983, 462-471

[S2] E.D. Sontag, *Stability and Stabilization: Discontinuities and the Effect of Disturbances*, in Nonlinear Analysis, Differential Equations and Control, F.H. Clarke and R.J. Stern editors, NATO Sciences Series, Series C: Mathematical and Physical Sciences, vol. 528, Kluwer Academic Publishers, 1999

[SS1] E.D. Sontag and H. Sussmann, *Remarks on continuous feedback control*, in Proc. IEEE Conf. Decision and Control, Albuquerque 1980, 916-921

[SS2] E.D. Sontag and H. Sussmann, *Nonsmooth Control Lyapunov Functions*, Proc. IEEE Conf. Decision and Control, New Orleans, Dec. 1995, IEEE Publications, 1995, 2799-2805

[S] H. Sussmann, *Subanalytic sets and feedback control*, J. Differential Equations, 31, 1979, 371-393

[VL] P.P. Varaiya and R. Liu, *Bounded-input Bounded-output Stability of Nonlinear Time-varying Differential Systems*, SIAM Journal Control, 4, 1966, 698-704

[V] M. Valadier, *Lignes de descente de fonctions lipschitziennes non pathologiques*, Sem. d'Anal. Convex, Montpellier, 1988, N.9

[Y] J.A. Yorke, *Differential Inequalities and Non-Lipschitz Scalar Functions*, Math. Syst. Theory, 4, 1970, 140-153

[Y1] T. Yoshizawa, *On the Stability of Solutions of a System of Differential Equations*, Memoirs of the College of Sciences, University of Kyoto, Series A, 29, 1955, 27-33

[Y2] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan, 1966