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(Article begins on next page)

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Bound states of the NLS equation on metric graphs with localized nonlinearities

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Abstract

We investigate the existence of multiple bound states of prescribed mass for the nonlinear Schrödinger equation on a noncompact metric graph. The main feature is that the nonlinearity is localized only in a compact part of the graph. Our main result states that for every integer k, the equation possesses at least k solutions of prescribed mass, provided that the mass is large enough. These solutions arise as constrained critical points of the NLS energy functional. Estimates for the energy of the solutions are also established.

AMS Subject Classification: 35R02, 35Q55, 81Q35, 49J35, 58E05, 46T05

Keywords: Minimax methods, metric graphs, nonlinear Schrödinger Equation, localized nonlinearity, Krasnosel'skii genus

1 Introduction

In this paper we discuss the existence of multiple bound states of *prescribed mass* for the NLS equation on metric graphs with localized nonlinearities.

Here we limit ourselves to a rather informal description of the results: the precise setting of the problem with all the required definitions will be given in detail in Section 2.

We consider a *noncompact connected metric graph*, that is, a connected graph $\mathcal{G} = (\mathrm{V}, \mathrm{E})$ where each edge is associated with a (possibly unbounded) interval of the real line. The subgraph $\mathcal{K}$ of $\mathcal{G}$ consisting of all the bounded edges is called the *compact core* of $\mathcal{G}$ (see Definition 2.1). In Figure 1 is depicted a typical metric graph.

Given a number $\mu > 0$, a bound state of *mass* μ for the NLS equation on $\mathcal{G}$, with nonlinearity localized in the compact core, is a function $u \in H^1(\mathcal{G})$ satisfying

$$\|u\|^2_{L^2(\mathcal{G})} = \mu \tag{1}$$

and that solves, for a suitable $\lambda \in \mathbb{R}$, the stationary NLS equation

$$u'' + \kappa(x)|u|^{p-2}\,u = \lambda u \tag{2}$$

on every edge of $\mathcal{G}$ with homogeneous Kirchhoff conditions at the vertices of the compact core $\mathcal{K}$. The function κ is the characteristic function of the compact core (this explains the term *localized* referred to the nonlinearity).

Many different boundary conditions can be imposed at the vertices of the compact core, giving rise to different models (see e.g. [9]). In this paper we choose the Kirchhoff conditions, since they are the most commonly used in the applications. See Definition 2.2 for the precise statement.

Bound states satisfy therefore a *double regime*: linear in the unbounded edges and nonlinear in the compact core.

In this paper we only treat nonlinearities with *subcritical* growth, namely, we always assume that $2 < p < 6$.

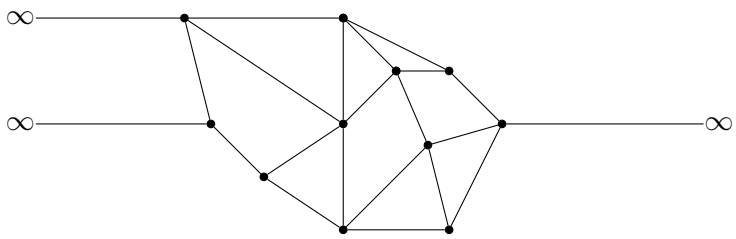

Figure 1: a metric graph with 3 half–lines and 19 bounded edges. The bounded edges form the *compact core* of the graph, where the nonlinearity is localized.

The existence of solutions with prescribed mass to stationary NLS type equations is a classical subject, both in dimension one and in higher dimensions, and the literature is huge (see e.g [12, 13, 24] and references therein). From a variational point of view, bound states of prescribed mass (i.e. the solutions of (1)–(2)) can be found as critical points of the energy functional

$$E(u) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{K})} = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{K}} |u|^p\,dx, \tag{3}$$

constrained on the manifold

$$M = \{u \in H^1(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} = \mu\}. \tag{4}$$

The principal obstacle that one has to face when looking for critical points of functionals like E is given by the lack of compactness caused by the unboundedness of the domain. As is well known, this may result in the failure of convergence of minimizing or Palais–Smale sequences for E. Typical devices used to gain compactness, such as restricting to functions enjoying some kind of symmetry, do not work on general non symmetric graphs like ours.

A further difficulty, typical of problems on graphs, is the impossibility of obtaining solutions of prescribed mass by *scaling*. For instance, when the spatial domain is $\mathbb{R}$ or $\mathbb{R}^n$, a possible approach to find (multiple) solutions consists in setting up suitable min–max procedures and then obtaining solutions of prescribed mass by scaling critical points to adjust their L^2 norm. In these cases one may even expect infinitely many solutions of prescribed mass (see e.g. [7, 8]). In order to carry out this program, it is of course necessary that the domain be invariant under homotheties. In our case however the presence of bounded edges of fixed length prevents this possibility. A striking example of this phenomenon can be found in [25], where it is shown that when $p \in [4, 6)$, the functional (3) does not even admit a global *minimum* on M for small values of the mass μ.

The scheme of the procedure we use to find multiple bound states can be roughly summarized as follows. We consider the restriction E_M of the functional E to the manifold M defined in (4). First we prove that E_M satisfies the Palais–Smale condition at a level c if and only if $c < 0$ (Section 4). Then we construct, under suitable assumptions on μ, multiple min-max classes working at negative levels (Section 5). The construction of these classes takes advantage of the evenness of E_M and is based on the Krasnosel'skii genus. More precisely, we show that for every $k \in \mathbb{N}$, some *negative* sublevels of E_M have genus at least k provided the mass μ is large enough. This yields the existence of at least k (pairs of) critical points for E_M, by standard results in Critical Point Theory.

A fundamental role in the construction of the min–max classes is played by the properties of the *solitons* φ_μ, namely, the minimizers of the functional

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} |u'|^2\,dx - \frac{1}{p}\int_{\mathbb{R}} |u|^p\,dx, \qquad p \in (2, 6), \tag{5}$$

over the set of $u \in H^1(\mathbb{R})$ such that $\|u\|^2_{L^2(\mathbb{R})} = \mu$ (Section 3). These will be used, after being appropriately scaled, cut off and placed in the compact core of $\mathcal{G}$, to show that some negative sublevel sets of E_M have the required genus.

Remark 1.1. The topological argument that we use (detection of sublevels of high genus) works also when the nonlinearity is present on the half–lines. In this case however it can be proved that the Palais–Smale condition fails at infinitely many negative levels. Thus any variational argument for the everywhere nonlinear problem must avoid the "bad levels". This seems to be out of reach at the moment.

Our main result is the following.

Theorem 1.2. *Let $\mathcal{G}$ be a noncompact connected metric graph with nonempty compact core, and let $p \in (2, 6)$. For every $k \in \mathbb{N}$, there exists $\mu_k > 0$ such that for all $\mu \geq \mu_k$ there exist at least k distinct pairs $(\pm u_j)$ of bound states*

of mass μ. Moreover, for every $j = 1, \dots, k$,

$$E_M(u_j) \leq \frac{1}{j^{2\beta}}\,\mathcal{E}(\varphi_\mu) + \sigma_k(\mu) < 0, \qquad \beta = \frac{p-2}{6-p}, \tag{6}$$

where $\sigma_k(\mu) \to 0$ (exponentially fast) as $\mu \to \infty$. Finally, for each j, the Lagrange multiplier λ_j relative to u_j is positive.

This result shows, in particular, that the localization of the nonlinearity in the compact core is essential. Consider for instance the simple case where $\mathcal{G}$ is an interval with two half–lines attached at its endpoints. In the degenerate case where the interval is reduced to a point equation (2) becomes linear and there are of course no bound states, for any $\mu > 0$. Also in the other degenerate case, where the interval is the whole real line, the functional E reduces to $\mathcal{E}$ of (5) and this has infinitely many ground states (the solitons), but no bound states at higher levels. It is thus the presence of the nonlinearity on a compact portion of the graph only that generates the bound states.

Remark 1.3. We observe that since by the properties of solitons (Section 3), $\frac{1}{j^{2\beta}}\,\mathcal{E}(\varphi_\mu) = j\mathcal{E}(\varphi_{\mu/j})$, the level estimate (6) means, asymptotically in μ, that the "j-th" bound state u_j behaves, energetically, not worse than the superposition of j solitons of mass μ/j, far apart from each other.

Remark 1.4. As we will see in Section 5, the solution corresponding to $j = 1$ is a global minimum of E_M, namely a *ground state*. This completes and clarifies some results of [25]: in some cases ground states exist if and only if their mass is large enough and their level, when $\mu \to \infty$, is asymptotically lower than that of a soliton of the same mass.

Remark 1.5. We point out that we have stated our main result, Theorem 1.2, under the assumption that the nonlinearity be localized in the *whole* compact core of the graph. We have done so because this is the most meaningful case in the applications. However, the technique of the proofs allows us to obtain the same result by assuming that the nonlinearity is localized in a *subset* of the compact core, even in a single edge (the case treated numerically in [16]). No change is necessary to carry out the proof in this more general case.

Remark 1.6. Some of the solutions we find might vanish identically on one or more half–lines, which is compatible with our definition of bound states. This type of solutions have been first found in [10] on the tadpole graph for the everywhere nonlinear problem. Ground states, however, cannot have this property, because being of constant sign they would violate the Kirchoff condition at the vertices of the half–lines where they vanish (see [25]).

Driven by the applications, the study of dynamics on quantum graphs has gained popularity in recent years since graphs emulate accurately complex mesoscopic and optical networks and allow one to reproduce properties of quantum chaotic systems (see [15, 18]).

In particular, [16] highlights some specific motivations for studying the existence of bound states for the NLS equation on metric graphs with nonlinearity localized on the compact core and [22] proposes the analysis of the problem from a rigorous point of view.

The investigation of these topics seems to play an important role in the analysis of the effects of nonlinearities on the transmission through a complex network of one–dimensional leads (such as *optical fibers*) and on the properties of the *Bose–Einstein condensates* in non–regular traps.

For an overview on the Schrödinger equation with localized or concentrated nonlinearities in standard domains we refer the reader to [2, 5, 11, 17, 21].

Finally, for the sake of completeness, we also recall that a general investigation of the existence of *ground* states in the "everywhere nonlinear" problem (namely, the problem with the nonlinearity located on the whole graph $\mathcal{G}$) is developed in [1, 3, 4].

2 Functional setting

We start by recalling briefly some basic facts and defintions concerning metric graphs (for a modern account on the topic we refer the reader to [20, 9, 3] and references therein).

In this paper, by *connected metric graph* $\mathcal{G} = (\mathrm{V}, \mathrm{E})$, we mean a connected *multigraph* (that is, multiple edges between the same vertices and self–loops are allowed), where each edge is a (possibly half–infinite) segment of line. The edges are joined at their endpoints — the vertices of $\mathcal{G}$ — according to the topology of the graph. Each edge e is represented by either a closed bounded interval $I_e = [0, \ell_e]$, $\ell_e > 0$, or a closed half–line $I_e = [0, +\infty)$, and a coordinate x_e is chosen in the corresponding interval. When the interval is bounded, the orientation of x_e is arbitrary while, in case of an unbounded edge, the half–line always starts at $x_e = 0$. In this way the graph $\mathcal{G}$ turns into a locally compact metric space, the metric being given by the shortest distance measured along the edges. In the sequel, for the sake of simplicity, we identify an edge e with its corresponding interval I_e, since this does not give rise to misunderstandings.

A metric graph is *compact* if and only if it does not contain any half–line. An important role in this paper is played by the following notion, introduced in [4].

Definition 2.1. If $\mathcal{G}$ is a metric graph, we define its *compact core* $\mathcal{K}$ as the metric subgraph of $\mathcal{G}$ consisting of all the bounded edges of $\mathcal{G}$.

Obviously the compact core $\mathcal{K}$ of any graph $\mathcal{G}$ is compact and, when $\mathcal{G}$ is connected (as we always assume throughout), $\mathcal{K}$ is also connected.

A function $u : \mathcal{G} \to \mathbb{R}$ can be regarded as a family of functions $(u_e)_{e\in \mathrm{E}}$, where $u_e : I_e \to \mathbb{R}$ is the restriction of u to the edge (represented by) I_e. The usual L^p spaces can be defined over $\mathcal{G}$ in the natural way, with norm

$$\|u\|^p_{L^p(\mathcal{G})} := \sum_{e\in \mathrm{E}} \|u_e\|^p_{L^p(I_e)},$$

while $H^1(\mathcal{G})$ is the set of continuous $u : \mathcal{G} \to \mathbb{R}$ such that $u_e \in H^1(I_e)$ for every $e \in \mathrm{E}$ and

$$\|u\|^2_{H^1(\mathcal{G})} := \|u'\|^2_{L^2(\mathcal{G})} + \|u\|^2_{L^2(\mathcal{G})}.$$

We notice that continuity at a vertex v is to be interpreted as a no–jump condition among different components u_e meeting at v. Further details can be found in [3]. From now on we denote the space $H^1(\mathcal{G})$ simply by H and its norm by $\|\cdot\|$. Also, H' denotes the dual of H.

As we outlined in the Introduction, the object of this paper is to prove existence of multiple bound states of prescribed mass for the NLS equation on a noncompact metric graph $\mathcal{G}$, where the nonlinearity is localized *only* in the compact core $\mathcal{K}$. Of course we always assume that $\mathcal{K}$ is not empty.

The precise definition of bound state of prescribed mass is the following.

Definition 2.2. Let $\mathcal{G} = (\mathrm{V}, \mathrm{E})$ be a connected metric graph with nonempty compact core $\mathcal{K}$ and let $\kappa : \mathcal{G} \to \{0, 1\}$ be the characteristic function of the compact core. We say that a function $u \in H$ is a *bound state of mass μ for the NLS equation on $\mathcal{G}$ with nonlinearity localized on $\mathcal{K}$ and Kirchhoff conditions* if:

(i) $\|u\|^2_{L^2(\mathcal{G})} = \mu$,

(ii) there exists $\lambda \in \mathbb{R}$ such that for every $e \in \mathrm{E}$

$$u_e'' + \kappa(x)|u_e|^{p-2}\, u_e = \lambda u_e,$$

(iii) for every vertex v in the compact core $\mathcal{K}$,

$$\sum_{e\succ \mathrm{v}} \frac{du_e}{dx_e}(\mathrm{v}) = 0.$$

The symbol $e \succ \mathrm{v}$ means that the sum is extended to all edges e incident at v, while $\frac{du_e}{dx_e}(\mathrm{v})$ is a shorthand notation for $u_e'(0)$ or $-u_e'(\ell_e)$, according to the fact that x_e is equal to 0 or ℓ_e at v.

For simplicity, we refer to a function u satisfying Definition 2.2 simply as a *bound state of mass μ*.

Throughout this paper we always suppose that $p \in (2, 6)$, the so-called L^2–subcritical case.

The problem of the search of bound states of mass μ has a variational structure that we now describe and that we will exploit to prove a multiplicity result.

Let $E : H \to \mathbb{R}$ be the functional

$$E(u) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{K})} = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{K}} |u|^p\,dx.$$

Plainly, $E \in C^1(H;\mathbb{R})$ and

$$E'(u)v = \int_{\mathcal{G}} u'v'\,dx - \int_{\mathcal{K}} |u|^{p-2}\,uv\,dx \qquad \forall v \in H.$$

For $\mu > 0$, define the manifold

$$M = \{u \in H : \|u\|^2_{L^2(\mathcal{G})} = \mu\}. \tag{7}$$

This is a submanifold of H of codimension 1 with the structure of Finsler manifold (see e.g. [8, 14]) and its tangent space at a given point u is the closed subspace of H defined by

$$T_uM = \left\{ v \in H : \int_{\mathcal{G}} uv\,dx = 0 \right\}.$$

We denote its dual space $(T_uM)'$ by T'_uM; the norms in T_uM and in T'_uM are those induced by the norms in H and H', respectively.

Finally, let $E_M : M \to \mathbb{R}$ denote the restriction of E to M; of course E_M is C^1 on M and

$$E'_M(u)v = E'(u)v \qquad \forall v \in T_uM.$$

The following characterization follows immediately from the Lagrange multiplier theorem.

Proposition 2.3. *A function $u \in H$ is a bound state of mass μ if and only if $u \in M$ and $E'_M(u) = 0$.*

We are thus led to the search of critical points of the functional E_M. To simplify some computations in the next sections, we set

$$\lambda = \lambda(u) := -\frac{1}{\mu}E'(u)u = \frac{1}{\mu}\left(\int_{\mathcal{K}} |u|^p\,dx - \int_{\mathcal{G}} |u'|^2\,dx\right) \tag{8}$$

and define, for every $u \in M$, a linear functional $J(u) : H \to \mathbb{R}$ as

$$J(u)v = \int_{\mathcal{G}} u'v'\,dx - \int_{\mathcal{K}} |u|^{p-2}\,uv\,dx + \lambda\int_{\mathcal{G}} uv\,dx.$$

Note that

$$J(u)v = E'_M(u)v \qquad \forall v \in T_uM \tag{9}$$

and, by definition of λ,

$$J(u)u = 0 \qquad \forall u \in H. \tag{10}$$

It is common to identify $E'_M(u)$ with $J(u)$. However, as pointed out in Remark 8.2 of [8], the identification is not always correct, away from critical points. The precise relation between $E'_M(u)$ and $J(u)$ that we are going to use is given in the following statement, discussed abstractly in [8]. We give a proof for completeness.

Lemma 2.4. *Let $u \in M$. Then*

$$\|E'_M(u)\|_{T'_uM} \le \|J(u)\|_{H'} \le \left(1 + \mu^{-1/2}\|u\|\right) \|E'_M(u)\|_{T'_uM}. \tag{11}$$

Proof. By (9),

$$\|E'_M(u)\|_{T'_uM} \le \|J(u)\|_{H'},$$

proving the first inequality in (11). Let $\pi_u : H \to T_uM$ be the restriction to H of the L^2 orthogonal projection onto T_uM. Then, every $v \in H$ can be written as $v = \left(\mu^{-1} \int_{\mathcal{G}} uv\,dx\right) u + \pi_u v$ and

$$\|\pi_u v\| \le \left(1 + \mu^{-1/2}\|u\|\right) \|v\|. \tag{12}$$

By (9) and (10), for every $v \in H$,

$$J(u)v = J(u)\left(\left(\mu^{-1}\int_{\mathcal{G}} uv\,dx\right)u + \pi_u v\right) = J(u)\pi_u v = E'_M(u)\pi_u v.$$

Then

$$|J(u)v| \le \|E'_M(u)\|_{T'_uM}\|\pi_u v\| \qquad \forall v \in H.$$

In view of (12), this proves also the second inequality in (11). □

Remark 2.5. From (11) we see that $u \in M$ is a critical point for E_M if and only if $J(u) = 0$. Also, the preceding lemma allows us to say that if $(u_n) \subset M$ is a *bounded* sequence such that $\|E'_M(u_n)\|_{T'_{u_n}M} \to 0$, then $\|J(u_n)\|_{H'} \to 0$, and vice–versa. Note that if the boundedness assumption is dropped, this need no longer be true.

3 Some properties of solitons

Let $\mu > 0$ and $p \in (2, 6)$. The *solitons of mass* μ are the minimizers of the functional

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} |u'|^2\,dx - \frac{1}{p}\int_{\mathbb{R}} |u|^p\,dx$$

on the set $\{u \in H^1(\mathbb{R}) : \|u\|^2_{L^2(\mathbb{R})} = \mu\}$. It is well known ([12, 26]) that, up to a change of sign, these are the translates of a unique function φ_μ

(the dependence on p is understood) which is positive, even and radially decreasing. Precisely, and in order to enlighten the dependence on μ,

$$\varphi_\mu(x) = \mu^\alpha \varphi_1(\mu^\beta x), \qquad \alpha = \frac{2}{6-p}, \quad \beta = \frac{p-2}{6-p}, \tag{13}$$

where

$$\varphi_1(x) := C_p \mathrm{sech}^{\alpha/\beta}(c_p x), \qquad \text{with} \quad C_p, c_p > 0. \tag{14}$$

A direct computation shows that

$$\mathcal{E}(\varphi_1) < 0 \qquad \text{and} \qquad \mathcal{E}(\varphi_\mu) = \mu^{2\beta+1}\mathcal{E}(\varphi_1). \tag{15}$$

The function φ_μ is the unique (positive and radially decreasing) solution of the stationary NLS equation

$$u'' + |u|^{p-2}\, u = \lambda u \qquad \text{on } \mathbb{R}$$

with

$$\lambda = \frac{1}{\mu}\left(\|u\|^p_{L^p(\mathbb{R})} - \|u'\|^2_{L^2(\mathbb{R})} \right) > 0 \tag{16}$$

in the class of functions $u \in H^1(\mathbb{R})$ with $\|u\|^2_{L^2(\mathbb{R})} = \mu$. The conservation of energy for φ_μ reads

$$\frac{1}{2}|\varphi_\mu'(x)|^2 + \frac{1}{p}|\varphi_\mu(x)|^p = \frac{\lambda}{2}|\varphi_\mu(x)|^2 \qquad \forall x \in \mathbb{R}. \tag{17}$$

Let $\mathcal{L}_\mu(x)$ be the Lagrangian density of $\mathcal{E}(\varphi_\mu)$, namely

$$\mathcal{L}_\mu(x) = \frac{1}{2}|\varphi_\mu'(x)|^2 - \frac{1}{p}|\varphi_\mu(x)|^p.$$

The following simple property will be used below.

Lemma 3.1. *For every $\mu > 0$ there exists a unique point $x_\mu > 0$ such that*

$$\mathcal{L}_\mu(x) \geq 0 \iff |x| \geq x_\mu.$$

In particular, $x_\mu = \mu^{-\beta} x_1$, so that x_μ is a decreasing function of μ and tends to zero as $\mu \to \infty$.

Proof. By (13), $\mathcal{L}_\mu(x) = \mu^{\alpha p}\mathcal{L}_1(\mu^\beta x)$ (note that $2\alpha + 2\beta = \alpha p$). The inequality

$$\mathcal{L}_1(t) \geq 0,$$

in view of (17), is equivalent to

$$\frac{2}{p}|\varphi_1(t)|^p \leq \frac{\lambda}{2}|\varphi_1(t)|^2,$$

namely,

$$|\varphi_1(t)|^{p-2} \le \frac{p\lambda}{4}.$$

Notice that here λ no longer depends on μ, being given by (16) with $u = \varphi_1$. As $\mathcal{L}_1(0) < 0$, we see that $|\varphi_1(0)|^{p-2} > p\lambda/4$, so that, since φ_1 is radially decreasing, there is a unique point x_1 such that

$$|\varphi_1(t)|^{p-2} \le \frac{p\lambda}{4} \iff |t| \ge x_1.$$

Setting $x_\mu = \mu^{-\beta} x_1$, the proof is complete. □

The following proposition is crucial for our purposes. Roughly, it says that when the mass is large, the energy of a soliton can be approximated using functions with small support.

Proposition 3.2. *For every $\ell > 0$ there exists $\mu_\ell > 0$ such that for all $\mu \ge \mu_\ell$, there exists $\psi = \psi_\mu \in H^1(\mathbb{R})$ satisfying*

(i) $\|\psi\|^2_{L^2(\mathbb{R})} = \mu$,

(ii) $\operatorname{supp}\psi \subset [0, \ell]$,

(iii) $\mathcal{E}(\psi) \le \mathcal{E}(\varphi_\mu) + r_\ell(\mu) < 0$,

where $r_\ell(\mu) \to 0$ (exponentially fast) as $\mu \to \infty$.

Proof. Given $\ell > 0$, let $\tau = \ell/2$. We take, using Lemma 3.1, μ_1 so large that for every $\mu \ge \mu_1$, the point x_μ of Lemma 3.1 satisfies $x_\mu \le \tau$. In this way,

$$\mathcal{L}_\mu(x) \ge 0 \qquad \text{for every } |x| \ge \tau \text{ and every } \mu \ge \mu_1. \tag{18}$$

Then, for each $\mu \ge \mu_1$, we define the function

$$v_\mu(x) = \big(\varphi_\mu(x) - \varphi_\mu(\tau)\big)^+,$$

so that $\operatorname{supp} v_\mu \subset [-\tau, \tau]$. By definition of $\mathcal{L}_\mu$ and (18) we see that

$$\begin{aligned}
\mathcal{E}(v_\mu) &= \frac{1}{2}\int_{-\tau}^{\tau} |\varphi_\mu'|^2\,dx - \frac{1}{p}\int_{-\tau}^{\tau} |\varphi_\mu - \varphi_\mu(\tau)|^p\,dx \\
&= \int_{-\tau}^{\tau} \mathcal{L}_\mu\,dx + \frac{1}{p}\int_{-\tau}^{\tau} (|\varphi_\mu|^p - |\varphi_\mu - \varphi_\mu(\tau)|^p)\,dx \\
&\le \mathcal{E}(\varphi_\mu) + \frac{1}{p}\int_{-\tau}^{\tau} (|\varphi_\mu|^p - |\varphi_\mu - \varphi_\mu(\tau)|^p)\,dx.
\end{aligned}$$

Now, by the Mean Value Theorem,

$$0 \le |\varphi_\mu(x)|^p - |\varphi_\mu(x) - \varphi_\mu(\tau)|^p \le p\varphi_\mu(\tau)|\varphi_\mu(x)|^{p-1} \qquad \forall x \in [-\tau, \tau],$$

and thus, by (13),

$$\begin{aligned}\mathcal{E}(v_\mu) &\le \mathcal{E}(\varphi_\mu) + \varphi_\mu(\tau)\int_{-\tau}^{\tau} |\varphi_\mu(x)|^{p-1}\,dx \\ &= \mathcal{E}(\varphi_\mu) + \mu^{\alpha p-\beta}\varphi_1(\tau\mu^\beta)\int_{-\tau\mu^\beta}^{\tau\mu^\beta} |\varphi_1(y)|^{p-1}\,dy \\ &\le \mathcal{E}(\varphi_\mu) + \mu^{2\beta+1}\varphi_1(\tau\mu^\beta)\int_{-\infty}^{+\infty} |\varphi_1(y)|^{p-1}\,dy \\ &\le \mathcal{E}(\varphi_\mu) + C\mu^{2\beta+1}\varphi_1(\tau\mu^\beta), \qquad (19)\end{aligned}$$

having set $y = \mu^\beta x$ in the second line (we have also written $\alpha p - \beta$ as $2\beta+1$, which is the same). Now, by the form of φ_1 given in (14), it is easy to check that

$$\begin{aligned}\mu^{2\beta+1}\varphi_1(\tau\mu^\beta) &= \frac{1}{\tau^{2+1/\beta}}(\tau\mu^\beta)^{2+1/\beta}\varphi_1(\tau\mu^\beta) \\ &\le \frac{C}{\tau^{2+1/\beta}}e^{-C\tau\mu^\beta} = \frac{C}{\ell^{2+1/\beta}}e^{-C\ell\mu^\beta},\end{aligned}$$

where all the constants C depend only on p. Setting

$$r_\ell(\mu) := \frac{C}{\ell^{2+1/\beta}}e^{-C\ell\mu^\beta}, \qquad (20)$$

we can write (19) as

$$\mathcal{E}(v_\mu) \le \mathcal{E}(\varphi_\mu) + r_\ell(\mu),$$

which then holds for every $\mu \ge \mu_1$. The quantity on the right–hand side is negative as soon as μ_1 is large enough, since as $\mu \to \infty$, $r_\ell(\mu) \to 0$ and $\mathcal{E}(\varphi_\mu) \to -\infty$ by (15) . Finally, setting

$$\psi(x) := \frac{\sqrt{\mu}}{\|v_\mu\|_{L^2(\mathbb{R})}}v_\mu(x-\tau),$$

we find that $\psi \in H^1(\mathbb{R})$, $\|\psi\|^2_{L^2(\mathbb{R})} = \mu$, $\operatorname{supp}\psi \subset [0, 2\tau] = [0, \ell]$ and

$$\mathcal{E}(\psi) \le \mathcal{E}(v_\mu) \le \mathcal{E}(\varphi_\mu) + r_\ell(\mu),$$

because $\sqrt{\mu} > \|v_\mu\|_{L^2(\mathbb{R})}$ and $\mathcal{E}(v_\mu)$ is negative. This completes the proof. □

Remark 3.3. For future reference we note that the quantity $r_\ell(\mu)$ defined in (20) is decreasing both in μ and in ℓ.

4 The Palais–smale condition

In this section we analyze the Palais–Smale condition for the functional E_M. To be precise, we recall the following definition.

Definition 4.1. Let $c \in \mathbb{R}$. We say that a sequence $(u_n) \subset M$ is a *Palais–Smale sequence* for E_M at level c if, as $n \to \infty$,

(i) $E_M(u_n) \to c$,

(ii) $\|E'_M(u_n)\|_{T'_{u_n}M} \to 0$.

We say that E_M satisfies the *Palais–Smale condition* at level c (shortly, E_M satisfies $(\mathrm{PS})_c$) if every Palais–Smale sequence at level c admits a subsequence converging in M.

The analysis of the Palais–Smale condition takes advantage of the following Gagliardo–Nirenberg inequality on graphs (for a proof see [4, 25]).

Proposition 4.2. *For every $p \in [2, \infty)$ there exists a constant $\mathcal{C}_p > 0$ (depending only on p) such that*

$$\|u\|^p_{L^p(\mathcal{G})} \le \mathcal{C}_p \|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})} \qquad \forall u \in H \tag{21}$$

This inequality implies that every Palais–Smale sequence is bounded in H. Indeed (and more generally), from (21),

$$E_M(u) \ge \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\mathcal{C}_p \mu^{\frac{p+2}{4}} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})} \qquad \forall u \in M.$$

Since $p \in (2, 6)$,

$$\|u\|^2 \le C(1 + E_M(u)) \qquad \forall u \in M, \tag{22}$$

(for some C depending on p and μ), showing that the sublevel sets of E_M are bounded.

Remark 4.3. Since every Palais–Smale sequence u_n is bounded, in view of Remark 2.5, the second condition in Definition 4.1 can be written more conveniently as $J(u_n) \to 0$ in H'. We will do so in the next proposition.

Proposition 4.4. *For every $\mu > 0$, the functional E_M satisfies $(PS)_c$ if and only if $c < 0$.*

Proof. We first show that the Palais–Smale condition does not hold at non-negative levels. For every $c, \mu > 0$, let $a > 0$ and $c_n > 0$ ($n \in \mathbb{N}$) be defined by

$$a^2 = 2c/\mu, \qquad c_n^2 = \sqrt{2c\mu}/n\pi.$$

Identify as usual one of the half–lines of $\mathcal{G}$ with the interval $[0, +\infty)$, and define the sequence

$$u_n(x) = \begin{cases} c_n \sin(ax) & \text{if } x \in [0, 2n\pi/a] \\ 0 & \text{otherwise on } \mathcal{G}. \end{cases}$$

Elementary computations show that u_n is a Palais–Smale sequence for E_M at level $c > 0$ that has no converging subsequences in M, since $u_n \to 0$ uniformly on $\mathcal{G}$. Similarly, if $\xi \in C_0^1(\mathbb{R}^+)$ with $\|\xi\|^2_{L^2(\mathbb{R}^+)} = \mu > 0$ and

$$u_n(x) = \begin{cases} n^{-1/2}\xi(x/n) & \text{if } x \in [0, +\infty) \\ 0 & \text{otherwise on } \mathcal{G}, \end{cases}$$

it easy to see that u_n is a Palais–Smale sequence for E_M at level 0 that again has no converging subsequences in M.

The situation changes at negative levels. Let $(u_n) \subset M$ be a Palais–Smale sequence at level $c < 0$. Since it is bounded in H, up to subsequences, $u_n \rightharpoonup u$ in H and $u_n \to u$ in $L^q_{loc}(\mathcal{G})$, for all $q \in [1, \infty]$. As a consequence, $u_n \to u$ in $L^p(\mathcal{K})$ and, by weak lower semicontinuity,

$$E(u) \le \liminf_n E_M(u_n) = c < 0.$$

Since $E(0) = 0$, this shows that $u \not\equiv 0$. The boundedness of u_n also implies, via (8), that λ_n is bounded; therefore, up to subsequences, $\lambda_n \to \lambda$ as $n \to \infty$. By (10), $J(u_n)u_n = 0$ for every n, so that

$$\begin{aligned} c + o(1) = E_M(u_n) - \frac{1}{p} J(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} |u_n'|^2 \, dx - \frac{\mu}{p}\lambda_n \\ &\ge -\frac{\mu}{p}\lambda_n = -\frac{\mu}{p}\lambda + o(1). \end{aligned}$$

Hence, $\lambda > 0$. Let now $A(u) : H \to \mathbb{R}$ be defined by

$$A(u)v = \int_{\mathcal{G}} u'v' \, dx + \lambda \int_{\mathcal{G}} uv \, dx.$$

By weak convergence of u_n we have $A(u)(u_n - u) \to 0$ as $n \to \infty$. Therefore

$$\begin{aligned} o(1) &= (J(u_n) - A(u))\,(u_n - u) \\ &= \int_{\mathcal{G}} |u_n' - u'|^2 \, dx - \int_{\mathcal{K}} |u_n|^{p-2} \, u_n(u_n - u) \, dx + \lambda_n \int_{\mathcal{G}} u_n(u_n - u) \, dx \\ &\quad - \lambda \int_{\mathcal{G}} u(u_n - u) \, dx \\ &= \int_{\mathcal{G}} |u_n' - u'|^2 \, dx + \lambda \int_{\mathcal{G}} |u_n - u|^2 \, dx + o(1), \end{aligned}$$

by strong convergence of u_n to u in $L^p(\mathcal{K})$ and convergence of λ_n to λ. Since λ is positive, this shows that $u_n \to u$ in M. □

5 Proof of the main result

We are going to prove Theorem 1.2 by means of a classical result in Critical Point Theory (e.g. Proposition 10.8 in [6]). First, we recall the notion of *genus* due to M. A. Krasnosel'skii (for a complete discussion, see [19, 23]).

Definition 5.1. Let $\mathcal{A}$ be the family of sets $A \subset H\backslash\{0\}$ such that A is closed and symmetric (namely, $u \in A \Rightarrow -u \in A$). For every $A \in \mathcal{A}$, the *genus* of A is the natural number defined by

$$\gamma(A) := \min\{n \in \mathbb{N} : \exists \varphi : A \to \mathbb{R}^n\backslash\{0\},\ \varphi \text{ continuous and odd}\}. \tag{23}$$

If no φ as in (23) exists, one sets $\gamma(A) = +\infty$.

Given $\mu > 0$, let M be as in (7). Define, for $j \in \mathbb{N}$,

$$\Gamma_j := \{A \subset M : A \in \mathcal{A},\ A \text{ is compact and } \gamma(A) \geq j\}$$

and

$$c_j := \inf_{A \in \Gamma_j} \max_{u \in A} E_M(u).$$

Note that $-\infty < c_1 \leq c_2 \leq \ldots$, the first inequality following from the fact that E_M is bounded below by (22) and the others by definition of the Γ_j's. By the quoted result in [6], each c_j is a critical level of E_M provided that the Palais–Smale condition holds at c_j. Also, if $c_j = c_{j+1} = \cdots = c_{j+q}$ for some $q \geq 1$, then the set $K = \{u \in M : E_M(u) = c_j,\ E'_M(u) = 0\}$ satisfies $\gamma(K) \geq q + 1$ and, in particular, is infinite.

Now, since the functional E_M satisfies the $(\mathrm{PS})_c$ conditions only if $c < 0$, in order to conclude we have to show that some of the min–max classes Γ_j work at negative levels, and this is where the choice of μ will become essential.

We start with a simple lemma, whose proof is omitted.

Lemma 5.2. *For every $k \in \mathbb{N}$, let $S^{k-1} = \{\theta \in \mathbb{R}^k : |\theta| = 1\}$. Then,*

$$\min_{\theta \in S^{k-1}} \sum_{j=1}^{k} |\theta_j|^p = k^{1-p/2} \qquad \textit{for every} \quad p \geq 2.$$

We are now ready to carry out the proof of the main result.

Proof of Theorem 1.2. Let e be any edge of $\mathcal{K}$, identified with the interval $[0, L]$. For $k \in \mathbb{N}$, let $\mu_{L/k}$ be the number provided by Proposition 3.2 (with the choice $\ell = L/k$), and define

$$\mu_k = k\mu_{L/k}. \tag{24}$$

Now, let M be the manifold (7) corresponding to any choice of $\mu \geq \mu_k$ and, accordingly, let E_M be the functional E restricted to M. Since $\mu/k \geq \mu_k/k = \mu_{L/k}$, Proposition 3.2 yields a function $\psi \in H^1(\mathbb{R})$ such that

(i) $\|\psi\|^2_{L^2(\mathbb{R})} = \mu/k$,

(ii) $\operatorname{supp}\psi \subset [0, L/k]$,

(iii) $\mathcal{E}(\psi) \le \mathcal{E}(\varphi_{\mu/k}) + r_{L/k}(\mu/k) < 0$.

We now construct, with the help of ψ, a compact symmetric set $A \subset M$ such that $\gamma(A) \ge k$ and $\max_{u\in A} E_M(u) < 0$ (this will show that for all $j \le k$, $\Gamma_j \neq \emptyset$ and $c_j < 0$). For every $j = 1, \dots, k$, define $\psi_j : \mathcal{G} \to \mathbb{R}$ as

$$\psi_j(x) = \begin{cases} \psi(x - (j-1)L/k) & \text{if } x \in [(j-1)L/k, jL/k] \\ 0 & \text{otherwise on } \mathcal{G}. \end{cases}$$

Each function ψ_j is therefore supported in the subset of the edge e identified with the subinterval $[(j-1)L/k, jL/k]$. Note that $\psi_i(x)\psi_j(x) = 0$ for every $x \in \mathcal{G}$ and every $i \neq j$. Clearly, $\psi_j \in H$ and

$$\|\psi_j\|^2_{L^2(\mathcal{G})} = \|\psi\|^2_{L^2(\mathbb{R})} = \mu/k, \qquad j = 1, \dots, k.$$

Consider the map $h : S^{k-1} \to H$ defined by

$$h(\theta)(x) = \sqrt{k} \sum_{j=1}^{k} \theta_j \psi_j(x). \tag{25}$$

Obviously, h is continuous, odd, $0 \notin h(S^{k-1})$, and $h(\theta)$ is supported in $[0, L]$. Moreover, for every $\theta \in S^{k-1}$,

$$\begin{aligned} \int_{\mathcal{G}} |h(\theta)(x)|^2\,dx = \int_0^L |h(\theta)(x)|^2\,dx &= k \sum_{j=1}^{k} \theta_j^2 \int_{(j-1)L/k}^{jL/k} |\psi_j(x)|^2\,dx \\ &= k \sum_{j=1}^{k} \theta_j^2 \int_0^{L/k} |\psi(x)|^2\,dx = k\|\psi\|^2_{L^2(\mathbb{R})} \sum_{j=1}^{k} \theta_j^2 = \mu \end{aligned}$$

by (i). Therefore h maps S^{k-1} into M. Similarly,

$$\int_{\mathcal{G}} |h(\theta)'(x)|^2\,dx = k \int_0^{L/k} |\psi'(x)|^2\,dx \sum_{j=1}^{k} \theta_j^2 = k \int_{\mathbb{R}} |\psi'(x)|^2\,dx$$

and

$$\int_{\mathcal{K}} |h(\theta)(x)|^p\,dx = k^{p/2} \int_{\mathbb{R}} |\psi(x)|^p\,dx \sum_{j=1}^{k} |\theta_j|^p \ge k \int_{\mathbb{R}} |\psi(x)|^p\,dx$$

by Lemma 5.2. Therefore, for every $\theta \in S^{k-1}$,

$$\begin{aligned} E_M(h(\theta)) &= \frac{1}{2}\int_{\mathcal{G}} |h(\theta)'(x)|^2\,dx - \frac{1}{p}\int_{\mathcal{K}} |h(\theta)(x)|^p\,dx \\ &\le \frac{k}{2}\int_{\mathbb{R}} |\psi'(x)|^2\,dx - \frac{k}{p}\int_{\mathbb{R}} |\psi(x)|^p\,dx \\ &= k\mathcal{E}(\psi) \le k(\mathcal{E}(\varphi_{\mu/k}) + r_{L/k}(\mu/k)) < 0, \end{aligned} \tag{26}$$

where the last inequalities follow from (iii). Set now $A = h(S^{k-1})$. Then A is compact, symmetric and $A \subset M$. By well–known properties of the genus (see e.g. Chapter 7 in [23])

$$\gamma(A) \ge \gamma(S^{k-1}) = k,$$

so that $A \in \Gamma_k$. By (26), $\max_{u\in A} E_M(u) < 0$, showing that

$$c_k = \inf_{A\in\Gamma_k} \max_{u\in A} E_M(u) < 0.$$

Thus E_M satisfies the Palais–Smale conditions at all levels $c_1 \le \cdots \le c_k$. We then apply Proposition 10.8 in [6] and find, for every $j = 1, \dots, k$, at least one pair of critical points $\pm u_j$ for E_M at level c_j. If two or more of the c_j's coincide, by the quoted result we deduce that E_M has infinitely many critical points.

Next we show that the estimates (6) are satisfied. The computations carried out so far for c_k can be repeated, exactly in the same way, for each c_j with $j < k$. Hence, working with any S^{j-1}, (26) reads

$$E_M(h(\theta)) \le j(\mathcal{E}(\varphi_{\mu/j}) + r_{L/j}(\mu/j)), \qquad j = 1, \dots, k,$$

where $h : S^{j-1} \to H$ is defined according to (25). By (15),

$$\mathcal{E}(\varphi_{\mu/j}) = \frac{1}{j^{2\beta+1}}\mathcal{E}(\varphi_\mu),$$

so that, for every $\theta \in S^{j-1}$,

$$E_M(h(\theta)) \le \frac{1}{j^{2\beta}}\mathcal{E}(\varphi_\mu) + j r_{L/j}(\mu/j)).$$

By Remark 3.3

$$j r_{L/j}(\mu/j)) \le k r_{L/k}(\mu/k)) := \sigma_k(\mu) \qquad j = 1, \dots, k,$$

hence we obtain

$$c_j \le \frac{1}{j^{2\beta}}\mathcal{E}(\varphi_\mu) + \sigma_k(\mu), \qquad j = 1, \dots, k,$$

and also the required level estimate is proved.

Finally, the sign of the Lagrange multiplier λ_j associated to u_j can be immediately deduced from the fact that the energy of u_j is negative via the definition of λ_j:

$$\lambda\mu = \int_{\mathcal{K}} |u_j|^p \, dx - \int_{\mathcal{G}} |u_j'|^2 \, dx > \frac{2}{p} \int_{\mathcal{K}} |u_j|^p \, dx - \int_{\mathcal{G}} |u_j'|^2 \, dx = -2E_M(u_j) > 0.$$

□

A few comments are in order. First, we note that the mass threshold μ_k defined in (24) is a decreasing function of the length L of the edge chosen for the construction carried out in the proof. Therefore, in order to minimize μ_k, and obtain the result for masses as small as possible, it is convenient to choose the longest edge in the compact core of the graph. Alternatively, with a minor modification of the argument, one could also "place" the k copies of the cut–off soliton used for the construction of the set A on *different* edges of the compact core, when this is preferable in order to minimize μ_k.

These considerations also show that there is no need to require that the nonlinearity be present on the whole compact core. A single edge is sufficient in order to obtain the same result, as anticipated in Remark 1.5.

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