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Regularity of the steering control for systems with persistent memory

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Abstract

The following fact is known for large classes of distributed control systems: when the target is regular, there exists a regular steering control. This fact is important to prove convergence estimates of numerical algorithms for the approximate computation of the steering control.

In this paper we extend this property to a class of systems with persistent memory (of Maxwell/Boltzmann type) and we show that it is possible to construct such smooth control via the solution of an optimization problem.

Keyword: Equations with persistent memory, controllability, regularity

MSC: 45K05, 93B03, 93B05, 93C22

1 Introduction

We study the following system where $x \in (0, \pi)$ and $t > 0$:

$$\begin{cases} w''(x, t) = w_{xx}(x, t) + \int_0^t M(t-s)w_{xx}(x, s) ds, & w(0, t) = f(t), \quad w(\pi, t) = 0 \\ w(x, 0) = 0, \quad w'(x, 0) = 0. \end{cases} \quad (1)$$

We assume $M(t) \in H^2(0, T)$ and $f(t) \in L^2(0, T)$ for every $T > 0$. As proved for example in [5], $w(x, t) \in C([0, T]; L^2(0, \pi)) \cap C^1([0, T]; H^{-1}(0, \pi))$ and, for every $(\xi, \eta) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ and $T > 2\pi$, there exists $f \in L^2(0, T)$ such that $w(T) = \xi$, $w'(T) = \eta$. We prove:

Theorem 1. *The following properties hold:*

1. Let $(\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$ and let $T > 2\pi$. There exists a steering control $f \in H_0^1(0, T)$.

2. *One of the smooth steering controls is the integral of the function g which realizes the minimum of a suitable quadratic functional introduced in Sect. 3.*

The statement 1 is proved in Section 2 while statement 2 is in Section 3.

We conclude this introduction with few comments. First we note that system (1) is often encountered in the study of viscoelasticity and diffusion equations with memory. When $M(t) = 0$ of course it reduces to the string equation. In the case of the wave equation (even when x in regions of \mathbb{R}^d , $d > 1$) theorem 1 is known. The proof that we give, based on moment methods, shows in particular controllability (in $H_0^1(0, \pi) \times L^2(0, \pi)$) of the cascade connection of system (1) with an integrator. We refer to [7, Ch. 11] and references therein for this idea and to [8] for a precise analysis of the reachable set using smooth controls in the case of the wave equation.

For memoryless systems, a result analogous to Theorem 1 is the key for a numerical analysis of the construction of steering controls via optimization methods, see [1].

Finally, it is easy to guess that Theorem 1 can be extended to the case $\dim x > 1$ and to higher regularity degree of the target. This will be the subject of a future analysis.

2 The moment problem and the proof of Theorem 1 item 1

The following computations are a bit simplified if we integrate the first equation of (1) on $[0, t]$ and we write it in the equivalent form (here $N(t) = 1 + \int_0^t M(s) ds$)

$$w'(x, t) = \int_0^t N(t-s)w_{xx}(x, s) ds, \quad w(x, 0) = 0, \quad w(0, t) = f(t), \quad w(\pi, t) = 0. \quad (2)$$

We use the orthonormal basis of $L^2(0, \pi)$ whose elements are $\Phi_n = \sqrt{(2/\pi)} \sin nx$, $n \in \mathbb{N}$, and we expand

$$w(x, t) = \sum_{n \in \mathbb{N}} \Phi_n(x)w_n(t), \quad w_n(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi \Phi_n(x)w(x) dx.$$

Then $w_n(x, t)$ must satisfy

$$w'_n(t) = -n^2 \int_0^t N(t-s)w_n(s) ds + n \int_0^t N(t-s) \left(\sqrt{2/\pi} f(s) \right) ds.$$

The function $\sqrt{2/\pi}f$ will be renamed f .

Let $z_n(t)$ solve

$$z'_n(t) = -n^2 \int_0^t N(t-s)z_n(s) ds, \quad z_n(0) = 1. \quad (3)$$

We have (see [3])

$$\begin{aligned} w_n(t) &= n \int_0^t \left(\int_0^{t-s} N(t-s-\tau)z_n(\tau)d\tau \right) f(s) ds = \\ &= \frac{1}{n} \int_0^t \left(\frac{d}{ds} z_n(t-s) \right) f(s) ds, \end{aligned} \quad (4)$$

$$w'_n(t) = n \int_0^t \left(-\frac{d}{ds} \int_0^{t-s} N(t-s-\tau)z_n(\tau)d\tau \right) f(s) ds. \quad (5)$$

We require that a target $(\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$ is reached at time T , i.e. we require $(w(T), w'(T)) = (\xi, \eta)$.

The Fourier expansion of the targets is

$$\xi = \sum_{n=1}^{+\infty} \frac{\xi_n}{n} \Phi_n, \quad \text{and} \quad \eta = \sum_{n=1}^{+\infty} \eta_n \Phi_n, \quad (\{\xi_n\}, \{\eta_n\}) \in l^2(\mathbb{N}) \times l^2(\mathbb{N}).$$

So, controllability to (ξ, η) at time T is equivalent to the existence of a control $f \in L^2(0, T)$ such that $w_n(T) = \xi_n/n$, $w'_n(T) = \eta_n$ for every n . The expression we found for $w_n(t)$ and $w'_n(t)$ suggest that we investigate whether is it possible to solve this problem with

$$f(t) = \int_0^t g(s) ds, \quad g \in L^2(0, T). \quad (6)$$

If this is possible then we have the existence of an H^1 -steering control, and we get a steering control in $H_0^1(0, T)$ if we can find g which satisfies the additional condition

$$\int_0^T g(s) ds = 0. \quad (7)$$

We replace the expression (6) in $w_n(T)$ and $w'_n(T)$ and we integrate by parts. We see that f is an H^1 steering control to (ξ, η) if the following *moment problem* is solvable:

$$\xi_n = \int_0^T g(r) dr - \int_0^T z_n(T-s)g(s) ds, \quad (8)$$

$$\eta_n = \int_0^T \left[n \int_0^{T-s} N(T-s-r)z_n(r) dr \right] g(s) ds = \int_0^T g(T-s) \left(\frac{-z_n'(s)}{n} \right) ds. \quad (9)$$

We multiply equation (9) by i and we sum to (8). Furthermore we impose the additional condition (7). We find the moment problem:

$$\int_0^T Z_n(s)g(T-s) ds = c_0, \quad c_n = \begin{cases} -\xi_n - i\eta_n & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \quad (10)$$

and $Z_n(t) = (z_n(s) + \frac{i}{n}z_n'(s))$ if $n > 0$, $Z_0(t) = 1$. In order to prove statement 1 of Theorem 1, we prove solvability of the moment problem (10).

We note that $\{c_n\}_{n>0}$ is an arbitrary *complex valued* $l^2(\mathbb{N})$ sequence while g is real (when ξ and η are real). We reformulate the moment problem (10) with $n \in \mathbb{Z}$. We proceed as follows: for $n < 0$ we define:

$$z_n(t) = z_{-n}(t), \quad \Phi_n(x) = \Phi_{-n}(x), \quad Z_{-n}(t) = \bar{Z}_n(t).$$

As in [5, Lemma 5.1], we see that the moment problem (10) can be equivalently studied with $n \in \mathbb{Z}$ and g complex valued.

Our goal is the proof that the moment problem (10), $n \in \mathbb{Z}$, is solvable. Even more, we prove that $\{Z_n(t)\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $L^2(0, T)$, provided that $T > 2\pi$.

Remark 2. *The fact that $\{Z_n(t)\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $L^2(0, T)$ implies the following additional information: **1)** the transformation from $g \in L^2(0, T)$ (and so also from $f \in H_0^1(0, T)$) to $(w(T), w'(T)) \in H_0^1(0, \pi) \times L^2(0, \pi)$ is linear and continuous; **2)** the solution $g \in L^2(0, T)$ of minimal norm of the moment problem depends continuously on the target $(\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$. Integrating this function g as in (6) we get the steering control f of minimal norm in $H_0^1(0, T)$ and so the solution $f \in H_0^1(0, T)$ of minimal norm depends continuously on the target $(\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$; **3)** any solution g of the moment problem belongs to $L_{0,T}^2 = \left\{ h \in L^2(0, T) : \int_0^T h(s) ds = 0 \right\}$.*

2.1 The proof that $\{Z_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $L^2(0, T)$, $T > 2\pi$

The proof that $\{Z_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $L^2(0, T)$, $T > 2\pi$, is divided in two steps: in the first one we show that the sequence $\{Z_n\}_{n \in \mathbb{Z} \setminus \{0\}}$ is a Riesz sequence in $L^2(0, T)$. Then we will prove that $\{Z_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $L^2(0, T)$ too. In the proof we use the following definitions and results (see [5, Chp. 3]): a sequence $\{x_n\}$ in a Hilbert space H is:

- a *Riesz sequence* when it is the image of an orthonormal sequence under a linear bounded and boundedly invertible transformation;
- ω -*independent* when the following holds: if $\{\alpha_n\} \in l^2$ and if $\sum_{n=1}^{+\infty} \alpha_n x_n = 0$ (convergence in the norm of H) then $\{\alpha_n\} = 0$.

Let $\{x_n\}$ be a Riesz sequence in the Hilbert space H and let $\{y_n\}$ be quadratically close to $\{x_n\}$, i.e. $\sum \|x_n - y_n\|_H^2 < +\infty$. Then there exists N such that $\{y_n\}_{|n| > N}$ is a Riesz sequence. If furthermore $\{y_n\}$ is ω -independent then it is a Riesz sequence too.

We introduce the notation and $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$.

Step 1: $\{Z_n\}_{n \in \mathbb{Z}'}$ is a Riesz sequence in $L^2(0, T)$, $T > 2\pi$ This part of the proof is contained in [5]. The proof in [5] is quite complex since there $x \in \Omega \subseteq \mathbb{R}^d$, $d \geq 1$. When $d = 1$ the proof is much simplified and goes as we sketch here for completeness.

We put $N'(0) = \gamma$. Using [4, Lemmas 5.2 and 5.5] we get that for every $T > 0$ there exists C such that

$$\sum_{n \in \mathbb{Z}'} \|Z_n(t) - e^{\gamma t} e^{int}\|_{L^2(0, T)}^2 \leq C. \quad (11)$$

Then there exists $N > 0$ such that $\{Z_n\}_{|n| \geq N}$ is a Riesz sequence in $L^2(0, T)$.

We prove that $\{Z_n\}_{n \in \mathbb{Z}'}$ is ω -independent i.e. we prove that $\{\alpha_n\}_{n \in \mathbb{Z}'} = 0$ when $\{\alpha_n\} \in l^2(\mathbb{Z}')$ and

$$\sum_{n \in \mathbb{Z}'} \alpha_n Z_n = 0 \quad \text{i.e.} \quad \sum_{n \in \mathbb{Z}'} \alpha_n \left(z_n + \frac{i}{n} z_n' \right) = 0. \quad (12)$$

Using $T > 2\pi$ and [5, Lemma 3.4] applied twice it is possible to prove that $\alpha_n = \frac{\gamma_n}{n^2}$ with $\{\gamma_n\} \in l^2(\mathbb{Z}')$ (see also [3]). This fact justifies the termwise differentiation of the series (12). Using

$$z_n''(t) = -n^2 N(t) - n^2 \int_0^t N(t-s) z_n'(s) ds \quad (13)$$

we get

$$\int_0^t N(t-s) \left[\sum_{n \in \mathbb{Z}'} \gamma_n \left(z_n(s) + \frac{i}{n} z'_n(s) \right) \right] ds - iN(t) \left[\sum_{n \in \mathbb{Z}'} \frac{\gamma_n}{n} \right] = 0. \quad (14)$$

Computing with $t = 0$ we see that $\sum_{n \in \mathbb{Z}'} n \alpha_n = \sum_{n \in \mathbb{Z}'} \frac{\gamma_n}{n} = 0$ and so, using $N(0) \neq 0$, we get

$$\sum_{n \in \mathbb{Z}'} [n^2 \alpha_n z_n(s) + i n \alpha_n z'_n(s)] = 0 \quad \text{hence} \quad \sum_{n \neq \pm 1, n \in \mathbb{Z}'} \alpha_n (n^2 - 1) \left[z_n + \frac{i z'_n}{n} \right] = 0.$$

Note that $\{\alpha_n (n^2 - 1)\} = \{\alpha_n^{(1)}\} \in l^2(\mathbb{Z}')$. Hence we can start a bootstrap argument and repeat this procedure. After at most $2N$ iteration of the process we get

$$\sum_{|n| > N} \alpha_n^{(N)} Z_n = 0$$

and so $\alpha_n^{(N)} = 0$ when $|n| > N$ since we noted that $\{Z_n\}_{|n| > N}$ is a Riesz sequence in $L^2(0, T)$. We have $\alpha_n^{(N)} = 0$ if and only if $\alpha_n = 0$ and this shows that the series (12) is a finite sum, $\sum_{n \in \mathbb{Z}', |n| \leq N} \alpha_n Z_n = 0$. The proof is now finished since it is easy to prove, as in [5, 6], that *the sequence $\{Z_n(t)\}_{n \in \mathbb{Z}'}$ is linearly independent.*

Step 2: $\{Z_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence Of course, $\{Z_n\}_{n \in \mathbb{Z}}$ is quadratically close to $\{e^{\gamma t} e^{int}\}_{n \in \mathbb{Z}}$. It remains to prove ω -independence, when $T > 2\pi$. We prove $\{\alpha_n\}_{n \in \mathbb{Z}} = 0$ when $\{\alpha_n\} \in l^2(\mathbb{Z})$ and

$$\alpha_0 + \sum_{n \in \mathbb{Z}'} \alpha_n Z_n = 0. \quad (15)$$

Using that constant functions belong to H^1 and [5, Lemma 3.4] applied twice we see that $\alpha_n = \gamma_n/n^2$, $\{\gamma_n\} \in l^2$. So, we can compute termwise the derivatives of both the sides of (15) and we get

$$\sum_{n \in \mathbb{Z}'} \alpha_n \left(z'_n(t) + \frac{i}{n} \left[-n^2 N(t) - n^2 \int_0^t N(t-s) z'_n(s) ds \right] \right) = 0. \quad (16)$$

Computing with $t = 0$ we get $\sum_{n \in \mathbb{Z}'} \alpha_n n = 0$. Then (using (3)) the equation (16) becomes

$$\int_0^t N(t-s) \left[\sum_{n \in \mathbb{Z}'} (\alpha_n n^2 z_n(s) + i \alpha_n n z'_n(s)) \right] ds = 0$$

so that (using again $N(0) \neq 0$ and $\{\alpha_n n^2\} \in l^2$)

$$\sum_{n \in \mathbb{Z}'} \alpha_n n^2 \left[z_n(t) + i \frac{1}{n} z_n'(t) \right] = \sum_{n \in \mathbb{Z}'} \alpha_n n^2 Z_n(t) = 0. \quad (17)$$

The fact that $\{Z_n(t)\}_{n \in \mathbb{Z}'}$ is a Riesz sequence implies that $\{\alpha_n\} = 0$ and so also $\alpha_0 = 0$, as we wanted to prove.

This ends the proof of Statement 1 in Theorem 1.

3 Variational characterization of the steering control and the proof of item 2 of Theorem 1

The fact that $\{Z_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence implies that the moment problem (10) admits solutions $g \in L^2(0, T)$ when $T > 2\pi$. Each one of these functions, once integrated, provides a steering control $f \in H_0^1(0, T)$. In this section we give a variational characterization of a solution g of the moment problem (10) as the minimizer of a quadratic functional, as in [2].

We recall the following definition from Remark 2:

$$L_{0,T}^2 = \left\{ h \in L^2(0, T), \int_0^T h(s) ds = 0 \right\} \subseteq L^2(0, T)$$

and we consider the problem

$$\begin{cases} w''(x, t) = w_{xx}(x, t) + \int_0^t M(t-s) w_{xx}(x, s) ds, \\ y'(t) = g(t) \in L_{0,T}^2, \\ w(0, t) = y(t), \quad w(\pi, t) = 0, \\ w(x, 0) = 0, \quad w'(x, 0) = 0, \quad y(0) = 0. \end{cases} \quad (18)$$

We proved that $(w(T), w'(T)) = (\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$ (and $y(T) = 0$) if and only if g solves the moment problem (10) with $n \in \mathbb{Z}$ (note that the condition $y(T) = 0$ comes for free, implied by $g \in L_{0,T}^2$). The first statement in Remark 2 implies that

$$\Lambda_T \in \mathcal{L}(L_{0,T}^2, H_0^1(0, \pi) \times L^2(0, \pi)) \quad \text{where } \Lambda_T g = (w(T), w'(T)). \quad (19)$$

Let (W_0, W_1) be any element of $L^2(0, \pi) \times H^{-1}(0, \pi)$ and consider

$$\begin{cases} W''(x, t) = W_{xx}(x, t) + \int_0^t M(t-s)W_{xx}(x, s) ds, \\ Y'(t) = \int_0^t M(t-s)W_x(0, s) ds + W_x(0, t), \\ W(0, t) = W(\pi, t) = 0, \\ W(x, 0) = W_0 = \sum_{n=1}^{+\infty} W_n^0 \Phi_n, \quad W'(x, 0) = W_1 = \sum_{n=1}^{+\infty} (nW_n^1) \Phi_n, \quad Y(0) = 0 \end{cases} \quad (20)$$

(note that $\{W_n^0\}, \{W_n^1\}$ belong to l^2).

We introduce the notations $\langle \cdot, \cdot \rangle$ and (\cdot, \cdot) to denote respectively the duality pairing of $H_0^1(0, \pi)$ and $H^{-1}(0, \pi)$ and the inner product in $L^2(0, \pi)$. Assuming first $g \in \mathcal{D}(0, T)$, W_0, W_1 in $\mathcal{D}(0, \pi)$ we multiply the first equation of (18) with $W(x, T-t)$ and the second one with $Y(T-t)$. Then we integrate respectively on $(0, \pi) \times (0, T)$ and on $(0, T)$ and we sum. Standard integrations by parts show that

$$(w'(T), W_0) + \langle w(T), W_1 \rangle = \int_0^T g(s)Y(T-s) ds \quad (21)$$

Using statement 1) in Remark 2, i.e. the continuous dependence of $(w(T), w'(T)) \in H_0^1(0, \pi) \times L^2(0, \pi)$ on $g \in L_{0,T}^2$, we see that

$$\begin{aligned} \left| \int_0^T g(s)Y(T-s) ds \right| &= |(w'(T), W_0) + \langle w(T), W_1 \rangle| \leq \\ &\leq |w'(T)|_{L^2(0,\pi)} |W_0|_{L^2(0,\pi)} + |w(T)|_{H_0^1(0,\pi)} |W_1|_{H^{-1}(0,\pi)} \leq \\ &\leq M [|W_0|_{L^2(0,\pi)} + |W_1|_{H^{-1}(0,T)}] |g|_{L_{0,T}^2}. \end{aligned}$$

So, the transformation $(W_0, W_1) \rightarrow Y(\cdot) \in L^2(0, T)$ admits a continuous extension to $L^2(0, \pi) \times H^{-1}(0, \pi)$ and we see also that $g \in L_{0,T}^2$ steers the solution of (18) to the target $(w(T), w'(T)) = (\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$ if and only if the following equality holds for every $W_0 \in L^2(0, \pi)$, $W_1 \in H^{-1}(0, \pi)$:

$$(\eta, W_0) + \langle \xi, W_1 \rangle = \int_0^T g(s)Y(T-s) ds = \int_0^T g(s) (P_0 Y(T-\cdot)) ds \quad (22)$$

where P_0 is the orthogonal projection of $L^2(0, T)$ onto $L_{0,T}^2$ (easily computed from cosine Fourier expansion).

We introduce the duality pairing of $H_0^1(0, \pi) \times L^2(0, \pi)$ and its dual $H^{-1}(0, \pi) \times L^2(0, \pi)$:

$$\langle\langle (\xi, \eta), (W_1, W_0) \rangle\rangle = (\eta, W_0) + \langle \xi, W_1 \rangle$$

so that Equality (21) takes the form

$$\langle\langle \Lambda_T g, (W_1, W_0) \rangle\rangle = \int_0^T g(s) (P_0 Y(T - \cdot)) \, ds, \text{ hence } \Lambda_T^*(W_1, W_0) = P_0 Y(T - \cdot).$$

Similar to [2], we consider the quadratic functional $(W_1, W_0) \mapsto \mathcal{J}(W_1, W_0)$ on $H^{-1}(0, \pi) \times L^2(0, \pi)$ defined by

$$\begin{aligned} \mathcal{J}(W_1, W_0) &= \frac{1}{2} \int_0^T |P_0 Y(T - \cdot)|^2 \, dt - (\eta, W_0) - \langle \xi, W_1 \rangle = \\ &= \frac{1}{2} \int_0^T |\Lambda_T^*(W_0, W_1)|^2 \, dt - \langle\langle (\xi, \eta), (W_1, W_0) \rangle\rangle. \end{aligned}$$

Computing the Fréchet derivative of \mathcal{J} we see that $(\hat{W}_1, \hat{W}_0) \in H^{-1}(0, \pi) \times L^2(0, \pi)$ is a stationary point if and only if

$$\int_0^T (P_0 Y(T - \cdot)) (P_0 \hat{Y}(T - \cdot)) \, dt - (\eta, W_0) - \langle \xi, W_1 \rangle = 0 \quad \forall (W_1, W_0) \in H^{-1}(0, \pi) \times L^2(0, \pi)$$

(here Y and \hat{Y} are the functions computed from (20) and initial conditions respectively $(W_0, W_1, 0)$ and $(\hat{W}_0, \hat{W}_1, 0)$). We see from here that if (\hat{W}_1, \hat{W}_0) is a stationary point of \mathcal{J} then $\hat{g}(t) = P_0 \hat{Y}(T - \cdot)$ realizes the equality (22), and so it is a steering control.

In order to complete the proof of item 2 of Theorem 1 we note the following result, which implies that \mathcal{J} has a unique stationary point in $H^{-1}(0, \pi) \times L^2(0, \pi)$, which is a minimum point.

Theorem 3. *The functional \mathcal{J} is continuous, coercive and strictly convex on $H^{-1}(0, \pi) \times L^2(0, \pi)$.*

Proof: Convexity is obvious and continuity follows since (19) implies $\Lambda_T^* \in \mathcal{L}(H^{-1}(0, \pi) \times L^2(0, \pi), L^2_{0,T})$. The proof of strict convexity is the same as in [2].

The operator $\Lambda_T g = (w(T), w'(T))$ from $g \in L^2_{0,T}$ to $H^1_0(0, \pi) \times L^2(0, \pi)$ is surjective so that its adjoint Λ_T^* is coercive. So, we have coercivity of the quadratic part of \mathcal{J} , hence of the functional \mathcal{J} itself. ■

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