Achievable Sum Rate of Linear MIMO Receivers with Multiple Rayleigh Scattering

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***Abstract*—We study the performance of multiple-input multiple-output (MIMO) wireless systems employing linear minimum mean-squared error (MMSE) or zero-forcing (ZF) processing at the receiver. In particular, we focus on a source-destination pair communicating through a multiple scattering channel affected by Rayleigh fading. This is an especially relevant case, as it can well represent the communication between a pico-base station and a user in 5G cellular networks. In this scenario, we investigate the system performance in terms of achievable sum rate. In the case of MMSE receiver, we provide a closed-form expression, exploiting the relationship derived by McKay *et al.* [1] between the achievable sum rate and the ergodic mutual information corresponding to optimal nonlinear receivers. For ZF receivers, instead, we leverage the results derived by Matthaiou *et al.* [2] and Jiang *et al.* [3], and derive compact upper and lower bounds to the sum rate. We validate the obtained expressions through numerical results.**

I. Introduction

Linear processing at the receiver side of a MIMO system is a suitable strategy to limit computational burden, while achieving close-to-optimal performance, especially in certain signal-to-noise (SNR) ranges. In spite of their practical relevance, information-theoretic characterization of linear detectors is yet to be performed in closed form but for some results regarding the minimum mean-squared error (MMSE) receiver [1], [4], under the assumption of Rayleigh/Rayleigh-product or uncorrelated Rician fading. Zero-forcing (ZF) receive processing has been investigated by Matthaiou *et al.* in [2], [5] providing bounds to the sum rate in the presence of Rayleigh fading, with and without large-scale Lognormal fading component. Furthermore, an upper bound to the single-branch signal-to-interference-plus-noise-ratio (SINR) in the case of ZF receivers and Rayleigh fading has been given by Jiang *et al.* in [3].

Finding a closed-form expression for the sum rate of MIMO communications in presence of fading and suboptimal receive processing is more difficult than the characterization in the case of optimal reception, due to the expression of the SINR. In [1], the authors unveiled a relationship between the sum rate for linear MIMO receivers and the mutual information conveyed by the same channel with optimal processing at the receiving end. The strategy proposed in [1] finds its easiest application when the channel matrix has independent columns, but the approach can be conveniently extended to the case of channels modeled by a product of independent matrices. A first step in this direction has been made in [4], where the performance of Rayleigh-product channel is investigated along the lines of McKay's result [1]. Throughout our paper, we further extend the analysis to multiple Rayleigh scattering MIMO channels, with an arbitrary number of scattering stages (clusters) and of transmit/receive antennas. Such a fading model is suitable for pico-cellular communication channels [6], foreseen as one of the viable solutions for 5G. We provide first an analysis of the spectral properties of the multiple-scattering channel matrix. Then, relying on [1], we provide a closed-form expression for the sum rate of a MIMO MMSE receiver. Additionally, borrowing results from [2] and [3], we analyze the ZF case and derive an upper and a lower bound to the sum rate.

II. Notation

Boldface uppercase and lowercase letters denote matrices ad vectors, respectively. The identity matrix is indicated by $\mathbf{I}$. The determinant and the conjugate transpose of the generic matrix $\mathbf{A}$ are denoted by $|\mathbf{A}|$ and $\mathbf{A}^{\mathsf{H}}$, respectively, while the (i,j)-th element of $\mathbf{A}$ is indicated by $[\mathbf{A}]_{i,j}$. Moreover, $\mathbb{E}_a[\cdot]$ represents the average operator with respect to the random variable a.

For any $m \times m$ Hermitian matrix $\mathbf{A}$ with eigenvalues $a_1, \dots, a_m$, the Vandermonde determinant is defined as [7, eq. (2.10)]:

$$V(\mathbf{A}) = \prod_{1 \le \ell < k \le m} (a_k - a_\ell)\,. \tag{1}$$

$G^{c,d}_{a,b}(\cdot|\cdot)$, with integer parameters a, b, c, d, denotes the Meijer-G function [8, Ch. 8].

The probability density function of the random variable a is denoted by $f_a(a)$, while its cumulative distribution function (CDF) by $F_a(a)$.

III. System Model

Let us consider a source-destination pair of nodes communicating through a wireless MIMO channel with $N-1$ independent scattering stages, hereinafter referred to as clusters (see Figure 1). Let us denote by n_0 and n_N the number of antennas at the source and destination, respectively. The signal received at the destination can be written as

$$\mathbf{y} = \sqrt{\alpha}\mathbf{H}\mathbf{x} + \mathbf{n} \tag{2}$$

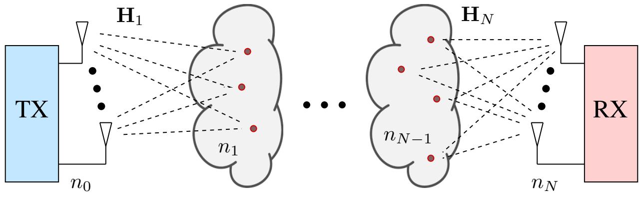

Fig. 1. Scattering channel.

where $\mathbf{y}$ and $\mathbf{x}$ are vectors of size equal to n_N and n_0, respectively. Assuming no CSI at the transmitter, the available transmit power is uniformly distributed over all the n_0 antennas, hence $\mathbf{x}$ is modeled as a random vector with covariance $\mathbb{E}_{\mathbf{x}}[\mathbf{x}\mathbf{x}^{\mathsf{H}}] = \frac{\mathcal{E}_s}{n_0}\mathbf{I}$. Note that the total transmitted energy is $\mathbb{E}_{\mathbf{x}}[\mathbf{x}^{\mathsf{H}}\mathbf{x}] = \mathcal{E}_s$. $\mathbf{H}$ is an $n_N \times n_0$ random channel matrix, hereinafter also referred to as *multiple-scattering channel matrix*. α is a normalization constant defined as

$$\alpha = \frac{n_0 n_N}{\mathsf{Tr}\{\mathbb{E}_{\mathbf{H}}[\mathbf{H}\mathbf{H}^{\mathsf{H}}]\}} \,. \tag{3}$$

As an example, if $N = 1$ and $\mathbf{H}$ has i.i.d. Gaussian complex entries with zero mean and unit variance, we have $\alpha = 1$. Finally, $\mathbf{n}$ is a vector of additive white Gaussian noise with covariance $\mathbb{E}_{\mathbf{n}}[\mathbf{n}\mathbf{n}^{\mathsf{H}}] = \mathcal{N}_0\mathbf{I}$. Under such assumptions, the overall SNR of the system is $\rho = \mathcal{E}_s/\mathcal{N}_0$.

We assume that $\mathbf{x}$ and $\mathbf{H}$ are independent and that the communication between source and destination is affected by Rayleigh fading. Also, each cluster is composed of n_i independent scatterers, $i = 1, \ldots, N-1$. The random channel matrix, $\mathbf{H}$, can be thus expressed as

$$\mathbf{H} = \mathbf{H}_N \ldots \mathbf{H}_i \ldots \mathbf{H}_1 \,, \tag{4}$$

where matrices $\mathbf{H}_i$, independent across i due to the independence of the scattering stages, have size $n_i \times n_{i-1}$, and are complex random with i.i.d. entries whose real and imaginary parts are independent and have a standard normal distribution. Given the communication system under study, in this work we consider $n_0 \le n_1 \le \ldots \le n_N$. Under such assumptions, the normalization constant α is given by

$$\alpha = \frac{n_0 n_N}{\mathsf{Tr}\{\mathbb{E}_{\mathbf{H}}[\mathbf{H}\mathbf{H}^{\mathsf{H}}]\}} = \frac{n_0 n_N}{\prod_{i=0}^{N} n_i} = \prod_{i=1}^{N-1} \frac{1}{n_i} \,. \tag{5}$$

When perfect CSI is available at the receiver, the ergodic mutual information achieved by optimal receive processing is given by:

$$\begin{aligned}
\mathcal{I}(\rho, n_0) &= \mathbb{E}_{\mathbf{H}}\left[\ln\left|\mathbf{I} + \frac{\rho\alpha}{n_0}\mathbf{H}^{\mathsf{H}}\mathbf{H}\right|\right] \\
&= \mathbb{E}_{\mathbf{\Lambda}}\left[\ln\left|\mathbf{I} + \frac{\rho\alpha}{n_0}\mathbf{\Lambda}\right|\right] \\
&= n_0\mathbb{E}_{\lambda}\left[\ln\left(1 + \frac{\rho\alpha}{n_0}\lambda\right)\right] \\
&\overset{(a)}{=} n_0 \int_0^{\infty} \ln\left(1 + \delta\lambda\right) f_{\lambda}(\lambda)\, \mathrm{d}\lambda \,,
\end{aligned} \tag{6}$$

where $\mathbf{\Lambda}$ and λ are, respectively, the diagonal matrix of eigenvalues and an unordered eigenvalue of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$. As far as the equality (a) is concerned, we defined

$$\delta = \frac{\rho\alpha}{n_0} \,. \tag{7}$$

We remark that, although $\mathcal{I}$ depends on several system parameters, for simplicity in (6) we highlighted only the dependency on the SNR, ρ, and on the number of transmit antennas, n_0. The distribution of λ, f_λ too depends on n_0, as highlighted in the last line of (6).

Assuming to employ a linear receiver instead of the optimal one, the system incurs some performance loss. The relationship between the optimal ergodic mutual information and the sum rate achieved by the MMSE receiver has been unveiled in [1]. Therein, compact expressions for achievable rates have been derived in the case of Rayleigh and Rician-faded MIMO channels, under various assumptions on the spatial correlation.

In this work we extend the analysis to the multiple-scattering channel matrix in (4). Furthermore, we analyse the case of ZF receiver for which no closed-form results on the sum rate are available yet. Thus, in this case we derive upper and lower bounds by exploiting the approaches proposed by Matthaiou *et al.* in [2], [5] and by Jiang *et al.* in [3].

IV. Mathematical Background

Hereinafter we list some results on the statistics of multiple-scattering channel matrices, which are useful in our analysis.

Given a multiple-scattering matrix with $N-1$ clusters as in (4), the joint law of the entries of matrices $\mathbf{H}_i$, $i = 1, \ldots, N$, is given by [9]:

$$f_{\mathbf{H}_i}(\mathbf{H}_i) = \mathrm{e}^{-\mathsf{Tr}\{\mathbf{H}_i{}^{\mathsf{H}}\mathbf{H}_i\}}\pi^{-n_i n_{i-1}} \,.$$

We further define the set of auxiliary variables $\nu_i = n_i - n_0$, $i = 1, \ldots, N$. Since we assume $n_0 \le n_1 \le \ldots \le n_N$, such variables are non-negative integers. It is worth mentioning, however, that this assumption can be relaxed based on the observations in [10, Sec. V].

The joint and marginal eigenvalue distributions of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ have been characterized, respectively, in [11] and in [10]. In particular, the joint law of the n_0 eigenvalues of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ can be written as [11]

$$f_{\mathbf{\Lambda}}(\mathbf{\Lambda}) = \frac{V(\mathbf{\Lambda})}{\mathcal{Z}}|\mathbf{G}(\mathbf{\Lambda})| \,, \tag{8}$$

where the normalizing constant $\mathcal{Z}$ is given by [10, Eq.(21)]

$$\mathcal{Z} = n_0! \prod_{i=1}^{n_0}\prod_{\ell=0}^{N} \Gamma(i + \nu_\ell) \,,$$

and $\mathbf{G}(\mathbf{\Lambda})$ is an $n_0 \times n_0$ matrix such that

$$[\mathbf{G}(\mathbf{\Lambda})]_{i,j} = G^{N,0}_{0,N}\left(\begin{matrix} - \\ \nu_N, \ldots, \nu_2, \nu_1 + i - 1 \end{matrix} \,\middle|\, \lambda_j \right) \,,$$

for $i, j = 1, \ldots, n_0$.

Let us now define the $n_0 \times n_0$ matrix $\mathbf{A}_h$ (with $h \in \mathbb{Z}$) with entries

$$[\mathbf{A}_h]_{i,j} = \Gamma(\nu_1 + i + j + h - 1) \prod_{\ell=2}^{N} \Gamma(\nu_\ell + j + h)\,. \quad (9)$$

Then, drawing on [12, Theorem I], the following proposition holds.

Proposition 4.1: The marginal density of a single, unordered eigenvalue λ of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ is given by:

$$f_\lambda(\lambda) = \sum_{i,j=1}^{n_0} \frac{[\mathbf{D}]_{i,j} G_{0,N}^{N,0}\left(\begin{array}{c} - \\ \nu_N, \ldots, \nu_2, \nu_1{+}i{-}1 \end{array} \middle| \lambda \right)}{\lambda^{1-j}\Gamma^{-1}(n_0)\mathcal{Z}} \quad (10)$$

where $[\mathbf{D}]_{i,j}$ is the (i,j)-th entry of the cofactor matrix of $\mathbf{A}_0$.

Proof: The proof is provided in the Appendix. ■

Clearly, f_λ depends on $n_0, n_1, \ldots, n_N$, however, for simplicity, we highlighted the dependency on n_0 only. The above expression differs from that in [10, Formula (52)], which is normalized to the number of eigenvalues n_0, and is based on the classical approach of k-point correlation functions for the density of an arbitrary subset of $k < n_0$ eigenvalues of a given random matrix. In particular, while (10) is a double sum of terms where a single Meijer function appears, the expression in [10] involves products of two Meijer functions. Thus, although equivalent, we chose to use the more compact expression in [12, Theorem I], particularized to our channel model, and to complete it by expliciting the normalizing constant therein.

The CDF of the minimum eigenvalue of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ is also provided, relying on [13], in the following proposition.

Proposition 4.2: The CDF of the minimum eigenvalue $\lambda_{\min}$ of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ is given by:

$$F_{\lambda_{\min}}(\lambda) = 1 - \frac{n_0!|\widetilde{\mathbf{G}}(\lambda)|}{\mathcal{Z}}, \quad (11)$$

where

$$[\widetilde{\mathbf{G}}(\lambda)]_{i,j} = G_{1,N+1}^{N+1,0}\left(\begin{array}{c} 1 \\ 0, \nu_N{+}j, \ldots, \nu_2{+}j, \nu_1{+}i{+}j{-}1 \end{array} \middle| \lambda \right).$$

Proof: The proof is provided in the Appendix. ■

Finally, the Shannon transform of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ is defined as $\mathcal{V}(\delta, n_0) = \mathbb{E}_\lambda[\ln(1{+}\delta\lambda)]$ [14, Def. 2.12], where δ is a positive real number. Its expression for the multiple-scattering channel can be obtained by replacing (10) in the above definition, by writing $\ln(1 + \delta\lambda)$ in terms of a Meijer-G, and by exploiting the properties of the Meijer-G functions [8]:

$$\mathcal{V}(\delta, n_0) = \sum_{i,j=1}^{n_0} \frac{\Gamma(n_0)[\mathbf{D}]_{i,j}}{\mathcal{Z}\delta^j} \cdot G_{2,N+2}^{N+2,1}\left(\begin{array}{c} -j, 1-j \\ -j, -j, \nu_N, \ldots, \nu_2, \nu_1{+}i{-}1 \end{array} \middle| \frac{1}{\delta} \right). \quad (12)$$

Using the definition of the Shannon transform and (6), we can write:

$$\mathcal{I}(\rho, n_0) = n_0 \mathcal{V}\left(\frac{\rho\alpha}{n_0}, n_0\right)\,. \quad (13)$$

V. Preliminary results

The positive and negative moments of λ are given in [10]. Here we provide the expression of the moments of the determinant of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$, which we will exploit later in our analysis. We also derive the first moment of $\ln|\mathbf{H}^{\mathsf{H}}\mathbf{H}|$, which is largely used in MIMO performance analysis (see e.g. [15, and references therein]).

Proposition 5.1: The moments of $|\mathbf{H}^{\mathsf{H}}\mathbf{H}|$ can be expressed as

$$\mathbb{E}_{\mathbf{H}}[|\mathbf{H}^{\mathsf{H}}\mathbf{H}|^h] = \frac{n_0!}{\mathcal{Z}}|\mathbf{A}_h| \qquad h \in \mathbb{N} \quad (14)$$

The proof is given in the Appendix.

Corollary 5.1:

$$\mathbb{E}_{\mathbf{H}}[\ln|\mathbf{H}^{\mathsf{H}}\mathbf{H}|] = \frac{n_0!}{\mathcal{Z}} \sum_{k=1}^{n_0} |\mathbf{A}_0^{(k)}|\,, \quad (15)$$

with $\mathbf{A}_0^{(k)}$ a square matrix of size n_0, whose elements coincide with those of $\mathbf{A}_0$, but for the k-th column, for which [16]

$$[\mathbf{A}_0^{(k)}]_{i,k} = [\mathbf{A}_0]_{i,k}\left[-\gamma + \sum_{t=1}^{\nu_1+i+k-2}\frac{1}{t} + \sum_{\ell=2}^{N}\left(-\gamma + \sum_{t=1}^{\nu_\ell+k-1}\frac{1}{t}\right)\right], \quad (16)$$

where γ is the Euler's constant.

The proof is given in the Appendix.

VI. Communication-theoretic Analysis

Let us consider the MIMO communication channel described in (2). Assuming to employ a linear filter at the receiver output and independent decoding, the MIMO channel can be decomposed into n_0 parallel subchannels. Let ρ_k denote the instantaneous SINR corresponding to the k-th subchannel. Then the achievable sum rate can be written as

$$R \triangleq \sum_{k=1}^{n_0} \mathbb{E}_{\rho_k}[\ln(1 + \rho_k)]\,. \quad (17)$$

The expression of ρ_k depends on the adopted receiving strategy (e.g., MMSE or ZF). Below we provide the exact closed-form expression for the achievable sum rate in the case of MMSE receiver, and an upper and a lower bound in the case of ZF receiver. Notice that the results we present below are based on the eigenanalysis of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$, rather than on the (cumbersome) statistics of ρ_k.

A. MMSE receiver

The MMSE filter for the signal in (2) is given by $\mathbf{F} = \mathbf{H}^{\mathsf{H}}(\mathbf{H}\mathbf{H}^{\mathsf{H}} + \mathbf{I}/\delta)^{-1}$, where δ is as in (7). The k-th component of the filtered signal $\mathbf{F}\mathbf{y}$, has SINR, ρ_k, given by [17, Ch. 6]:

$$\rho_k = \frac{1}{\left[\left(\mathbf{I} + \delta\mathbf{H}^{\mathsf{H}}\mathbf{H}\right)^{-1}\right]_{k,k}} - 1\,. \quad (18)$$

An explicit expression for the pdf of (18) is only available in the canonical Rayleigh case, i.e., when $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ is a central, uncorrelated Wishart matrix with n_N degrees of freedom [18]. However this problem can be circumvented by writing the term $[(\mathbf{I}+\delta\mathbf{H}^{\mathsf{H}}\mathbf{H})^{-1}]_{k,k}$ as [19]

$$\left[\left(\mathbf{I}+\delta\mathbf{H}^{\mathsf{H}}\mathbf{H}\right)^{-1}\right]_{k,k} = \frac{\left|\mathbf{I}+\delta\mathbf{H}^{(k)\mathsf{H}}\mathbf{H}^{(k)}\right|}{\left|\mathbf{I}+\delta\mathbf{H}^{\mathsf{H}}\mathbf{H}\right|} \tag{19}$$

where $\mathbf{H}^{(k)}$ is the matrix obtained by removing the k-th column from $\mathbf{H}$. By using (19) and (18) in (17) (as done also in [4]), we obtain

$$\begin{aligned} R^{\mathrm{MMSE}} =& \sum_{k=1}^{n_0} \mathbb{E}_{\mathbf{H}}\left[\ln\left|\mathbf{I}+\delta\mathbf{H}^{\mathsf{H}}\mathbf{H}\right|\right] \\ & - \sum_{k=1}^{n_0} \mathbb{E}_{\mathbf{H}^{(k)}}\left[\ln\left|\mathbf{I}+\delta\mathbf{H}^{(k)\mathsf{H}}\mathbf{H}^{(k)}\right|\right] . \end{aligned} \tag{20}$$

By using (6), the first term on the r.h.s. of (20) can be written as $n_0\mathcal{I}(\rho,n_0)$. As far as the second term is concerned, this depends on the distribution of the matrix $\mathbf{H}^{(k)}$, which has size $n_N \times n_0 - 1$. By using the definition of $\mathbf{H}$ in (4), $\mathbf{H}^{(k)}$ can be rewritten as

$$\mathbf{H}^{(k)} = \mathbf{H}_N \cdots \mathbf{H}_i \cdots \mathbf{H}_1^{(k)}$$

where $\mathbf{H}_1^{(k)}$ is the matrix obtained by removing the k-th column from $\mathbf{H}_1$. Since the entries of $\mathbf{H}_1$ are i.i.d., we conclude that the term $W=\mathbb{E}_{\mathbf{H}^{(k)}}[\ln|\mathbf{I}+\delta\mathbf{H}^{(k)\mathsf{H}}\mathbf{H}^{(k)}|]$ does not depend on k. Note that W is equivalent to the ergodic mutual information of the linear system $\tilde{\mathbf{y}}=\sqrt{\alpha}\mathbf{H}^{(k)}\tilde{\mathbf{x}}+\tilde{\mathbf{n}}$ where $\mathbb{E}_{\tilde{\mathbf{x}}}[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{\mathsf{H}}]=\frac{\mathcal{E}_s}{n_0}\mathbf{I}$ and $\mathbb{E}_{\tilde{\mathbf{n}}}[\tilde{\mathbf{n}}\tilde{\mathbf{n}}^{\mathsf{H}}]=\mathcal{N}_0\mathbf{I}$. In particular, note that, according to (3), the normalization constant α is the same for both matrices $\mathbf{H}$ and $\mathbf{H}^{(k)}$. It follows that

$$W = \mathcal{I}\left(\rho, n_0-1\right) .$$

In conclusion,

$$\begin{aligned} R^{\mathrm{MMSE}} &= n_0\mathcal{I}(\rho,n_0) - n_0\mathcal{I}\left(\rho,n_0-1\right) \\ &= n_0^2\mathcal{V}(\delta,n_0) - n_0(n_0-1)\mathcal{V}\left(\delta,n_0-1\right) . \end{aligned} \tag{21}$$

From (21) immediately follows that the availability of an explicit expression for the Shannon transform of the channel matrix allows for a closed-form evaluation of the sum rate in the MMSE case.

B. ZF receiver

When the ZF filter is employed at the receiver, the SNR on the k-th sub-channel is given by,

$$\rho_k = \frac{\delta}{\left[\left(\mathbf{H}^{\mathsf{H}}\mathbf{H}\right)^{-1}\right]_{k,k}} . \tag{22}$$

In absence of an exact expression for the sum rate of a MIMO communication with ZF receiver, we work toward bounding R^{ZF}. At first, we exploit the bounds provided in [2], directly derived with reference to the sum rate, rather than on ρ_k, and collect related results in the following proposition.

Proposition 6.1: The sum rate achievable with a ZF receiver over a MIMO channel affected by Rayleigh fading, in presence of multiple scattering, is upper bounded by [2, Thm.1][1]:

$$\begin{aligned} R^{\mathrm{ZF}} \quad \leq & \quad n_0 \ln\left(\mathbb{E}_\lambda\left[\frac{1}{\lambda}\right]+\delta\right) + n_0\mathbb{E}_{\mathbf{H}}[\ln|\mathbf{H}^{\mathsf{H}}\mathbf{H}|] \\ & - \sum_{k=1}^{n_0}\mathbb{E}_{\mathbf{H}^{(k)}}[\ln|\mathbf{H}^{(k)\mathsf{H}}\mathbf{H}^{(k)}|] \\ = & \quad n_0 \ln\left(\mathbb{E}_\lambda\left[\frac{1}{\lambda}\right]+\delta\right) + n_0\mathbb{E}_{\mathbf{H}}[\ln|\mathbf{H}^{\mathsf{H}}\mathbf{H}|] \\ & - n_0\mathbb{E}_{\mathbf{H}^{(k)}}[\ln|\mathbf{H}^{(k)\mathsf{H}}\mathbf{H}^{(k)}|] \end{aligned} \tag{23}$$

where recall that matrix $\mathbf{H}^{(k)}$ is obtained from $\mathbf{H}$ by removing the k-th column. Also, due to the independence and identical distribution of the columns of $\mathbf{H}^{(k)}$, the average $\mathbb{E}_{\mathbf{H}^{(k)}}[\ln|\mathbf{H}^{(k)\mathsf{H}}\mathbf{H}^{(k)}|]$ does not depend on k. Its value can be computed by exploiting Corollary 5.1 and by noting that $\mathbf{H}^{(k)\mathsf{H}}\mathbf{H}^{(k)}$ has size $(n_0-1)\times(n_0-1)$. The expression of the first negative moment of λ can be found in [10, Eq. (59)].

The sum rate is lower bounded by [2, Thm.3]:

$$\begin{aligned} R^{\mathrm{ZF}} \quad \geq & \quad \sum_{k=1}^{n_0} \ln\left(1+\delta \mathrm{e}^{\phi_k}\right) \\ = & \quad n_0 \ln\left(1+\delta \mathrm{e}^{\phi_k}\right) \end{aligned} \tag{24}$$

where for any $k \in \{1,\dots,n_0\}$,

$$\phi_k = \mathbb{E}_{\mathbf{H}}[\ln|\mathbf{H}^{\mathsf{H}}\mathbf{H}|] - \mathbb{E}_{\mathbf{H}^{(k)}}[\ln|\mathbf{H}^{(k)\mathsf{H}}\mathbf{H}^{(k)}|] .$$

An explicit expression of (24) for the channel model at hand is obtained by replacing (15) in the ϕ_k's.

For sake of completeness of our analysis, we also report the upper [3, Eq. (6)] and lower [20, Eq. (8)] bounds to the SINR, both related to the smallest eigenvalue of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$. I.e.,

$$\lambda_{\min}\delta \leq \rho_k \leq \frac{\lambda_{\min}\delta}{u}, \tag{25}$$

where u is a Beta random variable[2], hence $f_u(u)=(n_0-1)(1-u)^{n_0-2}$, $0\leq u\leq 1$. From (25) and the fact that the bounds are independent of k, it follows that

$$R^{\mathrm{ZF}} \quad \geq \quad n_0\mathbb{E}_{\lambda_{\min}}\ln\left(1+\lambda_{\min}\delta\right), \tag{26}$$

while

$$R^{\mathrm{ZF}} \quad \leq \quad n_0\mathbb{E}_{\lambda_{\min},u}\ln\left(1+\frac{\lambda_{\min}\delta}{u}\right) . \tag{27}$$

With regard to the upper bound, due to the independence of $\lambda_{\min}$ and u, (27) can be further expressed as:

$$\begin{aligned} R^{\mathrm{ZF}} \quad \leq & \quad n_0\left\{\mathbb{E}_{\lambda_{\min}}\left[\ln\left(\lambda_{\min}\delta\right)\right]+\mathcal{H}_{n_0-1}\right\} \\ + & \quad \mathbb{E}_{\lambda_{\min}}\left[\frac{{}_2F_1\left(1,1;n_0+1;-\frac{1}{\lambda_{\min}\delta}\right)}{\lambda_{\min}\delta}\right] \end{aligned} \tag{28}$$

[1]This bound explicitly depends on the first negative moment of an unordered eigenvalue of the channel matrix; in case it does not exist, one can resort to the upper bound [2, Thm.2], which holds irrespectively from the availability of $\mathbb{E}_\lambda[\lambda^{-1}]$.

[2]With reference to the proof technique of [3, Lemma V.I], we notice that the rightmost inequality (25) holds for any unitarily invariant matrix, and thus in particular to (4). As to the leftmost one, it holds also for MMSE receivers.

with

$$\mathcal{H}_{n_0-1} = \sum_{\ell=1}^{n_0-1} \frac{1}{\ell}.$$

Thus, an upper bound to the sum rate can be evaluated via numerical integration over the law of $\lambda_{\min}$.

VII. Numerical results

Here we validate the expressions of the mutual information and of the rates derived in the previous sections, against numerical (i.e., Monte Carlo) simulations.

Figure 2 shows the mutual information $\mathcal{I}(\rho, n_0)$, the sum-rates R^{MMSE} and R^{ZF}, and the upper and lower bounds to R^{ZF} plotted against the SNR ρ. In this scenario, we consider a channel with one scattering cluster ($N = 2$), 4 transmit antennas ($n_0 = 4$), 5 scatterers ($n_1 = 5$), and 6 receive antennas ($n_2 = 6$). In the plot, the lines represent the results obtained by evaluating the expressions in (13), (21), (23), (24) and (28). Note that the lower bound in (26) is not shown, as it results to be quite loose. The markers, instead, refer to the results obtained by averaging over $M = 1000$ randomly generated samples of the matrix $\mathbf{H}$. In particular,

- square markers have been obtained by computing

$$\bar{\mathcal{I}}(\rho, n_0) = \frac{1}{M} \sum_{m=1}^{M} \ln |\mathbf{I} + \delta \mathbf{H}^{[m]\mathsf{H}} \mathbf{H}^{[m]}|$$

- circles have been obtained by computing

$$\bar{R}^{\text{MMSE}} = -\frac{1}{M} \sum_{m=1}^{M} \sum_{k=1}^{n_0} \ln \left[\left(\mathbf{I} + \delta \mathbf{H}^{[m]\mathsf{H}} \mathbf{H}^{[m]} \right)^{-1} \right]_{k,k}$$

- triangles have been obtained by computing

$$\bar{R}^{\text{ZF}} = -\frac{1}{M} \sum_{m=1}^{M} \sum_{k=1}^{n_0} \ln \left(1 + \frac{\delta}{\left[\left(\mathbf{H}^{[m]\mathsf{H}} \mathbf{H}^{[m]} \right)^{-1} \right]_{k,k}} \right)$$

where $\mathbf{H}^{[m]}$ is the m-th realization of random matrix $\mathbf{H}$.

The figure shows a perfect match between Monte Carlo and analytical results for the MMSE case.

As far as the ZF case is concerned, upper and lower bounds based on Proposition 6.1 are very tight for high SNR, while at low SNR the upper bound exhibits a floor. In this last SNR range, the upper bound (27) is to be preferred. This is in contrast with the Rayleigh fading case, for which the upper bound in [2] was generally tight over a wide range of SNR. An intuitive explanation of this behavior is provided by the spectral density analysis in [10, Sec. IV]. Therein, it is shown that the marginal eigenvalue density for a non-trivial (i.e., $N \geq 2$) product model exhibits a quite different behavior with respect to the Rayleigh case. Indeed, while (23) depends on the statistics of an unordered eigenvalue of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$, (27) relies on the minimum eigenvalue. On the contrary, the lower bound based on $\lambda_{\min}$, and herein not depicted in any figure, is quite loose in the presence of multiple scattering. A refinement of the lower bounding techniques will be subject of future investigation.

In Figure 3, we compare the sum-rates achieved by the MMSE and ZF (the latter is numerical) filters in the case where $N = 1, 2, 3$ and $n_i = 4$, for $i = 0, \dots, N$. Note that for $N = 1$ the channel reduces to a classical Rayleigh MIMO without scattering clusters. The figure also reports the lower and upper bounds to R^{ZF}. We observe that as N increases, the performance of the system decreases. Also, for the ZF case, the gap between the numerical curve and the upper bound decreases while the lower bound to R^{ZF} tends to become looser .

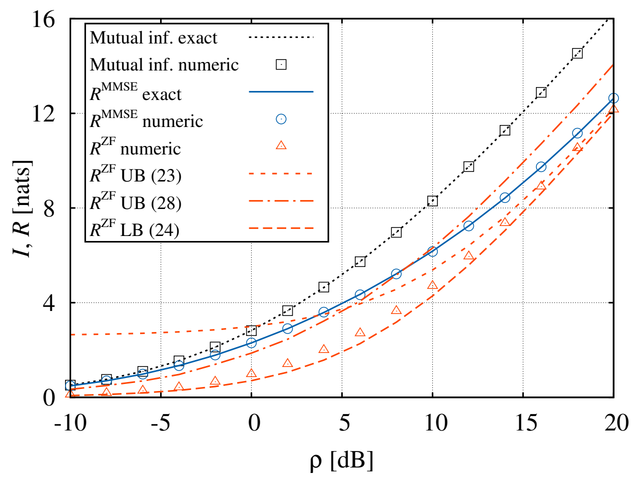

Fig. 2. Ergodic mutual information, sum rate and bounds as functions of the SNR ρ, for $N = 2$, $n_0 = 4$, $n_1 = 5$, and $n_2 = 6$.

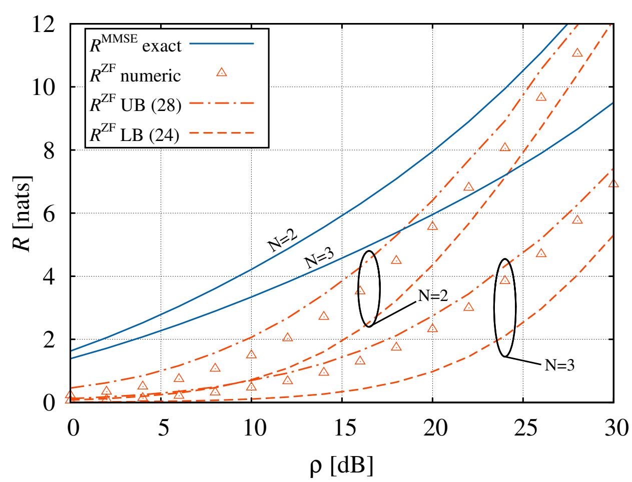

Fig. 3. Sum rates achieved by the MMSE and ZF filters plotted versus the SNR ρ, for $N = 1, 2, 3$, $n_i = 4$, $i = 0, \dots, N$.

VIII. Conclusions

We studied the performance of a MIMO communication system in presence of Rayleigh fading and a multiple-scattering channel between source and destination. We derived the exact closed-form expression for the achievable sum rate in the case of MMSE receivers. When ZF receiver is adopted, we provided a lower and an upper bound to the achievable sum rate by leveraging results available in the literature. Our analysis has been validated by numerical results. Future work will address lower-bounding techniques for ZF receivers, the case of correlated channels and a possible extension to multi-level MIMO relay systems.

APPENDIX A
PROOF OF PROPOSITION 4.1

The marginal density of the unordered eigenvalue of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ can be obtained by applying [12, Theorem I] to the joint pdf in (8), i.e.,

$$f_\lambda(\lambda) = \sum_{i,j=1}^{n_0} \frac{[\mathbf{D}]_{i,j} G_{0,N}^{N,0}\left(\begin{matrix} - \\ \nu_N, \ldots, \nu_2, \nu_1+i-1 \end{matrix} \middle| \lambda \right)}{\lambda^{1-j} K \mathcal{Z}} \quad (29)$$

where K is a proper normalization constant and $[\mathbf{D}]_{i,j}$ is the (i, j)-th entry of the cofactor matrix of $\mathbf{A}_0$.

In order to derive K, we impose $\int f_\lambda(\lambda)\,\mathrm{d}\lambda = 1$. Using the Laplace determinant expansion (as done in the proof of [12, Theorem I]) and applying [22, Corollary I], we obtain:

$$K = \frac{1}{(n_0-1)!} = \frac{1}{\Gamma(n_0)}\,. \quad (30)$$

By replacing (30) in (29), we get the assertion.

APPENDIX B
PROOF OF PROPOSITION 4.2

The CDF of the minimum eigenvalues of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ can be obtained by following the same steps as in [13]. In order to get an expression for $F_{\lambda_{\min}}(\lambda)$, one exploits first [13, Eq. (7)] and then, using (8), applies [22, Corollary I]. By doing so, we get:

$$F_{\lambda_{\min}}(\lambda) = 1 - \frac{n_0! |\widetilde{\mathbf{G}}(\lambda)|}{\mathcal{Z}}, \quad (31)$$

where

$$[\widetilde{\mathbf{G}}(\lambda)]_{i,j} = \int_\lambda^{+\infty} x^{j-1} G_{0,N}^{N,0}\left(\begin{matrix} - \\ \nu_N, \ldots, \nu_2, \nu_1+i-1 \end{matrix} \middle| x \right) \mathrm{d}x,$$

can be written in closed form via [8, (7.811.3)], as reported in the proposition statement.

APPENDIX C
PROOF OF PROPOSITION 5.1 AND COROLLARY 5.1

In order to prove Proposition 5.1, recall that $|\mathbf{H}^{\mathsf{H}}\mathbf{H}| = \prod_{\ell=1}^{n_0} \lambda_\ell$. Then, using (8), we have:

$$\begin{aligned} \mathbb{E}_{\mathbf{H}}[|\mathbf{H}^{\mathsf{H}}\mathbf{H}|^h] &= \frac{1}{\mathcal{Z}} \int_{[0,+\infty)^{n_0}} V(\mathbf{\Lambda}) |\mathbf{G}(\mathbf{\Lambda})| \prod_{i=1}^{n_0} \lambda_i^h \,\mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_{n_0} \\ &= \frac{n_0!}{\mathcal{Z}} |\mathbf{A}_h|, \end{aligned}$$

by virtue of [22, Corollary I]. Note that

$$[\mathbf{A}_h]_{i,j} = \int_{[0,+\infty)} \lambda^{j-1+h} [\mathbf{G}]_{i,j} \,\mathrm{d}\lambda$$

which results to be equal to the expression in (9) [8, 7.811.4].

In order to prove Corollary 5.1, we can write:

$$\begin{aligned} \mathbb{E}_{\mathbf{H}}[\ln|\mathbf{H}^{\mathsf{H}}\mathbf{H}|] &= \frac{\mathrm{d}}{\mathrm{d}s} \mathbb{E}_{\mathbf{H}}[\exp(s \ln|\mathbf{H}^{\mathsf{H}}\mathbf{H}|)] \Big|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s} \mathbb{E}_{\mathbf{H}}[|\mathbf{H}^{\mathsf{H}}\mathbf{H}|^s] \Big|_{s=0} \\ &= \frac{n_0!}{\mathcal{Z}} \frac{\mathrm{d}}{\mathrm{d}s} |\mathbf{A}_s| \Big|_{s=0} \end{aligned} \quad (32)$$

where in the last line we exploited the above Proposition. To compute the derivative of a matrix determinant, we apply the result in [16, Eq. (1)] and write:

$$\frac{\mathrm{d}}{\mathrm{d}s} |\mathbf{A}_s| = \sum_{k=1}^{n_0} |[\mathbf{a}_{s1}, \ldots \dot{\mathbf{a}}_{sk}, \ldots\ldots \mathbf{a}_{sn_0}]| \quad (33)$$

where $\mathbf{a}_{sk}$ is the k-th column of matrix $\mathbf{A}_s$ and $\dot{\mathbf{a}}_{sk}$ denotes the derivative of $\mathbf{a}_{sk}$. The derivative of the generic i-th entry of $\mathbf{a}_{sk}$ is given by:

$$\begin{aligned} [\dot{\mathbf{a}}_{sk}]_i &= \frac{\mathrm{d}}{\mathrm{d}s} \Gamma(\nu_1+i+k+s-1) \prod_{\ell=2}^{N} \Gamma(\nu_\ell+k+s) \\ &= \Gamma(\nu_1+i+k+s-1) \prod_{\ell=2}^{N} \Gamma(\nu_\ell+k+s) \cdot \\ &\quad \left[-\gamma + \sum_{t=1}^{\nu_1+i+k+s-2} \frac{1}{t} + \sum_{\ell=2}^{N} \left(-\gamma + \sum_{t=1}^{\nu_\ell+k+s-1} \frac{1}{t} \right) \right] \\ &= [\mathbf{A}_s]_{i,k} \left[-\gamma + \sum_{t=1}^{\nu_1+i+k+s-2} \frac{1}{t} + \sum_{\ell=2}^{N} \left(-\gamma + \sum_{t=1}^{\nu_\ell+k+s-1} \frac{1}{t} \right) \right] \end{aligned} \quad (34)$$

where γ is Euler's constant. By computing (33) and (34) for $s = 0$ and using the results in (32), we obtain the assertion.

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