BV-norm continuity of sweeping processes driven by a set with constant shape

BV-NORM CONTINUITY OF SWEEPING PROCESSES DRIVEN BY A SET WITH CONSTANT SHAPE

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ABSTRACT. We prove the BV-norm well posedness of *sweeping processes* driven by a moving convex set with constant shape, namely the BV-norm continuity of the so called *play operator* of elasto-plasticity.

1. Introduction

Mathematical models of material memory are often based on the following evolution variational inequality (cf. [15, 36]). Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{Z} \subseteq \mathcal{H}$ be a closed convex subset. Given $T > 0$ and $u : [0,T] \longrightarrow \mathcal{H}$, find $y : [0,T] \longrightarrow \mathcal{H}$ such that

$$\langle z - u(t) + y(t), y'(t) \rangle \leq 0 \qquad \forall z \in \mathcal{Z}, \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T], \tag{1.1}$$

$$u(t) - y(t) \in \mathcal{Z} \qquad \forall t \in [0,T], \tag{1.2}$$

with a given initial condition

$$u(0) - y(0) = z_0 \in \mathcal{Z}, \tag{1.3}$$

where y' denotes the time derivative of y and $\mathcal{L}^1$ is the one dimensional Lebesgue measure. Variational inequalities of the form (1.1)–(1.3) play an important role in elasto-plasticity, ferromagnetism, and phase transitions. It is worth noting that in the new unknown function $x := u - y$, inequality (1.1) can be equivalently formulated as the first order differential inclusion

$$x'(t) + \partial I_{\mathcal{Z}}(x(t)) \ni u'(t) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T], \tag{1.4}$$

$\partial I_{\mathcal{Z}}$ being the subdifferential of the indicator function $I_{\mathcal{Z}}$: $I_{\mathcal{Z}}(x) := 0$ if $x \in \mathcal{Z}$, $I_{\mathcal{Z}}(x) := \infty$ otherwise (precise definitions and formulations will be given in Sections 2 and 3). Problem (1.4)–(1.3) can be solved by using classical tools from the theory of evolution equations governed by maximal monotone operators (cf. [6]). In particular it turns out that for every $u \in W^{1,1}(0,T;\mathcal{H})$, the space of $\mathcal{H}$-valued absolutely continuous maps, there exists a unique $y \in W^{1,1}(0,T;\mathcal{H})$ satisfying (1.1)–(1.3) almost everywhere. The resulting solution operator $\mathsf{P} : W^{1,1}(0,T;\mathcal{H}) \longrightarrow W^{1,1}(0,T;\mathcal{H}) : u \longmapsto y$ is also called *(vector) play operator* following [18, p. 6, p. 151] (see also [33, p. 294]). The suggestive terms *input* and *output* are used for u and y respectively. On the other hand inequality (1.1) can also be interpreted as the time dependent gradient flow

$$y'(t) + \partial I_{u(t)-\mathcal{Z}}(y(t)) \ni 0 \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T]. \tag{1.5}$$

This is a particular case of *sweeping process*, which can be described as follows. Let us denote by $\mathscr{C}_{\mathcal{H}}$ the family of nonempty convex closed subsets of $\mathcal{H}$ and endow it with the Hausdorff metric:

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given $y_0 \in \mathcal{H}$ and $\mathcal{C} \in AC([0,T]\,;\mathscr{C}_{\mathcal{H}})$, the space of $\mathscr{C}_{\mathcal{H}}$-valued absolutely continuous maps, find a function $y \in W^{1,1}(0,T;\mathcal{H})$ such that

$$y(t) \in \mathcal{C}(t) \qquad \forall t \in [0,T]\,, \tag{1.6}$$

$$y'(t) + \partial I_{\mathcal{C}(t)}(y(t)) \ni 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T], \tag{1.7}$$

$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0), \tag{1.8}$$

where $\mathrm{Proj}_{\mathcal{K}}$ denotes the projection operator on a closed convex set $\mathcal{K}$. This problem was studied and solved in [31, 32, 33]. More generally in [34] the important case when $\mathcal{C} \in BV^r([0,T]\,;\mathscr{C}_{\mathcal{H}})$, the space of right continuous maps with bounded variation, is considered. In this case the formulation has to be generalized and one has to find $y \in BV^r([0,T]\,;\mathcal{H})$, a right continuous $\mathcal{H}$-valued function of bounded variation, such that there exists a positive measure μ and a μ-integrable function $v : [0,T] \longrightarrow \mathcal{H}$ satisfying

$$y(t) \in \mathcal{C}(t) \qquad \forall t \in [0,T]\,, \tag{1.9}$$

$$\mathrm{D}y = v\mu, \tag{1.10}$$

$$v(t) + \partial I_{\mathcal{C}(t)}(y(t)) \ni 0 \qquad \text{for } \mu\text{-a.e. } t \in [0,T], \tag{1.11}$$

$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0), \tag{1.12}$$

where $\mathrm{D}y$ denotes the distributional derivative of y. This in particular defines the solution operator $\mathsf{M} : BV^r([0,T]\,;\mathscr{C}_{\mathcal{H}}) \longrightarrow BV^r([0,T]\,;\mathcal{H})$ associating with $\mathcal{C} \in BV^r([0,T]\,;\mathscr{C}_{\mathcal{H}})$ the solution y of (1.9)–(1.12). Moreover if $\mathcal{C}$ is continuous then $\mathsf{M}(\mathcal{C})$ is also continuous, and if $\mathcal{C} \in AC([0,T]\,;\mathscr{C}_{\mathcal{H}})$ then $y = \mathsf{M}(\mathcal{C}) \in W^{1,1}(0,T;\mathcal{H})$ and y satisfies (1.6)–(1.8). Usually one says that (1.11) is the sweeping process driven by the moving convex set $\mathcal{C}$. For the theory of sweeping processes and some of their applications we also refer, e.g., to [30, 10, 2, 35, 11, 45, 16, 3, 4, 5, 13, 46] and their references.

A relevant feature of sweeping processes is their good behavior with respect to the change of time variable (cf. [34, Proposition 2i]): if M is the solution operator of the sweeping process, associating with $\mathcal{C}$ the solution y of (1.9)-(1.12), then we have

$$\mathsf{M}(\mathcal{C} \circ \gamma) = \mathsf{M}(\mathcal{C}) \circ \gamma \tag{1.13}$$

for every continuous increasing time-reparametrization γ. This property is also called *rate independence*. For the theory of rate independent operators and systems we refer, e.g., to [18, 9, 20, 47, 28, 29]. If $\mathcal{C} \in BV([0,T]\,;\mathscr{C}_{\mathcal{H}}) \cap C([0,T]\,;\mathscr{C}_{\mathcal{H}})$, a natural reparametrization of time is given by the (normalized) arc length $\ell_{\mathcal{C}} : [0,T] \longrightarrow [0,T]$ defined by

$$\ell_{\mathcal{C}}(t) := \frac{T}{\mathrm{V}(\mathcal{C},[0,T])}\,\mathrm{V}(\mathcal{C},[0,t]), \qquad t \in [0,T]\,,$$

where $\mathrm{V}(\mathcal{C},[0,t])$ denotes the variation of $\mathcal{C}$ over $[0,t]$. Therefore there exists a Lipschitz continuous mapping $\widetilde{\mathcal{C}}$ such that $\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$, thus the rate independence property yields

$$\mathsf{M}(\mathcal{C}) = \mathsf{M}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}, \tag{1.14}$$

and $\mathsf{M}(\widetilde{\mathcal{C}}) \in \mathit{Lip}([0,T]\,;\mathcal{H})$, the space of Lipschitz functions. This reparametrization method was used by Moreau in [31, 33] in order to reduce the sweeping process driven by an absolutely continuous moving set $\mathcal{C}(t)$ to the Lipschitz continuous case, while the reduction from $BV([0,T]\,;\mathscr{C}_{\mathcal{H}}) \cap C([0,T]\,;\mathscr{C}_{\mathcal{H}})$ to $\mathit{Lip}([0,T]\,;\mathscr{C}_{\mathcal{H}})$ is performed in [40, 42].

Let us observe that if $u \in W^{1,1}([0,T]\,;\mathcal{H})$ and $\mathcal{C}_u \in AC^r([0,T]\,;\mathscr{C}_{\mathcal{H}})$ is defined by $\mathcal{C}_u(t) := u(t) - \mathcal{Z}$, then we have $\mathsf{P}(u) = \mathsf{M}(\mathcal{C}_u)$. This remark naturally leads to the definition of the BV-play operator $\mathsf{P} : BV^r([0,T]\,;\mathcal{H}) \longrightarrow BV^r([0,T]\,;\mathcal{H})$: $\mathsf{P}(u) := \mathsf{M}(\mathcal{C}_u)$ for any $u \in BV^r([0,T]\,;\mathcal{H})$. We can say that *the play operator is a sweeping process driven by a moving convex set with constant shape*. There are other ways to define the play operator for BV inputs: we will provide a proof that P admits an integral variational formulation, to be more precise $y = \mathsf{P}(u)$ is the unique function such that (1.2) and (1.3) hold together with

$$\int_{[0,T]} \langle z(t) - u(t) + y(t), \mathrm{d}\mathrm{D}y(t)\rangle \le 0 \qquad \forall z \in BV^{\mathrm{r}}([0,T]\,;\mathcal{Z}), \tag{1.15}$$

where the integral is computed with respect to the Lebesgue-Stieltjes measure $\mathrm{D}y$. An analogous formulation is given in [22] where the Young integral is used. Of course the play operator enjoys of the rate independence property which reads $\mathsf{P}(u \circ \gamma) = \mathsf{P}(u) \circ \gamma$ for every $u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ and every continuous increasing reparametrization γ of time. In particular if $u \in BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$ and $\ell_u(t) := T\,\mathrm{V}(u,[0,t])/\,\mathrm{V}(u,[0,T])$, $t \in [0,T]$, we have

$$\mathsf{P}(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u, \tag{1.16}$$

where $\widetilde{u} \in \mathit{Lip}([0,T]\,;\mathcal{H})$ is such that $u = \widetilde{u} \circ \ell_u$.

The well-posedness of problem (1.1)–(1.3), i.e. the continuity of the operator P with respect to various topologies, is a fundamental issue both from a theoretical and applicative point of view. The behavior of $\mathsf{P} : BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) \longrightarrow BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ with respect to the topology of uniform convergence can be deduced, e.g., from the general results in [34] (cf. Thereom 3.2 below). The continuity of P with respect to the BV strict topology restricted to $BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$ was proved in [20, Proposition 4.11, p. 46] if the boundary of $\mathcal{Z}$ satisfies suitable regularity conditions, and in [39, Theorem 3.7] for arbitrary $\mathcal{Z}$. In general P is not BV-strict continuous on the whole $BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$, it was proved in [39] that the continuity in the strict topology holds if and only if $\mathcal{Z} = \{x \in \mathcal{H} \ : \ -\alpha \le \langle f, x\rangle \le \beta\}$ for some $f \in \mathcal{H} \smallsetminus \{0\}$ and $\alpha, \beta \in [0,\infty]$. In the one dimensional case it turns out P is always BV-strict continuous on $BV^{\mathrm{r}}([0,T]\,;\mathbb{R})$ (see also [47, 9, 37, 38]).

In this paper we address the problem of the continuity of P with respect to the classical BV-norm topology induced by the norm $\|u\|_{BV} := \|u\|_\infty + \mathrm{V}(u,[0,T])$, $u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$. For absolutely continuous inputs the BV-topology is exactly the standard $W^{1,1}$-topology, and the continuity of the restriction of P to $W^{1,1}(0,T;\mathcal{H})$ was proved in [19] for finite dimensional $\mathcal{H}$ and in [20] for separable Hilbert spaces. For such spaces $\mathcal{H}$, the continuity of P in $BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ (and in $BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$) is known only when $\mathcal{Z}$ has a smooth boundary (cf. [8, 26]), in this case P is even locally Lipschitz continuous. Anyway this regularity assumption turns out be restrictive in many applications.

In the present paper we prove that $\mathsf{P} : BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) \longrightarrow BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ is continuous with respect to the BV-norm topology for *every* arbitrary nonempty closed convex set $\mathcal{Z}$ (and with no separability assumptions on $\mathcal{H}$).

In order to describe what kind of difficulties arise in proving the general BV-norm continuity of P, let us briefly examine the known proofs in the more regular cases.

If the input u belongs to $W^{1,1}(0,T;\mathcal{H})$ and $x(t)$ solves (1.4), then $y'(t) = (\mathsf{P}(u))'(t)$ is a normal vector and $x'(t)$ is a tangential vector to $\mathcal{Z}$ at $x(t)$ in the sense of convex analysis. The proof given in [19] is strongly based on the orthogonal decomposition $u'(t) = x'(t) + y'(t)$. In the general BV case the distributional and the pointwise derivatives are different, so this procedure does not work.

If the input u is an arbitrary BV function, but $\mathcal{Z}$ is smooth, then the proof provided in [26] relies upon an explicit formula for the (unique) unit normal vector to the boundary of $\mathcal{Z}$. If $\mathcal{Z}$ is not smooth there can be several unit normal vectors at a boundary point and this argument cannot be used.

These considerations, together with the rate independence property, suggest to try to use formula (1.16), at least for the continuous case, and somehow "reduce" the problem to the Lipschitz continuous case: indeed if $u \in BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$ then $\mathsf{P}(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u$ and

$\widetilde{u}, \mathsf{P}(\widetilde{u}) \in \mathit{Lip}([0,T]\,;\mathcal{H})$, therefore one can try to get information on the BV-norm continuity of $\mathsf{P}(u)$ by using the above orthogonal decomposition for the arc length reparametrization $\widetilde{u}$.

We are going to show that this procedure is actually possible, thus we are left with the discontinuous case and one can try to extend the previous reparametrization procedure. If $u \in BV^r([0,T]\,;\mathcal{H})$, then the reparametrization $\widetilde{u}$ is a Lipschitz function defined on the image $\ell_u([0,T])$, therefore we need to extend the definition of $\widetilde{u}$ to the whole $[0,T]$, in other words we have to fill in the jumps of u. It is very natural to use segments, i.e. to define the Lipschitz continuous function $\widetilde{u} : [0,T] \longrightarrow \mathcal{H}$ to be affine on every interval $[\ell_u(t-), \ell_u(t)]$ and of course we still have $u = \widetilde{u} \circ \ell_u$. The length function ℓ_u is not continuous anymore, so rate independence does not apply, but anyway one may be tempted to use the formula $\mathsf{P}(\widetilde{u}) \circ \ell_u$. The issue here is that $\mathsf{P}(\widetilde{u}) \circ \ell_u \neq \mathsf{P}(u)$, as shown in [39] (see [24, 25] for a detailed comparison of these two operators). We overcome this problem by considering the more general framework of sweeping processes: we consider the driving moving set $\mathcal{C}_u(t) = u(t) - \mathcal{Z}$ and we fill in the jumps of $\mathcal{C}_u$, (i.e. of u) with a suitable $\mathscr{C}_{\mathcal{H}}$-valued function, indeed using "segments" $(1-t)\mathcal{A} + t\mathcal{B}$ would produce the "wrong" output $\mathsf{P}(\widetilde{u}) \circ \ell_u$. The proper choice is connecting two sets $\mathcal{A}$ and $\mathcal{B}$ by geodesics of the form $\mathcal{C}(t) := (\mathcal{A} + D_{t\rho}) \cap (\mathcal{B} + D_{(1-t)\rho})$, where ρ is the Hausdorff distance between $\mathcal{A}$ and $\mathcal{B}$, and D_r is the closed ball with center 0 and radius r. Indeed in the paper [43] it is proved that if $\mathcal{C} \in BV([0,T]\,;\mathscr{C}_{\mathcal{H}})$ and if $\widetilde{\mathcal{C}} \in \mathit{Lip}([0,T]\,;\mathcal{H})$ is the unique function such that $\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$ and

$$\widetilde{\mathcal{C}}(\ell_{\mathcal{C}}(t-)(1-\lambda) + \ell_{\mathcal{C}}(t)\lambda) = (\mathcal{C}(t-) + D_{\lambda\rho_t}) \cap (\mathcal{C}(t) + D_{(1-\lambda)\rho_t}), \tag{1.17}$$

for $t \in [0,T]$, $\lambda \in [0,1]$, with $\rho_t := d_{\mathscr{H}}(\mathcal{C}(t-), \mathcal{C}(t))$, then $\mathsf{M}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}$ is actually the solution of the sweeping process driven by $\mathcal{C}$, i.e. the formula $\mathsf{M}(\mathcal{C}) = \mathsf{M}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}$ holds.

In our particular situation if $u \in BV^r([0,T]\,;\mathcal{H})$ it follows that $\mathsf{M}(\widetilde{\mathcal{C}_u}) \in \mathit{Lip}([0,T]\,;\mathcal{H})$ is the play operator $\mathsf{P}(\widetilde{u}) \in \mathit{Lip}([0,T]\,;\mathcal{H})$ on the set $\ell_u([0,T])$, where the pointwise derivative can be exploited, while outside of $\ell_u([0,T])$, on the "jump set", we can analyze $\mathsf{P}(u)$ by means of formula (1.17). As a consequence, if $u_n \to u$ in $BV^r([0,T]\,;\mathcal{H})$ then the behaviour of the variation of $\mathsf{P}(u_n) = \mathsf{M}(\mathcal{C}_{u_n})$ can be studied with the help of the formula $\mathsf{P}(u_n) = \mathsf{M}(\mathcal{C}_{u_n}) = \mathsf{M}(\widetilde{\mathcal{C}}_{u_n}) \circ \ell_{\mathcal{C}_{u_n}}$ and the BV-norm continuity can be eventually proved by using tools from vector measure theory.

The paper is organized as follows. In the next section we present some preliminaries and in Section 3 we state our main continuity result. The reparametrization technique for convex-valued functions is adapted to our situation in Section 4 and it is exploited in Section 5 to prove the integral representation for P. In Section 6 we reduce our problem to a Lipschitz continuous sweeping process. All the results of these sections are used in Section 7 to prove the main theorem.

2. Preliminaries

In this section we recall the main definitions and tools needed in the paper. We denote by $\mathbb{N}$ the set of natural numbers (without 0). If E is a Banach space and $x, x_n \in E$ for every $n \in \mathbb{N}$, then the symbol $x_n \rightharpoonup x$ indicates that x_n is weakly convergent to x (cf., e.g., [7]). Given a subset S of the real line $\mathbb{R}$, if $\mathscr{B}(S)$ denotes the family of Borel sets in S, $\mu : \mathscr{B}(S) \longrightarrow [0,\infty]$ is a measure, $p \in [1,\infty]$, then the space of E-valued functions which are p-integrable with respect to μ will be denoted by $L^p(S,\mu;E)$ or simply by $L^p(\mu;E)$. We do not identify two functions which are equal μ-almost everywhere (μ-a.e.). For the theory of integration of vector valued functions we refer, e.g., to [27, Chapter VI]. When $\mu = \mathscr{L}^1$, the one dimensional Lebesgue measure, we write $L^p(S;E) := L^p(S,\mu;E)$.

2.1. **Functions with values in a metric space.** In this subsection we assume that

$$(X,d) \text{ is a complete metric space,} \tag{2.1}$$

where we admit that d is an extended metric, i.e. X is a set and $d : X \times X \longrightarrow [0, \infty]$ satisfies the usual axioms of a distance, but may take on the value ∞. The notion of completeness remains unchanged and the topology induced by d is defined in the usual way by means of the open balls $B_r(x_0) := \{x \in X \ : \ d(x, x_0) < r\}$ for $r > 0$ and $x_0 \in X$, so that it satisfies the first axiom of countability. The general topological notions of interior, closure and boundary of a subset $Y \subseteq X$ will be respectively denoted by $\operatorname{int}(Y)$, $\operatorname{cl}(Y)$ and ∂Y. If $x \in X$ and $Y \subseteq X$, we also set $d(x, Y) := \inf_{y \in Y} d(x, y)$.

If $I \subseteq \mathbb{R}$ is an interval and $f \in X^I$ (the space of X-valued functions defined on I), then $\operatorname{Cont}(f)$ denotes the continuity set of f, and $\operatorname{Discont}(f) := I \smallsetminus \operatorname{Cont}(f)$. The set of X-valued continuous functions defined on I is denoted by $C(I; X)$. For $S \subseteq I$ we write $\operatorname{Lip}(f, S) := \sup\{d(f(s), f(t))/|t - s| \ : \ s, t \in S, \ s \neq t\}$, $\operatorname{Lip}(f) := \operatorname{Lip}(f, I)$, the Lipschitz constant of f, and $Lip(I; X) := \{f \in X^I \ : \ \operatorname{Lip}(f) < \infty\}$, the set of X-valued Lipschitz continuous functions on I. We recall now the notion of BV function with values in a metric space (see, e.g., [1, 48]).

Definition 2.1. *Given an interval $I \subseteq \mathbb{R}$, a function $f \in X^I$, and a subinterval $J \subseteq I$, the* (pointwise) variation of f on J *is defined by*

$$\mathrm{V}(f, J) := \sup\left\{ \sum_{j=1}^{m} d(f(t_{j-1}), f(t_j)) \ : \ m \in \mathbb{N}, \ t_j \in J \ \forall j, \ t_0 < \cdots < t_m \right\}.$$

If $\mathrm{V}(f, I) < \infty$ *we say that* f is of bounded variation on I *and we set* $BV(I; X) := \{f \in X^I \ : \ \mathrm{V}(f, I) < \infty\}$.

It is well known that the completeness of X implies that every $f \in BV(I; X)$ admits one sided limits $f(t-), f(t+)$ at every point $t \in I$, with the convention that $f(\inf I-) := f(\inf I)$ if $\inf I \in I$, and that $f(\sup I+) := f(\sup I)$ if $\sup I \in I$. We set

$$BV^{\mathrm{l}}(I; X) := \{f \in BV(I; X) \ : \ f(t-) = f(t) \quad \forall t \in I\},$$
$$BV^{\mathrm{r}}(I; X) := \{f \in BV(I; X) \ : \ f(t) = f(t+) \quad \forall t \in I\}.$$

If I is bounded we have $Lip(I; X) \subseteq BV(I; X)$.

Definition 2.2. *Assume that $p \in [1, \infty]$. A mapping $f : I \longrightarrow X$ is called* AC^p-absolutely continuous *if there exists $m \in L^p(I; \mathbb{R})$ such that*

$$d(f(s), f(t)) \leq \int_s^t m(\sigma)\, \mathrm{d}\sigma \qquad \forall s, t \in [0, T], \ s \leq t. \tag{2.2}$$

The set of AC^p-absolutely continuous functions is denoted by $AC^p(I; X)$.

Clearly $AC^\infty(I; X) = Lip(I; X)$. If I is bounded then $AC^p(I; X) \subseteq BV(I; X) \cap C(I; X)$ for every $p \in [1, \infty[$, and $f \in AC^1(I; X)$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{j=1}^m d(f(s_k), f(t_k)) < \varepsilon$ whenever $m \in \mathbb{N}$ and $(]s_k, t_k[)_{k=1}^m$ is a family of mutually disjoint intervals with $\sum_{j=1}^m |t_k - s_k| < \delta$. In the next definition we recall two natural metrics in $BV(I; X)$.

Definition 2.3. *For every $f, g \in BV(I; X)$ we set*

$$\mathrm{d}_\infty(f, g) := \sup_{t \in I} d(f(t), g(t)), \tag{2.3}$$
$$\mathrm{d}_{us}(f, g) := \mathrm{d}_\infty(f, g) + |\,\mathrm{V}(f, I) - \mathrm{V}(g, I)|. \tag{2.4}$$

The metric d_∞ and d_{us} are called respectively uniform metric on $BV(I; X)$ *and* uniform strict metric on $BV(I; X)$. *We say that* $u_n \to u$ uniformly strictly on I *if $\mathrm{d}_{us}(u_n, u) \to 0$ as $n \to \infty$. Let us recall that d_{us} is not complete and the topology induced by d_{us} is not linear if X is a Banach space.*

Now we recall the notion of geodesic.

Definition 2.4. *Assume that $x, y \in X$ and $\mathrm{d}(x,y) < \infty$. A function $g \in Lip([0,1]\,;X)$ is called* a geodesic connecting x to y *if $g(0) = x$, $g(1) = y$, and $\mathrm{V}(g,[0,1]) = d(x,y)$.*

2.2. **Some convex analysis.** Let us assume that

$$\begin{cases} \mathcal{H} \text{ is a real Hilbert space with inner product } (x,y) \longmapsto \langle x, y\rangle, \\ \|x\| := \langle x, x\rangle^{1/2}, \end{cases} \tag{2.5}$$

and we endow $\mathcal{H}$ with the natural metric defined by $d(x,y) := \|x-y\|$, $x, y \in \mathcal{H}$. We also use the notation

$$D_r := \{x \in \mathcal{H} \ : \ \|x\| \le r\}, \qquad r \ge 0,$$

and we set

$$\mathscr{C}_{\mathcal{H}} := \{\mathcal{K} \subseteq \mathcal{H} \ : \ \mathcal{K} \text{ nonempty, closed and convex}\}.$$

If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $x \in \mathcal{H}$, then $\mathrm{Proj}_{\mathcal{K}}(x)$ denotes the projection on $\mathcal{K}$, i.e. $y = \mathrm{Proj}_{\mathcal{K}}(x)$ is the unique point such that $d(x,\mathcal{K}) = \|x-y\|$, or equivalently $y \in \mathcal{K}$ and y satisfies the variational inequality

$$\langle x - y, v - y\rangle \le 0 \qquad \forall v \in \mathcal{K}.$$

If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $x \in \mathcal{K}$, then $N_{\mathcal{K}}(x)$ denotes the *(exterior) normal cone of $\mathcal{K}$ at x*:

$$N_{\mathcal{K}}(x) := \{u \in \mathcal{H} \ : \ \langle u, v - x\rangle \le 0 \ \forall v \in \mathcal{K}\} = \mathrm{Proj}_{\mathcal{K}}^{-1}(x) - x. \tag{2.6}$$

We endow the set $\mathscr{C}_{\mathcal{H}}$ with the Hausdorff distance. Here is the definition.

Definition 2.5. *The* Hausdorff distance $\mathrm{d}_{\mathscr{H}} : \mathscr{C}_{\mathcal{H}} \times \mathscr{C}_{\mathcal{H}} \longrightarrow [0,\infty]$ *is defined by*

$$\mathrm{d}_{\mathscr{H}}(\mathcal{A},\mathcal{B}) := \max\left\{\sup_{a\in\mathcal{A}} d(a,\mathcal{B}), \sup_{b\in\mathcal{B}} d(b,\mathcal{A})\right\}, \qquad \mathcal{A},\mathcal{B} \in \mathscr{C}_{\mathcal{H}}. \tag{2.7}$$

Now we recall the notion of subdifferential (cf. [6, Chapter 2]). If $\Psi : \mathcal{H} \longrightarrow [0,\infty]$ is convex and lower semicontinuous and $D(\Psi) := \{x \in \mathcal{H} \ : \ \Psi(x) \ne \infty\} \ne \varnothing$, then for $x \in \mathcal{H}$ the *subdifferential of ϕ at x* is defined by $\partial\Psi(x) := \{y \in \mathcal{H} \ : \ \langle y, v - x\rangle + \Psi(x) \le \Psi(v) \ \forall v \in \mathcal{H}\}$. The *domain* of $\partial\Psi$ is defined by $D(\partial\Psi) := \{x \in \mathcal{H} \ : \ \partial\Psi(x) \ne \varnothing\}$. If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $I_{\mathcal{K}}$, the *indicator function of $\mathcal{K}$*, is defined by $I_{\mathcal{K}}(x) = 0$ if $x \in \mathcal{K}$ and $I_{\mathcal{K}}(x) = \infty$ if $x \notin \mathcal{K}$, then $\partial I_{\mathcal{K}}(x) = N_{\mathcal{K}}(x)$ for every $x \in D(I_{\mathcal{K}}) = D(\partial I_{\mathcal{K}}) = \mathcal{K}$.

2.3. **Differential measures.** Let E be a Banach space with norm $\|\cdot\|_E$ and let $I \subseteq \mathbb{R}$ be an interval. We recall that a *(Borel) vector measure* on I is a map $\mu : \mathscr{B}(I) \longrightarrow E$ such that $\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ whenever (B_n) is a sequence of mutually disjoint sets in $\mathscr{B}(I)$. The *total variation of μ* is the positive measure $|\mu| : \mathscr{B}(I) \longrightarrow [0,\infty]$ defined by

$$|\mu|(B) := \sup\left\{\sum_{n=1}^{\infty} \|\mu(B_n)\|_E \ : \ B = \bigcup_{n=1}^{\infty} B_n, \ B_n \in \mathscr{B}(I), \ B_h \cap B_k = \varnothing \text{ if } h \ne k\right\}.$$

The vector measure μ is said to be *with bounded variation* if $|\mu|(I) < \infty$. In this case the equality $\|\mu\| := |\mu|(I)$ defines a norm on the space of measures with bounded variation (see, e.g. [14, Chapter I, Section 3]).

If $\nu : \mathscr{B}(I) \longrightarrow [0,\infty]$ is a positive bounded Borel measure and if $g \in L^1(I,\nu;E)$, then $g\nu$ will denote the vector measure defined by $g\nu(B) := \int_B g \,\mathrm{d}\nu$ for every $B \in \mathscr{B}(I)$. In this case $|g\nu|(B) = \int_B \|g(t)\|_E \,\mathrm{d}\nu$ for every $B \in \mathscr{B}(I)$ (see [14, Proposition 10, p. 174]).

Let E_j, $j = 1,2,3$, be Banach spaces with norms $\|\cdot\|_{E_j}$ and let $E_1 \times E_2 \longrightarrow E_3 : (x_1,x_2) \longmapsto x_1 \bullet x_2$ be a bilinear form such that $\|x_1 \bullet x_2\|_{E_3} \le \|x_1\|_{E_1}\|x_2\|_{E_2}$ for every $x_j \in E_j$, $j = 1,2$. Assume that $\mu : \mathscr{B}(I) \longrightarrow E_2$ is a vector measure with bounded variation and let $f : I \longrightarrow E_1$ be a *step map with respect to μ*, i.e. there exist $f_1, \ldots, f_m \in E_1$ and $A_1, \ldots, A_m \in \mathscr{B}(I)$ mutually disjoint such that $|\mu|(A_j) < \infty$ for every j and $f = \sum_{j=1}^m \mathbb{1}_{A_j} f_j$, where $\mathbb{1}_S$ is the characteristic function of a set S, i.e. $\mathbb{1}_S(x) := 1$ if $x \in S$ and $\mathbb{1}_S(x) := 0$ if $x \notin S$. For such f we define $\int_I f \bullet \mathrm{d}\mu := \sum_{j=1}^m f_j \bullet \mu(A_j) \in E_3$. If $St(|\mu|; E_1)$ is the set of E_1-valued maps with respect to μ, then the map $St(|\mu|; E_1) \longrightarrow E_3 : f \longmapsto \int_I f \bullet \mathrm{d}\mu$ is linear and continuous when $St(|\mu|; E_1)$ is endowed with the L^1-semimetric $\|f-g\|_{L^1(|\mu|;E_1)} := \int_I \|f-g\|_{E_1} \mathrm{d}|\mu|$. Therefore it admits a unique continuous extension $\mathsf{I}_\mu : L^1(|\mu|; E_1) \longrightarrow E_3$ and we set

$$\int_I f \bullet \mathrm{d}\mu := \mathsf{I}_\mu(f), \qquad f \in L^1(|\mu|; E_1).$$

We will use the previous integral in two particular cases, namely when

a) $E_1 = \mathbb{R}$, $E_2 = E_3 = \mathcal{H}$, $\lambda \bullet x := \lambda x$ ($\int_I f \bullet \mathrm{d}\mu = \int_I f \,\mathrm{d}\mu$, integral of a real function with respect to a vector measure);

b) $E_1 = E_2 = \mathcal{H}$, $E_3 = \mathbb{R}$, $x_1 \bullet x_2 := \langle x_1, x_2\rangle$ ($\int_I f \bullet \mathrm{d}\mu = \int_I \langle f, \mathrm{d}\mu\rangle$, integral of a vector function with respect to a vector measure).

In the situation (b) with $\mu = g\nu$, ν bounded positive measure and $g \in L^1(\nu; \mathcal{H})$, arguing first on step functions, and then taking limits, it is easy to check that $\int_I \langle f, \mathrm{d}(g\nu)\rangle = \int_I \langle f, g\rangle \,\mathrm{d}\nu$ for every $f \in L^\infty(\mu; \mathcal{H})$. The following results (cf., e.g., [14, Section III.17.2-3, pp. 358-362]) provide a connection between functions with bounded variation and vector measures.

Theorem 2.1. *For every $f \in BV(I; \mathcal{H})$ there exists a unique vector measure of bounded variation $\mu_f : \mathscr{B}(I) \longrightarrow \mathcal{H}$ such that*

$$\mu_f(]c,d[) = f(d-) - f(c+), \qquad \mu_f([c,d]) = f(d+) - f(c-),$$
$$\mu_f([c,d[) = f(d-) - f(c-), \qquad \mu_f(]c,d]) = f(d+) - f(c+).$$

whenever $c < d$ and the left hand side of each equality makes sense.

Vice versa if $\mu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ is a vector measure with bounded variation, and if $f_\mu : I \longrightarrow \mathcal{H}$ is defined by $f_\mu(t) := \mu([\inf I, t[\cap I)$, then $f_\mu \in BV(I; \mathcal{H})$ and $\mu_{f_\mu} = \mu$.

Proposition 2.1. *Let $f \in BV(I; \mathcal{H})$, let $g : I \longrightarrow \mathcal{H}$ be defined by $g(t) := f(t-)$, for $t \in \mathrm{int}(I)$, and by $g(t) := f(t)$, if $t \in \partial I$, and let $V_g : I \longrightarrow \mathbb{R}$ be defined by $V_g(t) := \mathrm{V}(g, [\inf I, t] \cap I)$. Then $\mu_g = \mu_f$ and $|\mu_f| = \mu_{V_g} = \mathrm{V}(g, I)$.*

The measure μ_f is called *Lebesgue-Stieltjes measure* or *differential measure* of f. Let us see the connection with the distributional derivative. If $f \in BV(I; \mathcal{H})$ and if $\overline{f} : \mathbb{R} \longrightarrow \mathcal{H}$ is defined by

$$\overline{f}(t) := \begin{cases} f(t) & \text{if } t \in I \\ f(\inf I) & \text{if } \inf I \in \mathbb{R},\ t \notin I,\ t \le \inf I \\ f(\sup I) & \text{if } \sup I \in \mathbb{R},\ t \notin I,\ t \ge \sup I \end{cases}, \tag{2.8}$$

then, as in the scalar case, it turns out (cf. [39, Section 2]) that $\mu_f(B) = \mathrm{D}\overline{f}(B)$ for every $B \in \mathscr{B}(\mathbb{R})$, where $\mathrm{D}\overline{f}$ is the distributional derivative of $\overline{f}$, i.e.

$$-\int_{\mathbb{R}} \varphi'(t)\overline{f}(t)\,\mathrm{d}t = \int_{\mathbb{R}} \varphi \,\mathrm{d}\mathrm{D}\overline{f} \qquad \forall \varphi \in C_c^1(\mathbb{R}; \mathbb{R}),$$

$C_c^1(\mathbb{R}; \mathbb{R})$ being the space of real continuously differentiable functions on $\mathbb{R}$ with compact support. Observe that $\mathrm{D}\overline{f}$ is concentrated on I: $\mathrm{D}\overline{f}(B) = \mu_f(B \cap I)$ for every $B \in \mathscr{B}(I)$, hence in the remainder of the paper, if $f \in BV(I, \mathcal{H})$ then we will simply write

$$\mathrm{D}f := \mathrm{D}\overline{f} = \mu_f, \qquad f \in BV(I; \mathcal{H}), \tag{2.9}$$

and from the previous discussion it follows that

$$\|\mathrm{D}f\| = |\mathrm{D}f|([0,T]) = \|\mu_f\| \qquad \forall f \in BV(I; \mathcal{H}). \tag{2.10}$$

If I is bounded and $p \in [1,\infty]$, then the classical Sobolev space $W^{1,p}(I;\mathcal{H})$ consists of those functions $f \in C(I;\mathcal{H})$ such that $\mathrm{D}f = g\mathcal{L}^1$ for some $g \in L^p(I;\mathcal{H})$ and we have $W^{1,p}(I;\mathcal{H}) = AC^p(I;\mathcal{H})$. Let us also recall that if $f \in W^{1,1}(I;\mathcal{H})$ then the derivative $f'(t)$ exists for $\mathcal{L}^1$-a.e. in $t \in I$, $\mathrm{D}f = f'\mathcal{L}^1$, and $\mathrm{V}(f,I) = \int_I \|f'(t)\|\,\mathrm{d}t$ (cf., e.g. [6, Appendix]).

In [39, Lemma 6.4 and Theorem 6.1] it is proved that

Proposition 2.2. *Assume that $J \subseteq \mathbb{R}$ is a bounded interval and $h : I \longrightarrow J$ is nondecreasing.*

(i) $\mathrm{D}h(h^{-1}(B)) = \mathcal{L}^1(B)$ *for every* $B \in \mathscr{B}(h(\mathrm{Cont}(h)))$.

(ii) *If $f \in Lip(J;\mathcal{H})$ and $g : I \longrightarrow \mathcal{H}$ is defined by*

$$g(t) := \begin{cases} f'(h(t)) & \text{if } t \in \mathrm{Cont}(h) \\ \dfrac{f(h(t+)) - f(h(t-))}{h(t+) - h(t-)} & \text{if } t \in \mathrm{Discont}(h) \end{cases},$$

then $f \circ h \in BV(I;\mathcal{H})$ and $\mathrm{D}(f \circ h) = g\,\mathrm{D}h$. This result holds even if f' is replaced by any of its $\mathcal{L}^1$-representatives.

In the remainder of the paper we address our attention to left and right continuous functions of bounded variation on a compact interval $[a,b]$, $(-\infty < a < b < \infty)$. In this case we have

$$\|\mathrm{D}f\| = \|\mu_f\| = \mathrm{V}(f,]a,b[) + \|f(a+) - f(a)\| = \mathrm{V}(f,[a,b]) \qquad \forall f \in BV^l([a,b]\,;\mathcal{H}), \tag{2.11}$$

$$\|\mathrm{D}f\| = \|\mu_f\| = \mathrm{V}(f,]a,b[) + \|f(b) - f(b-)\| = \mathrm{V}(f,[a,b]) \qquad \forall f \in BV^r([a,b]\,;\mathcal{H}), \tag{2.12}$$

therefore when we consider left (resp. right) continuous functions we are essentially dealing with Lebesgue equivalence classes of functions with a special view on the initial point a (resp. final point b), allowing us to take into account Dirac masses at a or b. We will consider on $BV([a,b]\,;\mathcal{H})$ the classical BV-norm defined by

$$\|f\|_{BV} := \|f\|_\infty + \mathrm{V}(f,[a,b]), \qquad f \in BV([a,b]\,;\mathcal{H}), \tag{2.13}$$

which is equivalent to the norm defined by $|||f|||_{BV} := \|f(0)\| + \mathrm{V}(f,[a,b])$, $f \in BV([a,b]\,;\mathcal{H})$. Observe that we have

$$\|f\|_{BV} = \|f\|_\infty + \|\mathrm{D}f\| \qquad \forall f \in BV^l([a,b]\,;\mathcal{H}) \cup BV^r([a,b]\,;\mathcal{H}).$$

The topology induced by d_{us} is clearly weaker than the one induced by $\|\cdot\|_{BV}$.

3. Statement of the main result

In this section we state the main theorem of the present paper. To this aim we recall the well known existence results about sweeping processes and the play operator.

We assume that

$$\mathcal{Z} \in \mathscr{C}_{\mathcal{H}}, \tag{3.1}$$

$$0 < T < \infty. \tag{3.2}$$

Let us start with the general existence result for sweeping processes proved in [34].

Theorem 3.1. *If $\mathcal{C} \in BV^r([0,T]\,;\mathscr{C}_{\mathcal{H}})$ and $y_0 \in \mathcal{H}$, then there is a unique $\mathsf{M}(y_0,\mathcal{C}) := y \in BV^r([0,T]\,;\mathcal{H})$, such that there exist a measure $\mu : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$ and a function $v \in L^1(\mu;\mathcal{H})$ satisfying*

$$y(t) \in \mathcal{C}(t) \qquad \forall t \in [0,T]\,, \tag{3.3}$$

$$\mathrm{D}y = v\mu, \tag{3.4}$$

$$v(t) + \partial I_{\mathcal{C}(t)}(y(t)) \ni 0 \qquad \text{for } \mu\text{-a.e. } t \in [0,T], \tag{3.5}$$

$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0). \tag{3.6}$$

The resulting solution operator $\mathsf{M} : \mathcal{H} \times BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}}) \longrightarrow BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *is continuous when* $BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}})$ *is endowed with the topology induced by* d_{us} *and* $BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *is endowed with the topology induced by* d_{∞}. *We have* $\mathsf{M}(\mathcal{H} \times BV([0,T]\,;\mathscr{C}_{\mathcal{H}}) \cap C([0,T]\,;\mathscr{C}_{\mathcal{H}})) \subseteq BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$. *For every* $p \in [1,\infty]$ *we have* $\mathsf{M}(\mathcal{H} \times AC^p([0,T]\,;\mathscr{C}_{\mathcal{H}})) \subseteq W^{1,p}(0,T;\mathcal{H})$, *and if* $\mathcal{C} \in AC^p([0,T]\,;\mathscr{C}_{\mathcal{H}})$ *then* $y = \mathsf{M}(y_0,\mathcal{C})$ *is the unique function satisfying* (3.3), (3.6), *and*

$$y'(t) + \partial I_{\mathcal{C}(t)}(y(t)) \ni 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T]. \tag{3.7}$$

In this case it holds

$$\mathrm{Lip}(y) \le \mathrm{Lip}(\mathcal{C}). \tag{3.8}$$

Remark 3.1. The uniqueness and existence statements of the previous theorem are provided in [34, Proposition 3a, Proposition 3b]. The continuity of M is proved in [34, Proposition 2g]. The regularity statements and (3.8) are proved in [34, Corollary 2c] (for $p \in \,]1,\infty[$ see [40, Proposition 3.2]), while (3.7) is shown in [34, Proposition 3c].

The differential inclusion (3.5) is usually called *sweeping process driven by the moving convex set* $\mathcal{C}$. Now we recall the definition of the *play operator*, that is the operator solution of the sweeping process driven by a moving convex set with constant shape.

Definition 3.1. *For any* $u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *define* $\mathcal{C}_u \in BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}})$ *by*

$$\mathcal{C}_u(t) := u(t) - \mathcal{Z}, \qquad t \in [0,T]\,. \tag{3.9}$$

The operator $\mathsf{P} : \mathcal{Z} \times BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) \longrightarrow BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *defined by*

$$\mathsf{P}(z_0,u) := \mathsf{M}(u(0) - z_0, \mathcal{C}_u), \qquad z_0 \in \mathcal{Z},\ u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H}), \tag{3.10}$$

is called play operator (with characteristic $\mathcal{Z}$).

The following theorem will be proved in Section 4.3 and summarizes the main properties of P inherited by Theorem 3.1. It also provides an integral variational characterization of P.

Theorem 3.2. *The operator* $\mathsf{P} : \mathcal{Z} \times BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) \longrightarrow BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *is continuous if* $BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *is endowed with the topology induced by* d_{us} *in the domain and by* d_{∞} *in the codomain. We have* $\mathsf{P}(\mathcal{Z} \times BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})) \subseteq BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$. *Moreover if* $z_0 \in \mathcal{Z}$ *and* $u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$, *then* $\mathsf{P}(z_0,u) = y \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *is the unique function such that*

$$u(t) - y(t) \in \mathcal{Z} \qquad \forall t \in [0,T], \tag{3.11}$$

$$\int_{[0,T]} \langle z(t) - u(t) + y(t), \mathrm{d}\mathrm{D}y(t) \rangle \le 0 \qquad \forall z \in BV^{\mathrm{r}}([0,T]\,;\mathcal{Z}), \tag{3.12}$$

$$u(0) - y(0) = z_0. \tag{3.13}$$

Equivalently $\mathsf{P}(z_0,u) = y \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *is the unique function satisfying* (3.11), (3.13), *and*

$$\int_{[0,T]} \langle z(t) - u(t) + y(t), \mathrm{d}\mathrm{D}y(t) \rangle \le 0 \qquad \forall z \in L^{\infty}([0,T]\,;\mathcal{H}),\ z([0,T]) \subseteq \mathcal{Z}. \tag{3.14}$$

For every $p \in [1,\infty]$ *we have* $\mathsf{P}(\mathcal{Z} \times W^{1,p}(0,T;\mathcal{H})) \subseteq W^{1,p}(0,T;\mathcal{H})$ *and if* $u \in W^{1,p}(0,T;\mathcal{H})$ *then* $\mathsf{P}(z_0,u) = y$ *is the unique function satisfying* (3.11), (3.13), *and*

$$\langle z(t) - u(t) - y(t), y'(t) \rangle \le 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T]\,,\ \forall z \in \mathcal{Z}. \tag{3.15}$$

Remark 3.2. The integral formulation (3.12) (or (3.14)) is analogous to the formulation used in [22] where the Young integral is used. Our proof in Section 4.3 is completely independent and uses only tools from differentiation theory.

Here is the main theorem that we will prove in Section 7.

Theorem 3.3. *The play operator* $\mathsf{P} : \mathcal{Z} \times BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) \longrightarrow BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *is continuous if* $BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *is endowed with the topology induced by the* BV*-norm* (2.13).

Remark 3.3. If we deal with left continuous functions the formulations of our problem need to be modified accordingly. More precisely if $\mathcal{C} \in BV^{\mathrm{l}}([0,T]\,;\mathscr{C}_{\mathcal{H}})$ and $y_0 \in \mathcal{H}$ then there is a unique $\mathsf{M}(y_0,\mathcal{C}) := y \in BV^{\mathrm{l}}([0,T]\,;\mathcal{H})$ such that there exist a measure $\mu : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$ and a function $v \in L^1(\mu;\mathcal{H})$ satisfying (3.3), (3.4), (3.6) and

$$v(t) + \partial I_{\mathcal{C}(t+)}(y(t+)) \ni 0 \qquad \text{for } \mu\text{-a.e. } t \in [0,T].$$

This can be easily proved by adapting the proof of [34] to the left continuous case, or one can argue by reducing to Lipschitz inputs by using exacly the same argument as in [43, Theorem 6.1]. The play operator $\mathsf{P} : \mathcal{Z} \times BV^{\mathrm{l}}([0,T]\,;\mathcal{H}) \longrightarrow BV^{\mathrm{l}}([0,T]\,;\mathcal{H})$ is defined by $\mathsf{P}(z_0,u) := \mathsf{M}(u(0) - z_0, \mathcal{C}_u)$ for $z_0 \in \mathcal{Z}$, $u \in BV^{\mathrm{l}}([0,T]\,;\mathcal{H})$, where $\mathcal{C}_u \in BV^{\mathrm{l}}([0,T]\,;\mathscr{C}_{\mathcal{H}})$ is given by $\mathcal{C}_u(t) := u(t) - \mathcal{Z}$, $t \in [0,T]$. Moreover $\mathsf{P}(z_0,u) = y \in BV^{\mathrm{l}}([0,T]\,;\mathcal{H})$ is the unique function satisfying (3.11), (3.13), and

$$\int_{[0,T]} \big\langle z(t) - u(t+) + y(t+), \mathrm{d}\mathrm{D}y(t) \big\rangle \le 0 \qquad \forall z \in BV^{\mathrm{l}}([0,T]\,;\mathcal{Z}).$$

A motivation of the use of left continuous functions is, e.g., the fact that the viscous regularizations of rate independent processes converge to a left continuous function for the viscosity coefficient approaching zero (cf. [23, Theorem 2.4]). Modifying our proofs in the obvious way we get the following

Theorem 3.4. *The play operator* $\mathsf{P} : \mathcal{Z} \times BV^{\mathrm{l}}([0,T]\,;\mathcal{H}) \longrightarrow BV^{\mathrm{l}}([0,T]\,;\mathcal{H})$ *is continuous if* $BV^{\mathrm{l}}([0,T]\,;\mathcal{H})$ *is endowed with the topology induced by the* BV*-norm* (2.13).

4. Reparametrizations

In this section we recall the notion of a reparametrization by the arclength of a $\mathscr{C}_{\mathcal{H}}$-valued BV-function introduced in [43]. This will be the key tool for the proof of our main theorem. We start with a more general notion of reparametrization in a general metric space setting.

Assume that (2.1) holds and set

$$\Phi_X := \{(x,y) \in X \times X \ : \ 0 < d(x,y) < \infty\}.$$

4.1. **Reparametrization associated to a family of geodesics.** Let us recall [43, Proposition 5.1].

Proposition 4.1. *For* $f \in BV^{\mathrm{r}}([0,T]\,;X)$*, let* $\ell_f : [0,T] \longrightarrow [0,T]$ *be defined by*

$$\ell_f(t) := \begin{cases} \dfrac{T}{\mathrm{V}(f,[0,T])}\,\mathrm{V}(f,[0,t]) & \textit{if } \mathrm{V}(f,[0,T]) \neq 0 \\ \\ 0 & \textit{if } \mathrm{V}(f,[0,T]) = 0 \end{cases} \qquad t \in [0,T]\,.$$

(i) *We have that* ℓ_f *is nondecreasing,* $\mathrm{Discont}(f) = \mathrm{Discont}(\ell_f)$*, and*

$$\ell_f([0,T]) = [0,T] \smallsetminus \bigcup_{t \in \mathrm{Discont}(f)}]\ell_f(t-), \ell_f(t)]\,. \tag{4.1}$$

Moreover there is a unique $F : \ell_f([0,T]) \longrightarrow X$ *such that*

$$f = F \circ \ell_f, \qquad \mathrm{Lip}(F) \le \frac{\mathrm{V}(f,[0,T])}{T}.$$

(ii) *Let* $\mathscr{G} = (g_{(x,y)})_{(x,y)\in\Phi}$ *be a family of geodesics connecting* x *to* y *for every* $(x,y) \in \Phi_X$. *If* $\widetilde{f} : [0,T] \longrightarrow X$ *is defined by*

$$\widetilde{f}(\sigma) := \begin{cases} F(\sigma) & \text{if } \sigma \in \ell_f([0,T]) \\ \\ g_{(f(t-),f(t))}\left(\dfrac{\sigma - \ell_f(t-)}{\ell_f(t) - \ell_f(t-)}\right) & \text{if } \sigma \in \,]\ell_f(t-), \ell_f(t)],\ t \in \mathrm{Discont}(f) \end{cases}, \tag{4.2}$$

then

$$f = \widetilde{f} \circ \ell_f, \tag{4.3}$$

$$\mathrm{V}(\widetilde{f}, [0,T]) = \mathrm{V}(f, [0,T]), \tag{4.4}$$

$$\mathrm{Lip}(\widetilde{f}) = \mathrm{Lip}(F) \le \frac{\mathrm{V}(f,[0,T])}{T}, \tag{4.5}$$

and

$$\widetilde{f}([0,T]) = f([0,T]) \bigcup \left(\bigcup_{t \in \mathrm{Discont}(f)} g_{(f(t-),f(t))}([0,1]) \right).$$

4.2. **The Hilbert case.** Let us consider Proposition 4.1 with $X = \mathcal{H}$. In this case the family $\mathscr{G} = (g_{(x,y)})_{(x,y)\in\Phi_{\mathcal{H}}}$ is defined a fortiori by $g_{(x,y)}(t) := (1-t)x + ty$, $t \in [0,1]$. Therefore for every $u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ there exists a unique $\widetilde{u} \in \mathit{Lip}([0,T]\,;\mathcal{H})$ such that $\mathrm{Lip}(\widetilde{u}, [0,T]) \le \mathrm{V}(u, [0,T])/T$ and

$$u = \widetilde{u} \circ \ell_u \tag{4.6}$$

$$\widetilde{u}(\ell_u(t-)(1-\lambda) + \ell_u(t)\lambda) = (1-\lambda)u(t-) + \lambda u(t) \qquad \forall t \in [0,T]\,,\ \forall \lambda \in [0,1]\,. \tag{4.7}$$

Moreover from [39, Lemma 4.3, Proposition 4.10] we immediately infer the following

Proposition 4.2. *For* $u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *we have that*

$$\|\widetilde{u}'(\sigma)\|_{\mathcal{H}} = \frac{\mathrm{V}(u,[0,T])}{T} \qquad \text{for } \mathcal{L}^1\text{-a.e. } \sigma \in [0,T]. \tag{4.8}$$

If $u_n \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ *for every* $n \in \mathbb{N}$ *and* $\mathrm{d}_{us}(u_n, u) \to 0$ *then* $\widetilde{u}_n \to \widetilde{u}$ *in* $W^{1,p}(0,T;\mathcal{H})$ *for every* $p \in [1, \infty[$.

4.3. **Reparametrization of "convex-valued" curves.** Let us consider the situation of Proposition 4.1 in the case when $X = \mathscr{C}_{\mathcal{H}}$, and the family $\mathscr{G} = \big(\mathcal{G}_{(\mathcal{A},\mathcal{B})}\big)$ is defined by

$$\mathcal{G}_{(\mathcal{A},\mathcal{B})}(t) := (\mathcal{A} + D_{t\rho}) \cap (\mathcal{B} + D_{(1-t)\rho}), \quad t \in [0,1]\,, \quad \rho := d_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) < \infty. \tag{4.9}$$

The mapping $\mathcal{G}_{(\mathcal{A},\mathcal{B})}$ is actually a geodesic in $\mathscr{C}_{\mathcal{H}}$ (cf. [43, Proposition 4.1] and [44, Theorem 1]), therefore if $\mathcal{C} \in BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}})$, then there exists a unique $\widetilde{\mathcal{C}} \in \mathit{Lip}([0,T]\,;\mathscr{C}_{\mathcal{H}})$ such that $\mathrm{Lip}(\widetilde{\mathcal{C}}, [0,T]) \le \mathrm{V}(\mathcal{C}, [0,T])/T$ and

$$\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}} \tag{4.10}$$

$$\begin{aligned} \widetilde{\mathcal{C}}(\ell_{\mathcal{C}}(t-)(1-\lambda) + \ell_{\mathcal{C}}(t)\lambda) &= (\mathcal{C}(t-) + D_{\lambda\rho_t}) \cap (\mathcal{C}(t) + D_{(1-\lambda)\rho_t}) \\ &\forall t \in [0,T]\,,\ \forall \lambda \in [0,1]\,,\ \text{with } \rho_t := d_{\mathcal{H}}(\mathcal{C}(t-), \mathcal{C}(t)). \end{aligned} \tag{4.11}$$

Moreover the following Proposition is proved in [43, Corollary 5.1].

Proposition 4.3. *If* $\mathcal{C}, \mathcal{C}_n \in BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}})$ *for every* $n \in \mathbb{N}$ *and* $\mathrm{d}_{us}(\mathcal{C}_n, \mathcal{C}) \to 0$ *as* $n \to \infty$, *then* $\mathrm{d}_{us}(\widetilde{\mathcal{C}}_n, \widetilde{\mathcal{C}}) \to 0$ *as* $n \to \infty$.

The family of geodesics (4.9) is studied in [43] in connection with sweeping processes. Indeed in [43, Theorem 6.1] the following result is proved.

Theorem 4.1. *If $y_0 \in \mathcal{H}$ then $\mathsf{M}(y_0, \mathcal{C}) = \mathsf{M}(y_0, \widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}$ for every $\mathcal{C} \in BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}})$.*

The previous Theorem 4.1 allows us to reduce any BV-sweeping process to a Lipschitz continuous one. In order to study $\mathsf{M}(y_0, \widetilde{\mathcal{C}})$ we need the following Lemma proved in [43, Lemma 4.4].

Lemma 4.1. *Let $\mathcal{A}, \mathcal{B} \in \mathscr{C}_{\mathcal{H}}$ be such that $\mathrm{d}_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) < \infty$ and let $\mathcal{G}_{(\mathcal{A},\mathcal{B})} : [0,1] \longrightarrow \mathscr{C}_{\mathcal{H}}$ be defined by (4.9). For $u_0 \in \mathcal{A}$ let $t_0 \in [0,1]$ be the unique number such that $\|u_0 - \mathrm{Proj}_{\mathcal{B}}(u_0)\| = (1-t_0)\mathrm{d}_{\mathcal{H}}(\mathcal{A}, \mathcal{B})$. Then*

$$\mathsf{M}(u_0, \mathcal{G}_{(\mathcal{A},\mathcal{B})})(t) := \begin{cases} u_0 & \text{if } t \in [0, t_0[\\ u_0 + \dfrac{t-t_0}{1-t_0}(\mathrm{Proj}_{\mathcal{B}}(u_0) - u_0) & \text{if } t \in [t_0, 1[\ . \\ \mathrm{Proj}_{\mathcal{B}}(u_0) & \text{if } t = 1 \end{cases} \tag{4.12}$$

5. Integral representation for P

The reparametrization by the arc length allows to give a simple

Proof of Theorem 3.2. We only have to prove the statements about the integral formulations of P, the remaining assertions following from Theorem 3.1. Assume that $y = \mathsf{P}(z_0, u)$, then (3.11), (3.13) hold, and there exist a measure $\mu : \mathscr{B}([0,T]) \longrightarrow [0, \infty[$ and a function $v \in L^1(\mu;\mathcal{H})$ such that $\mathrm{D}y = v\mu$. If $z \in L^\infty(\mu;\mathcal{H})$ and $z([0,T]) \subseteq \mathcal{Z}$ then from (3.5) it follows that $\langle z(t) - u(t) + y(t), v(t)\rangle \le 0$ for μ-a.e. $t \in [0,T]$. Thus integrating this inequality with respect to μ we find

$$0 \ge \int_{[0,T]} \langle z(t) - u(t) + y(t), v(t)\rangle \,\mathrm{d}\mu = \int_{[0,T]} \langle z(t) - u(t) + y(t), \mathrm{d}\mathrm{D}y(t)\rangle,$$

thus (3.14) and (3.12) hold. Vice versa let us assume that, $y \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ satisfies (3.11)–(3.13). Since $y = \widetilde{y} \circ \ell_y$, from Proposition 2.2 we get that $\mathrm{D}y = v\,\mathrm{D}\ell_y$, where $v : [0,T] \longrightarrow \mathcal{H}$ is defined by $v(t) := \widetilde{y}\,'(\ell_y(t))$ for $t \in \mathrm{Cont}(y)$ and $v(t) := (\widetilde{y}(\ell_y(t)) - \widetilde{y}(\ell_y(t-)))/(\ell_y(t) - \ell_y(t-))$ for $t \in \mathrm{Discont}(y)$. Now set $C := \{s \in \mathrm{Cont}(\ell_y) \ : \ \mathrm{D}\ell_y(]s-h, s+h[\, \cap [0,T]) \neq 0 \ \forall h > 0)\}$ (i.e. C is the set of continuity points of ℓ_y which do not lie in the interior of a constancy interval of ℓ_y) and observe that $\lim_{h \searrow 0} \mathrm{D}\ell_y(]s-h, s+h[\, \cap [0,T]) = \mathrm{D}\ell_y(\{s\}) = 0$ for every $s \in C$. Let us recall that for any Banach space E and any $f \in L^1(\mathrm{D}\ell_y; E)$ there exists a $\mathrm{D}\ell_y$-zero measure set Z such that $f([0,T] \smallsetminus Z)$ is separable (see, e.g., [27, Property M11, p. 124]), therefore from [17, Corollary 2.9.9., p. 155] it follows that

$$\lim_{h \searrow 0} \frac{1}{\ell_y(s+h) - \ell_y(s-h)} \int_{[s-h, s+h[\cap[0,T]} \|f(t) - f(s)\|_E \,\mathrm{d}\mathrm{D}\ell_y(t) = 0 \tag{5.1}$$

for $\mathrm{D}\ell_y$-a.e. $s \in C$. In [17] the points s satisfying (5.1) are called $\mathrm{D}\ell_y$-*Lebesgue points of f on C with respect to the Vitali relation* $V = \{[s-h, s+h[\, \cap C \ ; \ s \in C, \ h > 0\}$. Let L be the set of $\mathrm{D}\ell_y$-Lebesgue points for both $t \longmapsto v(t)$ and $t \longmapsto \langle u(t) - y(t), v(t)\rangle$ on C with respect to V, thus $\mathrm{D}\ell_y(C \smallsetminus L) = 0$. Now fix $s \in L$ and $\zeta \in \mathcal{Z}$. A straighforward computation shows that

$$\lim_{h \searrow 0} \frac{1}{\ell_y(s+h) - \ell_y(s-h)} \int_{[s-h, s+h[\cap[0,T]} \langle \zeta, v(t)\rangle \,\mathrm{d}\mathrm{D}\ell_y(t) = \langle \zeta, v(s)\rangle.$$

Taking $z(t) := \zeta \mathbb{1}_{[s-h,s+h[}(t) + (u(t)-y(t))\mathbb{1}_{[0,T]\smallsetminus[s-h,s+h[}(t)$ in (3.12) for $h > 0$ sufficiently small we get

$$\int_{[s-h,s+h[\cap[0,T]} \langle \zeta, v(t)\rangle \,\mathrm{d}\mathrm{D}\ell_y(t) \le \int_{[s-h,s+h[\cap[0,T]} \langle u(t)-y(t), v(t)\rangle \,\mathrm{d}\mathrm{D}\ell_y(t)$$

Dividing this inequality by $\ell_y(s+h) - \ell_y(s-h)$ and taking the limit as $h \searrow 0$ we get $\langle \zeta - u(s) + y(s), v(s)\rangle \le 0$, therefore

$$\langle \zeta - u(s) + y(s), v(s)\rangle \le 0 \qquad \text{for } \mathrm{D}\ell_y\text{-a.e. } s \in C. \tag{5.2}$$

Now let $s \in \mathrm{Discont}(\ell_y)$ and take $z(t) = \zeta \mathbb{1}_{[s,s+h[}(t) + (u(t)-y(t))\mathbb{1}_{[0,T]\smallsetminus[s,s+h[}(t)$ in (3.12): we get

$$\int_{[s,s+h[} \langle \zeta - u(t) + y(t), v(t)\rangle \,\mathrm{d}\mathrm{D}\ell_y(t) \le 0$$

and taking the limit as $h \searrow 0$, by the dominated convergence theorem we infer that

$$0 \ge \int_{\{s\}} \langle \zeta - u(t) + y(t), v(t)\rangle \,\mathrm{d}\mathrm{D}\ell_y(t) = \langle \zeta - u(s) + y(s), v(s)\rangle \,\mathrm{D}\ell_y(\{s\}),$$

hence

$$\langle \zeta - u(s) + y(s), v(s)\rangle \le 0 \qquad \forall s \in \mathrm{Discont}(\ell_y). \tag{5.3}$$

Collecting together (5.2)–(5.3) and the fact that $\mathrm{D}\ell_y(\mathrm{Cont}(\ell_y)\smallsetminus C) = 0$, we get (3.5) and we are done. $\square$

6. Reduction to Lipschitz sweeping processes

Within this section we consider $u \in BV^r([0,T];\mathcal{H})$ and the moving convex set $\mathcal{C}_u(t) = u(t) - \mathcal{Z}$, and we study the properties of the sweeping process driven by the reparametrized curve $\widetilde{\mathcal{C}}_u \in Lip([0,T];\mathscr{C}_{\mathcal{H}})$. In this way we will be able to get information on the play operator thanks to the formula $\mathsf{P}(z_0,u) = \mathsf{M}(u(0)-z_0, \mathcal{C}_u) = \mathsf{M}(u(0)-z_0, \widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}$. It is useful to introduce the operators

$$\mathsf{S} : \mathcal{Z} \times BV^r([0,T];\mathcal{H}) \longrightarrow BV^r([0,T];\mathcal{H}), \qquad \mathsf{Q} : \mathcal{Z} \times BV^r([0,T];\mathcal{H}) \longrightarrow BV^r([0,T];\mathcal{H}),$$

defined by

$$\mathsf{S}(z_0,u) := u - \mathsf{P}(z_0,u), \quad \mathsf{Q}(z_0,u) := \mathsf{P}(z_0,u) - \mathsf{S}(z_0,u), \qquad u \in BV^r([0,T];\mathcal{H}). \tag{6.1}$$

In the regular case, the derivatives of these operators have a useful geometric interpretation, indeed if $z_0 \in \mathcal{Z}$ and $u \in W^{1,1}(0,T;\mathcal{H})$ then it is easily seen (cf. [20, Proposition 3.9, p. 33]) that $\langle (\mathsf{S}(z_0,u))', (\mathsf{Q}(z_0,u))'\rangle = 0$ for $\mathscr{L}^1$-a.e. $t \in [0,T]$, hence $(\mathsf{Q}(z_0,u))'(t)$ and $u'(t)$ are the diagonals of the rectangle with sides $(\mathsf{S}(z_0,u))'(t)$ and $(\mathsf{P}(z_0,u))'(t)$: it follows that $\|(\mathsf{Q}(z_0,u))'(t)\| = \|u'(t)\|$ for $\mathscr{L}^1$-a.e., $t \in [0,T]$ and this is a fundamental fact in the proof of the BV-continuity of the play operator in $W^{1,1}([0,T];\mathcal{H})$. Such relation makes no sense in the BV framework, but we will see that the operators S and Q still play a role.

Lemma 6.1. *Let $\mathcal{C}_u$ be defined by* (3.9) *for every $u \in BV^r([0,T];\mathcal{H})$, and $\mathsf{Q} : \mathcal{Z} \times BV^r([0,T];\mathcal{H}) \longrightarrow BV^r([0,T];\mathcal{H})$ by* (6.1)*, i.e.*

$$\mathsf{Q}(z_0,u) := 2\mathsf{P}(z_0,u) - u, \qquad z_0 \in \mathcal{Z},\ u \in BV^r([0,T];\mathcal{H}). \tag{6.2}$$

Then

$$\mathrm{V}(u,[0,T]) = \mathrm{V}(\mathcal{C}_u,[0,T]), \qquad \ell_u = \ell_{\mathcal{C}_u} \qquad \forall u \in BV^r([0,T];\mathcal{H}), \tag{6.3}$$

and

$$\mathsf{Q}(z_0,u) = (2\mathsf{M}(u(0)-z_0, \widetilde{\mathcal{C}}_u) - \widetilde{u}) \circ \ell_u \qquad \forall u \in BV^r([0,T];\mathcal{H}). \tag{6.4}$$

Proof. Identity (6.3) follows from the fact that $d_{\mathscr{H}}(u(t)-\mathcal{Z}, u(s)-\mathcal{Z}) = \|u(t)-u(s)\|$ for every $t,s \in [0,T]$. If $u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ then from (3.10), Theorem 4.1, (4.6), and (6.3) we infer that

$$\begin{aligned}\mathsf{Q}(z_0,u) &= 2\mathsf{P}(z_0,u) - u = 2\mathsf{M}(u(0)-z_0,\mathcal{C}_u) - u = 2\mathsf{M}(u(0)-z_0,\widetilde{\mathcal{C}}_u)\circ\ell_{\mathcal{C}_u} - \widetilde{u}\circ\ell_u \\ &= 2\mathsf{M}(u(0)-z_0,\widetilde{\mathcal{C}}_u)\circ\ell_u - \widetilde{u}\circ\ell_u = (2\mathsf{M}(u(0)-z_0,\widetilde{\mathcal{C}}_u) - \widetilde{u})\circ\ell_u.\end{aligned}$$

□

As a consequence, from Proposition 4.3 we infer the following

Corollary 6.1. *Assume that $u, u_n \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$, $\mathcal{C}_u$ and $\mathcal{C}_{u_n}$ are defined as in (3.9) for every $n \in \mathbb{N}$. If $\|u_n - u\|_{BV} \to 0$, then $d_{us}(\widetilde{\mathcal{C}}_n, \widetilde{\mathcal{C}}) \to 0$ as $n \to \infty$.*

Lemma 6.2. *Assume that $u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$, $\mathcal{C}_u$ is defined by (3.9), $z_0 \in \mathcal{Z}$, and set $y_0 := u(0) - z_0$. If*

$$w := \mathsf{Q}(z_0,u) := 2\mathsf{P}(z_0,u) - u \tag{6.5}$$

and

$$\hat{w} := 2\mathsf{M}(y_0,\widetilde{\mathcal{C}}_u) - \widetilde{u}, \tag{6.6}$$

then there exists a function $\hat{v}_w \in L^\infty(0,T;\mathcal{H})$ such that

(a) *$\hat{v}_w$ is a Lebesgue representative of $\hat{w}'$;*

(b) *it holds*

$$\|\hat{v}_w(\sigma)\| = \|\widetilde{u}'(\sigma)\| = \frac{\mathrm{V}(u,[0,T])}{T} \qquad \text{for } \mathscr{L}^1\text{-a.e. } \sigma \in \ell_u(\mathrm{Cont}(u)) \tag{6.7}$$

(the case $\mathscr{L}^1(\ell_u(\mathrm{Cont}(u))) = 0$ is not excluded);

(c) *if $g_w : [0,T] \longrightarrow \mathcal{H}$ is defined by*

$$g_w(t) := \begin{cases} \dfrac{\hat{w}(\ell_u(t)) - \hat{w}(\ell_u(t-))}{\ell_u(t)-\ell_u(t-)} & \text{if } t \in \mathrm{Discont}(u), \\ \\ \hat{v}_w(\ell_u(t)) & \text{otherwise,} \end{cases} \tag{6.8}$$

then

$$\mathrm{D}w = \mathrm{D}(\hat{w}\circ\ell_u) = g_w\,\mathrm{D}\ell_u, \tag{6.9}$$

i.e. g_w is a density of $\mathrm{D}w = \mathrm{D}(\hat{w}\circ\ell_u)$ with respect to $\mathrm{D}\ell_u$.

Proof. If $\hat{y} := \mathsf{M}(y_0,\widetilde{\mathcal{C}}_u)$ then

$$\hat{y}(\sigma) \in \widetilde{\mathcal{C}}_u(\sigma) \qquad \forall \sigma \in [0,T]\,, \tag{6.10}$$

$$\hat{y}'(\sigma) + \partial I_{\widetilde{\mathcal{C}}_u(\sigma)}(y(\sigma)) \ni 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } \sigma \in [0,T], \tag{6.11}$$

$$\hat{y}(0) = u(0) - x_0 \tag{6.12}$$

and, since it is immediately seen that

$$\widetilde{\mathcal{C}}_u(\sigma) = \widetilde{u}(\sigma) - \mathcal{Z} \qquad \forall \sigma \in \ell_u([0,T]), \tag{6.13}$$

it follows from (6.11) that

$$\langle \hat{y}'(\sigma), z - \widetilde{u}(\sigma) + \hat{y}(\sigma)\rangle \le 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } \sigma \in \ell_u([0,T]) \tag{6.14}$$

(the case $\mathscr{L}^1(\ell_u([0,T])) = 0$ is not excluded). Let A be the set where $\hat{w}$ is differentiable, hence $\mathscr{L}^1([0,T]\setminus A) = 0$, and observe that (4.12) and (6.6) imply that $\hat{w}$ is affine on every interval of the form $]\ell_u(t-),\ell_u(t)[$ with $t \in \mathrm{Discont}(u)$, thus $B := \bigcup_{t\in\mathrm{Discont}(u)}]\ell_u(t-),\ell_u(t)[\subseteq A$. Now define C as the set of points $\sigma \in A \cap \ell_u(\mathrm{Cont}(u))$ such that there are two sequences $h_n, k_n \in \mathbb{R}$ such that $h_n \searrow 0$ and $k_n \searrow 0$ as $n \to \infty$ and $\sigma + h_n \in \ell_u([0,T])$ and $\sigma - k_n \in \ell_u([0,T])$ for every $n \in \mathbb{N}$. Let us notice that $C \cap B = \varnothing$ and take $z = \hat{x}(\sigma + h_n)$ (respectively $z = \hat{x}(\sigma - h_n)$) in (6.14), divide by h_n (resp. by k_n), and take the limit as $n \to \infty$: as a result we get $\langle \hat{y}'(\sigma), \widetilde{u}'(\sigma) - \hat{y}'(\sigma)\rangle = 0$. Therefore for every $\sigma \in C$ we have

$$\begin{aligned}\|\hat{w}'(\sigma)\|^2 &= \|\hat{y}'(\sigma) - (\widetilde{u}'(\sigma) - \hat{y}'(\sigma))\|^2 = \|\hat{y}'(\sigma)\|^2 + \|\widetilde{u}'(\sigma) - \hat{y}'(\sigma)\|^2 \\ &= \|\hat{y}'(\sigma) + (\widetilde{u}'(\sigma) - \hat{y}'(\sigma))\|^2 = \|\widetilde{u}'(\sigma)\|^2,\end{aligned}$$

i.e.

$$\|\hat{w}'(\sigma)\| = \|\widetilde{u}'(\sigma)\| \qquad \forall \sigma \in C. \tag{6.15}$$

Now let $\sigma \in D := (A \cap \ell_u(\mathrm{Cont}(u))) \smallsetminus C$. From (4.1) it follows that σ is the endpoint of an interval of the kind $]\ell_u(t-), \ell_u(t)[$ with $t \in \mathrm{Discont}(u)$, thus at most two possibilities can occur:

(a) $\sigma \in \ell_u(]t-\delta, t[)$ with $t \in \mathrm{Discont}(u)$, $\delta > 0$ and $\ell_u(s) = \ell_u(t-)$ for every $s \in]t-\delta, t[$: therefore $\mathrm{D}\ell_u(\ell_u^{-1}(\sigma)) = 0$;

(b) $\sigma = \ell_u(t)$ with $t \in \mathrm{Discont}(u)$ and $\ell_u^{-1}(\sigma) = [t, s[$ with $s \in \mathrm{Discont}(u)$ and ℓ_u constant on $[t, s[$: therefore $\mathrm{D}\ell_u(\ell_u^{-1}(\sigma)) = 0$.

It follows that, since $(]\ell_u(t-), \ell_u(t)[)_{t \in \mathrm{Discont}(u)}$ is a countable family, $\mathcal{L}^1(D) = 0$ and $\mathrm{D}\ell_u(\ell_u^{-1}(D)) = 0$. Therefore if $e \in \mathcal{H}$ is such that $\|e\| = 1$ (if $\mathcal{H} = \{0\}$ there is nothing to prove), then the function $\hat{v}_w : [0, T] \longrightarrow \mathcal{H}$ defined by

$$\hat{v}_w(\sigma) := \begin{cases} \hat{w}'(\sigma) & \text{if } \sigma \in B \cup C, \\ \\ \dfrac{\mathrm{V}(\widetilde{u}, [0,T])}{T} e & \text{otherwise} \end{cases} \tag{6.16}$$

satisfies the required properties. Now the last statement on g_w follows from Proposition 2.2 and from the fact that $\mathrm{D}\ell_u(\ell_u^{-1}(D)) = 0$. $\square$

7. Proof of the main Theorem

In this section we prove the main Thereom 3.3. First, for the reader's convenience we restate the weak compactness theorem for measures [12, Theorem 5, p. 105] in a form which is suitable to our purposes.

Theorem 7.1. *Let $I \subseteq \mathbb{R}$ be an interval and let M be a subset of the vector space of measures $\mu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ with bounded variation endowed with the norm $\|\mu\| := |\mu|(I)$. Assume that M is bounded. Then M is weakly sequentially precompact if and only if there exists a bounded positive measure $\nu : \mathscr{B}(I) \longrightarrow [0, \infty[$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ which satisfies the implication*

$$\forall \varepsilon > 0\ \exists \delta > 0 \quad : \quad \left(B \in \mathscr{B}(I),\ \nu(B) < \delta \implies \sup_{\mu \in M} |\mu|(B) < \varepsilon \right). \tag{7.1}$$

Theorem 7.1 is stated in [12, Theorem 5, p. 105] as a topological precompactness result. An inspection in the proof easily shows that this is actually a sequential precompatness theorem, since an isometric isomorphism reduces it to the well-known Dunford-Pettis weak sequential precompactness theorem in $L^1(\nu; \mathcal{H})$ (see, e.g., [12, Theorem 1, p. 101]).

The following lemma is a vector measure counterpart of a well-known weak derivative argument.

Lemma 7.1. *Let $I \subseteq \mathbb{R}$ be an interval, $w, w_n \in BV(I; \mathcal{H})$ for every $n \in \mathbb{N}$, and $\mu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ be a measure with bounded variation. If $w_n \to w$ uniformly on I and $\mathrm{D}w_n \rightharpoonup \mu$, then $\mathrm{D}w = \mu$.*

Proof. Let $\overline{w}$ and $\overline{w}_n$ be the extensions of w and w_n to $\mathbb{R}$ defined as in (2.8). We have that $\overline{w}_n \to \overline{w}$ uniformly on $\mathbb{R}$ and $\mathrm{D}\overline{w}$ and $\mathrm{D}\overline{w}_n$ are Borel measures of bounded variation on $\mathbb{R}$, concentrated on I. We also extend μ to the measure $\overline{\mu} : \mathscr{B}(\mathbb{R}) \longrightarrow \mathcal{H}$ defined by $\overline{\mu}(B) := \mu(B \cap I)$,

$B \in \mathscr{B}(\mathbb{R})$, thus we have $\mathrm{D}\overline{w}_n \rightharpoonup \overline{\mu}$. Let $x \in \mathcal{H}$ and $\varphi \in C_c^1(\mathbb{R};\mathbb{R})$. Then the mapping $\nu \longrightarrow \langle x, \int_{\mathbb{R}} \varphi(t) \,\mathrm{d}\nu\rangle$ is a linear continuous functional on the space of Borel measures with bounded variation on $\mathbb{R}$, therefore we have

$$\lim_{n\to\infty} \left\langle x, \int_{\mathbb{R}} \varphi \,\mathrm{d}\mathrm{D}\overline{w}_n \right\rangle = \left\langle x, \int_{\mathbb{R}} \varphi \,\mathrm{d}\overline{\mu} \right\rangle$$

On the other hand we have

$$\lim_{n\to\infty} \left\langle x, \int_{\mathbb{R}} \varphi \,\mathrm{d}\mathrm{D}\overline{w}_n \right\rangle = \lim_{n\to\infty} \left\langle x, -\int_{\mathbb{R}} \varphi'(t)\overline{w}_n(t) \,\mathrm{d}t \right\rangle = \left\langle x, -\int_{\mathbb{R}} \varphi'(t)\overline{w}(t) \,\mathrm{d}t \right\rangle$$

hence

$$\left\langle x, \int_{\mathbb{R}} \varphi \,\mathrm{d}\overline{\mu} \right\rangle = \left\langle x, -\int_{\mathbb{R}} \varphi'(t)\overline{w}(t) \,\mathrm{d}t \right\rangle$$

and from the arbitrariness of x it follows that

$$\int_{\mathbb{R}} \varphi \,\mathrm{d}\overline{\mu} = -\int_{\mathbb{R}} \varphi'(t)\overline{w}(t) \,\mathrm{d}t = \int_{\mathbb{R}} \varphi \,\mathrm{d}\mathrm{D}\overline{w},$$

thus $\overline{\mu} = \mathrm{D}\overline{w}$ by the arbitrariness of φ. Hence $\mu = \mathrm{D}w$. □

We are now in position to provide the

Proof of Theorem 3.3. Assume that $z_0, z_{0,n} \in \mathcal{Z}$, $u, u_n \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H})$ for every $n \in \mathbb{N}$ and that $z_{0,n} \to z_0$ and $\|u_n - u\|_{BV} \to 0$ as $n \to \infty$. Let us set $y_0 := u(0) - z_0$, $y_{0,n} := u_n(0) - z_{0,n}$ for every $n \in \mathbb{N}$. For simplicity we define $\mathcal{C}, \mathcal{C}_n \in BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}})$ by $\mathcal{C}(t) := \mathcal{C}_u(t) = u(t) - \mathcal{Z}$, $\mathcal{C}_n(t) := \mathcal{C}_{u_n}(t) = u_n(t) - \mathcal{Z}$, $t \in [0,T]$, and we set $\ell := \ell_u = \ell_{\mathcal{C}}$, $\ell_n := \ell_{u_n} = \ell_{\mathcal{C}_n}$ (cf. (6.3)) for every $n \in \mathbb{N}$. Hence Theorem 4.1 yields

$$\mathsf{P}(z_0, u) = \mathsf{M}(y_0, \widetilde{\mathcal{C}}) \circ \ell, \quad \mathsf{P}(z_{0,n}, u_n) = \mathsf{M}(y_{0,n}, \widetilde{\mathcal{C}}_n) \circ \ell_n \qquad \forall n \in \mathbb{N}. \tag{7.2}$$

We also define

$$w := \mathsf{Q}(z_0, u) = 2\mathsf{P}(z_0, u) - u, \qquad w_n := \mathsf{Q}(z_{0,n}, u_n) = 2\mathsf{P}(z_{0,n}, u_n) - u_n, \tag{7.3}$$

and

$$\hat{w} := 2\mathsf{M}(y_0, \widetilde{\mathcal{C}}) - \widetilde{u}, \qquad \hat{w}_n := 2\mathsf{M}(y_{0,n}, \widetilde{\mathcal{C}}_n) - \widetilde{u}_n. \tag{7.4}$$

Now, with these notations, let $g_w \in L^\infty(0,T;\mathcal{H})$ and $g_{w_n} \in L^\infty(0,T;\mathcal{H})$ be the density functions provided by Lemma 6.2 in formula (6.8), with w replaced by w_n in the case of g_{w_n}, and for simplicity set $g := g_w$, $g_n := g_{w_n}$. Therefore we have that

$$\mathrm{D}w = g\,\mathrm{D}\ell, \qquad \mathrm{D}w_n = g_n\,\mathrm{D}\ell_n. \tag{7.5}$$

We will prove that $\|w_n - w\|_{BV} \to 0$ as $n \to \infty$, and the conclusion follows from (7.3) and from the linearity of the BV-norm topology. From (7.2), (7.3), the uniform convergence of u_n to u, Corollary 6.1, and from the continuity property of M stated in Theorem 3.1, we infer that

$$\|w_n - w\|_\infty \to 0 \qquad \text{as } n \to \infty. \tag{7.6}$$

Moreover from the inequality $|\operatorname{V}(u_n,[s,t]) - \operatorname{V}(u,[s,t])| \le \operatorname{V}(u_n - u,[s,t])$, $0 \le s \le t \le T$, and from the triangle inequality we immediately get that

$$\left|\mathrm{D}(\ell_n - \ell)\right|([0,T]) = \operatorname{V}(\ell_n - \ell,[0,T]) \to 0 \qquad \text{as } n \to \infty. \tag{7.7}$$

Thanks to (6.8), (7.4), (3.8), (4.5), and (6.3), we have that for every $t \in \operatorname{Discont}(u)$ and for every $n \in \mathbb{N}$

$$\begin{aligned}\|g_n(t)\| &\le \operatorname{Lip}(\hat{w}_n) \le 2\operatorname{Lip}(\mathsf{M}(y_0,\widetilde{\mathcal{C}}_n)) + \operatorname{Lip}(\widetilde{u}) \\ &\le 2\operatorname{Lip}(\widetilde{\mathcal{C}}_n) + \operatorname{Lip}(\widetilde{u}) \le 2\operatorname{V}(\mathcal{C}_n,[0,T])/T + \operatorname{V}(u_n,[0,T])/T \\ &= 3\operatorname{V}(u_n,[0,T])/T,\end{aligned}$$

while from (6.8) and (6.7) we infer that $\|g_n(t)\| \le \mathrm{V}(u_n,[0,T])$ for every $t \in \mathrm{Cont}(u)$ and for every n. Hence there is a constant $C > 0$, independent of n, such that

$$\|g_n(t)\| \le C \qquad \forall n \in \mathbb{N}, \tag{7.8}$$

and

$$|\mathrm{D}w_n|\,(B) = \int_B \|g_n(t)\| \,\mathrm{d}\mathrm{D}\ell(t) \le C\,|\mathrm{D}\ell_n|\,(B) \qquad \forall B \in \mathscr{B}([0,T]). \tag{7.9}$$

Therefore, since in particular $\mathrm{D}\ell_n$ is weakly convergent to $\mathrm{D}\ell$, by the weak sequential compactness Dunford-Pettis Theorem 7.1 for vector measures, by (7.6), and by Lemma 7.1, we have that $\mathrm{D}w_n$ is weakly convergent to $\mathrm{D}w$, in particular if $\phi : [0,T] \longrightarrow \mathcal{H}$ is an arbitrary bounded Borel function then $\mu \longmapsto \int_{[0,T]}\langle \phi(t), \mathrm{d}\mu(t)\rangle$ is a continuous linear functional on the space of measures with bounded variation and we have

$$\lim_{n\to\infty} \int_{[0,T]} \langle \phi(t), \mathrm{d}\mathrm{D}w_n(t)\rangle = \int_{[0,T]} \langle \phi(t), \mathrm{d}\mathrm{D}w(t)\rangle,$$

i.e.

$$\lim_{n\to\infty} \int_{[0,T]} \langle \phi(t), g_n(t)\rangle \,\mathrm{d}\mathrm{D}\ell_n(t) = \int_{[0,T]} \langle \phi(t), g(t)\rangle \,\mathrm{d}\mathrm{D}\ell(t). \tag{7.10}$$

On the other hand, by (7.8), we have that there exists $z \in L^p(\mathrm{D}\ell;\mathcal{H})$ such that $g_n \rightharpoonup z$ in $L^p(\mathrm{D}\ell;\mathcal{H})$ for every $p \in \,]1,\infty[$, therefore if we set $\psi_n(t) := \langle \phi(t), g_n(t)\rangle$ and $\psi(t) := \langle \phi(t), z(t)\rangle$ for $t \in [0,T]$, we have that $\psi_n \rightharpoonup \psi$ in $L^p(\mathrm{D}\ell;\mathbb{R})$, $p \in \,]1,\infty[$, thus

$$\begin{aligned}
&\left|\int_{[0,T]} \psi_n(t)\,\mathrm{d}\mathrm{D}\ell_n(t) - \int_{[0,T]} \psi(t)\,\mathrm{d}\mathrm{D}\ell(t)\right| \\
&\le \int_{[0,T]} |\psi_n(t)|\,\mathrm{d}\,|\mathrm{D}(\ell_n-\ell)|\,(t) + \left|\int_{[0,T]} (\psi_n(t)-\psi(t))\,\mathrm{d}\mathrm{D}\ell(t)\right| \\
&\le \|\phi\|_\infty \|g_n\|_\infty\, |\mathrm{D}(\ell_n-\ell)|\,([0,T]) + \left|\int_{[0,T]} (\psi_n(t)-\psi(t))\,\mathrm{d}\mathrm{D}\ell(t)\right| \\
&\le C\|\phi\|_\infty \|u_n-u\|_{BV} + \left|\int_{[0,T]} (\psi_n(t)-\psi(t))\,\mathrm{d}\mathrm{D}\ell(t)\right| \to 0
\end{aligned}$$

as $n \to \infty$. This means that

$$\lim_{n\to\infty} \int_{[0,T]} \langle \phi(t), g_n(t)\rangle \,\mathrm{d}\mathrm{D}\ell_n(t) = \int_{[0,T]} \langle \phi(t), z(t)\rangle \,\mathrm{d}\mathrm{D}\ell(t),$$

hence, by (7.10),

$$\int_{[0,T]} \langle \phi(t), \mathrm{d}(g\,\mathrm{D}\ell)(t)\rangle = \int_{[0,T]} \langle \phi(t), \mathrm{d}(z\,\mathrm{D}\ell)(t)\rangle. \tag{7.11}$$

The arbitrariness of ϕ and (7.11) imply that $z\,\mathrm{D}\ell = g\,\mathrm{D}\ell$ (cf. [14, Proposition 35, p. 326]), hence $z(t) = g(t)$ for $\mathrm{D}\ell$-a.e. $t \in [0,T]$ and we have found that

$$g_n \rightharpoonup g \qquad \text{in } L^p(\mathrm{D}\ell;\mathcal{H}), \quad \forall p \in \,]1,\infty]. \tag{7.12}$$

Now observe that (6.8) and (6.7) yield

$$\lim_{n\to\infty} \|g_n(t)\| = \lim_{n\to\infty} \frac{V(u_n,[0,T])}{T} = \frac{V(u,[0,T])}{T} = \|g(t)\| \qquad \forall t \in \mathrm{Cont}(u). \tag{7.13}$$

Moreover $\ell_n \to \ell$ uniformly by [43, Proposition 5.2], while formula (7.4), Corollary 6.1, and Proposition 4.2 imply that $\|\hat{w}_n - \hat{w}\|_\infty \to 0$ as $n \to \infty$, thus

$$\lim_{n\to\infty} \left\| \frac{\hat{w}_n(\ell_n(t)) - \hat{w}_n(\ell_n(t-))}{\ell_n(t) - \ell_n(t-)} \right\| = \left\| \frac{\hat{w}(\ell(t)) - \hat{w}(\ell(t-))}{\ell(t) - \ell(t-)} \right\| \qquad \forall t \in \operatorname{Discont}(u). \tag{7.14}$$

From (7.13), (7.14), and (7.8) it follows that

$$\begin{aligned} \lim_{n\to\infty} \|g_n\|^p_{L^p(\mathrm{D}\ell;\mathcal{H})} &= \lim_{n\to\infty} \int_{[0,T]} \|g_n(t)\|^p \,\mathrm{d}\mathrm{D}\ell(t) \\ &= \int_{[0,T]} \|g(t)\|^p \,\mathrm{d}\mathrm{D}\ell(t) = \|g\|^p_{L^p(\mathrm{D}\ell;\mathcal{H})} \qquad \forall p \in \left]1,\infty\right[\,, \end{aligned}$$

therefore by the uniform convexity of $L^p(\mathrm{D}\ell;\mathcal{H})$ for $p \in \left]1,\infty\right[$ we have

$$g_n \to g \qquad \text{in } L^p(\mathrm{D}\ell;\mathcal{H}) \ \forall p \in \left]1,\infty\right[, \tag{7.15}$$

and, since $\mathrm{D}\ell([0,T]) = T < \infty$,

$$g_n \to g \qquad \text{in } L^1(\mathrm{D}\ell;\mathcal{H}). \tag{7.16}$$

Hence g_n has a subsequence, which we do not relabel, that is convergent to g for $\mathrm{D}\ell$-a.e. t, thus

$$\begin{aligned} V(w_n - w,[0,T]) = \|\,\mathrm{D}(w_n - w)\| &= \|\,\mathrm{D}w_n - \mathrm{D}w\| = \|g_n\,\mathrm{D}\ell_n - g\,\mathrm{D}\ell\| \\ &\le \|g_n\,\mathrm{D}(\ell_n - \ell)\| + \|(g_n - g)\,\mathrm{D}\ell\| \\ &\le C\|\,\mathrm{D}(\ell_n - \ell)\| + \int_{[0,T]} \|g_n(t) - g(t)\| \,\mathrm{d}\mathrm{D}\ell(t) \to 0 \end{aligned}$$

as $n \to \infty$ and we are done. □

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