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Langevin equation for systems with a preferred spatial direction

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In this paper, we generalize the theory of Brownian motion and the Onsager–Machlup theory of fluctuations for spatially symmetric systems to equilibrium and nonequilibrium steady-state systems with a preferred spatial direction, due to an external force. To do this, we extend the Langevin equation to include a bias, which is introduced by an external force and alters the Gaussian structure of the system's fluctuations. In addition, by solving this extended equation, we provide a physical interpretation for the statistical properties of the fluctuations in these systems. Connections of the extended Langevin equation with the theory of active Brownian motion are discussed as well.

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I. INTRODUCTION

The dynamical theory of fluctuations in physical systems began to assume its modern form with the seminal papers of Onsager and Machlup [1,2]. They proposed to describe the time evolution of the thermodynamic fluctuating quantities, as well as of the hydrodynamic and electrodynamic variables, by a stochastic Langevin equation [3, Chapters 1 and 2]. Originally Onsager and Machlup considered fluctuations only in equilibrium systems. Generalizations of the Langevin equation for fluctuations to nonequilibrium steady states followed; see, e.g., Refs. [4–6], as discussed later in this paper.

The formalism of the Langevin equation was first developed in the theory of Brownian motion [3, Chapters 1 and 2]. Later, Onsager and Machlup proposed [1,2] that the fluctuations of the thermodynamic quantities can be described by the *same* stochastic equation as used for the velocity fluctuations of a Brownian particle in an equilibrium system. That is, the time evolution of a fluctuating quantity $\alpha(t)$ obeys the following Langevin dynamics:[1]

$$d\alpha(t) = -A\alpha(t)dt + BdW(t). \tag{1}$$

Here, A and B are positive constants whose values and physical interpretation depends on the system under consideration, while t is the time and $dW(t)$ is white noise, defined as a differential of a Wiener process $W(t)$ [3, Chapter 1]:

$$W(t) = \int_0^t dW(t'). \tag{2}$$

The first term on the right-hand side of Eq. (1) is a damping force with a friction constant A, which ensures that the fluctuations of the quantity $\alpha(t)$ decay to a macroscopically observable average value $\langle\alpha(t)\rangle$. The second term, $BdW(t)$, represents physically a microscopic noise of constant intensity B. It has a Gaussian nature, since $W(t)$ in Eq. (2) is by definition a normally distributed random variable of zero mean and variance t.

By solving Eq. (1), Onsager and Machlup predicted a Gaussian structure of the fluctuations in equilibrium systems. However, they explicitly omitted in their treatment [1] rotating systems and systems subject to an external field, because these do not possess the property of microscopic reversibility.

In this paper, we treat the dynamical theory of fluctuations for a class of systems subject to an external field. This includes not only equilibrium systems in an external potential, such as a gravitational potential, but in addition also systems maintained in a nonequilibrium steady state by an external thermodynamic, hydrodynamic, or electrodynamic gradient.

Our theory is mainly motivated by recent studies [4,8–10], which report a non-Gaussian structure of fluctuations in the class of systems with an external force. We pay a particular attention to the asymmetry of such fluctuations, described by *a skewness*[2] of their probability distributions.

The above-mentioned theoretical and experimental studies indicate that the probability distribution of fluctuations is biased in the presence of a *preferred spatial direction*, which is induced by an externally applied force. In contrast to such systems, the original Langevin equation (1) has a peculiar symmetry, since it has no preferred spatial direction. For it assigns equal probabilities to both positive and negative fluctuations of $\alpha(t)$, i.e., neither positive nor negative fluctuations are favored. However, this symmetry is broken when an external field introduces a special direction and, as conjectured in Refs. [9,10], alters the microscopic noise in this class of

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[1]The differential equation (1) is first order with respect to time. A second-order version of the Langevin equation was considered in Ref. [2] for systems in which the fluctuations of the currents should be taken into account. This modifies merely the deterministic character of the resulting dynamics, while the steady-state probability of the fluctuations, studied in this paper, remain unchanged, cf. Ref. [7, Sec. II.3].

[2]Skewness is related to the third moment of a probability distribution, so that symmetric distributions, like the Gaussian, have zero skewness.

systems. A consequence of this is a non-Gaussian structure of their fluctuations.

This symmetry argument can be introduced formally by using the principle of dissymmetry due to Curie [11,12]. In the treatment of Onsager and Machlup [1] it was implicit, that the systems they considered belong to Curie's limiting point group of the highest spherical symmetry [11]. A skewness of the fluctuations was observed in systems, which lack some symmetry operations with respect to this point group. In all these cases the bias of the fluctuations is evidently due to a reduction of symmetry or, as introduced by Curie, due to a dissymmetry[3] with respect to the systems regarded by Onsager and Machlup. In this paper, we develop a Langevin equation for the class of physical systems with a polar direction [11] due to an external force.

We emphasize the important role of the spatial asymmetry, in contrast to the temporal asymmetry of microscopically irreversible systems, which are dealt with by the microscopic fluctuation theory[4] [13]. It was long thought that macroscopic irreversibility would be an immanent property of all systems in an external potential. However, it was shown in Ref. [14] that the presence of a magnetic field does not change the time-reversal symmetry of an equilibrium system, while, as discussed earlier, it alters the probability structure of its fluctuations [9]. In Sec. VI, we remark, although, that a magnetic field may not belong to the class of systems, which are liable to the theoretical arguments of this paper.

Recently it was suggested in Ref. [4] that the original Langevin equation of the Onsager–Machlup theory can be extended by adding a third stochastic term. This term acts as an external force and causes, together with white noise, a non-Gaussian behavior of the fluctuations in a nonequilibrium system. To make further progress, the authors of Refs. [5,6] *assumed* that this term is, in general, a compound Poisson process, cf. Refs. [15, Chapter 2], and added it to Eq. (1), which then reads

$$d\alpha(t) = -A\alpha(t)dt + BdW(t) + dP(t), \tag{3}$$

where $dP(t)$ is shot noise, the properties of which are defined by a compound Poisson process.

Apparently shot noise in Eq. (3) was motivated by its applications in the theory of electric conductance [5,16,17, Chapter 1]. The Poisson process is discrete and makes Eq. (3) singular, cf. Ref. [6]. Although in the theory of electric conductance this singularity is explained by the discrete nature of the electric charge [16], it is a rather curious aspect of Eq. (3) in the context of Langevin dynamics of classical statistical mechanics.

As a matter of fact, for a long time Eq. (3) has been widely accepted as a model for Brownian motion of *active matter* [18–21], which has a broad range of applications in the field of biophysics. The nonequilibrium aspect of active Brownian particles is due to their ability to propel themselves in a preferred spatial direction on account of their internal energy source. A common underlying assumption is that this energy is released in quanta, as typical of biological systems, according to the Poisson law. Shot noise can then be obtained as a limiting case of a dichotomous Markov process [18,22]. An asymmetry of fluctuations, although, was not in the focus of the studies on active Brownian motion. A possible reason, why Eq. (3) was only recently considered for physical fluctuations with an external field in general, is that the self-propagation is a mechanism peculiar to active matter [21]. Besides assumptions about this mechanism, the importance of a preferred spatial direction was clearly appreciated in the theory of active Brownian motion [20]. Curiously, as shown in Appendix C, approximate estimates of the cumulants for shot noise [18, Sec. 3.2.2] coincide with the exact results for the theory presented here, which requires solely the assumption of directed motion. Therefore the results of this paper provide a model, which may bring new insights for active Brownian motion and even supersede Eq. (3).

The Poisson process assigns a nonzero probability only to non-negative numbers, so that the role it plays in Eq. (3) is twofold. First, it acts as an external force and, second, it introduces a bias, which makes Eq. (3) consistent with the symmetry of the class of systems considered here. Also, the microscopic noise is not represented solely by white noise, but has an additional contribution due to the stochastic nature of the third term, $dP(t)$. Shot noise itself can be discrete, as in Ref. [5], or continuous [6,18,22], when its intensity is a real random variable.

In this paper we propose to replace shot noise in Eq. (3) by a different non-Gaussian stochastic term, so that the extended Langevin equation for the fluctuations in systems with a preferred spatial direction would read

$$d\alpha(t) = -A\alpha(t)dt + BdW(t) + CdE_\tau(t). \tag{4}$$

Here, C is a positive or negative constant, while $dE_\tau(t)$ is a time differential of a Gamma process $E_\tau(t)$ with a timescale parameter τ,[5] cf. [23–25, Chapter I]. We will call $dE_\tau(t)$ *exponential noise* for a reason that is clarified in Sec. IV.

The first improvement achieved by Eq. (4), with respect to Eq. (3), is its statistical foundation, which is comparable to that of the original Langevin equation. For unlike shot noise assumed in Eq. (3), both white noise and exponential noise in Eq. (4) can be deduced from simplified models of the physical systems. This approach, first proposed by Smoluchowski for white noise (cf. Ref. [26]) is applied here in a modified form to determine statistical properties of exponential noise. To do so, we will follow the formalism of Chandrasekhar in Ref. [7, Chapter I], which describes how the Wiener process can be obtained as a continuous limit of a simple symmetric random walk from a discrete physical model of microscopic noise. In this paper we model the effect of an external force with an asymmetric random walk, which in a similar continuous limit leads to the concept of exponential noise. To complete the analogy with Chandrasekhar's construction, we verify in Sec. IV that, like the Wiener process, the Gamma process also arises in a more elaborate model of a random flight.

[3]"C'est la dissymétrie qui crée le phénomène" (it is the dissymmetry that creates the phenomenon) [12].

[4]To reverse the evolution of such systems, the sign of the external force should be changed together with that of the velocities and of the time [13].

[5]The Gamma process is characterized by statistically independent increments, each having a Gamma probability distribution [23–25, Chapter I].

TABLE I. Summary of the statistical properties of the stochastic processes considered in the text. The reciprocal dual of a random variable x in the cumulant-generating function[a] is denoted by $\tilde{x}$, while constants c and τ label various parameters having a similar physical significance; $W(t)$, $E_\tau(t)$, $P_c(t)$, and $P_e(t)$ are, respectively, a Wiener process, a Gamma process with a timescale τ, a Poisson process of a rate τ^{-1}, and of a constant intensity c, and a compound Poisson process of a rate τ^{-1} and of a random intensity, which is exponentially distributed with a scale parameter c.

Random variable (x) Parameters	$cW(t)$ c	$cE_\tau(t)$ $c,\ \tau$	$P_c(t)$ $c,\ \tau$	$P_e(t)$ $c,\ \tau$
Support	Real line	Half real line (positive or negative)	Integer multiples of c (non-negative or nonpositive)	Half real line (positive or negative)
Probability density	$\propto \exp\left(-\frac{x^2}{2c^2t}\right)$	$\propto x^{t/\tau-1}\exp\left[-\frac{x}{c}\right]$	$\propto \frac{(t/\tau)^{-x/c}}{\Gamma(x/c+1)}$	Not elementary
Cumulant-generating function	$c^2t\tilde{x}/2$	$\ln(1-c\tilde{x})^{-t/\tau}$	$[\exp(c\tilde{x})-1]t/\tau$	$\frac{tc\tilde{x}}{\tau(1-c\tilde{x})}$
First cumulant (mean)	0	ct/τ	ct/τ	ct/τ
Second cumulant (variance)	c^2t	c^2t/τ	c^2t/τ	$2c^2t/\tau$
Third cumulant	0	$2c^3t/\tau$	c^3t/τ	$6c^3t/\tau$

[a] A cumulant-generating function is the natural logarithm of the Laplace transform of a probability density function.

The second advantage of Eq. (4) is that, since exponential noise is not singular, in contrast to shot noise, it fits more naturally into a stochastic differential equation. While the Poisson process is discrete, it has a highly nontrivial continuous counterpart [27], which is, nonetheless, not considered by the proponents of Eq. (3). As mentioned earlier, the discrete nature of the third term in Eq. (3) introduces a singularity. In contrast to this, the Gamma process, like the Wiener process, offers a continuous nonsingular and infinitely divisible [25, Chapter I] alternative to shot noise. Therefore, the theory and the treatment of the Langevin equation Eq. (4) is in principle simpler than that of Eq. (3).

Appendix C reviews some technical aspects of the compound Poisson process for a more detailed comparison with the model presented in this paper. In addition, Table I summarizes the difference between the Wiener process $W(t)$, the Gamma process $E_\tau(t)$, and the variations of the compound Poisson process with (i) a constant (c) intensity $P_c(t)$, considered in Ref. [5], and (ii) with an exponentially (e) distributed random intensity $P_e(t)$, considered in Refs. [6,19–21].

In Sec. V we will show that the statistical properties of the fluctuating quantity $\alpha(t)$, which evolves according to the extended Langevin equation (4), can be computed in terms of the same physical parameters, which characterize the macroscopic state of the systems studied in this paper. In particular, we confirm the non-Gaussian structure of the fluctuations by calculating their skewness. Moreover, the *sign* of the skewness depends on the external force in a manner which was already observed by an earlier experiment [10].

Finally, we note that, while the behavior exhibited by Eq. (3) is qualitatively very similar to that of Eq. (4), they differ in principle because shot noise and exponential noise have significantly different statistical properties, cf. Table I and Appendix C. Equation (3) may well be applicable to some systems, which are listed in Ref. [6] and which need a noise term of a discrete nature, e.g., systems of a small size or with a weak external force. Nonetheless, in this paper we argue that Eq. (4) is a most natural extension of the Langevin dynamics for a variety of physical systems studied by the classical statistical mechanics.

II. A SIMPLIFIED PHYSICAL EXAMPLE

To provide a physical insight into the dynamics described by a Langevin equation of the form of Eq. (3) or Eq. (4), we consider in this section a macroscopic system as an idealization of the systems studied by classical statistical mechanics, which are of interest in this paper. This will allow us to develop a decomposition of the random noise into two parts: a symmetric and asymmetric random processes, respectively. The latter will also incorporate the action of an external field. As discussed afterwards, such a decomposition is not obvious at the level of classical statistical mechanics, but it is much clearer in the example considered below or some biological systems.

First, consider a man in a boat on a lake. When the man just sits in the boat, the motion of the boat can be described by the Langevin equation (1), where the damping force would be due to the friction of the boat in the water and white noise would be caused by spontaneous fluctuations due to the waves on the water surface and the wind. The stochastic term is motivated by the symmetry of this physical system, which *a priori* does not favor any direction of motion, so that the excitations pushing the boat forward or backward are equally probable. As a result, the boat's velocity is distributed symmetrically around zero.

Now imagine that the man begins to paddle, so that the boat is propelled forward by impulses which are imparted by the oar at a certain rate. This rate will depend on the rowing rhythm, which is, in general, irregular. For instance, the man sometimes may row slower and other times faster. This irregularity of the rowing rhythm can be accounted for statistically if we regard the total force imparted by the rower to the boat as a *random variable*, which has some definite average value over a sufficiently long time interval and assumes *only non-negative values*. This random variable, when added to the original Langevin equation (1) as a third term, yields a stochastic dynamics of the form (3) or (4).

This new stochastic term, which represents an external force acting on the boat, has one important attribute which distinguishes it from the white noise term discussed earlier. Namely, the external force assumes *only non-negative* values, since the rower propels the boat always forward. Clearly, the average velocity of the boat will then be positive. However, due to the external force, the fluctuations of the velocity are amplified in the forward direction and suppressed in the backward direction. This introduces a *bias* for the forward fluctuations of the boat's velocity and, thus, reduces the symmetry of the system.

Here it is relevant to remark that, if the third term in Eq. (3) or in Eq. (4) were either a constant or another Wiener process, the resulting fluctuations would have a Gaussian structure. In fact, a certain change of variables would then transform these equations into the form of Eq. (1). Therefore, both the stochastic nature and the absence of negative values of the external force turn out to be crucial to reproduce the non-Gaussian nature of the fluctuations in the class of systems considered in this paper.

Generalizing the above argument, we assume that in *all* physical systems of interest for this paper, the random noise can be represented as a linear superposition of a symmetric term, being white noise, and *some* asymmetric term, which corresponds to the external force. The latter is asymmetric, because it never takes on negative values. Both models discussed in the Introduction, Eqs. (3) and (4), are constructed in this way.

The described decomposition of the random noise will be assumed, in spite of the fact that, in a real thermodynamic, hydrodynamic, or electrodynamic system, the microscopic noise and the external force cannot be easily separated. For example, a Brownian particle, which collides with the molecules of a fluid subject to a density gradient, will drift, on average, in a certain direction. Then, since both the microscopic noise and the external force acting on the Brownian particle are both caused by the collisions with the fluid molecules, it is not obvious that each of the two can be represented in the Langevin equation by a distinct separate term of stochastic nature. Nonetheless, some biological systems [28] or, more generally, the active matter [18], bear some similarity to the rower example.

III. WHITE NOISE

This section reviews a simple one-dimensional (1D) random walk, as used by Chandrasekhar [7, Chapter I] to motivate the white noise term for the Langevin dynamics described by Eqs. (1), (3), and (4). In a slightly modified form, the same approach will be adopted in the next section to deduce the form of the third term in Eq. (4).

Consider a particle which suffers displacements along a line in the form of discrete steps of equal length. The particle moves one step forward with probability $p(1) = 1/2$, while the probability of a backward step is $p(-1) = 1 - p(1) = 1/2$. Equal probabilities of backward and forward displacements do not favor any direction of the motion. This is consistent with the symmetry of the system, described in Sec. II, where a man sits in a boat without doing anything.

The problem is to find the probability $W_N(m)$ that the particle has moved to a point m after a series of N steps, $-N \leqslant m \leqslant N$. Without loss of generality, we assume that the initial position of the particle is at zero $m_0 = 0$, so that the total displacement $\Delta m = m - m_0 = m$ equals the final position of the particle. The exact solution is given by the binomial distribution [7, Chapter I]:

$$W_N(m) = \frac{N![p(1)]^{(N+m)/2}[p(-1)]^{(N-m)/2}}{[(N+m)/2]![(N-m)/2]!}. \tag{5}$$

As can be shown [7, Chapter I], the binomial distribution Eq. (5) with $p(1) = 1/2$ approaches asymptotically a Gaussian[6] $p_G(m)$:

$$W_N(m) \underset{N\to\infty}{\to} p_G(m) = (2\pi N)^{-1/2} \exp\left(-\frac{m^2}{2N}\right). \tag{6}$$

To obtain the continuous limit of Eq. (6), one introduces a density of sites accessible to the particle per unit length $\rho = \Delta m/\Delta x$ and the rate of displacements suffered per unit time $\nu = \Delta N/\Delta t$, where Δx and Δt are now, respectively, the continuous increments of coordinate and time. Then, using Eq. (6) for ρ and ν fixed in the limit $\Delta x \to 0$ and $\Delta t \to 0$, one finds from Eq. (6) the probability density of particle's displacement Δx within a time interval Δt [7, Chapter I]:

$$p_G(\Delta x, \Delta t) = \frac{1}{\sqrt{2B^2\Delta t}} \exp\left(-\frac{\Delta x^2}{2B^2\Delta t}\right), \tag{7}$$

where $B^2 = \nu/\rho^2$.

The coordinate Δx is thus a Gaussian random variable. Therefore the continuous limit, used to obtain Eq. (7), can be interpreted in terms of the Wiener process, cf. [7, Chapter II, Lemma I], which allows us then to pose that

$$\Delta x = BW(\Delta t) = B\int_0^{\Delta t} dW(t). \tag{8}$$

Instead of random displacements in the coordinate space, one can consider "displacements" in a velocity space, as in the problem of Brownian motion. This way one obtains white noise in the Langevin equation (1).

The equal length of each step in this simple random walk problem turns out to be insignificant, as shown in Ref. [7, Chapter I]. In particular, random flight models, where the size of each step is sampled from a variety of probability distributions, lead again to a Gaussian distribution of the particle's total displacement. The key aspect, therefore, is that the considered dynamics favors no particular direction of motion, since it assigns equal probabilities to the forward and backward displacements at each step.

IV. EXPONENTIAL NOISE

To motivate the third term of the Langevin dynamics (4), we need to exclude negative values of the external force it represents, as was suggested in Sec. II. This constraint can

[6]The Gaussian approximation $p_G(m)$ is accurate only around the mean value of m, cf. Ref. [29]. Nonetheless, the original theory of the Langevin equation is not concerned with corrections for large deviations from the mean, since they have vanishingly small probabilities.

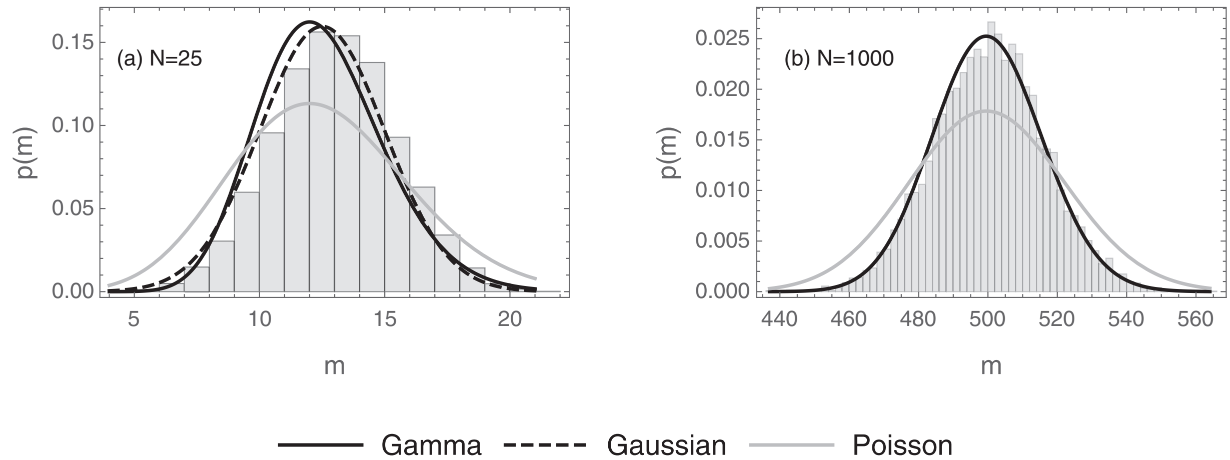

FIG. 1. Histograms of two random walk simulations: (a) $N = 25$, (b) $N = 1000$. In panel (b) the Gaussian approximation is not plotted, because it is indistinguishable from the graph of the Gamma distribution when N is so large. The Poisson distribution provides a poor approximation of the histogram data, since the probability of the forward step is not sufficiently small.

be implemented in the model of a simple random walk, reviewed in the preceding section, by a minor modification. Namely, the particle now will make *only forward* steps with the same probability $p(1) = 1/2$ or it will *remain at rest* with the probability $p(0) = 1 - p(1) = 1/2$. Then the particle's position can take on values in the integer range $0 \leqslant m \leqslant N$.

As in Sec. III, the problem is again to find the probability $W_N(m)$ that the particle moved from its initial position at zero to the position $m = \Delta m$ after N steps. The exact solution is again a binomial distribution, which can be obtained by the same argument as Eq. (5) [7, Chapter I]:

$$W_N(m) = \frac{N![p(1)]^m[p(0)]^{N-m}}{m!(N-m)!}. \tag{9}$$

As was done in Sec. III, the binomial probability mass[7] function (9) can be approximated by a Gaussian, which has a support $(-\infty,\infty)$. However, we emphasized earlier that the zero probability of all negative values has a *physical* significance because it represents an external force always acting in the forward direction. For that reason we have to abandon the Gaussian approximation, which holds only for small deviations from the mean, cf. Ref. [29].

Instead of the Gaussian approximation, it would be tempting to resort to the Poisson distribution, which is traditionally used as a limiting case of the binomial distribution (9) for $p(1) \to 0$ [30,31, Sec. 3-4], when a random variable of interest has its support on the half real line $[0,\infty)$. However, in accordance with the ideas developed in Sec. II, the Poisson model is restricted to weak external forces because it requires a vanishing probability of forward displacements, so that the particle mostly stays where it is. Fortunately, this rather restrictive assumption is irrelevant for the case $p(1) = 1/2$ of interest here, which actually is much better approximated by a different expression, as follows:

For the special case $p(1) = 1/2$, which is of interest here, we will demonstrate that the binomial distribution (9) can be approximated by a Gamma distribution p_Γ [32, Chapter 15] in the limit $N \to \infty$:

$$W_N(m) \underset{N\to\infty}{\to} p_\Gamma = \frac{m^{N-1}}{\theta^N \Gamma(N)} \exp(-m/\theta), \tag{10}$$

with an average value $\langle m\rangle = N\theta$, a variance $\mathrm{var}\{m\} = N\theta^2$, and the parameter $\theta = p(1) = p(0) = 1/2$, which is the mean rate of forward moves per step.

To the best of our knowledge, this work is the first to propose the Gamma approximation of the binomial distribution, which is motivated by the fact that the mean and the variance of $W_N(m)$ in Eq. (9) [32, Chapter 3] coincide with those of p_Γ [32, Chapter 15]. While a formal mathematical argument is given in Appendix A, below we illustrate the efficiency of Eq. (10) by the numerical simulations in Fig. 1. The Gamma approximation becomes indistinguishable from a Gaussian for a sufficiently large N, like in Fig. 1(b). An excellent agreement between the histograms and the Gamma probability distribution is evident for increasing N in Fig. 1, while the Poisson distribution gives a poor representation of the simulation data, as expected for a nonvanishing probability of the forward step $p(1) = 1/2$.

To specify the continuous counterparts of the discrete variables m and N in Eq. (10), we express the average displacement $\langle \Delta m\rangle$ in terms of the displacement rate $\nu = \Delta N/\Delta t$ per unit time and the density of positions $\rho = \Delta m/\Delta x$ per unit length, introduced in Sec. III, so that

$$\langle \Delta x\rangle = \langle \Delta m/\rho\rangle = \Delta N\theta/\rho = C\Delta t/\tau, \tag{11}$$

where $C = \theta/\rho$ and $\tau = 1/\nu$.

If we fix ρ and ν for $\Delta x \to 0$ and $\Delta t \to 0$ as in Sec. III, the continuous limit of Eq. (10) then follows from the property of infinite divisibility of the Gamma distribution [25, Chapter I]. This means, in particular, that Eq. (10) can be represented as a sum of N independent random variables distributed according to an *exponential law* of intensity θ [32, Chapter 15]. This property and Eq. (11) both motivate us to replace the sum over N in the continuous limit by a time integral of *exponential noise* $dE_\tau(t)$, which is then defined as the differential of the Gamma process $E_\tau(t)$, cf. Refs. [23,25], so that Δx obeys the

[7] The probability mass function is the discrete analog of the probability density function for continuous random variables.

probability law of a Gamma distribution:

$$p(\Delta x, \Delta t) = \frac{\Delta x^{\Delta t/\tau - 1}}{C^{\Delta t/\tau}\Gamma(\Delta t/\tau)} \exp\left(-\frac{\Delta x}{C}\right), \tag{12}$$

$$\Delta x = CE(\Delta t) = C\int_0^{\Delta t} dE_\tau(t). \tag{13}$$

Here, Eqs. (12) and (13) define the properties of exponential noise in the same manner as Eqs. (7) and (8), respectively, determine the properties of white noise.

We concluded Sec. III by pointing out that white noise also emerges in models of a random flight, where the length of the particle's displacements is sampled at each step from a symmetric probability distribution. Similarly, the random walk considered in this section can be generalized to a random flight with steps of a variable length. If the length of each displacement is sampled from an exponential probability distribution, the Gamma distribution of the particle's total displacement arises again as the sum of independent exponentially distributed random variables, a property already used above to deduce Eq. (12).

The analogy between the random walk problems of this and of the previous section is now complete. In Sec. III the Wiener process was obtained as the continuous limit of the symmetric random walk problem. By a similar argument, above we deduced the Gamma process from the continuous limit of an asymmetric random walk.

Finally, we conjecture that the generalization of the Langevin equation Eq. (1) to systems with a preferred spatial direction, induced by an external force, is given by Eq. (4). The new third term of that equation, i.e., $CdE_\tau(t)$, is proportional to exponential noise, defined by Eqs. (12) and (13). If instead of the coordinate space we considered the velocity space of a Brownian particle, C/τ would have a physical meaning of the mean external force, as discussed in the next section.

The skewness of Gamma process decreases with time, while the binomial distribution remains always symmetric. As pointed out in Appendix A in more detail, this discrepancy between $W_N(m)$ and $p_\Gamma(m)$ is acceptable for the limit $N \to \infty$. Note that the skewness, considered in Sec. I, is observed for the time-invariant probability distribution of physical fluctuations. As we discuss in Sec. V and, shortly, in Sec. VI, the extended Langevin dynamics, thanks to the frictional term, leads to a time-invariant probability measure, which reproduces the skewness of fluctuating variable.

V. SOLUTION OF THE EXTENDED LANGEVIN EQUATION

The extended Langevin equation (4) can be solved by a straightforward generalization of the method used in Ref. [7, Chapter II] for Eq. (1). To do this, we first denote by $\epsilon(t)$ the sum of the following two stochastic terms:

$$\epsilon(t) = BdW(t)/dt + CdE_\tau(t)/dt, \tag{14}$$

so that Eq. (4) can be rewritten as

$$d\alpha(t)/dt = -A\alpha(t) + \epsilon(t). \tag{15}$$

A formal solution of Eq. (15) was already given in Ref. [7, Chapter II], which we repeat here in our notation:

$$\alpha(t) = \alpha_0 \exp(-At) + \exp(-At)\int_0^t ds \exp(As)\epsilon(s), \tag{16}$$

where $\alpha_0 = \alpha(0)$ is an initial value condition.

Since in this paper we do not need a solution of Eq. (15) for a particular physical system, we focus our attention on the steady-state (SS) solution α_{SS}. This will suffice for our interest in the statistical nature of the fluctuations described by Eq. (4), as anticipated in Introduction. Taking the steady-state limit of Eq. (16) we have

$$\alpha_{SS} = \lim_{t\to\infty}\alpha(t) = \lim_{t\to\infty}\left[\int_0^t ds \exp[A(s-t)]\epsilon(s)\right]. \tag{17}$$

The decomposition of $\epsilon(t)$ in Eq. (14) splits the integral on the right-hand side of Eq. (17) into a sum of two terms:

$$B\int_0^t \exp[A(s-t)]dW(s) + C\int_0^t \exp[A(s-t)]dE_\tau(s). \tag{18}$$

The first integral in Eq. (18) is given by Lemma I of Ref. [7, Chapter II]. The result of integration is a normally distributed random variable $\beta(t)$, with a zero mean and a variance

$$\mathrm{var}\{\beta(t)\} = \frac{B^2}{2A}[1 - \exp(-2At)],$$

which in the steady-state limit becomes

$$\lim_{t\to\infty}\mathrm{var}\{\beta(t)\} = \frac{B^2}{2A}.$$

For the second integral in Eq. (18) we need another result established by Lemma 1 in Appendix B, which is analogous to the above cited Lemma I of Ref. [7, Chapter II], but applies to exponential noise. There we show that the second integral in Eq. (18) is a random variable given by a Gamma-mixture distribution and compute its mean, variance, and skewness. From now on we denote this random variable by $\gamma(t)$.

In summary, we found that $\alpha(t)$ is a sum of two independent random variables: a Gaussian $\beta(t)$ and a Gamma-mixture $\gamma(t)$. Therefore, the cumulant-generating function, cf. Appendix A, of $\alpha(t)$ is a sum of the Gaussian cumulant-generating function [32, Chapter 10] and the Gamma-mixture distribution, obtained in Appendix B. This allows us to calculate the mean, variance [var$\{\alpha(t)\}$], and the skewness [skw$\{\alpha(t)\}$] of $\alpha(t)$. Omitting straightforward computational details, we write immediately the final results for the steady-state solution α_{SS}

$$\langle\alpha_{SS}\rangle = \lim_{t\to\infty}\{\kappa_1[\beta(t)] + \kappa_1[\gamma(t)]\} = \frac{C}{\tau A}, \tag{19}$$

$$\mathrm{var}\{\alpha_{SS}\} = \lim_{t\to\infty}\{\kappa_2[\beta(t)] + \kappa_2[\gamma(t)]\} = \frac{B^2 + C^2/\tau}{2A}, \tag{20}$$

$$\mathrm{skw}\{\alpha_{SS}\} = \lim_{t\to\infty}\frac{\kappa_3[\beta(t)] + \kappa_3[\gamma(t)]}{\mathrm{var}\{\alpha(t)\}^{3/2}} \tag{21}$$

$$= \frac{4\sqrt{2A}C^3/\tau}{3(B^2 + C^2/\tau)^{3/2}}, \tag{22}$$

where κ_i stands for the ith cumulant.

Since the skewness of the steady-state solution does not vanish for $C > 0$, cf. Eq. (21), the structure of the fluctuations is non-Gaussian. Moreover, the skewness and the mean have the same sign as C, which is consistent with the experimental observations of Ref. [10]. Higher-order statistics than those in Eqs. (19)–(21), can also be computed from the cumulant-generating function.

The physical meaning of the parameters A, B, and C depends on the problem, modeled by the extended Langevin dynamics. For instance, for a Brownian particle, A is the friction constant, while the parameter B can be computed from the kinetic temperature[8] once A and C are known. Finally, as explained further, C/τ is the average magnitude of the external force. This can be seen if we take the steady-state average of both sides in Eq. (15), which corresponds to the macroscopic dynamics:

$$\langle d\alpha(t)/dt\rangle = -A\langle\alpha(t)\rangle + \langle\epsilon(t)\rangle = 0, \tag{23}$$

where the left-hand side must vanish in the steady state by definition. From Eqs. (14) and (23) one finds

$$\begin{aligned}\langle\alpha(t)\rangle &= A^{-1}\langle B dW(t)/dt + C dE_\tau(t)/d(t)\rangle \\ &= A^{-1}\langle C dE_\tau(t)/dt\rangle,\end{aligned} \tag{24}$$

since the average effect of white noise, $dW(t)$, vanishes. Finally, $\langle C dE_\tau(t)/dt\rangle = C/\tau$ because

$$d\left\langle C\int_0^t dE_\tau(t)\right\rangle \Big/ dt = d(Ct/\tau)/dt = C/\tau, \tag{25}$$

due to Eqs. (11)–(13).

Combining Eqs. (23)–(25), we have

$$\langle\alpha(t)\rangle = \frac{C}{\tau A},$$

which relates the terminal value $\langle\alpha(t)\rangle$ to the external force C/τ and the friction coefficient A. For the complete description of the extended Langevin dynamics, the timescale parameter τ, which is a new characteristic of a system, needs to be found as well.

In other words, all constants A, B, and C/τ are physical observables, which can be determined by measurements. In fact, these quantities are used to characterize physical systems in steady states, as was shown in the example above for Brownian motion.

VI. CONCLUSION

The Langevin dynamics of Eq. (1) was extended by a new term to obtain Eq. (4), which generalizes the theory of Brownian motion, as well as the Onsager–Machlup theory of fluctuations, from spatially symmetric equilibrium systems to equilibrium and nonequilibrium steady-state systems with a preferred spatial direction. We also provided statistical arguments in Sec. IV, which allowed us to deduce the form of the new term.

A method of solving the extended Langevin equation was demonstrated in Sec. V. In particular, we showed how the statistical properties of its steady-state solution can be computed from macroscopic physical observables. The steady-state probability distribution of the fluctuations is also characterized by the cumulant-generating function, which can be expressed by using the dilogarithm, a special mathematical function, cf. Appendix B. The corresponding probability density, which apparently cannot be expressed in terms of elementary functions, can be in principle approximated by the modulated Gaussian distribution [10] for practical applications.

The theory presented in this paper should be applicable to a variety of physical systems in classical statistical mechanics, such as an equilibrium fluid system in a gravitational potential or an electric current driven by a voltage difference. Applications of Eq. (4) to particular systems opens new perspectives for the future research in equilibrium and nonequilibrium statistical physics.

An important aspect of the presented theory is the steady-state limit for the variable of interest $\alpha(t)$, as in the case of the velocity of a Brownian particle. In contrast, the coordinate of a Brownian particle has a probability density, the variance of which is growing with time. Indeed, it is easy to show that, if the velocity of a Brownian particle obeys the extended Langevin dynamics, the dynamics of its position converges in time to the model of the drift diffusion. Consequently, the skewness, which fully develops in the steady-state probability distribution for the velocity of a Brownian particle under the action of external force, is decreasing with time for the distribution of its coordinate.

As shortly discussed in Sec. I and Appendix C, exponential noise also offers a new foundation for modeling of active Brownian motion. Some earlier estimates, while being approximate for shot noise, are exact in the framework built around exponential noise. Therefore it advances the earlier theory without need for a major revision of these results.

Finally, we would like to make a remark about the equilibrium systems in the magnetic field. The vector of a magnetic field has an *axial* nature, which means that it does not select a preferred direction but rather determines a sense of rotation in its normal plane. As discussed in Sec. I, such systems have a symmetry of the Curie's limiting point group ∞/m, which does not admit a preferred spatial direction [11]. This is in contrast to the forces, described by *polar* vectors considered here, e.g., the electric field, which belongs to the symmetry group $\infty \cdot m$ [11]. For this reason, systems subject to a magnetic field bear more similarity with the rotating systems, which still may need a further generalization of the Langevin equation.

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APPENDIX A: GAMMA APPROXIMATION OF BINOMIAL DISTRIBUTION

In this appendix we provide a formal mathematical argument for the Gamma approximation (10) of the binomial

[8] The kinetic temperature is proportional to the variance of the particle's linear momentum distribution, cf. Eq. (20).

distribution (9). For this we compare a *cumulant-generating function* [33,34, Sec. 26.1] of the Gamma distribution ($\mathcal{C}_\Gamma$) with that of the binomial distribution ($\mathcal{C}_B$).

We recall that a probability distribution of a random variable m is uniquely determined by its probability mass (or density) function or, equivalently, by its cumulant-generating function $\mathcal{C}(k)$:

$$\mathcal{C}(k) = \ln\langle \exp(km)\rangle_m = \sum_{i=1}^{\infty} \kappa_j \frac{k^j}{j!}, \tag{A1}$$

where k is the dual of m, while the angle brackets denote the average value. The Taylor coefficients κ_j in Eq. (A1) are the *cumulants* of m.

The first and second cumulants of a probability distribution are equal to its mean and its variance, respectively. The mean and the variance of the binomial distribution (9) are equal to those of the Gamma distribution (10), respectively, if $p(1) = \theta = 1/2$, cf. Ref. [32, Chapters 3 and 15]. It follows then that their cumulant-generating functions agree up to the third-order term in k, i.e.,

$$\mathcal{C}_B(k) - \mathcal{C}_\Gamma(k) = O(k^3),$$

because the first two cumulants cancel each other in the series expansion (A1) for $\mathcal{C}_B(k)$ and $\mathcal{C}_\Gamma(k)$, respectively.

For the binomial distribution (9), there are no asymptotic formulas of an accuracy higher than $O(k^3)$ with the support on the half real line.[9] The error of the third order is due to the skewness of the Gamma distribution. This property is inherent in all distributions which have support on the half real line, as a consequence of their obvious asymmetry. In other words, any asymptotic formula of Eq. (9) acquires skewness in the limit $N \to \infty$ if its support spreads over all non-negative reals $[0,N)_{N\to\infty}$, as considered in Sec. IV.

APPENDIX B: GAMMA-MIXTURE PROBABILITY DISTRIBUTION

When solving the extended Langevin equation in Sec. V, we had to evaluate a steady-state limit for a definite stochastic integral of the form

$$I = \int_0^t dE_\tau(s)\phi(s), \tag{B1}$$

where $\phi(s) = C\exp[A(s-t)]$ and $dE_\tau(s)$ is exponential noise with the timescale parameter τ.

Below we will consider a more general function $\phi(s)$. We obtain the cumulant-generating function of the random variable I [33,34, Sec. 26.1], cf. Appendix A, and compute some of its statistical moments, i.e., mean, variance, and skewness.

Lemma 1. Let I be a random variable given by

$$I = \int_0^t dE_\tau(s)\phi(s),$$

where $\phi(s)$ is some integrable function and $dE_\tau(s)$ is exponential noise with the timescale parameter τ. Then I has a Gamma-mixture distribution, which is described by a cumulant-generating function

$$\mathcal{C}(\tilde{I}) = -\int_0^t \frac{ds}{\tau} \ln[1-\phi(s)\tilde{I}],$$

where $\tilde{I}$ is the dual of I in the reciprocal Laplace space.

Proof. Partitioning the domain of integration $[0,t]$ into n subintervals of length Δt, so that $n\Delta t = t$, we express I as the limit of the following discrete sum S_n:

$$I \underset{n\to\infty}{\to} S_n = \sum_{j=0}^{n-1} \phi(j\Delta t)\int_{j\Delta t}^{(j+1)\Delta t} dE_\tau(s) = \sum_{j=0}^{n-1} r_j, \tag{B2}$$

where the index j runs through all subintervals.

By virtue of Eqs. (12) and (13), each term of the summation r_j in Eq. (B2) is an independent Gamma-distributed random variable. In other words, the probability distribution of I is a discrete *mixture of Gamma-distributed* random variables or, equivalently, a discrete Gamma-mixture distribution. The shape and scale parameters [32, Chapter 15] of each component r_j are, respectively, $\Delta t/\tau$ and $\phi(j\Delta t)$, while the cumulant-generating function of their sum is

$$\mathcal{C}(\tilde{S}_n) = -\sum_{j=0}^{n-1} \frac{\Delta t}{\tau} \ln[1-\phi(j\Delta t)\tilde{S}_n], \tag{B3}$$

where $\tilde{S}_n$ is the dual of S_n.

One should recognize in Eq. (B3) a Riemann sum, which in the limit $\Delta t \to 0$ ($n\to\infty$) becomes an integral. Then, from Eqs. (B2) and (B3), we conclude that

$$\mathcal{C}(\tilde{I}) = \lim_{n\to\infty} \mathcal{C}(\tilde{S}_n) = -\int_0^t \frac{ds}{\tau}\ln[1-\phi(s)\tilde{I}], \tag{B4}$$

which is the cumulant-generating function of the Gamma-mixture distribution.

The cumulants $\kappa_i(I)$, and hence the statistical moments of I, can be obtained either by differentiation of the cumulant-generating function given by Lemma 1, cf. Eq. (A1), or by using the calculus of cumulants. While the latter method was adopted in Sec. V to compute the skewness of the steady-state solution α_{SS}, cf. Eq. (21), in this section the former approach is more convenient.

Differentiating Eq. (B4) with respect to $\tilde{I}$ we find

$$\begin{aligned}
\langle I\rangle = \kappa_1(I) &= \left.\frac{d\mathcal{C}(\tilde{I})}{d\tilde{I}}\right|_{\tilde{I}=0} = \int_0^t \frac{ds}{\tau}\phi(s),\\
\mathrm{var}\{I\} = \kappa_2(I) &= \left.\frac{d^2\mathcal{C}(\tilde{I})}{d\tilde{I}^2}\right|_{\tilde{I}=0} = \int_0^t \frac{ds}{\tau}\phi(s)^2,\\
\kappa_3(I) &= \left.\frac{d^3\mathcal{C}(\tilde{I})}{d\tilde{I}^3}\right|_{\tilde{I}=0} = 2\int_0^t \frac{ds}{\tau}\phi(s)^3,
\end{aligned} \tag{B5}$$

from which the skewness can be computed by using its definition in terms of cumulants $\mathrm{skw}\{I\} = \kappa_3(I)/\kappa_2(I)^{3/2}$.

[9]However, there may exist approximations of Eq. (9) with the same order of accuracy as Eq. (10).

Now, returning to Eq. (B1), we plug $\phi(s) = C\exp[A(s-t)]$ into Eqs. (B4) and (B5). Then the cumulant-generating function, given by Eq. (B4), becomes:[10]

$$\mathcal{C}(\tilde{I}) = \frac{1}{\tau A}\{\mathrm{Li}_2(C\tilde{I}) - \mathrm{Li}_2[C\tilde{I}\exp(-At)]\}, \tag{B6}$$

where Li_2 stands for a dilogarithm function [34, Sec. 27.7]. Thus, that the steady-state limit of Eq. (B6) yields

$$\lim_{t\to\infty}\mathcal{C}(\tilde{I}) = \mathrm{Li}_2(C\tilde{I})/(\tau A). \tag{B7}$$

To evaluate Eq. (B5) for the form of $\phi(s)$, chosen above, it is convenient to consider first a general integral of the following form:

$$\begin{aligned}\int_0^t \frac{ds}{\tau}\phi(s)^n &= \int_0^t \frac{ds}{\tau} C^n \exp[nA(s-t)]\\ &= \frac{C^n}{n\tau A}[1-\exp(-nAt)],\end{aligned} \tag{B8}$$

for any integer n.

Using then Eq. (B8), we can calculate the statistics given in Eq. (B5), as well as their steady-state limits:

$$\begin{aligned}\lim_{t\to\infty}\langle I\rangle &= \lim_{t\to\infty}\left\{\frac{C}{\tau A}[1-\exp(-At)]\right\} = \frac{C}{\tau A},\\ \lim_{t\to\infty}\mathrm{var}\{I\} &= \lim_{t\to\infty}\left\{\frac{C^2}{2\tau A}[1-\exp(-2At)]\right\} = \frac{C^2}{2\tau A},\\ \lim_{t\to\infty}\kappa_3 &= \lim_{t\to\infty}\left\{\frac{2C^3}{3\tau A}[1-\exp(-3At)]\right\} = \frac{2C^3}{3\tau A}.\end{aligned} \tag{B9}$$

APPENDIX C: SHOT NOISE

In the literature there exist a few variations of shot noise. Here we review very briefly two of them, which are of concern in Refs. [5,6]. Eventually we also show that exponential noise can approximate both these variations of shot noise.

The simplest case is a shot noise of a constant intensity $dP_c(t)$, which is defined through the stochastic differential of the Poisson process $P_c(t)$, cf. Refs. [5,17, Chapter I], so that

$$P_c(t) = h\sum_{i=1}^{n(t)} H(t-t_i).$$

Here $H(t)$ is the Heaviside step function, h is the constant intensity of shot noise, while $n(t)$ is a random integer number from a Poisson distribution, so that on average one has $\langle n(t)\rangle = t/\tau$, with τ^{-1} being the rate of Poisson process.

Shot noise of constant intensity $dP_c(t)$ can be motivated by an asymmetric random walk, where the step length is a non-negative integer sampled from a Poisson distribution with one forward move per step on average. Then in the continuous limit with a fixed rate of forward displacement τ^{-1}, one obtains a Poisson process $P_c(t)/h$.

On the other hand, instead of the discrete Poisson distribution $P_c(t)/h$, one can approximate the random walk, described above, again by the continuous distribution p_Γ in Eq. (10) with the parameter $\theta = 1$. Indeed, since the first two cumulants of $P_c(t)/h$ and of Eq. (10) coincide, the argument of Appendix A ensures the error of this approximation within the third order.

Another version of shot noise is introduced by a compound Poisson process $P_e(t)$ [15, Sec. 2.5], with an intensity h_c sampled from an exponential distribution with a scale parameter c, cf. Refs. [6,18]:

$$P_e(t) = \sum_{i=1}^{n(t)} h_i H(t-t_i), \tag{C1}$$

where h_i is the ith realization of h_c, while $n(t)$, as before, is a Poisson-distributed random variable with a mean $\langle n(t)\rangle = t/\tau$.

Shot noise $dP_e(t)$ is a limiting case of a dichotomous Markov walk [18,22]. Its derivation is a bit more involved than random walks considered earlier and, therefore, we refer to Ref. [22] for details. Following Ref. [15, Sec. 2.5], the cumulant-generating function of the random variable $P = P_e(t)$, with $\tilde{P}$ being its reciprocal dual, is

$$\begin{aligned}\mathcal{C}(\tilde{P}) &= \ln\langle\exp(\tilde{P}P)\rangle_P\\ &= \ln\int_0^\infty dP\exp(\tilde{P}P)\sum_{i=1}^{n(t)} p_\tau(i)p_i(P),\end{aligned} \tag{C2}$$

where $p_\tau(i)$ is the probability density of the Poisson distribution with the mean t/τ, while $p_i(P)$ is the probability density of a sum of i realizations of h_c.

Since h_c is an exponentially distributed random variable, $p_i(P)$ is a Gamma distribution, cf. Sec. IV. By performing integration with respect to P in Eq. (C2), we further have from Eq. (C2)

$$\begin{aligned}\mathcal{C}(\tilde{P}) &= \ln\sum_{i=1}^{\infty} p_\tau(i)(1-c\tilde{P})^{-i}\\ &= \ln\sum_{i=1}^{\infty}\frac{\exp(-t/\tau)}{i!}\left[\frac{t}{\tau(1-c\tilde{P})}\right]^i\\ &= t/\tau[(1-c\tilde{P})^{-1}-1] = \frac{tc\tilde{P}}{\tau(1-c\tilde{P})}.\end{aligned} \tag{C3}$$

One can obtain an approximation of $P_e(t)$ by using the following simplification: For a sufficiently long time interval t, the sum in Eq. (C1) will have, on average, $\langle n(t)\rangle = t/\tau$ terms. In such a case, we can evaluate $P_e(t)$ roughly as a sum of t/τ exponentially distributed random variables with the scale parameter c. The result is again a Gamma process, which estimates $P_e(t)$ with the shape and scale parameters t/τ and c, respectively.

As noted in Sec. I, the approximate cumulants for $P_e(t)$, estimated from the Gamma approximation used above and differentiated with respect to time t, were reported earlier in Ref. [18, Sec. 3.2.2]. It appears, although, that the authors of Ref. [18] overlooked that their approximate result for shot noise is exact for the new kind of stochastic noise, i.e., the exponential noise proposed in this paper. The precise cumulants of $P_e(t)$, computed from Eq. (C3), are given in Table I for reference. Comparing them to the cumulants of a Gamma process, we conclude that the above approximation has an error of second order in $\tilde{P}$.

[10]We evaluated the integral in Eq. (B4) for $\phi(s) = C\exp[A(s-t)]$ by using a software for symbolic computations [35].

[1] L. Onsager and S. Machlup, Phys. Rev. **91**, 1505 (1953).

[2] S. Machlup and L. Onsager, Phys. Rev. **91**, 1512 (1953).

[3] W. T. Coffey and Y. P. Kalmykov, *The Langevin Equation: With Applications to Stochastic Problems in Physics, Chemistry and Electrical Engineering* (World Scientific, Singapore, 2012), Vol. 27.

[4] K. Kanazawa, T. G. Sano, T. Sagawa, and H. Hayakawa, Phys. Rev. Lett. **114**, 090601 (2015).

[5] K. Kanazawa, T. G. Sano, T. Sagawa, and H. Hayakawa, J. Stat. Phys. **160**, 1294 (2015).

[6] W. A. M. Morgado and S. M. D. Queirós, Phys. Rev. E **93**, 012121 (2016).

[7] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).

[8] S. Gustavsson, R. Leturcq, B. Simovič, R. Schleser, T. Ihn, P. Studerus, K. Ensslin, D. C. Driscoll, and A. C. Gossard, Phys. Rev. Lett. **96**, 076605 (2006).

[9] Y. Utsumi and K. Saito, Phys. Rev. B **79**, 235311 (2009).

[10] R. Belousov, E. G. D. Cohen, C.-S. Wong, J. A. Goree, and Y. Feng, Phys. Rev. E **93**, 042125 (2016).

[11] A. Shubnikov, Comput. Math. Appl. **16**, 357 (1988).

[12] P. Curie, J. Phys. Theor. Appl. **3**, 393 (1894).

[13] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Rev. Mod. Phys. **87**, 593 (2015).

[14] S. Bonella, G. Ciccotti, and L. Rondoni, Europhys. Lett. **108**, 60004 (2014).

[15] S. M. Ross *et al.*, *Stochastic Processes* (John Wiley and Sons, New York, 1996), Vol. 2.

[16] Y. Blanter and M. Büttiker, Phys. Rep. **336**, 1 (2000).

[17] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, Springer Series in Synergetics, 3rd ed. (Springer, Berlin, 2004).

[18] P. Romanczuk, M. Bär, W. Ebeling, B. Lindner, and L. Schimansky-Geier, Eur. Phys. J. Spec. Top. **202**, 1 (2012).

[19] P. Romanczuk and L. Schimansky-Geier, Phys. Rev. Lett. **106**, 230601 (2011).

[20] J. Strefler, W. Ebeling, E. Gudowska-Nowak, and L. Schimansky-Geier, Eur. Phys. J. B **72**, 597 (2009).

[21] H. G. Othmer, S. R. Dunbar, and W. Alt, J. Math. Biol. **26**, 263 (1988).

[22] C. Van Den Broeck, J. Stat. Phys. **31**, 467 (1983).

[23] J. Frenk and R. P. Nicolai, ERIM Report Series Reference No. ERS-2007-031-LIS (2007).

[24] F. Dufresne, H. U. Gerber, and E. S. W. Shiu, ASTIN Bull.: J. Int. Actuarial Assoc. **21**, 177 (1991).

[25] F. W. Steutel and K. Van Harn, *Infinite Divisibility of Probability Distributions on the Real Line* (CRC Press, 2003).

[26] J. Piasecki, Acta Phys. Pol., B **38**, 1623 (2007).

[27] A. Ilienko, Annales Univ. Sci. Budapest., Sect. Comp. **39**, 137 (2013).

[28] E. A. Codling, M. J. Plank, and S. Benhamou, J. R. Soc., Interface **5**, 813 (2008).

[29] J. B. Keller, Proc. Natl. Acad. Sci. USA **101**, 1120 (2004).

[30] I. J. Good, Stat. Sci. **1**, 157 (1986).

[31] A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 3rd ed. (McGraw-Hill, New York, 1991).

[32] K. Krishnamoorthy, *Handbook of Statistical Distributions with Applications* (CRC Press, Boca Raton, 2015).

[33] M. N. Berberan-Santos, J. Math. Chem. **41**, 71 (2006).

[34] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables* (U.S. Government Printing Office, Washington, 1964), Vol. 55.

[35] Wolfram Research, Inc., Mathematica, Version 10.3, Champaign, IL, 2015.