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COMBINATORIAL PRESENTATION OF MULTIDIMENSIONAL PERSISTENT HOMOLOGY

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ABSTRACT. A multifiltration is a functor indexed by $\mathbb{N}^r$ that maps any morphism to a monomorphism. The goal of this paper is to describe in an explicit and combinatorial way the natural $\mathbb{N}^r$-graded $R[x_1, \ldots, x_r]$-module structure on the homology of a multifiltration of simplicial complexes. To do that we study multifiltrations of sets and R-modules. We prove in particular that the $\mathbb{N}^r$-graded $R[x_1, \ldots, x_r]$-modules that can occur as R-spans of multifiltrations of sets are the direct sums of monomial ideals.

1. Introduction

Let $\mathbb{N}^r$ be the poset of r-tuples of natural numbers with partial order given by $(v_1, \ldots, v_r) \leq (w_1, \ldots, w_r)$ if and only if $v_i \leq w_i$ for all $1 \leq i \leq r$. Given a small category $\mathcal{C}$, a functor $F\colon \mathbb{N}^r \to \mathcal{C}$ is called a **multifiltration** if, for any $v \leq w$ in $\mathbb{N}^r$, the map $F(v \leq w)\colon F(v) \to F(w)$ is a monomorphism. Multifiltrations with values in the category of simplicial complexes, are the main objects we are studying in this article. By applying homology with coefficients in a ring R to a multifiltration of simplicial complexes $F\colon \mathbb{N}^r \to \text{Spaces}$ we obtain a functor $H_n(F, R)\colon \mathbb{N}^r \to R\text{-Mod}$ with values in the category of R-modules. The category of functors indexed by $\mathbb{N}^r$ with values in R-Mod is equivalent to the category of $\mathbb{N}^r$-graded modules over the polynomial ring $R[x_1, \ldots, x_r]$. One aim of this paper is to describe this $R[x_1, \ldots, x_r]$-module structure on $H_n(F, R)$ in a way that is suitable for calculations. One efficient way of doing it would be to give the minimal free presentation of $H_n(F, R)$ in terms of the multifiltration $F\colon \mathbb{N}^r \to \text{Spaces}$. This however we are unable to do directly. Instead we are going to describe two homomorphisms of finitely generated and free $\mathbb{N}^r$-graded $R[x_1, \ldots, x_r]$-modules $\boldsymbol{A} \to \boldsymbol{B} \to \boldsymbol{C}$ whose composition is the zero homomorphism (this sequence is a chain complex), and $H_n(F, R)$ is isomorphic to the homology of this complex. Since the modules involved are finitely generated and free and the homomorphisms preserve grading, these homomorphisms are simply given by matrices of elements in R. In our case the coefficients of the matrices are either 1, -1 or 0 and they can be explicitly expressed in terms of the multifiltration (we give a polynomial time procedure of how to do that in Section 5). One can then use standard computer algebra packages to study algebraic invariants of the module $H_n(F, R)$, in particular one can get its minimal free presentation as well as a minimal resolution, the set of Betti numbers and the Hilbert function [4, 5]. These invariants can be used then for topological

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data analysis according to the theory of multidimensional persistence (see for example [1, 2]). Our procedure reduces the computation of $H_n(F, R)$ to the computation of the homology of a chain complex of free $\mathbb{N}^r$-graded $R[x_1, \dots, x_r]$-modules. This is the starting point in [2] where the authors explain how to calculate this homology in polynomial time. One of our aims has been to show that such calculations can be done effectively for arbitrary multifiltrations of simplicial complexes with finite colimit and not only for the so called one-critical multifiltrations which are studied in [2].

In addition to the above description of persistence homology modules for arbitrary multifiltrations we have had another purpose for writing this paper. It has to do with the way persistence modules are obtained. As this has played an important role for us and our understanding of the subject, we decided to organise the entire paper around it. This paper is not just about the action of the polynomial ring. In topological data analysis one starts with representing data by functors with values in sets, arranged into a simplicial object. One then uses the R-span to obtain a functor with values in the category of chain complexes of R-modules and extracts homology out of it. It is our impression that typically one studies this entire process of obtaining persistent homology modules as one block. We however believe that it is worthwhile to understand the role of each of the above steps and study what is lost, added, and differences between persistence functors of sets, simplicial complexes, R-modules, and chain complexes. To understand these differences category theory provides a convenient language. This is not just a formalism for us since we believe it can help with understanding contributions of each step in the construction of persistence modules.

We start in Section 3 with discussing multifiltrations with values in the category of sets. We recall the structure of such multifiltrations, how can they be decomposed into indecomposable parts, how to classify the indecomposable pieces, and relation to monomial ideals. This is a rather standard material and for describing just the action of the polynomial ring on persistence modules most of it could be avoided. One could just prove directly that the persistence n-chains of a multifiltration is a direct sum of monomial ideals indexed by the set of n-simplices of the colimit of the multifiltration. However, it is our opinion that for both mathematical understanding and even computer implementations, it is advantageous not to go directly to the world of monomial ideals, but rather stay in the category of persistence sets, where such a splitting is much more natural. This leads directly to an algorithm for producing a free presentation of a multifiltration of sets.

In Section 4 we then study the effect of taking the R-span functor on multifiltrations of sets. We explain why the obtained multifiltrations of R-modules are rather special and prove that they are sums of monomial ideals. Since the R-span functor commutes with colimits, free presentations for multifiltrations of sets can be used to obtain free presentations of monomial ideals. In particular we obtain the first syzygy step of the Taylor resolution [4], and use it in Section 5.

Many functors which appear in topological data analysis are, by construction, the R-span of a functor with values in the category of sets. An example is the 0-th homology of a multifiltration of simplicial complexes. Our characterisation of the R-span of a multifiltration of sets allows to completely describe the functors representing n-simplices in a multifiltration of simplicial complexes. This is an essential step in order to obtain the desired combinatorial description of the $R[x_1,\dots,x_r]$-module structure on $H_n(F,R)$, see Section 5. A general principle motivated by the example above is the following: although topological data analysis might associate a complicated algebraic object to a data set the construction leading to such object or properties of the data itself can provide useful insight.

We conclude by pointing out, in Section 6, that for multifiltrations indexed by $\mathbb{N}^2$ a minimal free presentation of the module $H_n(F,R)$, as the cokernel of a homogeneous homomorphism between free modules, is an easier task. In this case, the kernel of $\boldsymbol{B}\to\boldsymbol{C}$ is free and therefore, given our previous results, it is sufficient to choose a set of free generators of this kernel to find a presentation of $H_n(F,R)$. The problem of identifying such a set of free generators in an algorithmic and combinatorial way is left as an open question.

2. NOTATION

2.1**.** The symbols Sets, Spaces, and R-Mod denote the categories of respectively sets, simplicial complexes, and R-modules, where we always assume that R is a commutative ring with identity. The R-linear span functor which assigns to a set S the free module $R(S)=\bigoplus_S R$ is denoted by $R\colon$ Sets $\to R$-Mod.

2.2**.** By definition a **simplicial complex** X is a collection of subsets of a set X_0 (called the set of vertices of X) such that: for any x in X_0, $\{x\}\in X$ and if $\sigma\in X$ and $\tau\subset\sigma$, then $\tau\in X$. An element σ in X is called a **simplex** of dimension $|\sigma|-1$. A complex is called **finite** if X_0 is a finite set. A morphism between two simplicial complexes $f\colon X\to Y$ is by definition a map of sets $f\colon X_0\to Y_0$ such that $f(\sigma)$ is a simplex in Y for any simplex σ in X. A morphism $f\colon X\to Y$ is a **monomorphism** if and only if the function $f\colon X_0\to Y_0$ is injective.

Let us choose an order $<$ on the set X_0. For $n\geq 0$, the symbol X_n denotes the set of strictly increasing sequences $x_0<\cdots<x_n$ of elements in X_0 for which the subset $\{x_0,\dots,x_n\}\subset X_0$ is a simplex in X. Such a sequence is called an **ordered** simplex of dimension n. For $0\leq i\leq n+1$, by forgetting the i-th element in a sequence $x_0<\cdots<x_{n+1}$ we get an element in X_n. The obtained map is denoted by $d_i\colon X_{n+1}\to X_n$. By applying the R-span functor and taking the alternating sum of the induced maps one gets:

$$RX_{n+1}\xrightarrow{\partial_{n+1}:=\sum_{i=0}^{n+1}(-1)^i Rd_i} RX_n\xrightarrow{\partial_n:=\sum_{i=0}^{n}(-1)^i Rd_i} RX_{n-1}$$

where for $n=0$, the R-module RX_{-1} is taken to be trivial. It is a standard fact that the composition $\partial_n\partial_{n+1}$ is the trivial map and hence the image $\mathrm{im}(\partial_{n+1})$ is a submodule of the kernel $\ker(\partial_n)$. The obtained complex is called the **ordered simplicial chain complex of X with coefficients in R**. The quotient $\ker(\partial_n)/\mathrm{im}(\partial_{n+1})$ is called the n-th **homology** of X and is denoted by $H_n(X,R)$. The isomorphism type of this module does not depend on the chosen ordering on X_0. Note that this is not a functor on the entire category of simplicial complexes. However if $f\colon X\to Y$ is a monomorphism, then we can choose first an ordering on Y_0 and then use it to induce an ordering on X_0 so the function $f\colon X_0\to Y_0$ is order preserving. With these choices, by applying f to ordered sequences element-wise, we obtain a map of sets $f_n\colon X_n\to Y_n$ which commutes with the maps d_i. In this way we get an induced map of homology modules that we denote by $H_n(f,R)\colon H_n(X,R)\to H_n(Y,R)$.

2.3**.** The symbol $R[x_1,\dots,x_r]$ denotes the $\mathbb{N}^r$-graded polynomial ring with coefficients in a ring R. The category of $\mathbb{N}^r$-graded $R[x_1,\dots,x_r]$-modules with the degree preserving homomorphisms is denoted by $R[x_1,\dots,x_r]$-**Mod** and we use bold face letters to denote such modules.

A monomial in $R[x_1,\dots,x_r]$ is a polynomial of the form $x_1^{v_1}\cdots x_r^{v_r}$. Its grade is given by $v=(v_1,\dots,v_r)$. Such a monomial is also written as x^v. An $\mathbb{N}^r$-graded ideal in $R[x_1,\dots,x_r]$ is called **monomial** if it is generated by monomials. An $\mathbb{N}^r$-graded $R[x_1,\dots,x_r]$-module isomorphic to the ideal of $R[x_1,\dots,x_r]$ generated by a single monomial x^v is called **free on one generator** in degree v and denoted by $\langle x^v\rangle$. An $\mathbb{N}^r$-graded $R[x_1,\dots,x_r]$-module which is isomorphic to a direct sum of free modules on one generator is called **free**. The R-module $\mathrm{Hom}(\langle x^v\rangle,\langle x^w\rangle)$ is either trivial if $v\not\geq w$, or is isomorphic to R if $v\geq w$. We use this to identify the R-module $\mathrm{Hom}(\oplus_{t\in T}\langle x^{v_t}\rangle,\oplus_{s\in S}\langle x^{w_s}\rangle)$ of homomorphisms between free modules with the set of $S\times T$ matrices of elements in R whose (s,t) entry is 0 if $v_t\not\geq w_s$. Thus to describe a degree preserving homomorphism between two finitely generated and free $\mathbb{N}^r$-graded $R[x_1,\dots,x_r]$-modules we need to specify:

- A matrix M of elements in R.
- Two functions, one that assigns to every row of M an element in $\mathbb{N}^r$ and the other that assigns to every column of M an element in $\mathbb{N}^r$. The values of these functions are called grades of the respective rows and columns. The grades of the columns correspond to the grades of the generators of the domain of the homomorphism and the grades of the rows correspond to the grades of the generators of the range of the homomorphism.
- The matrix M should satisfy the following property: the entry corresponding to a row with grade w and a column with grade v is zero if $v\not\geq w$.

2.4**.** Let I be a small category. The symbol $\mathrm{Fun}(I,\mathcal{C})$ denotes the category of functors indexed by I with values in a category $\mathcal{C}$ and natural transformations as morphisms. We use the symbol $\mathrm{Nat}_{\mathcal{C}}(F,G)$ to denote the set of natural transformations between two functors $F,G\colon I\to\mathcal{C}$. Recall [6] that the colimit of a functor $F\colon I\to\mathcal{C}$ is an object $\mathrm{colim}_I F$ in $\mathcal{C}$ together with morphisms $p_i\colon F(i)\to\mathrm{colim}_I F$, for any object i in I. These morphisms are required to satisfy the following universal property. First, for any $\alpha\colon i\to j$ in I, $p_jF(\alpha\colon i\to j)=p_i$. Second, if $q_i\colon F(i)\to X$ is a sequence of morphisms in $\mathcal{C}$ indexed by objects of I fulfilling the equality $q_jF(\alpha\colon i\to j)=q_i$ for any morphism α in I, then there is a unique $f\colon\mathrm{colim}_I F\to X$ such that $q_i=fp_i$ for any object i in I.

If I is the empty category, $\mathrm{colim}_I F$ is called the **initial** object and denoted by $\emptyset$. The initial object has the property that, for any object X in $\mathcal{C}$, the set of morphisms $\mathrm{mor}_C(\emptyset,X)$ has exactly one element. If I is a discrete category, then $\mathrm{colim}_I F$ is called the **coproduct** and denoted either by $\coprod_{i\in I}F(i)$ or $\bigoplus_{i\in I}F(i)$. The second notation is used only in the case the coproduct is taken in an additive or abelian category, as for example in R-Mod.

2.5**.** An object X in $\mathcal{C}$ is called **decomposable** if it is isomorphic to a sum $X_1\coprod X_2$ where neither X_1 nor X_2 is the initial object. It is **indecomposable** if it is neither initial nor decomposable. An object X is called **uniquely decomposable** if the following two conditions hold. First, it is isomorphic to a coproduct $\coprod_{i\in I}X_i$ where X_i is indecomposable for any i. Second, if X is isomorphic to $\coprod_{i\in I}X_i$ and to $\coprod_{j\in J}Y_j$, where X_i's and Y_j's are indecomposable, then there is a bijection $\phi\colon I\to J$ such that X_i and $Y_{\phi(i)}$ are isomorphic for any i in I.

In the category of sets the initial object is the empty set, the coproduct is the disjoint union, a set is decomposable if it contains at least two elements, and is indecomposable if it contains exactly one element. For $F\colon I \to \text{Sets}$, its colimit is the quotient of $\coprod_{i\in I} F(i)$ by the equivalence relation generated by x_i in $F(i)$ is related to x_j in $F(j)$ if there are morphisms $\alpha\colon i \to k$ and $\beta\colon j \to k$ in I for which $F(\alpha)(x_i) = F(\beta)(x_j)$.

2.6**.** The symbol $\mathbb{N}^r$ denotes the poset of r-tuples of natural numbers with partial order given by $(v_1, \ldots, v_r) \leq (w_1, \ldots, w_r)$ if and only if $v_i \leq w_i$ for all $1 \leq i \leq r$. The initial element $(0, \ldots, 0)$ in $\mathbb{N}^r$ is denoted simply by 0. Recall that the partial order on $\mathbb{N}^r$ is a lattice. This means that for any finite set of elements S in $\mathbb{N}^r$, there are elements $\min(S)$ and $\max(S)$ in $\mathbb{N}^r$ (not necessarily in S) with the following properties. First, for any v in S, $\min(S) \leq v \leq \max(S)$. Second, if u and w are elements in $\mathbb{N}^r$ for which $u \leq v \leq w$, for any v in S, then $u \leq \min(S)$ and $\max(S) \leq w$. Furthermore any non-empty subset S of $\mathbb{N}^r$ has an element v such that if $w < v$, then w is not in S. Such elements are called minimal in S and may not be unique. A functor indexed by the poset $\mathbb{N}^r$ that maps any morphism to a monomorphism is called a **multifiltration**. We will denote the colimit of a functor F indexed by $\mathbb{N}^r$ by $\operatorname{colim} F$. A multifiltration F with values in Sets or R-Mod is called **one critical** if for any element x in $\operatorname{colim} F$, the set $\{v \in \mathbb{N}^r \mid x \text{ is in the image of } p_v\colon F(v) \to \operatorname{colim} F\}$ has a unique minimal element which we denote by v_x and call the **critical coordinate** of x (see [2]). A functor $F\colon \mathbb{N}^r \to \text{Sets/Spaces}$ is called **compact** if $\operatorname{colim} F$ is a finite set/simplicial complex.

2.7**.** Let v be an element in $\mathbb{N}^r$. The functor $\text{mor}_{\mathbb{N}^r}(v, -)\colon \mathbb{N}^r \to \text{Sets}$ is called **free on one generator**. For example $\text{mor}_{\mathbb{N}^r}(0, -)\colon \mathbb{N}^r \to \text{Sets}$ is the constant functor with value the one point set. Since $\mathbb{N}^r$ is a poset, the values of a free functor on one generator are either the empty set, or the one point set. A functor $F\colon \mathbb{N}^r \to \text{Sets}$ is called **free** if it is isomorphic to a disjoin union of free functors on one generator. Note that any free functor is a multifiltration.

Composition with the R-span functor $R\colon \text{Sets} \to R\text{-Mod}$, is denoted by the same symbol $R\colon \text{Fun}(\mathbb{N}^r, \text{Sets}) \to \text{Fun}(\mathbb{N}^r, R\text{-Mod})$ and called by the same name the R-span functor. Recall that this R-span functor is the left adjoint to the forget the R-module structure functor. This implies that the R-span functor commutes with colimits, in particular it maps the initial object to the initial object and commutes with coproducts.

The functor $R\text{mor}_{\mathbb{N}^r}(v, -)\colon \mathbb{N}^r \to R\text{-Mod}$ is also called **free on one generator**. A functor $F\colon \mathbb{N}^r \to R\text{-Mod}$ is called **free** if it is isomorphic to the R-span of a free functor with values in Sets or equivalently, if it is isomorphic to a direct sum of free functors on one generator.

2.8**.** Recall that the category of functors $\text{Fun}(\mathbb{N}^r, R\text{-Mod})$ is equivalent to the category of $\mathbb{N}^r$-graded modules $R[x_1, \ldots, x_r]$-**Mod**. We are going to identify these categories using the following explicit equivalence which assigns to $F\colon \mathbb{N}^r \to R\text{-Mod}$, the $\mathbb{N}^r$-graded $R[x_1, \ldots, x_r]$-module given by $\boldsymbol{F} := \oplus_{v\in\mathbb{N}^r} F(v)$ with x_i acting on the component $F(v)$ via the map $F(v \leq v + e_i)$ where e_i is the i-th vector in the standard base. Via this identification, the free functor $R\text{mor}_{\mathbb{N}^r}(v, -)\colon \mathbb{N}^r \to R\text{-Mod}$ is mapped to the free module $\langle x^v \rangle$.

3. Functors with values in Sets

The aim of this section is to recall several basic properties of functors of the form $F\colon \mathbb{N}^r \to \text{Sets}$. Although many of these properties are well known, we decided to present their proofs for self containment and to set up comparison between set valued and R-module valued functors in the following section. We start with:

3.1. **Proposition.** *A functor $F\colon \mathbb{N}^r \to \text{Sets}$ is indecomposable (see 2.5) if and only if the set $\operatorname{colim} F$ contains exactly one element.*

Proof. If the values of F are not all empty, then $\operatorname{colim} F$ is not empty. Furthermore if $F = G \coprod H$, then $\operatorname{colim} F = (\operatorname{colim} G) \coprod (\operatorname{colim} H)$. This shows that if $\operatorname{colim} F$ contains exactly one point, then F is indecomposable. On the other hand we can decompose F as $\coprod_{x\in\operatorname{colim} F} F[x]$ where, for any point x in $\operatorname{colim} F$, $F[x]\colon \mathbb{N}^r \to \text{Sets}$ is the subfunctor of F whose values are given by $F[x](v) := \{y \in F(v) \mid p_v(y) = x\}$ (see 2.4). Observe that not all the values of $F[x]$ are empty. This describes F as a coproduct of indecomposable functors. Thus if F is indecomposable, then $\operatorname{colim} F$ has to contain only one element. $\square$

The argument in the above proof shows more:

3.2. **Corollary.** *Any functor $F\colon \mathbb{N}^r \to \text{Sets}$ is uniquely decomposable as:*

$$F = \coprod_{x\in\operatorname{colim} F} F[x].$$

In this paper we are not interested in all functors indexed by $\mathbb{N}^r$ with values in Sets, but those that map any morphism to a monomorphism. Such functors are called multifiltrations of sets (see 2.6) and here is their characterization:

3.3. **Proposition.** *A functor $F\colon \mathbb{N}^r \to \text{Sets}$ is a multifiltration if and only if the map $p_v\colon F(v) \to \operatorname{colim} F$ is a monomorphism for any v in $\mathbb{N}^r$.*

Proof. Recall that $\operatorname{colim} F$ is the quotient of $\coprod_{v\in\mathbb{N}^r} F(v)$ by the equivalence relation generated by x_v in $F(v)$ is related to x_w in $F(w)$, if there is $u \geq v$ and $u \geq w$ such that $F(v \leq u)(x_v) = F(w \leq u)(x_w)$. Note that since $\mathbb{N}^r$ is a lattice, the described relation is already an equivalence relation. Thus two elements of $F(v)$ are mapped to the same element in $\operatorname{colim} F$ if and only if they are mapped to the same element via $F(v \leq u)$ for some u and the proposition follows. $\square$

3.4. **Corollary.** *A functor $F\colon \mathbb{N}^r \to \text{Sets}$ is an indecomposable multifiltration if and only if the set $F(v)$ has at most one element for any v in $\mathbb{N}^r$ and there is u for which $F(u)$ is not empty.*

Proof. Assume first F is an indecomposable multifiltration. By Proposition 3.1, $\operatorname{colim} F$ is the one point set. The multifiltration assumption implies that $F(v)$ is a subset of $\operatorname{colim} F$ for any v (see 3.3). Consequently the set $F(v)$ can not contain more than one element. Since $\operatorname{colim} F$ is not empty, the values of F can not be all empty either. This shows one implication.

Recall that any element in $\operatorname{colim} F$ is of the form $p_v(x)$ for some v in $\mathbb{N}^r$ and x in $F(v)$. Assume that $\operatorname{colim} F$ has at least two elements, which we write as $p_v(x)$ and $p_w(y)$. The elements $F(v \leq \max\{v, w\})(x)$ and $F(w \leq \max\{v, w\})(y)$ therefore also have to be different. Consequently the set $F(\max\{v, w\})$ has more than one element. $\square$

Indecomposable multifiltrations of sets are therefore exactly the non empty sub-functors of the free functor $\mathrm{mor}_{\mathbb{N}^r}(0,-)$ on one generator given by the origin 0 in $\mathbb{N}^r$ (see 2.7).

Note that since there is a unique map from any set to the one point set, according to Corollary 3.4, if $F\colon \mathbb{N}^r \to \mathrm{Sets}$ is an indecomposable multifiltration, then, for any functor $G\colon \mathbb{N}^r \to \mathrm{Sets}$, there is at most one natural transformation $G \to F$. Thus the full subcategory of $\mathrm{Fun}(\mathbb{N}^r, \mathrm{Sets})$ given by the indecomposable multifiltrations is a poset. This is the inclusion poset of all the non empty sub-functors of the free functor $\mathrm{mor}_{\mathbb{N}^r}(0,-)$. Our next goal is to describe this poset. We do that using the notion of the **support** of a functor $F\colon \mathbb{N}^r \to \mathrm{Sets}$:

$$\mathrm{supp}(F) := \{v \in \mathbb{N}^r \mid F(v) \neq \emptyset\}$$

For example $\mathrm{supp}(\mathrm{mor}_{\mathbb{N}^r}(v,-)) = \{w \in \mathbb{N}^r \mid v \leq w\}$. Not all subsets of $\mathbb{N}^r$ can be a support. If v belongs to $\mathrm{supp}(F)$, then so does any $w \geq v$. Subsets of $\mathbb{N}^r$ that satisfy this property are called **saturated**.

3.5. **Proposition.** *The function* $(F\colon \mathbb{N}^r \to \mathrm{Sets}) \mapsto \mathrm{supp}(F)$ *is an isomorphism between the poset of indecomposable multifiltrations of sets and the inclusion poset of saturated non-empty subsets of* $\mathbb{N}^r$*.*

Proof. Observe first that if there is a natural transformation $F \to G$, then if $F(v)$ is not empty, then neither is $G(v)$. This means that $\mathrm{supp}(F) \subset \mathrm{supp}(G)$ which shows that the function $F \mapsto \mathrm{supp}(F)$ is a function of posets.

To define the inverse of the support function, choose a saturated subset S in $\mathbb{N}^r$ and an element v in $\mathbb{N}^r$. Set:

$$\Psi(S)(v) := \begin{cases} \{v\} & \text{if } v \in S \\ \emptyset & \text{if } v \notin S \end{cases}$$

Since S is saturated, if $\Psi(S)(v)$ is not empty, then neither is $\Psi(S)(w)$ for any $v \leq w$. We can therefore define $\Psi(S)(v \leq w)\colon \Psi(S)(v) \to \Psi(S)(w)$ to be the unique map. This defines a functor which by Corollary 3.4 is an indecomposable multifiltration. The construction Ψ gives a map of posets between the saturated subsets in $\mathbb{N}^r$ and indecomposable multifiltrations.

Note that $\mathrm{supp}(\Psi(S)) = S$. Furthermore, for any $F\colon \mathbb{N}^r \to \mathrm{Sets}$, there is a unique natural transformation $F \to \Psi(\mathrm{supp}(F))$ which becomes an isomorphism if F is an indecomposable multifiltration. This shows that Ψ is the inverse of the support function. $\square$

Our next step is to recall how the set of saturated subsets of $\mathbb{N}^r$ can be described. For any subset S of $\mathbb{N}^r$ define $\mathrm{gen}(S) := \{v \in S \mid \text{ if } w < v, \text{ then } w \notin S\}$ and call it the **minimal set of generators** of S. For example $\mathrm{gen}(\mathrm{supp}(\mathrm{mor}_{\mathbb{N}^r}(v,-))) = \{v\}$. Furthermore Proposition 3.5 implies that an indecomposable multifiltration $F\colon \mathbb{N}^r \to \mathrm{Sets}$ is free (necesarily on one generator) if and only if $\mathrm{gen}(\mathrm{supp}(F))$ consists of one element. This can be generalised to arbitrary multifiltrations:

3.6. **Proposition.** *A multifiltration* $F\colon \mathbb{N}^r \to$ *Sets is free if and only if it is one critical (see 2.6).*

Proof. We have a decomposition $F = \coprod_{x \in \mathrm{colim}\, F} F[x]$. Note that $\mathrm{supp}(F[x]) = \{v \in \mathbb{N}^r \mid x \text{ is in the image of } p_v\colon F(v) \to \mathrm{colim}\, F\}$. Thus by definition, F is one critical if and only if $\mathrm{gen}(\mathrm{supp}(F[x]))$ are one element sets, i.e., if the functors $F[x]$ are free on one generator, for every x in $\mathrm{colim}\, F$. $\square$

Directly from the definition of the minimal set of generators it follows that: (1) elements in $\mathrm{gen}(S)$ are not comparable; (2) any element in S is comparable to some element in $\mathrm{gen}(S)$. Furthermore Dickson's lemma states that $\mathrm{gen}(S)$ is finite (see [3], page 163).

Moreover the function $S \mapsto \mathrm{gen}(S)$ is a bijection between the set of saturated subsets of $\mathbb{N}^r$ and the set of all finite subsets of $\mathbb{N}^r$ whose elements are not comparable. The inverse of such function assigns to a subset T in $\mathbb{N}^r$ the saturated set, $\mathrm{sat}(T) := \{v \mid \text{there is } u \text{ in } T \text{ such that } v \geq u\}$. This together with Proposition 3.5 implies:

3.7. **Corollary.** *The following assignments are isomorphisms of posets:*

$$\begin{tikzcd}
\{\textit{finite incomparable subsets of } \mathbb{N}^r\} \arrow[d, bend right, "\mathrm{sat}"'] \\
\{\textit{saturated subsets of } \mathbb{N}^r\} \arrow[u, bend right, "\mathrm{gen}"'] \\
\{\textit{isomorphism classes of indecomposable multifiltrations of sets}\} \arrow[u, "\mathrm{supp}"'] \arrow[d, "\mathrm{ideal}"] \\
\{\text{monomial ideals of } R[x_1, ..., x_r]\}
\end{tikzcd}$$

where the monomial ideal asociated to an indecomposable multifiltration $F\colon \mathbb{N}^r \to \mathrm{Sets}$ *is given by* $\langle x^v \mid v \in \mathit{gen}(\mathit{supp}(F))\rangle$.

We have seen that the function $F \mapsto \mathrm{gen}(\mathrm{supp}(F))$ is a bijection between the set of indecomposable multifiltrations of sets and finite non-empty subsets of $\mathbb{N}^r$ whose elements are not comparable. We finish this section by giving a constructive formula for the inverse to this function. Let T be a subset of $\mathbb{N}^r$. Define $F_T\colon \mathbb{N}^r \to \mathrm{Sets}$ to be a functor given by the following coequalizer in $\mathrm{Fun}(\mathbb{N}^r, \mathrm{Sets})$:

$$F_T := \mathrm{colim}\left(\coprod_{v_0 \neq v_1 \in T} \mathrm{mor}_{\mathbb{N}^r}(\max\{v_0, v_1\}, -) \overset{\pi_0}{\underset{\pi_1}{\rightrightarrows}} \coprod_{v \in T} \mathrm{mor}_{\mathbb{N}^r}(v, -) \right)$$

where on the component indexed by $v_0 \neq v_1 \in T$, the map π_i is given by the unique natural transformation $\mathrm{mor}_{\mathbb{N}^r}(\max\{v_0, v_1\}, -) \to \mathrm{mor}_{\mathbb{N}^r}(v_i, -)$ induced by $v_i \leq \max\{v_0, v_1\}$.

3.8. **Proposition.** *If* $T \subset \mathbb{N}^r$ *is not empty, then the functor* F_T *is an indecomposable multifiltration whose support is given by* $\mathrm{sat}(T)$.

Proof. Let u be an element in $\mathbb{N}^r$. The set $F_T(u)$ is a quotient of $\coprod_{v \in T} \mathrm{mor}_{\mathbb{N}^r}(v, u)$ and hence $F_T(u) \neq \emptyset$ if and only if $\coprod_{v \in T} \mathrm{mor}_{\mathbb{N}^r}(v, u) \neq \emptyset$, implying the equality $\mathrm{supp}(F_T) = \mathrm{sat}(T)$. In particular if T is non-empty, then neither is $\mathrm{supp}(F_T)$.

Let $v_0 \leq u$ and $v_1 \leq u$ be two different elements in $\coprod_{v \in T} \mathrm{mor}_{\mathbb{N}^r}(v, u)$. These inequalities give an element $\max\{v_0, v_1\} \leq u$ in $\coprod_{v_0 \neq v_1 \in T} \mathrm{mor}_{\mathbb{N}^r}(\max\{v_0, v_1\}, -)$ which is mapped via π_i to $v_i \leq u$. The elements $v_0 \leq u$ and $v_1 \leq u$ are therefore sent, via the quotient map, to the same element in $F_T(u)$. The set $F_T(u)$ can therefore have at most one element and hence, according to 3.4, F_T is an indecomposable multifiltration. $\square$

3.9. **Corollary.** *If F is an indecomposable multifiltration, then it is isomorphic to $F_{\text{gen}(\text{supp}(F))}$.*

We can use the above construction to give a presentation of any multifiltration of sets. Here is a procedure of how to do that. Let $F\colon \mathbb{N}^r \to \text{Sets}$ be a multifiltration. For any v in $\mathbb{N}^r$, index elements of $F(v)$ by elements of $\operatorname{colim} F$ as follows: y in $F(v)$ has index x in $\operatorname{colim} F$ if $p_v(y) = x$. Let $F[x]$ be the subfunctor of F whose elements have index $x \in \operatorname{colim} F$ (see the proof of 3.1). It is an indecomposable multifiltration. Recall that $F = \coprod_{x\in\operatorname{colim} F} F[x]$. The functor F is then isomorphic to:

$$\coprod_{x\in\operatorname{colim} F} F_{\text{gen}(\text{supp}(F[x]))}$$

Since we are going to use this presentation, we need to introduce notation describing the involved functors.

- For any x in $\operatorname{colim} F$, define:
$$\mathcal{G}F[x] := \coprod_{v\in\text{gen}(\text{supp}(F[x]))} \text{mor}_{\mathbb{N}^r}(v, -)$$
$$\mathcal{K}F[x] := \coprod_{v_0\neq v_1\in\text{gen}(\text{supp}(F[x]))} \text{mor}_{\mathbb{N}^r}(\max\{v_0, v_1\}, -)$$
- Recall that there are natural transformations $\pi_0[x], \pi_1[x]\colon \mathcal{K}F[x] \to \mathcal{G}F[x]$ induced by $v_0 \leq \max\{v_0, v_1\}$ and $v_1 \leq \max\{v_0, v_1\}$.
- Since $F[x]$ is indecomposable, there is a unique natural transformation denoted by $p_{F,x}\colon \mathcal{G}F[x] \to F[x]$. This natural transformation has the universal property describing $F[x]$ as the colimit of the diagram:
$$\begin{tikzcd} \mathcal{K}F[x] \arrow[r, shift left, "\pi_0{[x]}"] \arrow[r, shift right, "\pi_1{[x]}"'] & \mathcal{G}F[x] \end{tikzcd}$$

By summing over all x in $\operatorname{colim} F$, we obtain functors $\mathcal{G}F\colon = \coprod_{x\in\operatorname{colim} F} \mathcal{G}F[x]$, $\mathcal{K}F\colon = \coprod_{x\in\operatorname{colim} F} \mathcal{K}F[x]$ and natural transformations $\pi_0, \pi_1\colon \mathcal{K}F \to \mathcal{G}F$ and $p_F\colon = \coprod_{x\in\operatorname{colim} F} p_{F,x}\colon \mathcal{G}F \to F$. The natural transformation p_F has the universal property describing F as the colimit of the diagram:

$$\begin{tikzcd} \mathcal{K}F \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & \mathcal{G}F \end{tikzcd}$$

Although the natural transformations $p_{F,x}$ are unique, the construction $F \mapsto \mathcal{G}F$ is not functorial. Nevertheless we attempt to define it also for a natural transformation $\alpha\colon F \to G$. Consider the map of sets $\operatorname{colim}\alpha\colon \operatorname{colim} F \to \operatorname{colim} G$. Since for any v in $\mathbb{N}^r$, the following square commutes, we get an inclusion $\alpha(F[x]) \subseteq G[\operatorname{colim}\alpha(x)]$

$$\begin{tikzcd} F(v) \arrow[r, "\alpha(v)"] \arrow[d, "p_v"'] & G(v) \arrow[d, "p_v"] \\ \operatorname{colim} F \arrow[r, "\operatorname{colim}\alpha"] & \operatorname{colim} G \end{tikzcd}$$

It follows that the set $\{w \in \text{gen}(\text{supp}(G[\operatorname{colim}\alpha(x)]) \mid w \leq v\}$ is not empty for any v in $\text{gen}(\text{supp}(F[x]))$. We can order this set using the lexicographical order and define $w_{\alpha,x,v}$ to be the smallest element of this set. Since $w_{\alpha,x,v} \leq v$, there is a unique natural transformation $\text{mor}_{\mathbb{N}^r}(v, -) \to \text{mor}_{\mathbb{N}^r}(w_{\alpha,x,v}, -)$. Define $\overline{\alpha}\colon \mathcal{G}F \to \mathcal{G}G$ to be the natural transformation which on the summand $\mathrm{mor}_{\mathbb{N}^r}(v,-)$ indexed by x in $\mathrm{colim}\, F$ and v in $\mathrm{gen}(\mathrm{supp}(F[x]))$ is given by the composition of $\mathrm{mor}_{\mathbb{N}^r}(v,-) \to \mathrm{mor}_{\mathbb{N}^r}(w_{\alpha,x,v},-)$ and the inclusion into $\mathcal{G}G$ of the summand $\mathrm{mor}_{\mathbb{N}^r}(w_{\alpha,x,v},-)$ indexed by $\mathrm{colim}\,\alpha(x)$ in $\mathrm{colim}\, G$ and $w_{\alpha,x,v}$ in $\mathrm{gen}(\mathrm{supp}(G[\mathrm{colim}\,\alpha(x)])$. Because of these choices we obtain a commutative diagram of natural transformations:

$$\begin{tikzcd} \mathcal{G}F \arrow[r, "\overline{\alpha}"] \arrow[d, "p_F"'] & \mathcal{G}G \arrow[d, "p_G"] \\ F \arrow[r, "\alpha"] & G \end{tikzcd}$$

Explicitly:

$$\begin{tikzcd} \mathrm{mor}_{\mathbb{N}^r}(v,-) \arrow[rr] \arrow[d, hook, "\text{summand indexed by } x \text{ and } v"'] & & \mathrm{mor}_{\mathbb{N}^r}(w_{\alpha,x,v},-) \arrow[d, hook, "\text{summand indexed by } \mathrm{colim}\,\alpha(x) \text{ and } w_{\alpha,x,v}"] \\ \coprod_{x\in \mathrm{colim}\, F} \coprod_{v\in \mathrm{gen}(\mathrm{supp}(F[x]))} \mathrm{mor}_{\mathbb{N}^r}(v,-) \arrow[rr, "\overline{\alpha}"] \arrow[d, "p_F"'] & & \coprod_{x\in \mathrm{colim}\, G} \coprod_{v\in \mathrm{gen}(\mathrm{supp}(G[x]))} \mathrm{mor}_{\mathbb{N}^r}(v,-) \arrow[d, "p_G"] \\ \coprod_{x\in \mathrm{colim}\, F} F[x] \arrow[rr, "\alpha"] & & \coprod_{x\in \mathrm{colim}\, G} G[x] \end{tikzcd}$$

It is important to point out that the assignment $(\alpha\colon F \to G) \mapsto (\overline{\alpha}\colon \mathcal{G}F \to \mathcal{G}G)$ is not a functor. It is not true in general that $\overline{\beta\,\alpha}$ equals $\overline{\beta}\,\overline{\alpha}$.

4. Set valued vs. R-Mod valued functors

Let R be a commutative ring with identity. Recall that we identify the category of functors $\mathrm{Fun}(\mathbb{N}^r, R\text{-Mod})$ with the category of $\mathbb{N}^r$-graded $R[x_1,\ldots,x_r]$-modules by assigning to $F\colon \mathbb{N}^r \to R$-Mod the $\mathbb{N}^r$-graded $R[x_1,\ldots,x_r]$-module given by $\boldsymbol{F} = \oplus_{v\in\mathbb{N}^r} F(v)$ (see 2.8). Via the above identification the free functor $R\mathrm{mor}_{\mathbb{N}^r}(0,-)$ (see 2.7) is mapped to the module $R[x_1,\ldots,x_r]$. Thus sub-functors of $R\mathrm{mor}_{\mathbb{N}^r}(0,-)$ are identified with $\mathbb{N}^r$-graded ideals in $R[x_1,\ldots,x_r]$. Among these sub-functors there are the R-spans of indecomposable multifiltrations of sets and among the $\mathbb{N}^r$-graded ideals in $R[x_1,\ldots,x_r]$ there are the monomial ideals. Note that for an indecomposable multifiltration of sets $F\colon \mathbb{N}^r \to \mathrm{Sets}$, the $\mathbb{N}^r$-graded ideal $\boldsymbol{RF} \subset R[x_1,\ldots,x_r]$ coincides with the monomial ideal $\langle x^v \mid v \in \mathrm{gen}(\mathrm{supp}(F))\rangle$ given in Corollary 3.7. It thus follows from this corollary that the sub-functors of $R\mathrm{mor}_{\mathbb{N}^r}(0,-)$ that are identified with monomial ideals are exactly the R-spans of indecomposable multifiltrations of sets. Since monomial ideals are indecomposable $R[x_1,\ldots,x_r]$-modules, then so are the R-spans of indecomposable multifiltrations of sets. These are the easiest indecomposable multifiltrations of R-modules. The following is a key fact about their finite sums:

4.1. **Proposition.** *Let $\{F_i\colon \mathbb{N}^r \to \mathrm{Sets}\}_{1\le i\le n}$ and $\{G_j\colon \mathbb{N}^r \to \mathrm{Sets}\}_{1\le j\le m}$ be two finite families of indecomposable multifiltrations of sets. If $\oplus_{i=1}^n RF_i\colon \mathbb{N}^r \to R\text{-Mod}$ and $\oplus_{j=1}^m RG_j\colon \mathbb{N}^r \to R\text{-Mod}$ are isomorphic as functors with values in R-Mod, then $n = m$, and there is a permutation σ of $\{1,\ldots,n\}$ for which $F_i\colon \mathbb{N}^r \to \mathrm{Sets}$ and $G_{\sigma(i)}\colon \mathbb{N}^r \to \mathrm{Sets}$ are isomorphic for any i.*

Proof. First note that if $F, G\colon \mathbb{N}^r \to \text{Sets}$ are indecomposable multifiltrations, then the map $R\text{Nat}_{\text{Sets}}(F,G) \to \text{Nat}_{R\text{-Mod}}(RF, RG)$, induced by the R-span functor, is an isomorphism of R modules (this is not true if F is a multifiltration but not indecomposable). Consequently the R module $\text{Nat}_{R\text{-Mod}}(RF, RG)$ is isomorphic to R if $\text{supp}(F) \subset \text{supp}(G)$ or it is trivial if $\text{supp}(F) \not\subset \text{supp}(G)$.

We proceed by induction on n to prove the proposition. Assume $n = 1$. Since RF and $\oplus_{j=1}^m RG_j$ are isomorphic, then so are their colimits which as R modules are isomorphic to respectively R and $\oplus_{j=1}^m R$. For commutative rings the rank of a free module is a well define invariant and hence $m = 1$. The functors RF and RG_1 are therefore isomorphic and by the discussion above $\text{supp}(F)$ and $\text{supp}(G)$ are the same subsets of $\mathbb{N}^r$. We can then use 3.5 to get F and G are isomorphic.

Assume $n > 1$. Consider the subsets $\text{supp}(F_i) \subset \mathbb{N}^r$ for $1 \le i \le n$ and choose among them a maximal one T with respect to the inclusion. By permuting we can assume that:

$$\text{supp}(F_i) = T, \text{ if } 1 \le i \le n' \qquad \text{and} \qquad \text{supp}(F_i) \ne T, \text{ if } n' < i \le n$$

Let $\phi\colon \oplus_{i=1}^n RF_i \to \oplus_{j=1}^m RG_j$ and $\psi\colon \oplus_{j=1}^m RG_j \to \oplus_{i=1}^n RF_i$ be inverse isomorphisms. Since the restriction of ϕ to F_1 is non trivial, there is j such that $T = \text{supp}(F_1) \subset \text{supp}(G_j)$. By the same argument, since the restriction of ψ to G_j is not trivial, there is l for which $\text{supp}(G_j) \subset \text{supp}(F_l)$. As we chose T to be a maximal among the supports of F_i's, we get $l \le n'$ and $\text{supp}(G_j) = T$. Again by permuting if necessary we can assume that:

$$T = \text{supp}(G_i), \text{ if } 1 \le i \le m' \qquad \text{and} \qquad T \not\subset \text{supp}(G_i), \text{ if } m' < i \le m$$

This means that ϕ maps the submodule $\oplus_{i=1}^{n'} RF_i \subset \oplus_{i=1}^n RF_i$ to the submodule $\oplus_{j=1}^{m'} RG_j \subset \oplus_{j=1}^m RG_j$. Furthermore the restriction of $\phi\colon \oplus_{i=1}^{n'} RF_i \to \oplus_{j=1}^{m'} RG_j$ is an isomorphism whose inverse is given by the restriction of ψ. We therefore get that their colimits $\oplus_{i=1}^{n'} R$ and $\oplus_{j=1}^{m'} R$ are also isomorphic and hence $n' = m'$. Moreover, by taking the quotients, we obtain an isomorphism between $\oplus_{i>n'}^n RF_i$ and $\oplus_{j>n'}^m RG_j$. The proposition now follows from the inductive assumption. $\square$

The above proposition can be restated in the form:

4.2. **Corollary.**

1. *Let $\{I_i\}_{1\le i\le n}$ and $\{J_j\}_{1\le j\le m}$ be monomial ideals in $R[x_1 \ldots, x_r]$. If the $\mathbb{N}^r$-graded $R[x_1, \ldots, x_r]$-modules $\oplus_{i=1}^n I_i$ and $\oplus_{j=1}^m J_j$ are isomorphic, then $m = n$ and there is a permutation σ of $\{1, \ldots, n\}$ for which $I_i = J_{\sigma(i)}$.*
2. *Let $F, G\colon \mathbb{N}^r \to$ Sets be compact multifiltrations (see 2.6). Then F and G are isomorphic if and only if their R-spans $RF, RG\colon \mathbb{N}^r \to R$-Mod are isomorphic.*

The statement 4.2.(2) is not true if the functors F and G are not multifiltrations:

4.3. **Example.** Let $F_1, F_2\colon \mathbb{N} \to$ Sets be functors with the same values $F_1(0) = F_2(0) = \{a, b, c, d\}$, $F_1(1) = F_2(1) = \{e, f\}$ and $F_1(n) = F_2(n) = \{g\}$ for $n \ge 2$, however with different maps which are given by the following diagrams:

$$
\begin{tikzcd}
F_1(0) \arrow[r] & F_1(1) \arrow[r] & F_1(2) \\
a \arrow[dr, mapsto] & & \\
b \arrow[r, mapsto] & e \arrow[r, mapsto] & g \\
c \arrow[r, mapsto] & f \arrow[ur, mapsto] & \\
d \arrow[ur, mapsto] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F_2(0) \arrow[r] & F_2(1) \arrow[r] & F_2(2) \\
a \arrow[dr, mapsto] & & \\
b \arrow[r, mapsto] & e \arrow[r, mapsto] & g \\
c \arrow[ur, mapsto] & f \arrow[ur, mapsto] & \\
d \arrow[ur, mapsto] & &
\end{tikzcd}
$$

Although the functors F_1 and F_2 are not isomorphic, their R-spans RF_1 and RF_2 are.

The following example illustrates the fact that not all (indecomposable) multifiltrations of R-modules are R-spans of (indecomposable) multifiltrations of sets.

4.4. **Example.** Consider the multifiltration $F : \mathbb{N}^2 \to R\text{-Mod}$ which on the square $\{v \le (2,2)\} \subset \mathbb{N}^2$ is given by the following commutative diagram:

$$
\begin{tikzcd}
R \arrow[r, "\alpha"] & R \oplus R \arrow[r, "\mathrm{id}"] & R \oplus R \\
0 \arrow[u] \arrow[r] & R \arrow[u, "\beta"'] \arrow[r, "\beta"] & R \oplus R \arrow[u, "\mathrm{id}"'] \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & R \arrow[u, "\gamma"']
\end{tikzcd}
$$

and for w in $\mathbb{N}^2 \setminus \{v \le (2,2)\}$, the map $F(\min(w,(2,2)) \le w)$ is an isomorphism. Assume further that α, β, and γ are monomorphisms and their images are pairwise different submodules of $R \oplus R$. Then this functor is not isomorphic to the R-span of any functor with values in Sets. Note further that in this case F is an indecomposable multifiltration of R-modules whose colimit is free of rank 2 (compare with Proposition 3.1).

Being one critical (see 2.6) for multifiltrations of sets is equivalent to being free (see 3.6). This is not true for multifiltrations of R-modules if $r > 2$:

4.5. **Example.** Consider the multi filtration $F\colon \mathbb{N}^3 \to R\text{-Mod}$ which on the cube $\{v \le (1,1,1)\} \subset \mathbb{N}^3$ is given by the following commutative diagram:

$$
\begin{tikzcd}
& R^2 \arrow[rr, "\alpha"] & & R^4 \\
0 \arrow[ur] \arrow[rr] & & R^2 \arrow[ur, "\beta"] & \\
& 0 \arrow[uu] \arrow[rr] & & R^2 \arrow[uu, "\gamma"'] \\
0 \arrow[uu] \arrow[ur] \arrow[rr] & & 0 \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

and for w in $\mathbb{N}^3 \setminus \{v \le (1,1,1)\}$ the map $F(\min(w,(1,1,1)) \le w)$ is an isomorphism. Then this functor is one critical, it is not free, and it is not the R-span of a multiflitration of sets.

For bifiltrations ($r = 2$) we have the following positive result:

4.6. **Proposition.** *Assume R is a field. A bifiltration $F\colon \mathbb{N}^2 \to R\text{-Mod}$ is free if and only if it is one critical.*

Proof. One implication holds more generally for all r. If $F\colon \mathbb{N}^r \to R\text{-Mod}$ is free, it is the R-span of a free functor $G : \mathbb{N}^r \to \text{Sets}$. Thus F is isomorphic to $\bigoplus_{x\in \operatorname{colim} G} R\text{mor}(v_x, -)$. Since the R-span functor commutes with colimits, we can identify $\operatorname{colim} F$ with $R(\operatorname{colim} G)$. Consider an element $y = \sum_{i=1}^{n} c_i x_i$ in $\operatorname{colim} F$ where x_i belongs to $\operatorname{colim} G$. Note that:

$$\{v \in \mathbb{N}^2 \mid y \in F(v)\} = \bigcap_{i=1}^{n} \{v \in \mathbb{N}^2 \mid x_i \in G(v)\}$$

It follows that this set has a unique minimal element given by $\max\{v_{x_i} \mid 1 \le i \le n\}$. This shows that F is one critical.

Assume now that $F\colon \mathbb{N}^2 \to R\text{-Mod}$ is one critical. To show that it is free it would be enough to prove that it is the R-span of a multifiltration of sets since in this case this multifiltration of sets would be also one critical and therefore free by Proposition 3.6. Define $G(0,0)$ to be a base of $F(0,0)$. Since $F((0,0) \le (1,0))\colon F(0,0) \to F(1,0)$ is an inclusion, we can extend that base of $F(0,0)$ to a base $G(1,0)$ of $F(1,0)$. We can proceed by induction on n and define in this way a sequence of sets

$$G(0,0) \subset G(1,0) \subset \cdots G(n,0) \subset \cdots$$

whose R-span gives the functor F restricted to $\mathbb{N} \times \{0\} \subset \mathbb{N}^2$. We continue again by induction. Assume that $k > 1$ and we have constructed a functor:

$$G\colon \mathbb{N} \times \{v \in \mathbb{N} \mid v < k\} \to \text{Sets}$$

whose R-span is isomorphic to the restriction of F. By the same argument as before, since $F((0, k-1) \le (0,k))\colon F(0,k-1) \to F(0,k)$ is an inclusion we can extend the base $G(0,k-1)$ of $F(0,k-1)$ to a base $G(0,k)$ of $F(0,k)$. Assume $n > 0$ and that we have defined a functor:

$$G\colon \mathbb{N} \times \{v \in \mathbb{N} \mid v < k\} \cup \{v \in \mathbb{N} \mid v < n\} \times \{v \in \mathbb{N} \mid v \le k\} \to \text{Sets}$$

whose R-span is isomorphic to the restriction of F. Since F is one critical the intersection of the images of $F(n-1,k)$ and $F(n,k-1)$ in $F(n,k)$ coincide with the image of $F(n-1,k-1)$. It follows that the induced map:

$$\operatorname{colim}(F(n,k-1) \hookleftarrow F(n-1,k-1) \hookrightarrow F(n-1,k)) \to F(n,k)$$

is an inclusion. Here the assumption $r = 2$ is crucial. We can then extend the subset:

$$\operatorname{colim}(G(n,k-1) \hookleftarrow G(n-1,k-1) \hookrightarrow G(n-1,k)) \hookrightarrow F(n,k)$$

to a base $G(n,k)$ of $F(n,k)$. In this way we get a desired functor

$$G\colon \mathbb{N} \times \{v \in \mathbb{N} \mid v \le k\} \to \text{Sets}$$

whose R-span is isomorphic to F. □

We finish this section with a procedure of obtaining a free presentation of the $\mathbb{N}^r$-graded $R[x_1, \ldots, x_r]$-module $\boldsymbol{RF}$ associated to the R-span of a multifiltration $F\colon \mathbb{N}^r \to \text{Sets}$. In the first 3 steps we recall from the end of Section 3 how to build a presentation of F.

- Decompose F into indecomposable components $\coprod_{x\in \operatorname{colim} F} F[x]$.

- For any x, find the set $T_x := \mathrm{gen}(\mathrm{supp}(F[x]))$.
- Recall that F can be described as the coequalizer of two natural transformations $\pi_0, \pi_1 \colon \mathcal{K}F \to \mathcal{G}F$ between free functors. Explicitly F is isomorphic to the colimit of the following diagram:

$$\coprod_{x \in \operatorname{colim} F} \left(\coprod_{v_0 \neq v_1 \in T_x} \mathrm{mor}_{\mathbb{N}^r}(\max\{v_0, v_1\}, -) \underset{\pi_1[x]}{\overset{\pi_0[x]}{\rightrightarrows}} \coprod_{v \in T_x} \mathrm{mor}_{\mathbb{N}^r}(v, -) \right)$$

  where on the component indexed by $v_0 \neq v_1 \in T_x$, the map π_i, is given by the unique natural transformation $\mathrm{mor}_{\mathbb{N}^r}(\max\{v_0, v_1\}, -) \to \mathrm{mor}_{\mathbb{N}^r}(v_i, -)$ induced by $v_i \leq \max\{v_0, v_1\}$.
- Since the R-span functor commutes with colimits, we get that the module $\boldsymbol{RF}$ is isomorphic to the coequalizer of the following two maps $\boldsymbol{\pi_0}$ and $\boldsymbol{\pi_1}$ between free $\mathbb{N}^r$-graded $R[x_1, \ldots, x_r]$-modules (see 2.3):

$$\bigoplus_{x \in \operatorname{colim} F} \left(\bigoplus_{v_0 \neq v_1 \in T_x} \langle x^{\max\{v_0, v_1\}} \rangle \underset{\boldsymbol{\pi_1[x]}}{\overset{\boldsymbol{\pi_0[x]}}{\rightrightarrows}} \bigoplus_{v \in T_x} \langle x^v \rangle \right)$$

  where $\boldsymbol{\pi_i[x]}$, on the component indexed by $v_0 \neq v_1 \in T_x$, is given by the inclusion $\langle x^{\max\{v_0, v_1\}} \rangle \hookrightarrow \langle x^{v_i} \rangle$. Thus the columns of the matrix representing $\boldsymbol{\pi_i[x]}$ have all entries zero except one which is one.
- The module $\boldsymbol{RF}$ is then isomorphic to the cokernel of the difference $\boldsymbol{\pi_0} - \boldsymbol{\pi_1}$. Note that the columns of the matrix $M(F_x)$ representing $\boldsymbol{\pi_0} - \boldsymbol{\pi_1}$ are vectors of the form: one entry is 1, one entry is -1, and all other entires are zero.

To summarize, with a multifiltration $F \colon \mathbb{N}^r \to \mathrm{Sets}$ we have associated the following invariants:

(1) a set $\operatorname{colim} F$;
(2) for any x in $\operatorname{colim} F$, a finite subset $T_x := \mathrm{gen}(\mathrm{supp}(F[x]))$ of $\mathbb{N}^r$;
(3) for any x in $\operatorname{colim} F$, a $|T_x| \times \binom{|T_x|}{2}$ matrix $M(F_x)$, representing the map $\boldsymbol{\pi_0[x]} - \boldsymbol{\pi_1[x]}$ whose columns are vectors of the form: one entry is 1, one entry is -1, and all other entires are zero.

These invariants can be used to get the $\mathbb{N}^r$-graded $R[x_1, \ldots, x_r]$-module associated to the R-span RF as the cokernel of the map:

$$\bigoplus_{x \in \operatorname{colim} F} \left(\bigoplus_{v_0 \neq v_1 \in T_x} \langle x^{\max\{v_0, v_1\}} \rangle \xrightarrow{M(F_x)} \bigoplus_{v \in T_x} \langle x^v \rangle \right)$$

Note that for any x in $\operatorname{colim} F$, the matrix $M(F_x)$ represents the map in the first syzygy step of the Taylor resolution of a monomial ideal with minimal generating set $\{x^v\}_{v \in T_x}$ (see [4], Exercise 17.11).

5. Functors with values in Spaces

Let $F : \mathbb{N}^r \to \mathrm{Spaces}$ be a multifiltration of simplicial complexes, $X := \operatorname{colim} F$, and R a commutative ring with identity. Let us choose an ordering on the set of vertices of X. Since F is a multifiltration, we can restrict this ordering to the set of vertices of $F(v)$, for any v in $\mathbb{N}^r$. In this way the maps $F(v \leq w)$ are order preserving and we can form a functor of ordered n-simplices to get a multifiltration of sets $F_n \colon \mathbb{N}^r \to \mathrm{Sets}$ (see 2.2) which assigns to any v in $\mathbb{N}^r$ the set $F(v)_n$ of ordered n-simplices in $F(v)$. These functors, for various n's, are connected via the natural transformations given by the maps $d_i : F_{n+1}(v) \to F_n(v)$ which forget the i-th element of an ordered simplex (see 2.2). By applying the R-span functor and taking the alternating sum of the induced maps we obtain a diagram of natural transformations in $\mathrm{Fun}(\mathbb{N}^r, R\text{-Mod})$:

$$RF_{n+1} \xrightarrow{\partial_{n+1}:=\sum_{i=0}^{n+1}(-1)^i d_i} RF_n \xrightarrow{\partial_n:=\sum_{i=0}^{n}(-1)^i d_i} RF_{n-1} \tag{1}$$

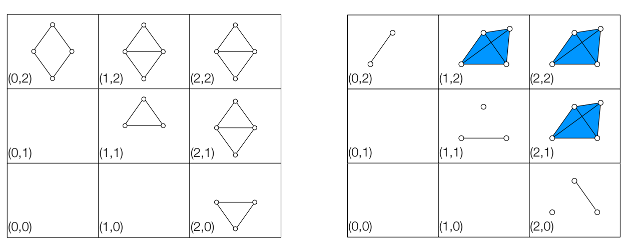
The composition of these maps is trivial and hence we can form a homology functor $H_n(F,R)\colon \mathbb{N}^r \to R$-Mod which in general may not be a multifiltration.

Let us consider the case of $n = 0$. Recall that since RF_{-1} is assumed to be the trivial functor (see 2.2), $H_0(F,R)$ is given by $\mathrm{coker}(d_0 - d_1\colon RF_1 \to RF_0)$. This cokernel is simply the coequalizer of the two maps $d_0, d_1\colon RF_1 \to RF_0$. As the R-span functor commutes with colimits, we then get an isomorphism between $H_0(F,R)$ and the R-span of the following functor with values in the category of sets:

$$\mathrm{colim}\left(F_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} F_0 \right)$$

This is a special property of the 0-th homology. If $n \geq 1$, then it is not true in general that the functors $H_n(F,R)$, $\mathrm{coker}(\partial_{n+1}\colon RF_{n+1} \to RF_n)$, and $\ker(\partial_n\colon RF_n \to RF_{n-1})$ are R-spans of functors with values in the category of sets, even if R is a field as the following example illustrates:

5.1. **Example.** Consider the two multifiltrations of simplicial complexes $F, G : \mathbb{N}^2 \to$ Spaces which on the square $\{v \leq (2,2)\} \subset \mathbb{N}^2$ are described in Figure 1 and for w in $\mathbb{N}^2 \setminus \{v \leq (2,2)\}$, the maps induced by $(\min\{w_1, 2\}, \min\{w_2, 2\}) \leq w$ are the identities.



$F : \mathbb{N}^r \to Spaces$ $\qquad$ $G : \mathbb{N}^r \to Spaces$

FIGURE 1. Multifiltrations with values in Spaces

In the multifiltration $F\colon \mathbb{N}^r \to$ Spaces there are no 2-simplices and hence $H_1(F,R) = \ker(\partial_1\colon RF_1 \to RF_0)$. On the square $\{v \leq (2,2)\} \subset \mathbb{N}^2$, the functors $\ker(\partial_1\colon RF_1 \to$

RF_0) and $\operatorname{coker}(\partial_2 \colon RG_2 \to RG_1)$ are given respectively by the diagrams:

$$\begin{tikzcd}
R \arrow[r, "\binom{1}{1}"] & R \oplus R \arrow[r, "\mathrm{id}"] & R \oplus R \\
0 \arrow[u] \arrow[r] & R \arrow[u, "\binom{1}{0}"] \arrow[r, "\binom{1}{0}"] & R \oplus R \arrow[u, "\mathrm{id}"'] \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & R \arrow[u, "\binom{0}{1}"']
\end{tikzcd}
,\qquad
\begin{tikzcd}
R \arrow[r, "\binom{1}{0}"] & R \oplus R \arrow[r, "\mathrm{id}"] & R \oplus R \\
0 \arrow[u] \arrow[r] & R \arrow[u, "\binom{0}{1}"] \arrow[r, "\binom{0}{1}"] & R \oplus R \arrow[u, "\mathrm{id}"'] \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & R \arrow[u, "\binom{-1}{-1}"']
\end{tikzcd}$$

By Example 4.3 both of the functors are not the R-span of a multifiltration of sets.

We now assume that $F : \mathbb{N}^r \to \text{Spaces}$ is a compact multifiltration. It follows that $X = \operatorname{colim} F$ is a finite complex and the functor $F_n \colon \mathbb{N}^r \to \text{Sets}$ is a compact multifiltration. By Corollary 3.2, F_n can be uniquely decomposed as a coproduct of indecomposable multifiltrations of sets indexed by the set of n-simplices in X, $F_n = \coprod_{\sigma \in X_n} F_n[\sigma]$. Applying the R-span functor, we obtain $RF_n = \bigoplus_{\sigma \in X_n} RF_n[\sigma]$. The R-span of an indecomposable multifiltration of sets is a monomial ideal (see Section 4). We have therefore decomposed $\boldsymbol{RF_n}$ as a direct sum of monomial ideals indexed by the set of n-simplices σ in X, $\boldsymbol{RF_n} = \bigoplus_{\sigma \in X_n} \boldsymbol{RF_n[\sigma]}$. Furthermore such monomial ideals can be completely described in terms of the multifiltration of simplicial complexes. The monomial ideal $\boldsymbol{RF_n}[\sigma]$ is in fact the R-span of the monomials x^v, such that $F(v)$ contains the simplex σ, $\boldsymbol{RF_n}[\sigma] = \langle x^v \mid \sigma \in F(v) \rangle$. A free presentation of $\boldsymbol{RF_n}$ therefore can be easily computed exploiting its decomposition as a sum of monomial ideals (see Section 4).

Since in general the functor $H_n(F, R)$ is not the R-span of a multifiltration of sets we cannot directly use the construction in Section 4 to compute a free presentation of the module $\boldsymbol{H_n(F, R)}$. Our goal in this section is to describe the module $\boldsymbol{H_n(F, R)}$ as the homology of a chain complex of $\mathbb{N}^r$-graded free $R[x_1, \ldots, x_r]$-modules of the form $A \to B \to C$. To compute the homology of a chain complex of $\mathbb{N}^r$-graded free $R[x_1, \ldots, x_r]$-modules one can use standard commutative algebra software [5] or an algorithm presented in [2] which often is faster. As it was pointed out in [2], the efficiency of such algorithm is a consequence of homogeneity and the fact that matrices involved are sparse with coefficients 1, -1, or 0.

We proceed as follows:

- Define the free functor $D_n := \coprod_{\sigma \in X_n} \operatorname{mor}_{\mathbb{N}^r}(0, -)$.
- Consider the decomposition $F_n = \coprod_{\sigma \in X_n} F_n[\sigma]$. Define $\phi_n \colon F_n \to D_n$ to be the coproduct of the unique inclusions $\phi_{n,\sigma} := F_n[\sigma] \hookrightarrow \operatorname{mor}_{\mathbb{N}^r}(0, -)$, $\phi_n = \coprod_{\sigma \in X_n} \phi_{n,\sigma}$.
- The presentations of the multifiltrations of sets F_n and natural transformations between them, given at the end of Section 3, together with the natural transformation defined above, can be organized into the following

commutative diagrams for any $0 \leq i \leq n+1$ and $0 \leq j \leq n$:

$$\begin{tikzcd}
 & \mathcal{G}F_{n+1} \arrow[r, "p_{F_{n+1}}"] \arrow[d, "\overline{d_i}"'] & F_{n+1} \arrow[d, "d_i"] & \\
\mathcal{K}F_n \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & \mathcal{G}F_n \arrow[r, "p_{F_n}"] \arrow[d, "\alpha_j"'] & F_n \arrow[d, "d_j"] & \\
 & \mathcal{G}F_{n-1} \arrow[r, "p_{F_{n-1}}"] & F_{n-1} \arrow[r, hook, "\phi_{n-1}"] & D_{n-1}.
\end{tikzcd}$$

- For $0 \leq j \leq n$ denote by α_j the composition $\alpha_j := \phi_{n-1} d_j p_{F_n}$ and consider the following diagram:

$$\begin{tikzcd}
 & \mathcal{G}F_{n+1} \arrow[d, shift right=2, "\overline{d_0}"'] \arrow[d, shift left=2, "\overline{d_{n+1}}"] \arrow[d, phantom, "\cdots"] \\
\mathcal{K}F_n \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & \mathcal{G}F_n \arrow[d, shift right=2, "\alpha_0"'] \arrow[d, shift left=2, "\alpha_n"] \arrow[d, phantom, "\cdots"] \\
 & D_{n-1}
\end{tikzcd}$$

- By applying the R-span functor and additivity we get two homomorphisms of $\mathbb{N}^r$-graded free $R[x_1, \ldots, x_r]$-modules:

$$\boxed{\boldsymbol{R\mathcal{K}F_n} \oplus \boldsymbol{R\mathcal{G}F_{n+1}} \xrightarrow{\left[\boldsymbol{\pi_0 - \pi_1} \ \sum_{i=0}^{n+1}(-1)^i \overline{\boldsymbol{d_i}}\right]} \boldsymbol{R\mathcal{G}F_n} \xrightarrow{\sum_{j=0}^{n}(-1)^j \boldsymbol{\alpha_j}} \boldsymbol{RD_{n-1}}.}$$

We will now prove that this diagram is the desired chain complex of free $\mathbb{N}^r$-graded $R[x_1, \ldots, x_r]$-modules whose homology is $\boldsymbol{H_n(F, R)}$. In doing this we explain the intuition leading to such a sequence starting from equation 1.

5.2. **Proposition.** *The composition of the above homomorphisms is trivial and the homology of this complex is isomorphic to* $\boldsymbol{H_n(F, R)}$.

Proof. Consider the complex whose homology is $\boldsymbol{H_n(F, R)}$:

$$\boldsymbol{RF_{n+1}} \xrightarrow{\boldsymbol{\partial_{n+1}}} \boldsymbol{RF_n} \xrightarrow{\boldsymbol{\partial_n}} \boldsymbol{RF_{n-1}}.$$

Since $\phi_{n-1} \colon F_{n-1} \hookrightarrow D_{n-1}$ is an inclusion and $p_{F_{n+1}} \colon \mathcal{G}F_{n+1} \to F_{n+1}$ is a surjection, the following sequence is also a complex whose homology is isomorphic to $\boldsymbol{H_n(F, R)}$

$$\boldsymbol{R\mathcal{G}F_{n+1}} \xrightarrow{\boldsymbol{\partial_{n+1} p_{F_{n+1}}}} \boldsymbol{RF_n} \xrightarrow{\boldsymbol{\phi_{n-1}\partial_n}} \boldsymbol{RD_{n-1}}.$$

Intuitively, thinking of $\boldsymbol{RF_{n+1}}$ as the cokernel of $\boldsymbol{\pi_0 - \pi_1} \colon \boldsymbol{R\mathcal{K}F_{n+1}} \to \boldsymbol{R\mathcal{G}F_{n+1}}$ we can replace $\boldsymbol{RF_{n+1}}$ with the free module $\boldsymbol{R\mathcal{G}F_{n+1}}$, while preserving the homology of the chain complex. The boundary operator $\boldsymbol{\partial_{n+1}}$ should now be expressed in terms of generators of $\boldsymbol{RF_{n+1}}$ and therefore replaced by $\boldsymbol{\partial_{n+1} p_{F_{n+1}}}$. Substituting the module $\boldsymbol{RF_{n-1}}$ with its ambient free module $\boldsymbol{RD_{n-1}}$ by composing $\boldsymbol{\partial_n}$ with $\boldsymbol{\phi_{n-1}}$ also obviously preserves homology.

Recall that $\boldsymbol{RF_n}$ is the cokernel of the map $\boldsymbol{\pi_0} - \boldsymbol{\pi_1}\colon \boldsymbol{R\mathcal{K}F_n} \to \boldsymbol{R\mathcal{G}F_n}$. The result now follows from the long exact homology sequence induced by the short exact sequence of chain complexes:

$$\begin{tikzcd}
\boldsymbol{R\mathcal{K}F_n} \arrow[r, "\boldsymbol{\pi_0}-\boldsymbol{\pi_1}"] \arrow[d] & \boldsymbol{\pi_0} - \boldsymbol{\pi_1}(\boldsymbol{R\mathcal{K}F_n}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\boldsymbol{R\mathcal{K}F_n} \oplus \boldsymbol{R\mathcal{G}F_{n+1}} \arrow[r, "{\left[\boldsymbol{\pi_0}-\boldsymbol{\pi_1}\ \overline{\boldsymbol{\partial_{n+1}}}\right]}"] \arrow[d, "\text{projection}"] & \boldsymbol{R\mathcal{G}F_n} \arrow[r, "\sum_{j=0}^{n}(-1)^j \boldsymbol{\alpha_j}"] \arrow[d, "\boldsymbol{p_{F_n}}"] & \boldsymbol{RD_{n-1}} \arrow[d, equal] \\
\boldsymbol{R\mathcal{G}F_{n+1}} \arrow[r, "\boldsymbol{\partial_{n+1} p_{F_{n+1}}}"] & \boldsymbol{RF_n} \arrow[r, "\boldsymbol{\phi_{n-1}\partial_n}"] & \boldsymbol{RD_{n-1}}
\end{tikzcd}$$

□

An important fact is that the above sequence of free $\mathbb{N}^r$-graded $R[x_1, \dots, x_r]$-modules that computes $\boldsymbol{H_n(F, R)}$ can be easily and explicitly described in terms of the original multifiltration of simplicial complexes. Here are the involved modules:

$$\boldsymbol{R\mathcal{K}F_n} = \bigoplus_{\sigma \in X_n} \bigoplus_{v_0 \neq v_1 \in \mathrm{gen}(\sigma)} \langle x^{\max\{v_0, v_1\}} \rangle$$

$$\boldsymbol{R\mathcal{G}F_n} = \bigoplus_{\sigma \in X_n} \bigoplus_{v \in \mathrm{gen}(\sigma)} \langle x^v \rangle$$

$$\boldsymbol{RD_{n-1}} = \bigoplus_{\sigma \in X_{n-1}} R[x_1, \dots, x_r]$$

We will now show how to compute the matrices associated to the maps in this sequence (see 2.3 for our convention to describe homomorphisms between free $\mathbb{N}^r$-graded $R[x_1, \dots, x_r]$-modules).

- Let σ be a simplex in X_n or X_{n+1}. Consider the set $\{v \in \mathbb{N}^r \mid \sigma \in F(v)\}$. This is a saturated set and hence admits a finite minimal set of generators which we denote by $\mathrm{gen}(\sigma)$. This set coincides with $\mathrm{gen}(\mathrm{supp}(F[\sigma]))$ and its elements are exactly the minimal elements of the set $\{v \in \mathbb{N}^r \mid \sigma \in F(v)\}$.
- The matrix $\left[\boldsymbol{\pi_0} - \boldsymbol{\pi_1}\ \sum_{i=0}^{n+1}(-1)^i \overline{\boldsymbol{d_i}}\right]$ is a concatenation of two matrices one for $\boldsymbol{\pi_0} - \boldsymbol{\pi_1}$ and one for $\sum_{i=0}^{n+1}(-1)^i \overline{\boldsymbol{d_i}}$.
- The matrix for $\boldsymbol{\pi_0} - \boldsymbol{\pi_1}$ is a block diagonal. The blocks are indexed by simplices in X_n and the block corresponding to σ in X_n is of the size $|\mathrm{gen}(\sigma)| \times \binom{|\mathrm{gen}(\sigma)|}{2}$. The entry in this block indexed by v in $\mathrm{gen}(\sigma)$ and $v_0 \neq v_1$ in $\binom{\mathrm{gen}(\sigma)}{2}$ has row grade v and column grade $\max\{v_0, v_1\}$. Its value is 1 if $v = v_0$, -1 if $v = v_1$, and 0 otherwise.
- The rows of the matrix for $\sum_{i=0}^{n+1}(-1)^i \overline{\boldsymbol{d_i}}$ are indexed in the same way and have the same grades as the rows of the matrix for $\boldsymbol{\pi_0} - \boldsymbol{\pi_1}$. The columns of the matrix for $\sum_{i=0}^{n+1}(-1)^i \overline{\boldsymbol{d_i}}$ are divided into blocks indexed by simplices in X_{n+1}. The columns in the block corresponding to σ in X_{n+1} are indexed by $\mathrm{gen}(\sigma)$. The corresponding element in $\mathrm{gen}(\sigma)$ is the grade of the column. Each column has exactly $n + 2$ non-zero entries which are either 1 or -1. For a column indexed by v in $\mathrm{gen}(\sigma)$, the non-zero entries occur in the row blocks corresponding to the simplices $d_i(\sigma)$. In each such block there

is only one non-zero entry and is equal to $(-1)^i$ and occurs in the row corresponding to the minimal element with respect to the lexicographical order in the set $\{w \in \operatorname{gen}(d_i(\sigma)) \mid w \leq v\}$.
- The matrix for $\sum_{j=0}^{n}(-1)^j \boldsymbol{\alpha_j}$ has rows indexed by simplices in X_{n-1}. All the rows have grade 0. The columns are divided into blocks indexed by simplices in X_n. The columns in the block corresponding to σ in X_n are indexed by $\operatorname{gen}(\sigma)$. The corresponding element in $\operatorname{gen}(\sigma)$ is the grade of the column. The entry in this matrix in the row indexed by τ in X_{n-1} and the column indexed by v in $\operatorname{gen}(\sigma)$ for σ in X_n has value $(-1)^i$ if $\tau = d_i(\sigma)$ and 0 otherwise. Note that in any row, the entries in the same column block have the same value but different grades.

We end this section showing our procedure to compute the module $\boldsymbol{H_1(F,R)}$ with an example.

5.3. **Example.** Consider the multifiltration $F : \mathbb{N}^2 \to \text{Spaces}$ which on the square $\{v \leq (2,2)\} \subset \mathbb{N}^2$ is described in Figure 2 and for w in $\mathbb{N}^2 \setminus \{v \leq (2,2)\}$, the maps induced by $\min(w,(2,2)) \leq w$ are the identities. The simplicial complex $X = \operatorname{colim} F$ is given by the complex $F(2,2)$ and we choose an ordering of its vertices as indicated in Figure 2.

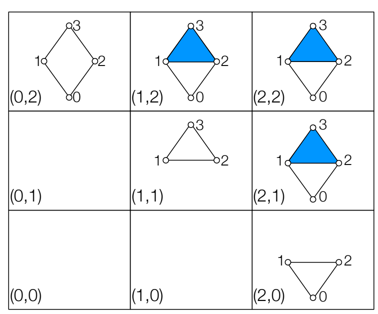


Figure 2. Multifiltration of Spaces with ordering on the set of vertices

The functors $H_1(F,R)\colon \mathbb{N}^2 \to R\text{-Mod}$ on the square $\{v \leq (2,2)\} \subset \mathbb{N}^2$ is given by the following commutative diagram:

$$\begin{tikzcd}
R \arrow[r, "1"] & R \arrow[r, "1"] & R \\
0 \arrow[r] \arrow[u] & R \arrow[r, "0"] \arrow[u, "0"'] & R \arrow[u, "1"'] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & R \arrow[u, "1"']
\end{tikzcd}$$

We will now go through the steps presented above and construct the elements needed to compute $\boldsymbol{H_1(F,R)}$:

- $X_0 = \{0,\ 1,\ 2,\ 3\}$, $X_1 = \{0<1,\ 0<2,\ 1<2,\ 1<3,\ 2<3\}$, $X_2 = \{1<2<3\}$, and $X_n = \emptyset$ for $n \geq 3$.
- For an ordered simplex σ in X, the minimal set of generators $\operatorname{gen}(\sigma)$, ordered by the lexicographical order, is given by the tables:

σ	0	1	2	3
gen(σ)	(0,2) (2,0)	(0,2) (1,1) (2,0)	(0,2) (1,1) (2,0)	(0,2) (1,1)

σ	$0<1$	$0<2$	$1<2$	$1<3$	$2<3$
gen(σ)	(0,2) (2,0)	(0,2) (2,0)	(1,1) (2,0)	(0,2) (1,1)	(0,2) (1,1)

σ	$1<2<3$
gen(σ)	(1,2) (2,1)

- We thus have:

$\boldsymbol{R\mathcal{K}F_1}$	$2\langle x^{(1,2)}\rangle \oplus \langle x^{(2,1)}\rangle \oplus 2\langle x^{(2,2)}\rangle$
$\boldsymbol{R\mathcal{G}F_1}$	$4\langle x^{(0,2)}\rangle \oplus 3\langle x^{(1,1)}\rangle \oplus 3\langle x^{(2,0)}\rangle$
$\boldsymbol{R\mathcal{G}F_2}$	$\langle x^{(1,2)}\rangle \oplus \langle x^{(2,1)}\rangle$
$\boldsymbol{RD_0}$	$4R[x_1,x_2]$

- The matrix associated to $\boldsymbol{\pi_0 - \pi_1} : \boldsymbol{R\mathcal{K}F_1} \to \boldsymbol{R\mathcal{G}F_1}$ with the block decomposition and the column and row grades is given by:

$$
\begin{array}{cc|c|c|c|c|c|}
 & & 0<1 & 0<2 & 1<2 & 1<3 & 2<3 \\
 & & (2,2) & (2,2) & (2,1) & (1,2) & (1,2) \\
0<1 & (0,2) & 1 & 0 & 0 & 0 & 0 \\
 & (2,0) & -1 & 0 & 0 & 0 & 0 \\
\hline
0<2 & (0,2) & 0 & 1 & 0 & 0 & 0 \\
 & (2,0) & 0 & -1 & 0 & 0 & 0 \\
\hline
1<2 & (1,1) & 0 & 0 & 1 & 0 & 0 \\
 & (2,0) & 0 & 0 & -1 & 0 & 0 \\
\hline
1<3 & (0,2) & 0 & 0 & 0 & 1 & 0 \\
 & (1,1) & 0 & 0 & 0 & -1 & 0 \\
\hline
2<3 & (0,2) & 0 & 0 & 0 & 0 & 1 \\
 & (1,1) & 0 & 0 & 0 & 0 & -1
\end{array}
$$

- The matrix associated to $\sum_{j=0}^{2}(-1)^j\overline{\boldsymbol{d_i}} : \boldsymbol{R\mathcal{G}F_2} \to \boldsymbol{R\mathcal{G}F_1}$ with the block decomposition and the column and row grades is given by:

$$
\begin{array}{cc|cc}
 & & \multicolumn{2}{c}{1<2<3} \\
 & & (1,2) & (2,1) \\
0<1 & (0,2) & 0 & 0 \\
 & (2,0) & 0 & 0 \\
\hline
0<2 & (0,2) & 0 & 0 \\
 & (2,0) & 0 & 0 \\
\hline
1<2 & (1,1) & 1 & 1 \\
 & (2,0) & 0 & 0 \\
\hline
1<3 & (0,2) & -1 & 0 \\
 & (1,1) & 0 & -1 \\
\hline
2<3 & (0,2) & 1 & 0 \\
 & (1,1) & 0 & 1
\end{array}
$$

- The matrix associated to $\sum_{j=0}^{1}(-1)^j\boldsymbol{\alpha_j} : \boldsymbol{R\mathcal{G}F_1} \to \boldsymbol{RD_0}$ with the block decomposition and the column and row grades is given by:

$$\begin{array}{cc}
 & \begin{array}{cc|cc|cc|cc|cc} \multicolumn{2}{c|}{0<1} & \multicolumn{2}{c|}{0<2} & \multicolumn{2}{c|}{1<2} & \multicolumn{2}{c|}{1<3} & \multicolumn{2}{c}{2<3} \\ (0,2) & (2,0) & (0,2) & (2,0) & (1,1) & (2,0) & (0,2) & (1,1) & (0,2) & (1,1) \end{array} \\
\begin{array}{cc} 0 & 0 \\ \hline 1 & 0 \\ \hline 2 & 0 \\ \hline 3 & 0 \end{array} & \left(\begin{array}{cc|cc|cc|cc|cc} -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)
\end{array}$$

The following observation is not exploited in our algorithm but might be useful in identifying other strategies to compute the homology of a multifiltration of simplicial complexes. The modules $\boldsymbol{RD_n}$ fit into the ordered simplicial chain complex of X with coefficients in $R[x_1, \dots, x_r]$ (see 2.2):

$$\boldsymbol{RD_{n+1}} \xrightarrow{\partial_{n+1}} \boldsymbol{RD_n} \xrightarrow{\partial_n} \boldsymbol{RD_{n-1}}.$$

Furthermore the natural transformations $\phi_n \colon \boldsymbol{RF_n} \hookrightarrow \boldsymbol{RD_n}$ define the inclusion of chain complexes:

$$\begin{tikzcd}
\boldsymbol{RF_{n+1}} \arrow[r, hook, "\boldsymbol{\phi_{n+1}}"] \arrow[d, "\partial_{n+1}"] & \boldsymbol{RD_{n+1}} \arrow[d, "\partial_{n+1}"] \\
\boldsymbol{RF_n} \arrow[r, hook, "\boldsymbol{\phi_n}"] \arrow[d, "\partial_n"] & \boldsymbol{RD_n} \arrow[d, "\partial_n"] \\
\boldsymbol{RF_{n-1}} \arrow[r, hook, "\boldsymbol{\phi_{n-1}}"] & \boldsymbol{RD_{n-1}}
\end{tikzcd}$$

Therefore, for any n in $\mathbb{N}$, the chain complex in equation 1 which by definition computes $H_n(X, R)$ is a subcomplex of the chain complex computing the n-th homology of X with coefficients in $R[x_1, \dots, x_r]$.

6. Presentations of bifiltrations

Assume that R is a field. For a general multifiltration of spaces, to get a presentation of its homology, one can apply a standard algebra software to the exact sequence given in Proposition 5.2. In the case of a bifiltration $F \colon \mathbb{N}^2 \to \text{Spaces}$ one can try to be more efficient. Instead of applying the software directly to the complex given in 5.2, one can first use the fact that the polynomial ring $R[x_1, x_2]$ has projective dimension 2. This implies that the kernel of any map between free modules is free. In particular the kernel $\boldsymbol{Z}$ of the map $\sum_{j=0}^{n}(-1)^j \boldsymbol{\alpha_j} \colon \boldsymbol{R\mathcal{G}F_n} \to \boldsymbol{RD_{n-1}}$ is free. Let $\phi \colon \boldsymbol{R\mathcal{K}F_n} \oplus \boldsymbol{R\mathcal{G}F_{n+1}} \to \boldsymbol{Z}$ be the map that fits into the following commutative diagram:

$$\begin{tikzcd}
\boldsymbol{R\mathcal{K}F_n} \oplus \boldsymbol{R\mathcal{G}F_{n+1}} \arrow[r, "\phi"] \arrow[rr, bend right, "{[\boldsymbol{\pi_0} - \boldsymbol{\pi_1} \;\; \sum_{i=0}^{n+1}(-1)^i \overline{\boldsymbol{d_i}}]}"'] & \boldsymbol{Z} \arrow[r, hook] & \boldsymbol{R\mathcal{G}F_n}
\end{tikzcd}$$

The map $\phi \colon \boldsymbol{R\mathcal{K}F_n} \oplus \boldsymbol{R\mathcal{G}F_{n+1}} \to \boldsymbol{Z}$ is a free presentation of $\boldsymbol{H_n(F, R)}$. To take a full advantage of this idea, one would need to be able to describe in an efficient way a set of free generators of $\boldsymbol{Z}$. As of writing this paper, we have not found a method for doing it.

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