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HOLOMORPHIC DIFFERENTIALS AND LAGUERRE DEFORMATION OF SURFACES

EMILIO MUSSO AND LORENZO NICOLODI

Abstract. A Laguerre geometric local characterization is given of L-minimal surfaces and Laguerre deformations (T-transforms) of L-minimal isothermic surfaces in terms of the holomorphicity of a quartic and a quadratic differential. This is used to prove that, via their L-Gauss maps, the T-transforms of L-minimal isothermic surfaces have constant mean curvature $H = r$ in some translate of hyperbolic 3-space $\mathbb{H}^3(-r^2) \subset \mathbb{R}^4_1$, de Sitter 3-space $\mathbb{S}^3_1(r^2) \subset \mathbb{R}^4_1$, or have mean curvature $H = 0$ in some translate of a time-oriented lightcone in $\mathbb{R}^4_1$. As an application, we show that various instances of the Lawson isometric correspondence can be viewed as special cases of the T-transformation of L-isothermic surfaces with holomorphic quartic differential.

1. Introduction

Many features of constant mean curvature (CMC) surfaces in 3-dimensional space forms, viewed as isothermic surfaces in Möbius space S^3, can be interpreted in terms of the transformation theory of isothermic surfaces.[1] For instance, the Lawson correspondence between CMC surfaces in space forms can be viewed as a special case of the classical T-transformation of isothermic surfaces. More specifically, it is shown that CMC surfaces in space forms arise in associated 1-parameter families as T-transforms of minimal surfaces in space forms [3, 13, 14, 15, 20, 22]. In addition to being isothermic, minimal surfaces in space forms are Willmore, that is, are critical points of the Willmore energy $\int (H^2 - K) dA$, where H and K are the mean and (extrinsic) Gauss curvatures, and dA is the induced area element of the surface [5, 11, 42]. By a classical result of Thomsen [42, 20, 22], a Willmore surface without umbilics is isothermic if and only if it is locally Möbius equivalent to a minimal surface in some space form. In [44], K. Voss obtained a uniform Möbius geometric characterization of Willmore surfaces

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[1] We recall that a surface is isothermic if it admits conformal curvature line coordinates away from umbilic points.

and CMC surfaces in space forms using the quartic differential $\mathcal{Q}$ introduced by Bryant [11] for Willmore surfaces (cf. also [9, 10, 22] for more details). Voss observed that $\mathcal{Q}$, which indeed may be defined for any conformal immersion of a Riemann surface M into S^3, is holomorphic if and only if, locally and away from umbilics and isolated points, the immersion is Willmore or has constant mean curvature in some space form embedded in S^3.

The purpose of this paper is to discuss the Laguerre geometric counterpart of the Möbius situation described above. One should consider that, already in the fundamental work of Blaschke and Thomsen [5], Möbius and Laguerre surface geometries were developed in parallel, as subgeometries of Lie sphere geometry. Another motivation is that several classical topics in Laguerre geometry, including Laguerre minimal surfaces and Laguerre isothermic surfaces, have recently received much attention in the theory of integrable systems, in discrete differential geometry, and in the applications to geometric computing and architectural geometry [6, 7, 8, 26, 27, 28, 29, 32, 37, 38, 40].

Let us begin by recalling some facts about the Laguerre geometry of surfaces in $\mathbb{R}^3$ to better illustrate our results. The group of Laguerre geometry consists of those transformations that map oriented planes in $\mathbb{R}^3$ to oriented planes, oriented spheres (including points) to oriented spheres (including points), and preserve oriented contact. As such, the Laguerre group is a subgroup of the group of Lie sphere transformations and is isomorphic to the 10-dimensional restricted Poincaré group [5, 17]. Any smooth immersion of an oriented surface into $\mathbb{R}^3$ has a Legendre (contact) lift to the space of contact elements $\Lambda = \mathbb{R}^3 \times S^2$. The Laguerre group acts on the Legendre lifts rather than on the immersions themselves, since it does not act by point-transformations. The *Laguerre space* is Λ as homogeneous space of the Laguerre group. The principal aim of Laguerre geometry is to study the properties of an immersion which are invariant under the action of the Laguerre group on Legendre surfaces. Locally and up to Laguerre transformation, any Legendre immersion arises as a Legendre lift.

A smooth immersed surface in $\mathbb{R}^3$ with no parabolic points is *Laguerre minimal* (*L-minimal*) if it is an extremal of the Weingarten functional

$$\int (H^2/K - 1) dA,$$

where H and K are the mean and Gauss curvatures of the immersion, and dA is the induced area element of the surface [4, 5, 28, 35]. The functional and so its critical points are preserved by the Laguerre group. A surface in $\mathbb{R}^3$ is *L-isothermic* if, away from parabolic and umbilic points, it admits a conformal curvature line parametrization with respect to the third fundamental form [5, 29]. L-isothermic surfaces are invariant under the Laguerre group. See [29, 32] for a recent study on L-isotermic surfaces and their transformations, including the analogues of the T-transformation and of the Darboux transformation in Möbius geometry.

In our discussion we will adopt the cyclographic model of Laguerre geometry [5, 17]. Accordingly, the space of oriented spheres and points in $\mathbb{R}^3$ is naturally identified with Minkowski 4-space $\mathbb{R}^4_1$ and the Laguerre space Λ is described as the space of isotropic (null) lines in $\mathbb{R}^4_1$. For a Legendre immersion $F = (f, n) : M \to \Lambda$, the *Laguerre Gauss* (*L-Gauss*) *map* of F, $\sigma_F : M \to \mathbb{R}^4_1$, assigns to each $p \in M$ the point of $\mathbb{R}^4_1$ representing the *middle sphere*, that is, the oriented sphere of $\mathbb{R}^3$ of radius H/K which is in oriented contact with the tangent plane of f at $f(p)$. Away from umbilics and parabolic points, σ_F is a spacelike immersion with isotropic mean curvature vector; moreover, σ_F has zero mean curvature vector in $\mathbb{R}^4_1$ if and only if F is L-minimal [5, 28].

In [28], for a Legendre immersion $F = (f, n) : M \to \Lambda$ which is L-minimal, we introduced a Laguerre invariant holomorphic quartic differential $\mathcal{Q}_F$ on M viewed as a Riemann surface with the conformal structure induced by $dn \cdot dn$. This quartic differential $\mathcal{Q}_F$ may be naturally defined for arbitrary nondegenerate[2] Legendre surfaces together with an invariant quadratic differential $\mathcal{P}_F$. The quadratic differential $\mathcal{P}_F$ is holomorphic when $\mathcal{Q}_F$ is holomorphic and vanishes if F is L-minimal (cf. Section 3).

The first main result of this paper provides a Laguerre geometric characterization of Legendre surfaces with holomorphic quartic differential.

Theorem A. *The quartic differential $\mathcal{Q}_F$ of a nondegenerate Legendre immersion $F : M \to \Lambda$ is holomorphic if and only if the immersion F is L-minimal, in which case $\mathcal{P}_F$ is zero, or is locally the T-transform of an L-minimal isothermic surface.*

The T-transforms in the statement of Theorem A can be seen as second order Laguerre deformations [29, 33] in the sense of Cartan's general deformation theory [16, 18, 21]. Using Theorem A we then characterize L-minimal isothermic surfaces and their T-transforms in terms of the differential geometry of their L-Gauss maps. We shall prove the following.

Theorem B. *Let $F : M \to \Lambda$ be a nondegenerate Legendre immersion. Then:*

1. *F is L-minimal and L-isothermic if and only if its L-Gauss map $\sigma_F : M \to \mathbb{R}^4_1$ has zero mean curvature in some spacelike, timelike, or (degenerate) isotropic hyperplane of $\mathbb{R}^4_1$.*
2. *F has holomorphic $\mathcal{Q}_F$ and non-zero $\mathcal{P}_F$ if and only if its L-Gauss map $\sigma_F : M \to \mathbb{R}^4_1$ has constant mean curvature $H = r$ in some translate of hyperbolic 3-space $\mathbb{H}^3(-r^2) \subset \mathbb{R}^4_1$, de Sitter 3-space $\mathbb{S}^3_1(r^2) \subset \mathbb{R}^4_1$, or has zero mean curvature in some translate of a time-oriented lightcone $\mathcal{L}^3_+ \subset \mathbb{R}^4_1$.*

In addition, if the L-Gauss map of F takes values in a spacelike (respectively, timelike, isotropic) hyperplane, then the L-Gauss maps of the T-transforms

[2] cf. Section 2.4 for the right definition.

of F take values in a translate of a hyperbolic 3-space (respectively, de Sitter 3-space, time-oriented lightcone).

As an application of the two theorems above, we show that the Lawson correspondence [24] between certain isometric CMC surfaces in different hyperbolic 3-spaces and, in particular, the Umehara–Yamada isometric perturbation [43] of minimal surfaces of $\mathbb{R}^3$ into CMC surfaces in hyperbolic 3-space, can be viewed as a special case of the T-transformation of L-isothermic surfaces with holomorphic quartic differential. (For a Möbius geometric interpretation of the Umehara–Yamada perturbation see [19, 34]). This interpretation also applies to the generalizations of Lawson's correspondence in the Lorentzian [36] and the (degenerate) isotropic situations, namely to the perturbation of maximal surfaces in Minkowski 3-space into CMC spacelike surfaces in de Sitter 3-space [1, 2, 23, 25], and that of zero mean curvature spacelike surfaces in a (degenerate) isotropic 3-space into zero mean curvature spacelike surfaces in a time-oriented lightcone of $\mathbb{R}^4_1$.

The paper is organized as follows. Section 2 recalls some background material about Laguerre geometry, develops the method of moving frames for Legendre surfaces, and briefly discusses L-isothermic surfaces and their T-transforms. Section 3 proves Theorem A. It constructs a quartic differential $\mathcal{Q}_F$ and a quadratic differential $\mathcal{P}_F$ on M from a nondegenerate Legendre immersion $F : M \to \Lambda$, and proves that $\mathcal{Q}_F$ is holomorphic if and only if F is L-minimal or is L-isothermic of a special type (cf. Proposition 3.5 and Section 3.2). Section 4 proves Theorem B. In particular, L-minimal isothermic surfaces and their T-transforms are characterized in terms of the differential geometry of their L-Gauss maps (cf. Propositions 4.1 and 4.4). Section 5 discusses the Laguerre deformation of surfaces with holomorphic $\mathcal{Q}$ in relation with various instances of the Lawson isometric correspondence for CMC spacelike surfaces.

2. Preliminaries and definitions

2.1. **The Laguerre space.** Let $\mathbb{R}^4_1$ denote Minkowski 4-space with its structure of affine vector space and a translation invariant Lorentzian scalar product $\langle\ ,\ \rangle$ which takes the form

$$\langle v, w\rangle = -(v^1 w^4 + v^4 w^1) + v^2 w^2 + v^3 w^3 = g_{ij} v^i w^j,$$

with respect to the standard basis $\epsilon_1, \dots, \epsilon_4$. We use the summation convention over repeated indices. A vector $v \in \mathbb{R}^4_1$ is *spacelike* if $\langle v, v\rangle > 0$, *timelike* if $\langle v, v\rangle < 0$, *lightlike* (or *null* or *isotropic*) if $\langle v, v\rangle = 0$ and $v \neq 0$. We fix a space orientation by requiring that the standard basis is positive, and fix a time-orientation by saying that a timelike or lightlike vector v is positive if $\langle v, \epsilon_1 + \epsilon_4\rangle < 0$. The corresponding positive lightcone is given by

$$\mathcal{L}^3_+ = \left\{ v \in \mathbb{R}^4_1 : \langle v, v\rangle = 0,\ \langle v, \epsilon_1 + \epsilon_4\rangle < 0 \right\}. \tag{2.1}$$

The *Laguerre group* L is the group of isometries of $\mathbb{R}^4_1$ which preserve the given space and time orientations. It is isomorphic to the semidirect product $\mathbb{R}^4 \rtimes G$, where G consists of elements $a = (a^i_j) \in \mathrm{GL}(4,\mathbb{R})$ such that

$$\det a = 1, \quad g_{hk} a^h_i a^k_j = g_{ij}, \quad a\epsilon_1, a\epsilon_4 \in \mathcal{L}^3_+. \tag{2.2}$$

L is isomorphic to the (restricted) *Poincaré group* $\mathbb{R}^4 \rtimes \mathrm{SO}_o(3,1)$.

By a *Laguerre frame* $(x; a_1, \ldots, a_4)$ is meant a position vector x of $\mathbb{R}^4_1$ and an oriented basis $(a_1, \ldots, a_4)$ of $\mathbb{R}^4_1$, such that

$$\langle a_i, a_j \rangle = g_{ij}, \quad a_1, a_4 \in \mathcal{L}^3_+. \tag{2.3}$$

L acts simply transitively on Laguerre frames and the manifold of all such frames may be identified, up to the choice of a reference frame, with L. For any $(x, a) \in L$, we regard x and $a_i = a\epsilon_i$ as $\mathbb{R}^4$-valued functions. There are unique 1-forms ω^i_0 and ω^i_j such that

$$dx = \omega^i_0 a_i, \tag{2.4}$$

$$da_i = \omega^j_i a_j. \tag{2.5}$$

Exterior differentiation of (2.3), (2.4) and (2.5) yields the structure equations:

$$0 = \omega^k_i g_{kj} + \omega^k_j g_{ki}, \tag{2.6}$$

$$d\omega^i_0 = -\omega^i_j \wedge \omega^j_0, \tag{2.7}$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j. \tag{2.8}$$

Any time-oriented isotropic line in $\mathbb{R}^4_1$ may be realized as $x+tv$ where $x \in \mathbb{R}^4_1$, $v \in \mathcal{L}^3_+$ and t ranges over $\mathbb{R}$, and will be denoted by $[x, v]$. The set of all isotropic lines

$$\Lambda = \left\{ [x, v] : x \in \mathbb{R}^4_1, v \in \mathcal{L}^3_+ \right\}$$

is called the *Laguerre space*. The group L acts transitively on Λ by

$$L \times \Lambda \to \Lambda, \quad ((x, a), [y, v]) \mapsto (x, a) \cdot [y, v] = [x + ay, av].$$

If we choose $[0, \epsilon_1] \in \Lambda$ as an origin, and let L_0 be the isotropy subgroup of L at $[0, \epsilon_1]$, the smooth map

$$\pi_L : L \to \Lambda, \quad \pi_L(A) = A \cdot [0, \epsilon_1] = [x, a_1]$$

is the projection map of a principal L_0-bundle over $\Lambda \cong L/L_0$. The elements of L_0 are matrices of the form

$$X(d; b; x) = \left(\begin{pmatrix} d_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} d_2 & \tilde{x}^1 & \tilde{x}^2 & \frac{d_2}{2} {}^t x x \\ 0 & b^1_1 & b^1_2 & x^1 \\ 0 & b^2_1 & b^2_2 & x^2 \\ 0 & 0 & 0 & \frac{1}{d_2} \end{pmatrix} \right),$$

where $b = (b^i_j) \in \mathrm{SO}(2)$, $d = (d_1, d_2)$, $d_2 > 0$, $x = {}^t(x^1, x^2) \in \mathbb{R}^2$, $(\tilde{x}^1, \tilde{x}^2) = d_2 {}^t x b$.

2.2. **The cyclographic model of Laguerre geometry ([5, 17, 37]).** The fundamental objects of Laguerre geometry in Euclidean space are *oriented planes* and *L-spheres* (or *cycles*). By an L-sphere is meant an oriented sphere or a point (a sphere of radius zero). The orientation is determined by specifying a unit normal vector for planes and a signed radius in the case of a sphere.

In the *cyclographic model* of Laguerre geometry, an L-sphere $\sigma(p, r)$, with center $p = {}^t(p^1, p^2, p^3)$ and signed radius $r \in \mathbb{R}$, is represented as the point of $\mathbb{R}^4_1$ given by

$$x(p, r) = {}^t\Big(\frac{r + p^1}{\sqrt{2}}, p^2, p^3, \frac{r - p^1}{\sqrt{2}}\Big).$$

An oriented plane $\pi(n, p)$ through p orthogonal to $n = (n^1, n^2, n^3) \in S^2 \subset \mathbb{R}^3$ is identified with the *isotropic* hyperplane through $x(p) \in \mathbb{R}^4_1$ with isotropic normal vector

$$v(n, p) = {}^t\Big(\frac{1 + n^1}{2}, \frac{n^2}{\sqrt{2}}, \frac{n^3}{\sqrt{2}}, \frac{1 - n^1}{2}\Big).$$

The oriented contact of L-spheres and oriented planes corresponds in $\mathbb{R}^4_1$ to the incidence of points and isotropic hyperplanes. Two oriented L-spheres represented by points x and y in $\mathbb{R}^4_1$ are in oriented contact if and only if

$$\langle x - y, x - y\rangle = 0.$$

In this case, $x - y$ is the normal vector of the isotropic hyperplane in $\mathbb{R}^4_1$ corresponding to the common tangent plane of the L-spheres represented by x and y.

This implies that to any time-oriented isotropic line ℓ corresponds a pencil of oriented spheres which are in oriented contact at $p(\ell) \in \mathbb{R}^3$ with a fixed plane π, where $p(\ell)$ represents the unique x on ℓ such that $\langle x, \epsilon_1 + \epsilon_4\rangle = 0$. In other words, Λ can be identified with the space $\mathbb{R}^3 \times S^2$ of oriented contact elements of $\mathbb{R}^3$ by the correspondence

$$(p, n) \in \mathbb{R}^3 \times S^2 \mapsto [x(p), v(n, p)] \in \Lambda. \tag{2.9}$$

By the structure equations of L, we see that the 1-form $-\langle dx, a_1\rangle$ defines an L-invariant contact distribution on the Laguerre space Λ. By (2.9), such a contact structure coincides with the standard contact structure on $\mathbb{R}^3 \times S^2$. In this way L can be seen as a 10-dimensional group of contact transformations acting on $\mathbb{R}^3 \times S^2$.

The points on a spacelike line ℓ in $\mathbb{R}^4_1$ represent L-spheres in Euclidean space which envelope a circular cone. For two points x and y on ℓ, we have

$$\langle x - y, x - y\rangle = d^2,$$

where d is the *tangential distance*, that is, the Euclidean distance between the points where any common oriented tangent plane touches the L-spheres corresponding to x and y. The tangential distance is zero precisely when the two L-spheres are in oriented contact.

The points on a timelike line ℓ in $\mathbb{R}^4_1$ represent L-spheres without any common oriented tangent plane. For two points x and y on ℓ,

$$\langle x-y, x-y\rangle = -d^2,$$

where d is the *parallel distance*, that is, the distance between the equally oriented parallel tangent planes to the two corresponding L-spheres.

The *spherical system of L-spheres* determined by a point $z \in \mathbb{R}^4_1$ and a constant $c \in \mathbb{R}$ is the set of all L-spheres represented by points $x \in \mathbb{R}^4_1$ satisfying the equation of the *pseudo-hypersphere*

$$\langle x-z, x-z\rangle = c. \tag{2.10}$$

The system consists of all L-spheres which have constant tangential or parallel distance $\sqrt{|c|}$ from the fixed L-sphere represented by z. For $c = 0$, the system consists of all L-spheres which are in oriented contact with the fixed L-sphere represented by z. The spherical system (pseudo-hypersphere) defined by (2.10) is *isotropic* if $c = 0$, *timelike* if $c > 0$, and *spacelike* if $c < 0$.

The *planar system of L-spheres* determined by a point $z \in \mathbb{R}^4_1$ and a vector $v \in \mathbb{R}^4_1$ is the set of all L-spheres represented by points $x \in \mathbb{R}^4_1$ satisfying the equation of the hyperplane

$$\langle x-z, v\rangle = 0. \tag{2.11}$$

The planar system (hyperplane) is called *isotropic* (respectively, *timelike*, *spacelike*) if the vector v is isotropic (respectively, spacelike, timelike).

2.3. **Laguerre surface geometry: the middle frame.** An immersed surface $f : M \to \mathbb{R}^3$, oriented by a unit normal field $n : M \to S^2$, induces a lift $F = (f, n)$ to Λ which is a Legendre (contact) immersion with respect to the canonical contact structure of Λ. More generally, a *Legendre surface* is an immersed surface $F = (f, n) : M \to \Lambda$ such that $df \cdot n = 0$. The additional condition $dn \cdot dn > 0$ will be assumed throughout. Moreover, we say that F is *nondegenerate* if the quadratic forms $df \cdot dn$ and $dn \cdot dn$ are everywhere linearly independent on M. We recall that two Legendrian immersions (M, F) and (M', F') are said to be *L-equivalent* if there exists a diffeomorphism $\phi : M \to M'$ and $A \in L$ such that $F' \circ \phi = AF$. Locally and up to L-equivalence, any Legendre surface arises as a Legendre lift. In particular, two immersed surfaces in $\mathbb{R}^3$ are L-equivalent if their Legendre lifts are L-equivalent.

Definition 2.1. A (local) *Laguerre frame field* along a Legendre immersion $F : M \to \Lambda$ is a smooth map $A = (a_0, a) : U \to L$ defined on an open subset $U \subset M$, such that $\pi_L(a_0, a) = [a_0, a_1] = F$.

For any Laguerre frame field $A : U \to L$ we let

$$\alpha = \big((\alpha^i_0), (\alpha^i_j)\big) = \big((A^*\omega^i_0), (A^*\omega^i_j)\big)\,.$$

We then have

$$\alpha^4_0 = 0, \quad \alpha^2_1 \wedge \alpha^3_1 \neq 0.$$

Any other Laguerre frame field $\hat{A}$ on U is given by $\hat{A} = AX(d; b; x)$, where $X = X(d; b; x) : U \to L_0$ is a smooth map, and $\hat{\alpha}$ and α are related by $\hat{\alpha} = X^{-1}\alpha X + X^{-1}dX$.

A Laguerre frame field A along F is called a *middle frame field* if there exist smooth functions p_1, p_2, p_3, q_1, $q_2 : U \to \mathbb{R}$ such that $\alpha = ((\alpha_0^i), (\alpha_j^i))$ takes the form

$$\left(\begin{pmatrix} 0 \\ \alpha_0^2 \\ \alpha_0^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2q_2\alpha_0^2 - 2q_1\alpha_0^3 & p_1\alpha_0^2 + p_2\alpha_0^3 & p_2\alpha_0^2 + p_3\alpha_0^3 & 0 \\ \alpha_0^2 & 0 & -q_1\alpha_0^2 - q_2\alpha_0^3 & p_1\alpha_0^2 + p_2\alpha_0^3 \\ -\alpha_0^3 & q_1\alpha_0^2 + q_2\alpha_0^3 & 0 & p_2\alpha_0^2 + p_3\alpha_0^3 \\ 0 & \alpha_0^2 & -\alpha_0^3 & -2q_2\alpha_0^2 + 2q_1\alpha_0^3 \end{pmatrix}\right), \tag{2.12}$$

with $\alpha_0^2 \wedge \alpha_0^3 > 0$; (α_0^2, α_0^3) is called the *middle coframe* of F. The existence of a middle frame field along F was proved in [28], under the nondegeneracy assumption. The smooth functions q_1, q_2, p_1, p_2, p_3 form a complete system of Laguerre invariants for F and satisfy the following structure equations:

$$d\alpha_0^2 = q_1\alpha_0^2 \wedge \alpha_0^3, \qquad d\alpha_0^3 = q_2\alpha_0^2 \wedge \alpha_0^3, \tag{2.13}$$

$$dq_1 \wedge \alpha_0^2 + dq_2 \wedge \alpha_0^3 = (p_3 - p_1 - {q_1}^2 - {q_2}^2)\alpha_0^2 \wedge \alpha_0^3, \tag{2.14}$$

$$dq_1 \wedge \alpha_0^3 - dq_2 \wedge \alpha_0^2 = -p_2\alpha_0^2 \wedge \alpha_0^3, \tag{2.15}$$

$$dp_1 \wedge \alpha_0^2 + dp_2 \wedge \alpha_0^3 = (-3q_1p_1 - 4q_2p_2 + q_1p_3)\alpha_0^2 \wedge \alpha_0^3, \tag{2.16}$$

$$dp_2 \wedge \alpha_0^2 + dp_3 \wedge \alpha_0^3 = (-3q_2p_3 - 4q_1p_2 + q_2p_1)\alpha_0^2 \wedge \alpha_0^3. \tag{2.17}$$

The vector a_0 is independent of the middle frame field $A = (a_0, a)$ and is therefore globally defined on M.

Definition 2.2. The *Laguerre Gauss map* (*L-Gauss map*) of the nondegenerate Legendre immersion $F : M \to \Lambda$ is the smooth map

$$\sigma_F : M \to \mathbb{R}^4_1$$

defined locally by $\sigma_F := a_0 : U \to \mathbb{R}^4_1$, where $A = (a_0, a) : U \to L$ is a middle frame field along F.

Remark 2.3. If $A = (a_0, a)$ is a middle frame field along F and U is connected, the only other middle frame field on U is given by

$$\tilde{A} = (a_0, a_1, -a_2, -a_3, a_4).$$

Under this frame change, the invariants p_1, p_2, p_3, q_1, q_2 transform by

$$\tilde{q}_1 = -q_1, \quad \tilde{q}_2 = -q_2, \quad \tilde{p}_1 = p_1, \quad \tilde{p}_2 = p_2, \quad \tilde{p}_3 = p_3.$$

Thus, there are well defined global functions J, $\mathrm{W} : M \to \mathbb{R}$ such that locally

$$\mathrm{J} = \frac{1}{2}(p_1 - p_3), \quad \mathrm{W} = \frac{1}{2}(p_1 + p_3).$$

We recall that a nondegenerate Legendrian immersion $F : M \to \Lambda$ is *L-minimal* if and only if $p_1 + p_3 = 0$ on M (cf. [28]).

2.4. **L-isothermic surfaces.** We recall that a nondegenerate Legendrian immersion $F : M \to \Lambda$ is *L-isothermic* if there exist local coordinates which simultaneously diagonalize the definite pair of quadratic forms $\langle da_0, da_0\rangle = (\alpha_0^2)^2 + (\alpha_0^3)^2$ and $\langle da_0, da_1\rangle = (\alpha_0^2)^2 - (\alpha_0^3)^2$ and which are isothermal with respect to $\langle da_0, da_0\rangle$. If $f : M \to \mathbb{R}^3$ is an immersed surface without umbilic and parabolic points, oriented by the unit normal field n, the L-isothermic condition amounts to the existence of isothermal (conformal) curvature line coordinates for the pair of quadratic forms $III = dn\cdot dn$ and $II = df\cdot dn$. In [29], it is shown that F is L-isothermic if and only if p_2 vanishes identically on M. In this case, there exist isothermal curvature line coordinates $z = x + iy$ such that the middle coframe (α_0^2, α_0^3) takes the form

$$\alpha_0^2 = e^u dx, \quad \alpha_0^3 = e^u dy,$$

for a smooth function u on M. The function $\Phi = e^u$ is called the *Blaschke potential* of F.

Accordingly, from (2.13), (2.14) and (2.15) it follows that

$$q_1 = -e^{-u}u_y, \qquad q_2 = e^{-u}u_x, \tag{2.18}$$

$$p_1 - p_3 \;=\; -e^{-2u}\Delta u. \tag{2.19}$$

Moreover, using (2.16) and (2.17) yields

$$\begin{aligned} d\left(e^{2u}(p_1+p_3)\right) &= -e^{2u}\left\{\left(e^{-2u}\Delta u\right)_x + 4u_x(e^{-2u}\Delta u)\right\}dx \\ &\quad + e^{2u}\left\{(e^{-2u}\Delta u)_y + 4u_y(e^{-2u}\Delta u)\right\}dy. \end{aligned} \tag{2.20}$$

The integrability condition of (2.20) is the so-called *Blaschke equation*,

$$\Delta\left(e^{-u}(e^u)_{xy}\right) = 0, \tag{2.21}$$

which can be viewed as the completely integrable (soliton) equation of L-isothermic surfaces [30, 31].

Conversely, let U be a simply connected domain in $\mathbb{C}$, and let $\Phi = e^u$ be a solution to the Blaschke equation (2.21). It follows that the right hand side of (2.20) is a closed 1-form, say η_Φ. Thus, $\eta_\Phi = dK$, for some function K determined up to an additive constant. If we let

$$\mathrm{w} = Ke^{-2u}, \quad \mathrm{j} = -\frac{1}{2}e^{-2u}\Delta u, \tag{2.22}$$

the 1-form defined by

$$\alpha = \left(\begin{pmatrix} 0 \\ e^u dx \\ e^u dy \\ 0 \end{pmatrix}, \begin{pmatrix} 2du & (\mathrm{w}+\mathrm{j})e^u dx & (\mathrm{w}-\mathrm{j})e^u dy & 0 \\ e^u dx & 0 & u_y dx - u_x dy & (\mathrm{w}+\mathrm{j})e^u dx \\ -e^u dy & -u_y dx + u_x dy & 0 & (\mathrm{w}-\mathrm{j})e^u dy \\ 0 & e^u dx & -e^u dy & -2du \end{pmatrix}\right)$$

satisfies the Maurer–Cartan integrability condition

$$d\alpha + \alpha\wedge\alpha = 0$$

and then integrates to a map $A = (x, a) : U \to L$, such that $dA = A\alpha$. The map $F : U \to \Lambda$ defined by

$$F = [x, a_1]$$

is a smooth Legendre immersion and A is a middle frame field along F. Thus, F is an L-isothermic immersion (unique up to Laguerre equivalence) and Φ is its Blaschke potential.

If, for any $m \in \mathbb{R}$, we let

$$\mathrm{W}_m = \mathrm{W} + me^{-2u}, \quad \mathrm{J}_m = \mathrm{J} = -\frac{1}{2}e^{-2u}\Delta u, \tag{2.23}$$

then the 1-form defined by

$$\alpha^{(m)} = \left(\begin{pmatrix} 0 \\ e^u dx \\ e^u dy \\ 0 \end{pmatrix}, \begin{pmatrix} 2du & (\mathrm{W}_m+\mathrm{J}_m)e^u dx & (\mathrm{W}_m-\mathrm{J}_m)e^u dy & 0 \\ e^u dx & 0 & u_y dx - u_x dy & (\mathrm{W}_m+\mathrm{J}_m)e^u dx \\ -e^u dy & -u_y dx + u_x dy & 0 & (\mathrm{W}_m-\mathrm{J}_m)e^u dy \\ 0 & e^u dx & -e^u dy & -2du \end{pmatrix} \right)$$

satisfies the Maurer–Cartan integrability condition

$$d\alpha^{(m)} + \alpha^{(m)} \wedge \alpha^{(m)} = 0,$$

so that there exists a smooth map $A^{(m)} = (x^{(m)}, a^{(m)}) : U \to L$, such that $dA^{(m)} = A^{(m)}\alpha^{(m)}$. The map $F_m : U \to \Lambda$, given by $F_m = [x^{(m)}, a_1^{(m)}]$, is a smooth Legendre immersion and $A^{(m)}$ is a middle frame field along F_m. Thus, F_m is an L-isothermic immersion (unique up to Laguerre equivalence) with the same Blaschke potential Φ as $F = F_0$. Then there exist a 1-parameter family of non-equivalent L-isothermic immersions F_m, all of which have the same Blaschke potential Φ.

Actually, any other nondegenerate L-isothermic immersion having Φ as Blaschke potential is Laguerre equivalent to F_m, for some $m \in \mathbb{R}$.

Definition 2.4. Two L-isothermic immersions $F, \tilde{F}$ which are not Laguerre equivalent are said to be *T-transforms* (*spectral deformations*) of each other if they have the same Blaschke potential.

Remark 2.5. The spectral family F_m constructed above describes all T-fransforms of $F = F_0$. In fact, any nondegenerate T-transform of F is Laguerre equivalent to F_m, for some $m \in \mathbb{R}$. Such a 1-parameter family of L-isothermic surfaces amounts to the family of second order Laguerre deformations of F in the sense of Cartan ([28, 29]).

2.5. **The geometry of the L-Gauss map.** Given the identification of the space of L-spheres with Minkowski 4-space $\mathbb{R}^4_1$, an immersion $\sigma : M \to \mathbb{R}^4_1$ of a surface M into $\mathbb{R}^4_1$ is called a *sphere congruence*. A Legendre immersion $F = (f, n)$ is said to envelope the sphere congruence σ if, for each $p \in M$, the L-sphere represented by $\sigma(p)$ and the oriented plane $\pi(n(p), f(p))$ are in oriented contact at $f(p)$. If σ is a spacelike immersion, there exist two enveloping surfaces [5].

For a nondegenerate Legendre immersion $F : M \to \Lambda$, the L-Gauss map

$$\sigma_F : M \to \mathbb{R}^4_1,\ p \mapsto \sigma_F(p) := a_0(p)$$

defines a spacelike immersion (cf. [28]) which corresponds to the classical *middle sphere congruence* of F (cf. [5, 28]).

The middle frame field A along F is adapted to the L-Gauss map σ_F. In fact, if $T\mathbb{R}^4_1$ denotes the tangent bundle of $\mathbb{R}^4_1$, then the bundle induced by σ_F over M splits into the direct sum

$$\sigma_F^*(T\mathbb{R}^4_1) = T(\sigma_F) \oplus N(\sigma_F),$$

where $T(\sigma_F) = \operatorname{span}\{a_2, a_3\}$ is the tangent bundle of σ_F and $N(\sigma_F) = \operatorname{span}\{a_1, a_4\}$ its normal bundle.

The first fundamental form of σ_F, i.e., the metric induced by σ_F on M, has the expression

$$g_\sigma = \langle d\sigma_F, d\sigma_F\rangle = (\alpha_0^2)^2 + (\alpha_0^3)^2,$$

and α_0^2, α_0^3 defines an orthonormal coframe field on M.

As $d\sigma_F(TM) = \operatorname{span}\{a_2, a_3\}$, it follows from (2.4) that

$$\alpha_0^1 = 0 = \alpha_0^4.$$

From the exterior derivative of these equations, we have

$$0 = d\alpha_0^1 = -\alpha_2^1 \wedge \alpha_0^2 - \alpha_3^1 \wedge \alpha_0^3,$$
$$0 = d\alpha_0^4 = -\alpha_2^4 \wedge \alpha_0^2 - \alpha_3^4 \wedge \alpha_0^3$$

and then, by Cartan's Lemma,

$$\alpha_i^\nu = h_{i2}^\nu \alpha_0^2 + h_{i3}^\nu \alpha_0^3, \quad h_{ij}^\nu = h_{ji}^\nu \quad \nu = 1, 4;\ i, j = 2, 3,$$

where the functions h_{ij}^ν are the components of the second fundamental form of σ_F,

$$\Pi = \sum_{i,j=2,3} h_{ij}^1 \alpha_0^i \alpha_0^j \otimes a_4 + \sum_{i,j=2,3} h_{ij}^4 \alpha_0^i \alpha_0^j \otimes a_1.$$

From (2.12), it follows that

$$(h_{ij}^1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}.$$

The mean curvature vector of σ_F is half the trace of Π with respect to g_σ,

$$2\mathbf{H} = (p_1 + p_3) a_1.$$

Remark 2.6. Note that $\mathbf{H}$ is a null section of the normal bundle $N(\sigma_F)$, i.e., $\langle \mathbf{H}, \mathbf{H}\rangle = 0$. Moreover, $\mathbf{H} \equiv 0$ on M if and only if $p_1 + p_3$ vanishes identically on M if and only if the Legendrian immersion $F : M \to \mathbf{\Lambda}$ is L-minimal (cf. [28]).

With respect to the null frame field $\{a_1, a_4\}$, the normal connection $\nabla^\perp$ in the normal bundle $N(\sigma)$ of σ is given by

$$\nabla^\perp a_1 = \alpha_1^1 \otimes a_1, \quad \nabla^\perp a_4 = -\alpha_1^1 \otimes a_4.$$

In particular, we have

$$2\nabla^\perp \mathbf{H} = \left[d(p_1 + p_3) + (p_1 + p_3)\, \alpha_1^1\right] a_1, \tag{2.24}$$

so that the parallel condition $\nabla^\perp \mathbf{H} = 0$ takes the form

$$d(p_1 + p_3) + 2(p_1 + p_3)(q_2 \alpha_0^2 - q_1 \alpha_0^3) = 0. \tag{2.25}$$

3. The proof of Theorem A

In this section, for any nondegenerate Legendre immersion $F : M \to \Lambda$, we introduce a quartic differential $\mathcal{Q}_F$ and a quadratic differential $\mathcal{P}_F$. Theorem A will be proved using some results about Legendre immersions with holomorphic $\mathcal{Q}_F$ (cf. Section 3.1) and the interpretation of such immersions as T-transforms of L-minimal isothermic surfaces (cf. Section 3.2).

3.1. **Holomorphic differentials for Legendre immersions.** Let M be an oriented surface and let $F : M \to \Lambda$ be a nondegenerate Legendrian immersion into the Laguerre space. Let $A : M \to L$ be the middle frame field along F and let $\alpha = A^{-1}dA$ denote its Maurer–Cartan form. The metric $(\alpha_0^2)^2 + (\alpha_0^3)^2$ and the area element $\alpha_0^2 \wedge \alpha_0^3$ induced by A determine on M an oriented conformal structure and hence, by the existence of isothermal coordinates, a unique compatible complex structure which makes M into a Riemann surface. In terms of the middle frame field A, the complex structure is characterized by the property that the complex-valued 1-form

$$\varphi = \alpha_0^2 + i\alpha_0^3 \tag{3.1}$$

is of type $(1, 0)$.

Definition 3.1. Let $F : M \to \Lambda$ be a nondegenerate Legendrian immersion. The complex-valued quartic differential form given by

$$\mathcal{Q}_F = Q\varphi^4, \quad Q := \frac{1}{2}(p_1 - p_3) - ip_2, \tag{3.2}$$

and the complex-valued quadratic differential form given by

$$\mathcal{P}_F = P\varphi^2, \quad P := p_1 + p_3 \tag{3.3}$$

are globally defined on the Riemann surface M.

Remark 3.2. The quartic differential $\mathcal{Q}_F$ was considered by the authors for L-minimal surfaces [28]. For L-minimal surfaces, $\mathcal{Q}_F$ is holomorphic. The quadratic differential $\mathcal{P}_F$ vanishes exactly for L-minimal surfaces.

We now collect some useful facts about these differentials. We begin with a simple observation.

Lemma 3.3. *The quartic differential $\mathcal{Q}_F$ is holomorphic if and only if*

$$dQ \wedge \varphi = -4(q_2\alpha_0^2 - q_1\alpha_0^3)Q \wedge \varphi. \tag{3.4}$$

Proof. Taking the exterior derivative of (3.1) and using the structure equations give

$$d\varphi = (q_2\alpha_0^2 - q_1\alpha_0^3) \wedge \varphi. \tag{3.5}$$

Let z be a local complex coordinate on M, so that

$$\varphi = \lambda dz, \quad \lambda \neq 0. \tag{3.6}$$

Then, locally,

$$\mathcal{Q}_F = Q\lambda^4(dz)^4.$$

Exterior differentiation of (3.6) and use of (3.5) give

$$\big(d\lambda - \lambda(q_2\alpha_0^2 - q_1\alpha_0^3)\big) \wedge \varphi = 0. \tag{3.7}$$

By (3.7), it is easily seen that condition (3.4) holds if and only if

$$d(Q\lambda^4) \wedge \varphi = \lambda^4 \left[dQ + 4(q_2\alpha_0^2 - q_1\alpha_0^3)Q\right] \wedge \varphi = 0,$$

that is, if and only if $\frac{\partial}{\partial \bar{z}}(Q\lambda^4) = 0$.

□

Next, we prove the following.

Proposition 3.4. *Let $F : M \to \Lambda$ be a nondegenerate Legendrian immersion. Then:*

1. *$\mathcal{Q}_F$ is holomorphic if and only if the L-Gauss map of F has parallel mean curvature vector.*
2. *If $\mathcal{Q}_F$ is holomorphic, then $\mathcal{P}_F$ is holomorphic.*
3. *If $\mathcal{Q}_F$ is holomorphic and $\mathcal{P}_F$ is non-zero, then F is L-isothermic.*
4. *If $\mathcal{Q}_F = Q\varphi^4$ is holomorphic and $\mathcal{P}_F = P\varphi^2 \neq 0$, then*

$$Q = cP^2,$$

for a real constant c.

Proof. (1) It suffices to prove that (3.4) is equivalent to the parallel condition equation (2.25). Writing out the left and right hand side of (3.4) using the structure equations, we get

$$\begin{aligned} dQ \wedge \varphi = &-\frac{1}{2}(dp_1 + dp_3) \wedge \alpha_0^2 + \frac{i}{2}(dp_1 + dp_3) \wedge \alpha_0^3 \\ &+ (q_1p_3 - 3q_1p_1 - 4q_2p_2)\alpha_0^2 \wedge \alpha_0^3 + i(3q_2p_3 + 4q_1p_2 - q_2p_1)\alpha_0^2 \wedge \alpha_0^3 \end{aligned}$$

and

$$\begin{aligned} -4(q_2\alpha_0^2 - q_1\alpha_0^3)Q \wedge \varphi = &\left[2q_1p_3 - 2q_1p_1 - 4q_2p_2\right] \alpha_0^2 \wedge \alpha_0^3 \\ &+ i\left[2q_2p_3 + 4q_1p_2 - 2q_2p_1)\right] \alpha_0^2 \wedge \alpha_0^3. \end{aligned}$$

Thus, (3.4) is equivalent to

$$\begin{aligned} -\frac{1}{2}(dp_1 + dp_3) \wedge \alpha_0^2 - (q_1p_1 + q_1p_3)\alpha_0^2 \wedge \alpha_0^3 &= 0, \\ \frac{1}{2}(dp_1 + dp_3) \wedge \alpha_0^3 + (q_2p_3 + q_2p_1)\alpha_0^2 \wedge \alpha_0^3 &= 0, \end{aligned}$$

which in turn is equivalent to the parallel condition

$$d(p_1 + p_3) + 2(p_1 + p_3)(q_2\alpha_0^2 - q_1\alpha_0^3) = 0,$$

as claimed.

(2) Observe that the exterior derivative of φ can be written as

$$d\varphi = (q_2\alpha_0^2 - q_1\alpha_0^3) \wedge \varphi. \tag{3.8}$$

By reasoning as above, $\mathcal{P}_F$ is holomorphic if and only if

$$dP \wedge \varphi = -2(q_2\alpha_0^2 - q_1\alpha_0^3)P \wedge \varphi. \tag{3.9}$$

The claim follows from the condition $dP + 2P(q_2\alpha_0^2 - q_1\alpha_0^3) = 0$, which amounts to the condition that $\mathcal{Q}_F$ be holomorphic.

(3) If $\mathcal{P}_F$ is non-zero, i.e., F is not L-minimal, it follows from (2.25) that $d\alpha_1^1 = 0$. On the other hand, the structure equations give

$$d\alpha_1^1 = 2p_2\alpha_0^2 \wedge \alpha_0^3,$$

which implies $p_2 = 0$. Thus F is L-isothermic.

(4) Under the given hypotheses, it follows from assertion (3) that $p_2 = 0$ and then that condition (3.4) can be written

$$dQ + 4\mu Q \equiv 0, \quad \text{mod } \varphi,$$

where $\mu = q_2\alpha_0^2 - q_1\alpha_0^3$ and $d\varphi = \mu \wedge \varphi$. Moreover, condition (3.9) that $\mathcal{P}_F$ be holomorphic can be written

$$dP + 2\mu P \equiv 0, \quad \text{mod } \varphi.$$

Actually, $dP + 2\mu P = 0$. It then follows that

$$d\left(\frac{Q}{P^2}\right) \equiv 0, \quad \text{mod } \varphi.$$

This proves that the real-valued function Q/P^2 is holomorphic, and hence a constant function, as claimed. □

We are now ready to prove our next result.

Proposition 3.5. *The quartic differential $\mathcal{Q}_F$ of a nondegenerate Legendre immersion $F : M \to \Lambda$ is holomorphic if and only if the immersion is L-minimal, in which case the quadratic differential $\mathcal{P}_F$ vanishes on M, or is L-isothermic with Blaschke potential $\Phi = e^u$ satisfying the second order partial differential equation*

$$\Delta u = ce^{-2u}, \tag{3.10}$$

where c is a real constant.

Proof. If $\mathcal{Q}_F$ is holomorphic and the holomorphic quartic differential $\mathcal{P}_F$ vanishes, then F is L-minimal. If instead $\mathcal{P}_F$ is nowhere vanishing, then F is L-isothermic by Proposition 3.4 (3). Let $z = x + iy : U \subset M \to \mathbb{C}$ be an isothermic chart, so that the middle coframing (α_0^2, α_0^3) takes the form $\alpha_0^2 = e^u dx$ and $\alpha_0^3 = e^u dy$, where $\Phi = e^u$ is the Blaschke potential (cf. Section 2.4).

From (2.18) and (2.19) we get

$$\mathcal{Q}_F = \frac{1}{2}(p_1 - p_3)\omega^4 = \text{I}\!\text{I}e^{4u}(dz)^4 = -\frac{1}{2}(e^{-2u}\Delta u)e^{4u}(dz)^4.$$

Since $\mathcal{Q}_F$ is holomorphic,

$$(e^{-2u}\Delta u)e^{4u} = c,$$

for a constant $c \in \mathbb{R}$, that is

$$\Delta u = ce^{-2u}.$$

Conversely, if we assume that $\Phi = e^u$ satisfy the equation (3.10), then the right hand side of (2.20) vanishes identically, which implies that $p_1 + p_3 = ke^{-2u}$, for a constant $k \in \mathbb{R}$. A direct computation shows that $p_1 + p_3 = ke^{-2u}$ satisfies the equation

$$d(p_1 + p_3) + 2(p_1 + p_3)(q_2\alpha_0^2 - q_1\alpha_0^3) = 0.$$

This expresses the fact that the L-Gauss map of F, $\sigma_F = a_0$, has parallel mean curvature vector, or equivalently, that the quartic differential Q_F is holomorphic. □

3.2. **Special L-isothermic surfaces and Laguerre deformation.**

Definition 3.6. A nondegenerate L-isothermic immersion $F : M \to \Lambda$ is called *special* if its Blaschke potential $\Phi = e^u$ satisfies the second order partial differential equation (3.10) of Theorem 3.5, i.e.,

$$\Delta u = ce^{-2u}, \quad c \in \mathbb{R}.$$

The constant c is called the *character* of the special L-isothermic surface F.

Example 3.7 (L-minimal isothermic surfaces)**.** In terms of the Laguerre invariants, L-minimal surfaces are characterized by the condition $p_1+p_3 = 0$ (cf. [28]). Therefore, if a nondegenerate L-isothermic immersion $F : M \to \Lambda$ is also L-minimal, the right hand side of (2.20) is identically zero. This implies

$$d\left(e^{2u}\Delta u\right) = 0,$$

and hence the following.

Proposition 3.8. *Any nondegenerate L-minimal isothermic immersion $F : M \to \Lambda$ is special L-isothermic.*

Other examples of L-minimal isothermic surfaces include L-minimal canal surfaces [27, 31].

3.2.1. *Special L-isothermic surfaces as T-transforms.* Let $F : M \to \Lambda$ be a special L-isothermic immersion. From the proof of Theorem 3.5, we have that the invariants J and W of F are given by

$$\mathrm{W} = ke^{-2u}, \quad \mathrm{J} = -\frac{1}{2}e^{-2u}\Delta u, \tag{3.11}$$

where k is a real constant.

Definition 3.9. The constant k will be referred to as the *deformation* (or *spectral*) *parameter* of the special L-isothermic immersion F.

We have the following.

Proposition 3.10. *Any special L-isothermic immersion in Laguerre space is the T-transform of an L-minimal isothermic immersion.*

Proof. According to Section 2.4, there exists, up to Laguerre equivalence, a unique L-isothermic immersion F with Blaschke potential $\Phi = e^u$ satisfying (3.10) and with invariant functions

$$\mathrm{W} = 0, \quad \mathrm{J} = -\frac{1}{2}e^{-2u}\Delta u = -\frac{c}{2}e^{-4u}.$$

Since $\mathrm{W} = 0$, we have that F is L-minimal. Next, let $\tilde{F}$ be a special L-isothermic immersion with the same Blaschke potential Φ as F and with deformation parameter k. The discussion in Section 2.4 implies that $\tilde{F}$ is a T_m-transform of F. The invariants of $\tilde{F}$ are then given by

$$\mathrm{W}_m = me^{-2u}, \quad \mathrm{J}_m = \mathrm{J} = -\frac{c}{2}e^{-4u}. \tag{3.12}$$

From (3.11) and (3.12), it follows that $m = k$. □

From Proposition 3.5, Proposition 3.8, and Proposition 3.10, we get the first main results of the paper.

Theorem A. *The quartic differential $\mathcal{Q}_F$ of a nondegenerate Legendre immersion $F : M \to \Lambda$ is holomorphic if and only if the immersion F is L-minimal, in which case $\mathcal{P}_F$ vanishes, or is locally the T-transform of an L-minimal isothermic surface.*

In particular, if F has holomorphic $\mathcal{Q}_F$ and zero $\mathcal{P}_F$, then F is L-isothermic if and only if it is L-minimal isothermic.

4. The proof of Theorem B

In this section we characterize L-minimal isothermic surfaces and their T-transforms (i.e., special L-isothermic surfaces with non-zero deformation parameter) in terms of the geometry of their L-Gauss maps. Theorem B will be proved using these characterizations, which are given, respectively, in Proposition 4.1 and Proposition 4.4.

4.1. The geometry of L-minimal isothermic surfaces.

The property of being L-minimal and L-isothermic is reflected in the differential geometry of the L-Gauss map of F. In the following result, the terminology used for hyperplanes of $\mathbb{R}^4_1$ is that introduced in Section 2.2.

Proposition 4.1. *A nondegenerate L-minimal immersion $F : M \to \Lambda$ is L-isothermic if and only if its L-Gauss map $\sigma_F : M \to \mathbb{R}^4_1$ is restricted to lie in the hyperplane of $\mathbb{R}^4_1$ defined by the equation*

$$\langle \sigma_F - O, v \rangle = 0,$$

for some point O and some constant vector v. In particular, the L-Gauss map σ_F of a nondegenerate L-minimal isothermic immersion F has zero mean curvature in some spacelike, timelike, or (degenerate) isotropic hyperplane of $\mathbb{R}^4_1$.

Proof. Let $A = (a_0, a)$ be a middle frame field along F and let $z = x + iy$ be an isothermic chart, so that $\alpha_0^2 = e^u dx$, $\alpha_0^3 = e^u dy$, where $\Phi = e^u$ is the Blaschke potential. Since $p_1 + p_3 = 0$ and $p_2 = 0$, by (2.16) and (2.17), we have

$$dp_1 + 2p_1\alpha_1^1 = 0. \tag{4.1}$$

Next, define

$$v := e^{2u}\left(-p_1 a_1 + a_4\right).$$

By exterior differentiation of v and use of (4.1), it is easily verified that $dv = 0$, i.e., v is a constant vector. This, combined with the fact that $d\sigma_F = \alpha_0^2 a_2 + \alpha_0^3 a_3$, gives

$$d\langle \sigma_F, v\rangle = \langle d\sigma_F, v\rangle = 0,$$

that is,

$$\langle \sigma_F - O, v\rangle = 0,$$

for some point $O \in \mathbb{R}_1^4$, which implies that σ_F actually lies in the hyperplane of $\mathbb{R}_1^4$ defined by O and the vector v. Depending on whether v is timelike, spacelike, or isotropic, σ_F lies in a spacelike, timelike, or (degenerate) isotropic hyperplane of $\mathbb{R}_1^4$.

Conversely, if $\langle \sigma_F - O, v\rangle = 0$, for some point $O \in \mathbb{R}_1^4$ and some constant vector v, then $\langle d\sigma_F, v\rangle = 0$, which implies $v = \ell_1 a_1 + \ell_4 a_4$, for some smooth functions ℓ_1, ℓ_4. Exterior differentiation of $v = \text{const}$ and use of the structure equations yields

$$\begin{aligned} d\ell_1 + \ell_1\alpha_1^1 = 0, \qquad & (\ell_1 + \ell_4 p_1)\alpha_0^2 + \ell_4 p_2 \alpha_0^3 = 0, \\ \ell_4 p_2 \alpha_0^2 - (\ell_1 + \ell_4 p_1)\alpha_0^3 = 0, \qquad & d\ell_4 - \ell_4\alpha_1^1 = 0, \end{aligned}$$

from which follows that $d\alpha_1^1 = 0$. On the other hand, by (2.12) and (2.15),

$$d\alpha_1^1 = 2p_2\alpha_0^2 \wedge \alpha_0^3,$$

which implies $p_2 = 0$, and hence F is L-isothermic.

The last claim follows from the fact that σ_F lies in a hyperplane, a totally geodesic submanifold, and from the fact that, being F L-minimal, σ_F has zero mean curvature vector, that is, $\mathbf{H} = 0$ (cf. also [2], Remark 3). □

Remark 4.2. From the previous proof, it follows that

$$\langle v, v\rangle = 2p_1 e^{4u},$$

and that the equation (3.10) satisfied by the Blaschke potential $\Phi = e^u$ becomes

$$\Delta u = -\langle v, v\rangle e^{-2u}. \tag{4.2}$$

Thus, according to whether p_1 is negative, positive, or zero, σ_F lies in some spacelike, timelike, or isotropic hyperplane of $\mathbb{R}_1^4$.

Remark 4.3. In the language of Section 2.2, Proposition 4.1 says that the L-spheres represented by the L-Gauss map of an L-minimal isothermic surface are restricted to lie in a planar system of L-spheres. The description of L-minimal isothermic surfaces goes back to the work of Blaschke (cf. [4] (1925) and [5], § 81), where it is proved that, up to L-equivalence, they either correspond to minimal surfaces in Euclidean space, surfaces whose middle sphere congruence is tangent to a fixed plane in Euclidean space, or surfaces whose middle spheres have centers lying on a fixed plane. More recently, it has been proved (cf. [41]) that L-minimal isothermic surfaces are locally Laguerre equivalent to surfaces with vanishing mean curvature in $\mathbb{R}^3$, $\mathbb{R}^3_1$, or a (degenerate) isotropic 3-space $\mathbb{R}^3_0$ of signature $(2,0)$. See also [39] for other results on L-minimal isothermic surfaces.

4.2. **The geometry of special L-isothermic surfaces.** We now characterize special L-isothermic surfaces with non-zero deformation parameter in terms of their L-Gauss maps. This is given by the following result.

Proposition 4.4. *Let $F : M \to \Lambda$ be a nondegenerate Legendre immersion. The following two statements are equivalent:*

1. *F has holomorphic $\mathcal{Q}_F$ and non-zero $\mathcal{P}_F$.*
2. *F is L-isothermic and its L-Gauss map σ_F is restricted to lie on the hypersurface of $\mathbb{R}^4_1$ defined by the equation*

$$\langle \sigma_F - O, \sigma_F - O\rangle = \text{costant}, \tag{4.3}$$

for some point O of $\mathbb{R}^4_1$.

Proof. Let us show that (2) implies (1). Let $A = (a_0, a)$ be a middle frame field along F. Differentiation of (4.3) yields

$$\langle d\sigma_F, \sigma_F - O\rangle = 0,$$

which implies

$$\sigma_F - O = \ell_1 a_1 + \ell_4 a_4, \tag{4.4}$$

for smooth functions ℓ_1, ℓ_2. Differentiation of (4.4) and use of the structure equations, taking into account that $p_2 = 0$, yields

$$\begin{aligned}\alpha^2_0\, a_2 + \alpha^3_0\, a_3 &= d\ell_1 a_1 + d\ell_4 a_4 + \ell_1 da_1 + \ell_4 da_4\\ &= \left(d\ell_1 + \ell_1\alpha^1_1\right) a_1 + (\ell_1 + \ell_4 p_1)\,\alpha^2_0\, a_2\\ &\quad + (-\ell_1 + \ell_4 p_3)\,\alpha^3_0\, a_3 + \left(d\ell_4 - \ell_4\alpha^1_1\right) a_4,\end{aligned}$$

which amounts to

$$\ell_1 + \ell_4 p_1 = 1, \qquad -\ell_1 + \ell_4 p_3 = 1, \tag{4.5}$$

$$d\ell_1 + \ell_1\alpha^1_1 = 0, \qquad d\ell_4 - \ell_4\alpha^1_1 = 0. \tag{4.6}$$

The consistency condition of (4.5), $p_1 + p_3 \neq 0$, implies $\mathcal{P}_F$ non-zero. Solving (4.5) for ℓ_1, ℓ_4, we get

$$\ell_1 = \frac{p_3 - p_1}{p_1 + p_3}, \quad \ell_4 = \frac{2}{p_1 + p_3}.$$

Now, it is readily seen that equation $d\ell_4 - \ell_4\alpha_1^1 = 0$ amounts the condition that $\mathcal{Q}_F$ be holomorphic. Using this and Proposition 3.4 (4), one checks that equation $d\ell_1 + \ell_1\alpha_1^1 = 0$ is identically satisfied.

Conversely, if (1) holds, by Proposition 3.4 (3), F is L-isothermic. Now, since $\mathcal{P}_F$ is non-zero, equations (4.5) are consistent and

$$\ell_1 = \frac{p_3 - p_1}{p_1 + p_3}, \quad \ell_4 = \frac{2}{p_1 + p_3}.$$

By Proposition 3.4 (4), we have $(p_3 - p_1) = c(p_1 + p_3)^2$, for a constant c, so that

$$d\ell_1 + \ell_1\alpha_1^1 = c\left[d(p_1 + p_3) + (p_1 + p_3)\alpha_1^1\right] = 0.$$

We also have

$$d\ell_4 - \ell_4\alpha_1^1 = \frac{-2}{(p_1 + p_3)^2}\left[d(p_1 + p_3) + (p_1 + p_3)\alpha_1^1\right] = 0,$$

which implies that equations (4.6) are identically satisfied. There exist then functions ℓ_1, ℓ_4 such that

$$d(\sigma_F - \ell_1 a_1 - \ell_4 a_4) = 0.$$

Thus,

$$\sigma_F - O = \ell_1 a_1 + \ell_4 a_4,$$

for some point $O \in \mathbb{R}^4_1$, which is equivalent to the condition (4.3), as required. □

Remark 4.5. Observe that, with the notation used above,

$$\langle \sigma_F - O, \sigma_F - O\rangle = \text{costant} = \frac{4(p_1 - p_3)}{(p_1 + p_3)^2} = \frac{2\mathrm{J}}{\mathrm{W}^2}. \tag{4.7}$$

Remark 4.6. In the language of Section 2.2, the previous proposition says that the L-spheres represented by the L-Gauss map of a nondegenerate Legendre immersion F with holomorphic $\mathcal{Q}_F$ and non-zero $\mathcal{P}_F$ are restricted to lie in a spherical system of L-spheres.

For $r > 0$ and some point $O \in \mathbb{R}^4_1$, we let

$$\mathbb{S}^3_1(O, r^2) = \left\{x \in \mathbb{R}^4_1 \,:\, \langle x - O, x - O\rangle = 1/r^2\right\}$$

denote the *timelike pseudo-hypersphere centered at* O, a translate of de Sitter 3-space $\mathbb{S}^3_1(r^2) \subset \mathbb{R}^4_1$. The Lorentzian metric on $\mathbb{R}^4_1$ restricts to a Lorentzian metric on $\mathbb{S}^3_1(O, r^2)$ having constant sectional curvature r^2.

In the same way, for $r > 0$ and some point $O \in \mathbb{R}^4_1$, we let

$$\mathbb{H}^3_0(O, -r^2) = \left\{x \in \mathbb{R}^4_1 \,:\, \langle x - O, x - O\rangle = -1/r^2\right\}$$

denote the *spacelike pseudo-hypersphere centered at* O. The Lorentzian metric on $\mathbb{R}^4_1$ restricts to a Riemannian metric on $\mathbb{H}^3_0(O, -r^2)$ of constant sectional curvature $-r^2$. The hyperquadric $\mathbb{H}^3_0(O, -r^2)$ consists of two components congruent to each other under an isometry of $\mathbb{R}^4_1$: the component $\mathbb{H}^3_+(O, -r^2)$ through $O + {}^t\left(\frac{1}{\sqrt{2}r}, 0, 0, \frac{1}{\sqrt{2}r}\right)$, and the component $\mathbb{H}^3_-(O, -r^2)$

through $O+{}^t\left(\frac{-1}{\sqrt{2}r},0,0,\frac{-1}{\sqrt{2}r}\right)$. The components $\mathbb{H}^3_+(O,-r^2)$ and $\mathbb{H}^3_-(O,-r^2)$ are translates, respectively, of the *future* and *past embeddings* of hyperbolic 3-space $\mathbb{H}^3(-r^2)$ in $\mathbb{R}^4_1$.

The *isotropic pseudo-hypersphere centered at* O is the affine lightcone at O defined by

$$\mathcal{L}^3(O)=\left\{x\in\mathbb{R}^4_1\,:\,\langle x-O,x-O\rangle=0\right\}.$$

Equivalently, $\mathcal{L}^3(O)=O+\mathcal{L}^3$, where $\mathcal{L}^3=\left\{x\in\mathbb{R}^4_1\,:\,\langle x,x\rangle=0\right\}$. The Lorentzian metric on $\mathbb{R}^4_1$ restricts to a degenerate metric of signature $(2,0)$ on $\mathcal{L}^3(O)\setminus\{O\}$. The two components $\mathcal{L}^3_+(O)$ and $\mathcal{L}^3_-(O)$ of the hypersurface $\mathcal{L}^3(O)\setminus\{O\}$ are translates of the time-oriended lightcones $\mathcal{L}^3_+$ and $\mathcal{L}^3_-$ of $\mathbb{R}^4_1$ (cf. (2.1)).

We are now in a position to prove our second main result.

Theorem B. *Let $F:M\to\Lambda$ be a nondegenerate Legendre immersion. Then:*

1. *F is L-minimal and L-isothermic if and only if its L-Gauss map $\sigma_F:M\to\mathbb{R}^4_1$ has zero mean curvature in some spacelike, timelike, or (degenerate) isotropic hyperplane of $\mathbb{R}^4_1$.*
2. *F has holomorphic $\mathcal{Q}_F$ and non-zero $\mathcal{P}_F$ if and only if its L-Gauss map $\sigma_F:M\to\mathbb{R}^4_1$ has constant mean curvature $H=r$ in some $\mathbb{H}^3_\pm(O,-r^2)\subset\mathbb{R}^4_1$, $\mathbb{S}^3_1(O,r^2)\subset\mathbb{R}^4_1$, or has zero mean curvature in some $\mathcal{L}^3_\pm(O)\subset\mathbb{R}^4_1$.*

In addition, if the L-Gauss map of F takes values in a spacelike (respectively, timelike, isotropic) hyperplane, then the L-Gauss maps of the T-transforms of F take values in a translate of a hyperbolic 3-space (respectively, de Sitter 3-space, time-oriented lightcone).

Proof. (1) is a consequence of Proposition 4.1. (2) From the preceding discussion and by Proposition 4.4, the L-Gauss map σ_F takes values in a component of some $\mathbb{H}^3_0(O,-r^2)$, in some $\mathbb{S}^3_1(O,r^2)$, or in a component of some $\mathcal{L}^3(O)\setminus\{O\}$, that is, σ_F lies in some (translate of) $\mathbb{H}^3(-r^2)$, $\mathbb{S}^3_1(r^2)$, or in some translate of a time-oriented lightcone of $\mathbb{R}^4_1$. From this and the fact that σ_F has isotropic mean curvature vector field, i.e., $\langle\mathbf{H},\mathbf{H}\rangle=0$, it follows that the mean curvature of σ_F is constant, of values $H=r$ or zero, as indicated (cf. also [2], Remark 3).

The last claim is a consequence of (3.12), (4.2) and (4.7). □

5. Laguerre deformation and Lawson correspondence

In [24], Lawson proved that there is an isometric correspondence between certain constant mean curvature surfaces in space forms. Let $\mathcal{M}^3(\kappa)$ denote the simply-connected, 3-dimensional space form of constant curvature κ. Let M be a simply-connected surface and let $f_1:M\to\mathcal{M}^3(\kappa_1)$ be an immersion of constant mean curvature H_1, with induced metric I and shape

operator S_1. Then, for each constant $\kappa_2 \leq H_1^2 + \kappa_1$, the pair I, $S_2 := S_1 + (H_2 - H_1)Id$ satisfies the Gauss and Codazzi equations for an immersion $f_2 : M \to \mathcal{M}^3(\kappa_2)$ of constant mean curvature $H_2 = \sqrt{H_1^2 + \kappa_1 - \kappa_2}$, which is isometric to f_1.[3] The isometric immersions f_1, f_2 are said to be related by the *Lawson correspondence*. When f_1 is a minimal immersion, f_2 is also referred to as a constant mean curvature *cousin* of f_1. In particular, minimal surfaces in $\mathbb{R}^3$ (respectively, S^3) correspond to constant mean curvature one surfaces in $\mathbb{H}^3(-1)$ (respectively, $\mathbb{R}^3$). For $\kappa_1 = \kappa_2$ and $H_1 = H_2$ we get the family of associated constant mean curvature H_1 surfaces. See [12, 43] for special cases of the Lawson correspondence.

In [36], Palmer proved that there exists a Lawson correspondence between certain constant mean curvature spacelike surfaces in Lorentzian space forms. In particular, there is a correspondence between maximal ($H = 0$) spacelike surfaces in Minkowski 3-space $\mathbb{R}^3_1$ and spacelike surfaces of constant mean curvature ± 1 in de Sitter 3-space $\mathbb{S}^3_1(1)$. See [1, 2, 25] for the discussion of special cases of this correspondence.

Example 5.1 (Deformation of special L-isotermic surfaces with $c > 0$)**.** Let $F : M \to \Lambda$ be a nondegenerate special L-isothermic surface with Blaschke potential $\Phi = e^u$ satisfying the equation

$$\Delta u = ce^{-2u},$$

with character $c > 0$, and deformation (spectral) parameter $k > 0$. This implies

$$\mathrm{W} = ke^{-2u}, \quad \mathrm{J} = -\frac{1}{2}e^{-2u}\Delta u.$$

According to Proposition 4.4 and (4.7), the L-Gauss map σ of F has constant mean curvature $H = \frac{k}{\sqrt{c}}$ into (a translate of) the hyperbolic 3-space of constant curvature $\kappa = -\frac{k^2}{c}$. For each $m \in \mathbb{R}_+$, consider the T_m-transform F_m of F. Again by Proposition 4.4 and (4.7), the L-Gauss map σ_m of F_m is restricted to lie on the hyperquadric centered at O given by

$$\langle \sigma_m - O, \sigma_m - O \rangle = \frac{2\mathrm{J}_m}{\mathrm{W}_m^2} = -\frac{c}{(m+k)^2},$$

for some $O \in \mathbb{R}^4_1$. Thus, σ_m has constant mean curvature $H_m = \frac{m+k}{\sqrt{c}}$ in (a translate of) the hyperbolic 3-space of curvature $\kappa_m = -\frac{(m+k)^2}{c}$. Note that

$$\kappa_m + H_m^2 = \kappa + H^2 = 0$$

does not depend on m. We have then established that the L-Gauss maps of the T-transforms of special L-isothermic surfaces with positive character and positive deformation parameter all have constant mean curvature in (a translate of) some hyperbolic 3-space. Moreover, since the metrics induced by σ_m do not depend on m, i.e., $g_{\sigma_m} = g_\sigma$, we may conclude that the

[3] Actually, there exists a 2π-periodic family of isometric immersions $f_{2,\theta} : M \to \mathcal{M}^3(\kappa_2)$, the classical associated family.

T-transformation of such special L-isothermic surfaces can be viewed, via their L-Gauss maps, as the Lawson correspondence between certain constant mean curvature surfaces in different hyperbolic 3-spaces.

If $F = F_0$ is L-minimal isothermic, its L-Gauss map σ_0 is minimal in (a translate of) Euclidean space $\mathbb{R}^3$. In this case, for each $m \in \mathbb{R}^*$, the L-Gauss map σ_m has constant mean curvature $m/\sqrt{c}$ in hyperbolic 3-space $\mathbb{H}^3(-m^2/c)$. The family $\{\sigma_m\}_{m\in\mathbb{R}^*}$ can be viewed as the 1-parameter family of isometric immersions associated with the minimal immersion σ_0 considered by Umehara–Yamada [43]. This provides a Laguerre geometric interpretation of the Umehara–Yamada isometric perturbation of minimal surfaces in Euclidean space into constant mean curvature surfaces in hyperbolic 3-space. A Möbius geometric interpretation of the Umehara–Yamada isometric perturbation was given in [19].

Example 5.2 (Deformation of special L-isotermic surfaces with $c < 0$)**.** If $F : M \to \Lambda$ is a nondegenerate special L-isothermic surface with Blaschke potential $\Phi = e^u$ satisfying the equation

$$\Delta u = ce^{-2u},$$

with character $c < 0$, and deformation (spectral) parameter $k > 0$, then

$$\mathrm{W} = ke^{-2u}, \quad \mathrm{J} = -\frac{1}{2}e^{-2u}\Delta u.$$

By Proposition 4.4 and (4.7), the L-Gauss map σ has constant mean curvature $H = \frac{k}{\sqrt{-c}}$ into (a translate of) the de Sitter 3-space of constant curvature $\kappa = -\frac{k^2}{c}$. For each $m \in \mathbb{R}_+$, the L-Gauss map σ_m of the T_m-transform F_m is restricted to lie on the hyperquadric centered at O given by

$$\langle \sigma_m - O, \sigma_m - O\rangle = \frac{2\mathrm{J}_m}{\mathrm{W}_m^2} = -\frac{c}{(m+k)^2},$$

for some $O \in \mathbb{R}^4_1$. Thus, σ_m has constant mean curvature $H_m = \frac{m+k}{\sqrt{-c}}$ in (a translate of) the de Sitter 3-space of curvature $\kappa_m = -\frac{(m+k)^2}{c}$. Note that

$$\kappa_m + H_m^2 = \kappa + H^2$$

does not depend on m. We have then established that the L-Gauss maps of the T-transforms of special L-isothermic surfaces with negative character and positive deformation parameter all have constant mean curvature in (a translate of) some de Sitter 3-space. As above, the metrics induced by σ_m do not depend on m, i.e., $g_{\sigma_m} = g_\sigma$. Thus, the T-transformation of special L-isothermic surfaces with negative character and positive deformation parameter can be viewed, via their L-Gauss maps, as the Lawson correspondence between certain constant mean curvature spacelike surfaces in different de Sitter 3-spaces.

If $F = F_0$ is L-minimal isothermic, σ_0 is maximal ($H_0 = 0$) in (a translate of) Minkowski 3-space $\mathbb{R}^3_1$. In this case, for each $m \in \mathbb{R}^*$, the L-Gauss map

σ_m has constant mean curvature $m/\sqrt{-c}$ in de Sitter 3-space $\mathbb{S}^3(-m^2/c)$. This provides a Laguerre geometric interpretation of the Lawson correspondence between maximal spacelike surfaces in Minkowski 3-space and constant mean curvature spacelike surfaces in de Sitter 3-space (cf. Remark 5.4 below).

Example 5.3 (Deformation of special L-isotermic surfaces with $c = 0$)**.** Similar considerations hold for special L-isothermic surfaces with character $c = 0$. By considering the L-Gauss maps, the T-transforms of a zero mean curvature spacelike surface in (a translate of) a time-oriented lightcone $\mathcal{L}^3_+ \subset \mathbb{R}^4_1$ all have zero mean curvature in (a translate of) $\mathcal{L}^3_+$. In particular, if σ_0 is a zero mean curvature spacelike surface in a (degenerate) isotropic hyperplane, then, for each $m \in \mathbb{R}^*$, the L-Gauss map σ_m has zero mean curvature in some translate of $\mathcal{L}^3_+$. As a by-product, the T-transformation establishes an isometric correspondence between zero mean curvature spacelike surfaces in a (degenerate) isotropic 3-space and zero mean curvature spacelike surfaces in a time-oriented lightcone of $\mathbb{R}^4_1$. For a brief introduction to isotropic geometry we refer to [38, 40].

Remark 5.4. The L-Gauss maps of special L-isothermic surfaces are examples of the so-called surfaces of Bryant type in $\mathbb{R}^4_1$ (cf. [2]): a spacelike immersion $\psi : M \to \mathbb{R}^4_1$ with isotropic mean curvature vector $\mathbf{H}$, i.e., $\langle \mathbf{H}, \mathbf{H} \rangle = 0$,[4] and flat normal bundle is called a *surface of Bryant type* in $\mathbb{R}^4_1$ if M is locally isometric to some minimal surface in $\mathbb{R}^3$ or to some maximal surface in $\mathbb{R}^3_1$. In the context of surfaces of Bryant type, [2] describes an isometric perturbation of constant mean curvature $H = r$ surfaces in $\mathbb{H}^3(-r^2)$ (respectively, $\mathbb{S}^3_1(r^2)$) to minimal (respectively, maximal) surfaces in $\mathbb{R}^3$ (respectively, $\mathbb{R}^3_1$), which generalizes that of Umehara-Yamada [43]. By the above discussion, these isometric deformations can be viewed as special cases of the Laguerre deformation of L-isothermic surfaces.

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