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Supermodular comparison of dependence models and multivariate processes, with applications

Esther Frostig[1] and Franco Pellerey[2]

Abstract. The supermodular order is a well-known tool to compare the intrinsic degree of dependence between random vectors or multivariate processes. In this note we describe a general framework for the supermodular comparisons of models incorporating individual and common factors. Examples are given on how to apply these models in comparing hitting times for multivariate processes of interest within risk analysis and reliability theory.

1. Introduction

In dealing with vectors of risks, or vectors of lifetimes, a common way to model the dependence among their components is to consider sets of independent random variables, some of them describing the individuality of the risks and some other describing factors that give rise to mutual partial dependence. Next step is to define the components of the vectors as functions of these factors. For example, in reliability analysis of multicomponent systems (see, e.g., [4]) vectors of lifetimes having Marshall-Olkin multivariate exponential distribution are often considered whose margins are defined as minimum between random values, appropriately chosen from a set of independent and exponentially distributed variables describing individual and common causes of failure. Similarly, multivariate processes are considered in different fields of applied probability. Their components are functions of independent processes, describing individual behaviors in some cases and in some other describing evolutions depending on common environmental conditions. This happens, for example, in the reliability field involving multivariate Poisson processes (see, e.g., [2]), or in risk analysis (like in [22]).

In this setup, sometimes is important to figure out the influence of common factors on the strength of positive dependence among vector components. One possible way is to use comparisons among random vectors based on monotone dependence concepts, or comparisons based on their copulas. Here we describe conditions under which two random vectors (or multivariate processes) defined as above, and having

[1]E. Frostig, University of Haifa, Department of Statistics, Rabin Building, 31905 Haifa, Israel; `frostig@stat.haifa.ac.il`

[2]F. Pellerey, Politecnico di Torino, Dipartimento di Scienze Matematiche, Corso Duca degli Abruzzi 24, 10129 Torino, Italy; `franco.pellerey@polito.it`

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distributions in the same Fréchet class, are comparable in the *supermodular* order. In recent literature these comparisons are considered the most well known positive dependence. As a result, we provide a tool to immediately recognize the effects of common factors, in spite of the fact that the direct computation of dependence through other tools, like their copulas, is in many cases not easy.

We provide examples of application: criteria to compare the positive dependence in default times, or to compare failure times in reliability systems of components subjected to wear, according to Lévy processes. We also provide criteria to compare some multivariate parametrical distributions in supermodular order.

This is the plan of the paper. Section 2 is devoted to the description of the general dependence model that incorporates individual and common factors, and to a brief description of the supermodular order. In two propositions of Section 3, sufficient conditions are given for the supermodular comparison of models, accordingly to our setup. In Section 4 we present some applications.

First, we give some conventions and notations used throughout the paper. The notation $=_{st}$ stands for equality in distribution. For any family of parameterized random variables $\{X_\theta \mid \theta \in \mathcal{T}\}$, such that $\mathcal{T} \subseteq \mathbb{R}$ is the support of a random variable Θ, then we denote by $X(\Theta)$ the mixture of the family $\{X_\theta \mid \theta \in \mathcal{T}\}$ with respect to Θ. For any random variable (or vector) X and an event A, $[X \,|A]$ denotes a random variable whose distribution is the conditional distribution of X given A. Also, throughout this paper we write "increasing" instead of "non-decreasing" and "decreasing" instead of "non-increasing".

2. Preliminaries

Useful definitions and preliminary results are recalled in this section. In particular, a description of the multivariate models employed in the following, together with the definition of supermodular order, is provided.

2.1. **A general model for multivariate risks and multivariate lifetimes with common and individual factors.** The model here presented is a natural generalization, in higher dimensions, of the one defined and studied in the bivariate setting, see [14].

Let $\odot$ denotes any binary increasing, commutative and associative operator between real numbers, and denote with $\bigodot$ the repeated application of this operator: $\bigodot_{i\in\{1,2,\cdots,n\}} x_i = x_1 \odot x_2 \odot \cdots \odot x_n$. For example, the operator $\odot$ can be minimum $\wedge$ or the maximum $\vee$, and in this case $\bigodot_{i\in\{1,2,\cdots,n\}} x_i = \bigwedge_{i\in\{1,2,\cdots,n\}} x_i$ or $\bigodot_{i\in\{1,2,\cdots,n\}} x_i = \bigvee_{i\in\{1,2,\cdots,n\}} x_i$, respectively, or it can be the sum or the product (restricted to non-negative real numbers, to preserve increasing property), so that in these cases $\bigodot_{i\in\{1,2,\cdots,n\}} x_i$ stands for $\sum_{i\in\{1,2,\cdots,n\}} x_i$ or $\prod_{i\in\{1,2,\cdots,n\}} x_i$.

Let now $I = \{1, \cdots, n\}$, and let $\mathbf{S} = \{S_j, j \in J \subseteq \mathbb{N}\}$ be a collection of subsets of I. Also, let $\{X_j, j \in J\}$ be a set of independent random variables describing possible factors influencing multivariate risks or lifetimes. Define the set $\Lambda_i = \{S_j : i \in S_j\}$ for $i = 1, \cdots, n$ and let $T_i = \bigodot_{\{j\,:\,S_j\in\Lambda_i\}} X_j$ be the i-th component in the vector $\mathbf{T} = (T_1, \cdots, T_n)$. For example, given $I = \{1, 2\}$ and $\mathbf{S} = \{S_1, S_2, S_3\} = \{\{1\}, \{2\}, \{1, 2\}\}$, the independent variables X_1, X_2, X_3 contributes to define the vector of lifetimes $\mathbf{T} = (T_1, T_2) = (X_1 \odot X_3, X_2 \odot X_3)$: the first lifetime is influenced by factors X_1 and X_3, while factors X_2 and X_3 act on the second lifetime. In this example X_1 and X_2 describe individual factors (that influence the first and the second component, respectively), while X_3 is a random factor acting to both the components of $\mathbf{T}$.

This model has been considered in reliability theory to describe the vector of lifetimes of a set of components subjected to common and individual shocks. In fact, X_j can be seen as the waiting time to the j-th shock event, that causes the failure of components indexed by S_j. Define $\Lambda_i = \{S_j \,:\, i \in S_j\}$ for $i = 1, \cdots, n$, and let $\odot$ be the minimum. Then $T_i = \min_{j\,:\,S_j \in \Lambda_i}\{X_j\}$ is the time to failure of component i and the joint distribution of $\mathbf{T} = (T_1, \cdots, T_n)$ is the *Generalized Marshall-Olkin (GMO)* distribution studied in [11] (see also the references therein). In the particular case, when X_j are exponentially distributed, the well-known multivariate exponential distribution given in [12] is recovered.

Similar models have been considered in risk theory or in multiple default problems to describe sets of dependent risks. Indeed, let again $\mathbf{S} = \{S_j, j \in J \subseteq \mathbb{N}\}$ be a collection of subsets of $I = \{1, \cdots, n\}$, and let $\{X_j, j \in J\}$ be a set of independent random variables. Assume that every X_j additively acts on all the components of index $i \in S_j$. Define the set $\Lambda_i = \{S_j \,:\, i \in S_j\}$ for $i = 1, \cdots, n$. Then one can consider the vector $\mathbf{T} = (T_1, \cdots, T_n)$ of dependent risks, where $T_i = \sum_{j\,:\,S_j \in \Lambda_i} X_j$ for all $i = 1, \cdots, n$, thus replacing the operator $\odot$ with the sum. For example, a vector of risks defined as $(T_1, T_2, T_3) = (X_1 + X_4, X_2 + X_5, X_3 + X_4 + X_5)$, where the X_i are independent each other, is defined as above, with $I = \{1, 2, 3\}$ and $\mathbf{S} = \{\{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}$. This kind of additive risks model has been recently considered in [16] (see also [17] and references therein).

2.2. **Supermodular order.** We recall here the supermodular order, which is one of most well know orders considered in the literature to compare the degree of positive dependence among components of random vectors.

Let $\prec$ denote the coordinatewise ordering in $\mathbb{R}^n$. Let us recall that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be *supermodular* if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ it is

$$\varphi(\mathbf{x}) + \varphi(\mathbf{y}) \prec \varphi(\mathbf{x} \wedge \mathbf{y}) + \varphi(\mathbf{x} \vee \mathbf{y}) \,, \tag{2.1}$$

where the operators $\wedge$ and $\vee$ denote respectively coordinatewise minimum and maximum.

Definition 2.1. Let $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, ..., Y_n)$ be two n-dimensional random vectors, then $\boldsymbol{X}$ is said to be *smaller than* $\boldsymbol{Y}$ *in the supermodular order* (denoted by $\boldsymbol{X} \prec_{\mathrm{sm}} \boldsymbol{Y}$) if

$$\mathbb{E}[\phi(\boldsymbol{X})] \leq \mathbb{E}[\phi(\boldsymbol{Y})] \,,$$

for every supermodular real-valued function ϕ defined on $\mathbb{R}^n$ for which the expectations exist.

To provide an intuitive interpretation of the supermodular order, which is actually a comparison dealing with concordance between components of random vectors, recall that two random variables are said to be *concordant* if large values of one correspond to large values of the other, and if small values of one correspond to small values of the other. Similarly, given two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$, we say that (x_1, y_1) and (x_2, y_2) are *concordant* if $x_1 < x_2$ and $y_1 < y_2$, or if $x_1 > x_2$

and $y_1 > y_2$, and we say that (x_1, y_1) and (x_2, y_2) are *discordant* if $x_1 < x_2$ and $y_1 > y_2$, or if $x_1 > x_2$ and $y_1 < y_2$. Observe that supermodular functions assign higher values along lines connecting concordant points, rather than along lines connecting discordant points. So expectations of supermodular functions of random vectors assume higher values when the probability mass of the random vector is concentrated along lines connecting concordant points. Therefore the probability of concordance is higher than the probability of discordance. Thus, $\boldsymbol{X} \prec_{\rm sm} \boldsymbol{Y}$ if the concordance (positive dependence) among the components of $\boldsymbol{Y}$ is greater (in the sense defined above) than the concordance (positive dependence) among the components of $\boldsymbol{X}$.

For example, as is well-known, if $\mathbf{X}$ is a multivariate normal random vector with mean vector $\mathbf{0}$ and variance-covariance matrix $\mathbf{S}$, and $\mathbf{Y}$ is a multivariate normal random vector with mean vector $\mathbf{0}$ and variance-covariance matrix $\mathbf{S}+\mathbf{D}$, where $\mathbf{D}$ is a matrix with zero diagonal elements such that $\mathbf{S}+\mathbf{D}$ is nonnegative definite, then $\boldsymbol{X} \prec_{\rm sm} \boldsymbol{Y}$ if, and only if, all the entries of $\mathbf{D}$ are nonnegative (see [21] page 400). Moreover, given the vectors $\mathbf{X}$ and $\mathbf{Y}$ assuming values on $\mathbb{R}^2$, $\boldsymbol{X} \prec_{\rm sm} \boldsymbol{Y}$ implies $r_{\mathbf{X}} \leq r_{\mathbf{Y}}$, $\rho_{\mathbf{X}} \leq \rho_{\mathbf{Y}}$ and $\tau_{\mathbf{X}} \leq \tau_{\mathbf{Y}}$, where r, ρ and τ denote, respectively, the Pearson's correlation coefficient, the Spearman's concordance index and the Kendall's concordance index.

The supermodular order has been considered in several applied contexts (see [20], [15], [7], [18], or [6], among others). For a complete description of the supermodular order and its properties see [21]. Among other properties of the supermodular order, the following one will be used in the next sections; the proof may be found for example in [20].

Lemma 2.1. *Let $X_1, \cdots, X_n$ be identically distributed random variables. Then*

$$(X_1, \cdots, X_n) \prec_{\rm sm} (X_1, \cdots, X_1) .$$

3. Sufficient conditions for supermodular comparisons

We present here two results describing conditions for the supermodular comparison of random vectors defined according to the dependence model introduced in the previous section. These results are not new, since they have been proved already in literature for specific choices of the operator $\odot$, like the minimum or the sum (see, e.g., [24]). Moreover, they can be proved by using known closure properties of the supermodular order, stated in [21]. However, the statements, and the proofs, given here are more general than those given in the literature, being comprehensive of all the possible choices of $\odot$.

Let us consider the random vector $\mathbf{T}$ described in Subsection 2.1, and a new random vector $\widetilde{\mathbf{T}}$ defined by means of the same set $\mathbf{S} = \{S_j, j \in J \subseteq \mathbb{N}\}$, but with different common and individual random factors $\widetilde{X}_j$.

Let $\mathbf{I} = \{I_1, \cdots, I_k\} \subseteq \mathbf{S}$ be such that the $I_r, r = 1, \cdots, k$, are disjoint subsets of $I = \{1, \cdots, n\}$ and also such that $\bigcup_{r=1}^k I_r \in \mathbf{S}$, and let $\mathcal{E}$ and $\mathcal{E}_r$, $r = 1, \cdots, k$ be independent and identically distributed random variables, independent of the X_j

and $\tilde{X}_j$. Let:

$$\begin{aligned}
\tilde{X}_j &=_{st} X_j && \text{if } S_j \notin \{I_1, \cdots, I_k, \cup_{r=1}^k I_r\} \ , \\
X_j &=_{st} \tilde{X}_j \odot \mathcal{E} && \text{if } S_j = \cup_{r=1}^k I_r \ , \\
\tilde{X}_j &=_{st} X_j \odot \mathcal{E}_r && \text{if } S_j \in \{I_1, \cdots, I_k\} \ .
\end{aligned}$$

Now consider the random vector $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \cdots, \widetilde{T}_n)$, where the $\widetilde{T}_i$ are defined as $\widetilde{T}_i = \bigodot_{\{j \,:\, S_j \in \Lambda_i\}} \widetilde{X}_j$. It is easy to observe that, under relations above, because of associativity of $\odot$, the vectors $\mathbf{T} = (T_1, \cdots, T_n)$ and $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \cdots, \widetilde{T}_n)$ have the same marginal distributions.

For example, given $\mathbf{T} = (T_1, T_2) = (X_1 \odot X_3, X_2 \odot X_3)$, a new vector having the same marginal distributions of $\mathbf{T}$ can be defined considering three new independent and identically distributed random variables $\mathcal{E}, \mathcal{E}_1$ and $\mathcal{E}_2$, observing that $\mathbf{T}$ is defined by letting $\mathbf{S} = \{S_1, S_2, S_3\} = \{\{1\}, \{2\}, \{1, 2\}\}$, and considering the independent variables $\widetilde{X}_j$ such that $\widetilde{X}_1 = X_1 \odot \mathcal{E}_1$, $\widetilde{X}_2 = X_2 \odot \mathcal{E}_2$ and $X_3 = \widetilde{X}_3 \odot \mathcal{E}$. The corresponding vector is $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \widetilde{T}_2) = (\widetilde{X}_1 \odot \widetilde{X}_3, \widetilde{X}_2 \odot \widetilde{X}_3)$, and the two vectors $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ have the same marginal distributions, being, e.g.,

$$T_1 = X_1 \odot X_3 = X_1 \odot \mathcal{E} \odot \widetilde{X}_3 =_{st} X_1 \odot \mathcal{E}_1 \odot \widetilde{X}_3 = \widetilde{X}_1 \odot \widetilde{X}_3 = \widetilde{T}_1 \ .$$

The following statement provides a comparison of the degree of dependence between the two vectors.

Proposition 3.1. *Let the vectors* $\mathbf{T}$ *and* $\widetilde{\mathbf{T}}$ *be defined as above. Then* $\widetilde{\mathbf{T}} \prec_{\mathrm{sm}} \mathbf{T}$.

Proof. For $i \in I = \{1, 2, \cdots, n\}$ let

$$Z_i = \bigodot_{\{j \,:\, S_j \in \Lambda_i, S_j \notin \{I_1, \cdots, I_k, \cup_{j=1}^k I_j\}\}} X_j \ .$$

Consider now the variables W_i defined as

$$W_i = \begin{cases} Z_i & \text{if } i \notin \cup_{r=1}^k I_r \ , \\ Z_i \odot \widetilde{X}_{\cup_{r=1}^k I_r} \odot X_{I_r} & \text{if } i \in I_r, \, r \in \{1, \cdots, k\} \ . \end{cases}$$

Then the components T_i of the vector $\mathbf{T}$ can be represented as

$$T_i = \begin{cases} W_i & \text{if } i \notin \cup_{r=1}^k I_r \\ W_i \odot \mathcal{E} & \text{if } i \in I_r, \ r \in \{1, \cdots, k\} \ . \end{cases} \tag{3.1}$$

Similarly, the components $\widetilde{T}_i$ of the vector $\widetilde{\mathbf{T}}$ can be represented as

$$\widetilde{T}_i = \begin{cases} W_i & \text{if } i \notin \cup_{r=1}^k I_r \\ W_i \odot \mathcal{E}_r & \text{if } i \in I_r, \in \{1, \cdots, k\} \ . \end{cases} \tag{3.2}$$

Let $|I_r| = m_r$, $r = 1, \cdots, k$, and $m = \sum_{r=1}^k m_r$. Without loss of generality, assume that the first m components are the ones in $\cup_{r=1}^k I_r$. Consider any supermodular function $\widetilde{g}$ from $\mathbb{R}^n$ to $\mathbb{R}$. Observe that for fixed $\mathbf{w} = (w_1, \cdots, w_n)$, the function $g(x_1, \cdots, x_m) = \widetilde{g}(x_1 \odot w_1, \cdots, x_m \odot w_m, w_{m+1}, \cdots, w_n)$ is a supermodular function from $\mathbb{R}^m$ to $\mathbb{R}$. Thus, to prove that $\mathbb{E}[\widetilde{g}(\widetilde{\mathbf{T}})] \leq \mathbb{E}[\widetilde{g}(\mathbf{T})]$, it enough to show that $\mathbb{E}[\widetilde{g}(\widetilde{\mathbf{T}})|\mathbf{W} = \mathbf{w}] \leq \mathbb{E}[\widetilde{g}(\mathbf{T})|\mathbf{W} = \mathbf{w}]$, or equivalently, that:

$$\mathbb{E}[g(\underbrace{\mathcal{E}_1, \cdots, \mathcal{E}_1}_{m_1}, \underbrace{\mathcal{E}_2, \cdots, \mathcal{E}_2}_{m_2}, \cdots, \underbrace{\mathcal{E}_k, \cdots, \mathcal{E}_k}_{m_k})] \leq \mathbb{E}[g(\underbrace{\mathcal{E}, \cdots, \mathcal{E}}_{m})] .$$

The last inequality follows from (3.1), (3.2) and Lemma 2.1.

□

Proposition 3.1 applies, for example, to the parametric h–transform of the Marshall-Olkin (MO) multivariate exponential distributions defined in [23]. Consider an MO exponential distribution defined by means of a set $\{X_j, j \in J\}$ of independent and exponentially distributed variables, with respective rates $L = \{\lambda_{S_j}, j \in J\}$. Assume that every X_j is the waiting time to the j-type shock, which causes the failure of the component in the set S_j. Define $\Lambda_i = \{S_j : i \in S_j\}$ for $i = 1, \cdots, n$ and let $T_i = \min_{j : S_j \in \Lambda_i}\{X_j\}$ be the time to failure of i-th component, having survival function $\overline{F}_{T_i}(t) = \exp(-l_i t)$, where $l_i = \sum_{j : S_j \in \Lambda_i} \lambda_{S_j}$. Then the vector $\mathbf{T} = (T_1, \cdots, T_n)$ has an MO exponential distribution and its joint survival function is defined as

$$\overline{F}_{\mathbf{T}}(t_1, \cdots, t_n) = P(T_1 > t_1, \cdots, T_n > t_n) = \prod_{j : S_j \in \mathbf{S}} \min_{i \in S_j}\{\exp(-\lambda_{S_j} t_i)\} . \tag{3.3}$$

For every fixed $\delta \in \mathbb{R}^+$ such that $\delta \leq \lambda_{S_j}$, $\forall j \in J$, and for every $\mathbf{I} = \{I_1, \cdots, I_k\} \subseteq \mathbf{S}$ as above, a new set $h^{\delta,\mathbf{I}}(L)$ of rates is defined letting

$$h^{\delta,\mathbf{I}}(\lambda_{S_j}) = \begin{cases} \lambda_{S_j} & \text{if } S_j \notin \{I_1, \cdots, I_k, \cup_{r=1}^k I_r\} , \\ \lambda_{S_j} - \delta & \text{if } S_j = \cup_{r=1}^k I_r , \\ \lambda_{S_j} + \delta & \text{if } S_j \in \{I_1, \cdots, I_k\} , \end{cases}$$

thus obtaining a new vector of lifetimes having a MO exponential distribution that has the same marginal distributions of the original one. From Proposition 3.1, letting $\odot$ to be the minimum and observing that the hazard rate of the minimum of two independent and exponentially distributed random variables is given by the sum of the corresponding rates, one immediately gets the following corollary.

Corollary 3.1. *Let $\mathbf{T} = (T_1, \cdots, T_n)$ have a MO exponential distribution with set parameters $L = \{\lambda_{S_j}, S_j \in \mathbf{S}\}$, and let $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \cdots, \widetilde{T}_n)$ be another random vector having MO exponential distribution with set parameters $h^{\delta,\mathbf{I}}(L) = \{h^{\delta,\mathbf{I}}(\lambda_{S_j}), S_j \in \mathbf{S}\}$. Then $\widetilde{\mathbf{T}} \prec_{\mathrm{sm}} \mathbf{T}$.*

Note that, actually, Proposition 3.1 applies to all random vectors, or multivariate distributions, defined as described above and satisfying

$$\widetilde{r}_j(t) = \begin{cases} r_j(t) & \text{if } S_j \notin \{I_1, \cdots, I_k, \cup_{r=1}^k I_r\} , \\ r_j(t) - \delta(t) & \text{if } S_j = \cup_{r=1}^k I_r , \\ r_j(t) + \delta(t) & \text{if } S_j \in \{I_1, \cdots, I_k\} , \end{cases}$$

for every $t \geq 0$, where $\widetilde{r}_j$ and r_j are the hazard rates of the variables $\widetilde{X}_j$ and X_j, respectively. Thus, $\prec_{\mathrm{sm}}$ can be applied to compare two vectors having multivariate Weibull distributions and identical shape parameter in supermodular order (see, e.g [13] for definition of multivariate Weibull distributions) or vectors having the

Generalized Marshall-Olkin distribution, as defined in [11]. A list of similar results on comparisons, but for other multivariate distributions, is provided in Section 4.

Using techniques similar to these described in the proof of Proposition 3.1, supermodular comparisons of random vectors can be provided where the dependence among components is described by means of more sets of commonalities. An example is provided in the following. Below, for any set $A \subseteq I = \{1, \cdots, n\}$ the notation $\overline{A}$ denotes the complement of A with respect to I.

Let $\mathbf{T}$ have joint survival function defined as in Subsection 2.1. Consider two sets of indexes $I_1, I_2 \subseteq I$ having nonempty intersection and such that

$$I_1 \cup I_2 \notin \{I_1, I_2\} \quad \text{and} \quad I_1, I_2, I_1 \cap I_2 \in \mathbf{S}\ .$$

Consider two independent and identically distributed random variables $\mathcal{E}_1$ and $\mathcal{E}_2$, independent also on the X_j. Then define two random vectors $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \cdots, \widetilde{T}_n)$ and $\hat{\mathbf{T}} = (\hat{T}_1, \cdots, \hat{T}_n)$ such that

$$\widetilde{T}_i = \begin{cases} T_i & \text{if } i \notin I_1 \cup I_2 \\ T_i \odot \mathcal{E}_1 & \text{if } i \in I_1 \cap \overline{I}_2 \\ T_i \odot \mathcal{E}_2 & \text{if } i \in \overline{I}_1 \cap I_2 \\ T_i \odot \mathcal{E}_1 \odot \mathcal{E}_2 & \text{if } i \in I_1 \cap I_2\ , \end{cases} \tag{3.4}$$

and

$$\hat{T}_i = \begin{cases} T_i & \text{if } i \notin I_1 \cup I_2 \\ T_i \odot \mathcal{E}_1 & \text{if } i \in (I_1 \cap \overline{I}_2) \cup (\overline{I}_1 \cap I_2) \\ T_i \odot \mathcal{E}_1 \odot \mathcal{E}_2 & \text{if } i \in I_1 \cap I_2\ . \end{cases} \tag{3.5}$$

For example, the two vectors

$$\widetilde{\mathbf{T}} = (X_1 \odot \mathcal{E}_1, X_2 \odot \mathcal{E}_1 \odot \mathcal{E}_2, X_3 \odot \mathcal{E}_2) \quad \text{and} \quad \hat{\mathbf{T}} = (X_1 \odot \mathcal{E}_1, X_2 \odot \mathcal{E}_1 \odot \mathcal{E}_2, X_3 \odot \mathcal{E}_1)$$

can be defined as above, where $\mathbf{S} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$ and $I_1 = \{1, 2\}$, $I_2 = \{2, 3\}$.

Constructions of random vectors by means of pairwise inclusion-exclusion transforms like the one described above are considered, for instance, in [24], in the context of Marshall-Olkin multivariate exponential distribution applied to compare reliabilities of multicomponent systems.

The two vectors defined in (3.4) and (3.5) can be compared in supermodular order, as stated in the following proposition.

Proposition 3.2. *Let $\widetilde{\mathbf{T}}$ and $\hat{\mathbf{T}}$ be as defined as above. Then $\widetilde{\mathbf{T}} \prec_{\text{sm}} \hat{\mathbf{T}}$.*

Proof. Without loss of generality we assume that the first m_1 components of $\hat{\mathbf{T}}$ and $\widetilde{\mathbf{T}}$ are in $I_1 \cap \overline{I}_2$, the next m_2 components are in $I_2 \cap \overline{I}_1$, the next m_3 components are in $I_1 \cap I_2$. Then $m = m_1 + m_2 + m_3$.

Let $\widetilde{g}$ be any supermodular function defined on $\mathbb{R}^n$. Let us prove that $\mathbb{E}[\widetilde{g}(\widetilde{\mathbf{T}})] \leq \mathbb{E}[\widetilde{g}(\hat{\mathbf{T}})]$. Observe that for any fixed $\mathbf{t} = (t_1, \cdots, t_n)$,

$$g(x_1, \cdots, x_m) = \widetilde{g}(t_1 \odot x_1, \cdots, t_m \odot x_m, t_{m+1}, \cdots, t_n)$$

is a supermodular function of $(x_1, \cdots, x_m)$. In order to prove the proposition, we need

$$\mathbb{E}[\widetilde{g}(\tilde{T}_1, \cdots, \tilde{T}_n)|\mathbf{T} = \mathbf{t}] \geq \mathbb{E}[\widetilde{g}(\hat{T}_1, \cdots, \hat{T}_n)|\mathbf{T} = \mathbf{t}] \ , \tag{3.6}$$

where $\mathbf{T} = (T_1, \cdots, T_n)$. In view of (3.4) and (3.5) inequality (3.6) is equivalent to

$$\mathbb{E}[g(\underbrace{\mathcal{E}_1, \cdots, \mathcal{E}_1}_{m_1+m_2} \ \underbrace{\mathcal{E}_1 \odot \mathcal{E}_2, \cdots, \mathcal{E}_1 \odot \mathcal{E}_2}_{m_3})] \geq$$

$$\geq \mathbb{E}[g(\underbrace{\mathcal{E}_1, \cdots, \mathcal{E}_1}_{m_1} \ \underbrace{\mathcal{E}_2, \cdots, \mathcal{E}_2}_{m_2} \ \underbrace{\mathcal{E}_2 \odot \mathcal{E}_2, \cdots, \mathcal{E}_1 \odot \mathcal{E}_2}_{m_3})] \ .$$

Let F be the cumulative distribution function of $\mathcal{E}_r$, $r = 1, 2$. Notice that

$$\mathbb{E}[g(\underbrace{\mathcal{E}_1, \cdots, \mathcal{E}_1}_{m_1+m_2} \ \underbrace{\mathcal{E}_1 \odot \mathcal{E}_2, \cdots, \mathcal{E}_1 \odot \mathcal{E}_2}_{m_3})] =$$

$$= \iint g(\underbrace{x_1, \cdots, x_1}_{m_1+m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3}) \, dF(x_1) \, dF(x_2) =$$

$$= \iint_{x_1 < x_2} \left(g(\underbrace{x_1, \cdots, x_1}_{m_1+m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3}) + \right.$$

$$\left. + \, g(\underbrace{x_2, \cdots, x_2}_{m_1+m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3}) \right) \, dF(x_1) \, dF(x_2) \ . \tag{3.7}$$

Similarly,

$$\mathbb{E}[g(\underbrace{\mathcal{E}_1, \cdots, \mathcal{E}_1}_{m_1} \ , \ \underbrace{\mathcal{E}_2, \cdots, \mathcal{E}_2}_{m_2} \ \underbrace{\mathcal{E}_1 \odot \mathcal{E}_2, \cdots, \mathcal{E}_1 \odot \mathcal{E}_2}_{m_3})] =$$

$$= \iint g(\underbrace{x_1, \cdots, x_1}_{m_1}, \underbrace{x_2, \cdots, x_2}_{m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3}) \, dF(x_1) \, dF(x_2) =$$

$$= \iint_{x_1 < x_2} \left(g(\underbrace{x_1, \cdots, x_1}_{m_1}, \underbrace{x_2, \cdots, x_2}_{m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3}) + \right.$$

$$\left. + \, g(\underbrace{x_2, \cdots, x_2}_{m_1}, \underbrace{x_1, \cdots, x_1}_{m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3}) \right) \, dF(x_1) \, dF(x_2) \ . \tag{3.8}$$

Since g is supermodular, then $x_1 < x_2$ implies that

$$g(\underbrace{x_1, \cdots, x_1}_{m_1+m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3}) + g(\underbrace{x_2, \cdots, x_2}_{m_1+m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3}) \geq$$

$$\geq g(\underbrace{x_1, \cdots, x_1}_{m_1}, \underbrace{x_2, \cdots, x_2}_{m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3}) +$$

$$+ \, g(\underbrace{x_2, \cdots, x_2}_{m_1}, \underbrace{x_1, \cdots, x_1}_{m_2}, \underbrace{x_1 \odot x_2, \cdots, x_1 \odot x_2}_{m_3})$$

and the result follows. Note that, the common term

$$\sum g(\underbrace{x, \cdots, x}_{m_1+m_2}, \underbrace{x \odot x, \cdots, x \odot x}_{m_3}) P[\mathcal{E}_1 = x] P[\mathcal{E}_2 = x]$$

should be added in equations (3.7) and (3.8).

□

4. Applications

Some examples of application of the results presented in the previous section are described here. In one of them, the well-known *usual stochastic order*, which is a stochastic comparison between univariate random variables, is considered. For this reason, its definition is recalled here.

Definition 4.1. Let X and Y be any two random variables. Then X is said to be *smaller than* Y *in the usual stochastic order* (denoted by $X \leq_{st} Y$) if $E[\phi(X)] \leq E[\phi(Y)]$ for every increasing function such that the expectations exist, or, equivalently, if $P[X > t] \leq P[Y > t]$ for all $t \in \mathbb{R}$.

The usual stochastic order, also known as first order stochastic dominance, is a stochastic comparison often considered in actuarial sciences and reliability. For a comprehensive discussion on properties and applications of the usual stochastic ordering we refer to [21].

4.1. **Applications to default models and system's lifetimes.** Consider n defaultable firms, and let $(\tau^1, \cdots, \tau^n)$ denote their default times. In recent years, a growing interest has been observed in describing dependence among default times and in separating the dependence structure and marginal default probabilities. This, because financial failures have been attributed to erroneous assessment of the degree of dependence between risks (see, e.g., [9] and [19]).

Thus, many different models have been defined and studied in recent literature to describe relationships among default times. For example, dependence among default times is introduced by considering time change of a univariate subordinator in [12], while dependence among the default times subordinating to hitting times for multivariate dependent subordinators is introduced in [22]. We introduce a default model in the same spirit, applying the results described in previous section to introduce dependence among the default times.

To this aim, recall that a Lévy process is a process with independent stationary increments, and that a subordinator is a Lévy process with non-decreasing sample paths. Also recall that, for $\theta > 0$, given a subordinator $\mathbf{X} = \{X(t), t \in \mathbb{R}^+\}$, then

$$\mathbb{E}[e^{-\theta X(t)}] = e^{-t\Psi(\theta)}, \Psi(\theta) = \mu\theta + \int_0^\infty (1 - e^{-\theta x})\, d\nu(x) ,$$

where $\mu \geq 0$ is the drift, and ν is the Lévy measure. The function Ψ is commonly called Laplace exponent of the Lévy process. For more details see [12], or standard references on Lévy processes. Similarly, we can define an n-dimensional subordinator $\mathbf{X} = (\mathbf{X}_1, \cdots, \mathbf{X}_n) = \{(X_1(t), \cdots, X_n(t)), t \in \mathbb{R}^+\}$, as an n-dimensional process with stationary independent increments and non-decreasing sample paths such that

$$\mathbb{E}[e^{-\langle \boldsymbol{\theta}, \mathbf{X}(t) \rangle}] = e^{-t\Psi(\boldsymbol{\theta})} ,$$

where $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^n a_i b_i$.

It is possible to introduce a dependence structure for $\mathbf{X}$ according to Propositions 3.1, or to Proposition 3.2. Consider, e.g., the scenario according to Proposition 3.1. Let $\{\mathbf{V}_j, j \in \mathbf{J}\}$ and $\{\widetilde{\mathbf{V}}_j, j \in \mathbf{J}\}$ be two sets of independent subordinators, with drifts $\eta_j \geq 0, \widetilde{\eta}_j > 0$ and Lévy measures $\Pi_j, \widetilde{\Pi}_j$, respectively. Similarly, let $\mathbf{Z}$ and $\mathbf{Z}_r, r = 1, \cdots, k$, be identically distributed independent subordinators independent of $\mathbf{V}_j$ and $\widetilde{\mathbf{V}}_j$, with drift ξ and a Lévy measure ς. Let $I_1, \cdots, I_k$ be disjoint sets in $\mathbf{S}$ and let

$$\begin{aligned} &\widetilde{\mathbf{V}}_j =_{st} \mathbf{V}_j && \text{if } S_j \notin \{I_1, \cdots, I_k, \cup_{r=1}^k I_r\}\ , \\ &\mathbf{V}_j =_{st} \widetilde{\mathbf{V}}_j + \mathbf{Z} && \text{if } S_j = \cup_{r=1}^k I_r\ , \\ &\widetilde{\mathbf{V}}_j =_{st} \mathbf{V}_j + \mathbf{Z}_r && \text{if } S_j \in \{I_1, \cdots, I_k\}\ . \end{aligned} \tag{4.1}$$

Finally, for $i = 1, \cdots, n$, let

$$\mathbf{X}_i = \sum_{j\,:\,i \in S_j} \mathbf{V}_j \quad \text{and} \quad \widetilde{\mathbf{X}}_i = \sum_{j\,:\,i \in S_j} \widetilde{\mathbf{V}}_j$$

(i.e., let $X_i(t) = \sum_{j\,:\,i\in S_j} V_j(t)$ and $\widetilde{X}_i(t) = \sum_{j\,:\,i\in S_j} \widetilde{V}_j(t)$ for all $t \in \mathbb{R}^+$).

Clearly, $\mathbf{X}_i$ and $\widetilde{\mathbf{X}}_i$ are subordinators with the same drift and the same Lévy measure. Thanks to Proposition 3.1, for any $t \in \mathbb{R}^+$ the following ordering holds

$$(X_1(t), \cdots, X_n(t)) \succ_{\text{sm}} (\widetilde{X}_1(t), \cdots, \widetilde{X}_n(t))\ .$$

By independence and stationarity of the increments, and by the closure of the supermodular order with respect to sums (see [20]), one can immediately verify that for any sequence $t_1, \cdots, t_n$, we also have

$$(X_1(t_1), \cdots, X_n(t_n)) \succ_{\text{sm}} (\widetilde{X}_1(t_1), \cdots, \widetilde{X}_n(t_n))\ . \tag{4.2}$$

Let now $\mathcal{E}_j, j = 1, \cdots, n$ be i.i.d exponentially distributed random variables with parameters 1. [22] defined the default time of the i-th firm, τ_i, is defined as follows:

$$\tau_i = \inf\{t \geq 0\ :\ X_i(t) \geq \mathcal{E}_i\}\ .$$

Then $\boldsymbol{\tau}$ has the multivariate Marshall-Olkin exponential distribution as proved in [22] where the parameters are given as functions of the drifts and of the Lévy measures of $\mathbf{X}$, through its Laplace exponent $\boldsymbol{\Psi}(\theta)$.

Let $\widetilde{\tau}_i$ be similarly defined with respect to $\widetilde{X}_i$. By (4.2), reasoning as in Proposition 2.6 in [6], (see also [5]), we immediately have

$$\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n) \succ_{\text{sm}} \widetilde{\boldsymbol{\tau}} = (\widetilde{\tau}_1, \cdots, \widetilde{\tau}_n) \tag{4.3}$$

that is, the degree of positive dependence is greater in $\boldsymbol{\tau}$ than in $\widetilde{\boldsymbol{\tau}}$.

To consider dependence comparison in the case $\mathcal{E}_i$ not exponentially distributed, we apply the time change procedure described in [12]. Assume that $\mathcal{E}_i$ has distribution G_i with $1 - G_i(t) = \exp(-H_i(t))$. Then $H_i(t)$ is strictly monotone and continuous, and grows to infinity as t goes to infinity. Assume that for $i = 1, \cdots, n$, the equation $\Psi_i(1) = \mathbb{E}[e^{-X_i(t)}] = 1$ holds. This can be achieved by controlling the drift (see [12], Example 2.3). Thus

$$\mathbb{P}[X_i(H_i(t)) \leq \mathcal{E}_i] = e^{-H_i(t)} = \mathbb{P}(\tau_i > t)\ .$$

Consider now the sequence $s_1 < \cdots < s_m$, such that $s_m \to \infty$ as $m \to \infty$, and $\sup\{s_i - s_{i-1} < \delta\}$. Similarly to the ideas in [6], we can define

$$\min\{\tau_i, m\} = \sum_{k=1}^{m} \prod_{l=1}^{k} \mathbf{1}_{(X_i(H_i(s_l) \le \mathcal{E}_i))} ,$$

where $\mathbf{1}_A$ denotes the indicator function of the event A. Again, by using the same arguments given in Proposition 2.7 (see [6]), taking into account (4.2), equation (4.3) is recovered.

It is interesting to observe that the same model, and the same results, can be applied in reliability analysis of multicomponent systems. In fact, a system having components subjected to wear and modelling the accumulated wears, for each component, can be considered through a set of processes $\{\mathbf{X}_i, i = 1, 2, ..., n\}$. Assuming the failure of a component whenever the accumulated wear reaches a fixed (or random) threshold, the vector $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ of the components lifetimes is defined similarly to the vector of default times above. And, reasoning as above, comparisons like (4.3) for vectors of lifetimes are recovered. This inequality is of interest in reliability analysis since lifetimes of series or parallel systems are compared by usual stochastic order. In fact, observing that the supermodular order implies the positive orthant dependence order (see [21], for details), the following result is a corollary

$$\begin{aligned} P[\min(\tau_1, \cdots, \tau_n) > t] &= P[\tau_1 > t, \cdots, \tau_n > t] \le \\ &\le P[\widetilde{\tau}_1 > t, \cdots, \widetilde{\tau}_n > t] = P[\min(\widetilde{\tau}_1, \cdots, \widehat{\tau}_n) > t] , \end{aligned}$$

for all $t \ge 0$, i.e.,

$$\min(\tau_1, \cdots, \tau_n) \le_{st} \min(\widetilde{\tau}_1, \cdots, \widehat{\tau}_n) ,$$

and

$$\begin{aligned} P[\max(\tau_1, \cdots, \tau_n) \le t] &= P[\tau_1 \le t, \cdots, \tau_n \le t] \le \\ &\le P[\widetilde{\tau}_1 \le t, \cdots, \widetilde{\tau}_n \le t] = P[\max(\widetilde{\tau}_1, \cdots, \widehat{\tau}_n) \le t] , \end{aligned}$$

for all $t \ge 0$, i.e.,

$$\max(\tau_1, \cdots, \tau_n) \ge_{st} \max(\widetilde{\tau}_1, \cdots, \widehat{\tau}_n) .$$

Thus, lifetimes of series and parallel systems can be compared by means of the usual stochastic order and making use of the results described in Section 3.

4.2. **Supermodular order for families of multivariate distributions.** Apart for the Generalized Marshall-Olkin distribution, and in particular for the Marshall Olkin exponential distribution, the results described in Section 3 can be used to provide simple conditions to compare random vectors having specific multivariate parametric distributions, in the supermodular order. Few examples are described in the following.

Example 4.1. Proposition 3.1 applies to *multivariate Gamma distributions*, whose interest in actuarial sciences is clearly described in [1]. Consider a collection $\mathbf{S} = \{S_j, j \in J \subseteq \mathbb{N}\}$ of subsets of $I = \{1, \cdots, n\}$, and let $\{X_j, j \in J\}$ be a set of independent and Gamma distributed random variables with shape parameters α_j, and scale parameters β_j, respectively. Consider the random vector $\mathbf{T} = (T_1, \cdots, T_n)$ defined by letting $T_i = \sum_{j\,:\,S_j \in \Lambda_i} \frac{\beta_j}{b_i} X_j$, for fixed $b_i > 0, i \in I$, i.e., according to the general model replacing $\odot$ with the sum. Then $\mathbf{T}$ has a multivariate Gamma distribution, whose margins are Gamma distributed with parameters (a_i, b_i), where $a_i = \sum_{j\,:\,S_j \in \Lambda_i} \alpha_j$. In particular, when $\mathbf{S} = \{I, \{1\}, \{2\}, \cdots, \{n\}\}$, then $\mathbf{T}$ has a Cherian and Ramabhadran multivariate Gamma distribution (see [10]), whose margins are Gamma distributed with parameters $(\alpha_0 + \alpha_i, b_i)$, where α_0 is the shape parameter associated to the set I and α_i the shape parameter associated with $\{i\}$.

Let $\mathbf{I} = \{I_1, \cdots, I_k\} \subseteq \mathbf{S}$ be such that the $I_r, r = 1, \cdots, k$ are disjoint sets in $\mathbf{I}$ and also such that $\bigcup_{r=1}^k I_r \in \mathbf{S}$. Consider any fixed $\delta \in \mathbb{R}^+$ such that $\delta \le \alpha_j$, $\forall j \in J$. A new multivariate Gamma distributed vector $\widetilde{\mathbf{T}}$ can be defined by means of a set $\{\widetilde{X}_j, j \in J\}$ of independent and Gamma distributed random variables with scale parameters β_j and shape parameters

$$\widetilde{\alpha}_j = \begin{cases} \alpha_j & \text{if } S_j \notin \{I_1, \cdots, I_k, \cup_{r=1}^k I_r\} , \\ \alpha_j - \delta & \text{if } S_j = \cup_{r=1}^k I_r , \\ \alpha_j + \delta & \text{if } S_j \in \{I_1, \cdots, I_k\} . \end{cases}$$

Observing that $\widetilde{\mathbf{T}}$ has the same marginal distributions of $\mathbf{T}$, from Proposition 3.1 one immediately gets the following corollary.

Corollary 4.1. *Let* $\mathbf{T}$ *and* $\widetilde{\mathbf{T}}$ *be two vectors having multivariate Gamma distributions defined as above. Then* $\widetilde{\mathbf{T}} \prec_{\mathrm{sm}} \mathbf{T}$.

Example 4.2. *Multivariate Pareto distributions* are also of remarkable interest in actuarial sciences. In [3], applications are described of multivariate Pareto of the second kind to the pricing problem both in life and non-life insurance contexts.

As shown in Proposition 2.2 in [3], a possible representation of vectors having these distributions is $\mathbf{X} = (X_1, \cdots, X_n) = (\mu_1 + \sigma_1(e^{Y_1} - 1), \cdots, \mu_n + \sigma_n(e^{Y_n} - 1))$, where $\mathbf{Y} = (Y_1, \cdots, Y_n)$ is a vector having MO exponential distribution, and joint survival function

$$\overline{F}_{\mathbf{Y}}(y_1, \cdots, y_n) = \exp(-\alpha_1 y_1 - \cdots - \alpha_n y_n - \alpha_0 \max(y_1, \cdots, y_n)) .$$

Consider now another vector $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \cdots, \widetilde{X}_n)$ similarly defined through a vector $\widetilde{\mathbf{Y}}$ having MO exponential distribution. If the parameters α_i are such that $\widetilde{\alpha}_0 = \alpha_0 - \delta$ and $\widetilde{\alpha}_i = \alpha_i + \delta$ for all $i = 1, \cdots, n$ and for any $\delta < \alpha_0$ then $\mathbf{Y} \prec_{\mathrm{sm}} \widetilde{\mathbf{Y}}$ holds from Corollary 3.1. If $\widetilde{\mu}_i = \mu_i, \widetilde{\sigma}_i = \sigma_i$ and the same conditions on the $\widetilde{\alpha}_i$ and α_i are true, then $\mathbf{X} \prec_{\mathrm{sm}} \widetilde{\mathbf{X}}$ follows from Theorem 9.A.9 in [21]. Applications in comparisons of premiums can be derived (see, again, [3]). Note that in the case $\widetilde{\mu}_i \neq \mu_i$ and/or $\widetilde{\sigma}_i \neq \sigma_i$, the vectors $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ are no more ordered in supermodular order, since they have different marginal distributions.

Example 4.3. *Multivariate Poisson distributions* are a third simple case, where the results in Section 3 can be used to define a criteria for dependence comparisons among vectors having multivariate parametric distributions by using comparisons among their parameters. In fact, they also admit a representation as in our model, replacing $\odot$ with the sum (see, e.g., [8]). Consider a collection $\mathbf{S} = \{S_j, j \in J \subseteq \mathbb{N}\}$ of subsets of $I = \{1, \cdots, n\}$, and let $\{X_j, j \in J\}$ be a set of independent and Poisson distributed random variables with parameters λ_j, respectively. Consider the random vector $\mathbf{N} = (N_1, \cdots, N_n)$ defined by letting $N_i = \sum_{j\,:\,S_j \in \Lambda_i} X_j$; then $\mathbf{N}$ has a multivariate Poisson distribution, whose margins are Poisson distributed with parameters l_i, where $l_i = \sum_{j\,:\,S_j \in \Lambda_i} \lambda_j$.

Let $\mathbf{I} = \{I_1, \cdots, I_k\} \subseteq \mathbf{S}$ be such that the $I_r, r = 1, \cdots, k$, are disjoint sets in $\mathbf{I}$ and also such that $\bigcup_{r=1}^k I_r \in \mathbf{S}$, and consider any fixed $\delta \in \mathbb{R}^+$ such that $\delta \leq \lambda_j, \leq \forall j \in J$. A new multivariate Poisson distributed vector $\widetilde{\mathbf{N}}$ can be defined by means of a set $\{\widetilde{X}_j, j \in J\}$ of independent and Poisson distributed random variables having parameters

$$\widetilde{\lambda}_j = \begin{cases} \lambda_j & \text{if } S_j \notin \{I_1, \cdots, I_k, \cup_{r=1}^k I_r\} , \\ \lambda_j - \delta & \text{if } S_j = \cup_{r=1}^k I_r , \\ \lambda_j + \delta & \text{if } S_j \in \{I_1, \cdots, I_k\} . \end{cases}$$

Observe that $\widetilde{\mathbf{N}}$ has the same marginal distributions of $\mathbf{N}$. Moreover $\widetilde{\mathbf{N}} \prec_{\mathrm{sm}} \mathbf{N}$ follows from Proposition 3.1.

Obviously, the same considerations can be done dealing with *multivariate Poisson processes* instead of multivariate Poisson distributions, i.e., replacing the set $\{X_j, j \in J\}$ with a set of independent univariate Poisson processes.

Actually, Proposition 3.1 can be similarly applied to any multivariate distribution whose margins are infinite-divisible distributions.

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References

[1] D.H. Alai, Z. Landsman & M. Sherris, *Lifetime dependence modelling using a truncated multivariate Gamma distribution*, Insurance Math. Econom., 52(2013), 542–549.

[2] E. Arjas, C.K. Hansen & P. Thyregod, *Heterogeneous part quality as a source of reliability improvement in repairable systems*, Technometrics, 33(1991), 1–12.

[3] A.V. Asimit, E. Furman & R. Vernic, *On a multivariate Pareto distribution*, Insurance Math. Econom., 46(2010), 308–316.

[4] R.E. Barlow & F. Proschan, *Statistical theory of reliability and life testing. Probability models to begin with.*, Silver Spring, MD(1981).

[5] N. Bäuerle, A. Blatter & A. Müller, *Dependence properties and comparison results for Lévy processes*, Math. Methods Oper. Res., 67(2008), 161–186.

[6] M. Denuit, E. Frostig & B. Levikson, *Supermodular comparison of time-to-ruin random vectors*, Methodol. Comput. Appl. Probab., 9(2007), 41–54.

[7] E. Frostig, *Ordering ruin probabilities for dependent claim streams*, Insurance Math. Econom., 32(2003), 93–114.

[8] N. Johnson, S. Kotz & N. Balakrishnan, *Discrete multivariate distributions*, Wiley Series in Probability and Statistics (1997).

[9] S. Jones, *The formula that felled Wall St.*, Financial Times, April 24, 2009.

[10] S. Kotz, N. Balakrishnan & N. Johnson, *Continuous multivariate distributions*, Vol. 1, Wiley Series in probability and statistics, (2000).

[11] X. Li & F. Pellerey, *Generalized Marshall-Olkin distributions and related bivariate aging properties*, J. Multivariate Anal., 102(2011), 1399–1409.

[12] J. Mai & M. Scherer, *A tractable multivariate default model based on a stochastic time-change*, Internat. J. Theoret. Appl. Finance, 2(2009), 227–249.

[13] A.W. Marshall & I. Olkin, *A multivariate exponential distribution*, J. Amer. Statist. Assoc., 62(1967), 30–41.

[14] P. Muliere & M. Scarsini, *Characterization of a Marshall-Olkin type class of distributions*, Ann. Inst. Statist. Math., 39(1987), 429–441.

[15] A. Müller, *Stop-loss order for portfolios of dependent risks*, Insurance Math. Econom., 21(1997), 219–223.

[16] S. Nadarajah & S. Kotz, *On the linear combinations of exponential and Gamma random variables*, Entropy, 7(2005), 161–171.

[17] F. Pellerey, M. Shaked & S. Yasaei Sekeh, *Comparisons of concordance in additive models*, Statist. Probab. Lett., 582(2012), 2059–2067.

[18] L. Rüschendorf, *Comparison of multivariate risks and positive dependence*, Adv. Appl. Prob., 41(2004), 391–406.

[19] F. Salmon, *Recipe for disaster: the formula that killed Wall Street*, Wired Magazine, 17(2009).

[20] M. Shaked & J.G. Shanthikumar, *Supermodular stochastic orders and positive dependence of random vectors*, J. Multivariate Anal., 61(1997), 86–101.

[21] M. Shaked & J.G. Shanthikumar, *Stochastic orders*, Springer Verlag, New York (2007).

[22] Y. Sun, R. Mendoza-Arriaga & V. Linetsky, *Marshall-Olkin distributions, subordinators, efficient simulation, and applications to credit risk*, available at SSRN: `http://ssrn.com/abstract=1702087` or `http://dx.doi.org/10.2139/ssrn.1702087`, July 21, 2014.

[23] S.H. Xu, *Structural analysis of a queueing system with multi–classes of correlated arrivals and blocking*, Oper. Res., 47(1999), 263–276.

[24] S.H. Xu & H. Li, *Majorization of weighted trees: a new tool to study correlated stochastic systems*, Math. Oper. Res., 25(2000), 298–323.