Order preserving SUPG stabilization for the Virtual Element formulation of advection-diffusion problems

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Abstract

In the framework of the discretization of advection-diffusion problems by means of the Virtual Element Method, we consider stabilization issues. Herein, stabilization is pursued by adding a consistent SUPG-like term. For this approach we prove optimal rates of convergence. Numerical results clearly show the stabilizing effect of the method up to very large Péclet numbers and are in very good agreement with the expected rate of convergence.

Keywords: Virtual Element Methods, Advection-diffusion problem, SUPG, stability, convergence
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1. Introduction

Recently, a new discretization approach has been developed, named the Virtual Element Method (VEM), that allows the use of general polygonal and polyhedral meshes [1, 2].

The VEM has been applied in a wide number of contexts, such as plate bending problems [3], elasticity problems [4, 5], Stokes problems [6] and the Steklov eigenvalue problem [7]. A non-conforming formulation has been devised in [8]. Recently, the VEM has been also used in the treatment of fluid dynamics models involving underground flow simulations [9–11]: in that context, the application of the VEM was driven by the need of circumventing mesh generation problems. In these applications, the primal problem is solved to compute the Darcy velocity field, that can be used afterwards to simulate the transport of a dispersed, passive pollutant in a geological basin. The flow regimes in underground transport phenomena are usually transport-dominated, due to the very low diffusivity of the pollutant into the bulk fluid, thus calling for a stabilization of the VEM.

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Many strategies have been devised to obtain a stable solution for standard Finite Element discretizations, involving, for example, local projections [12] or suitably built bubble functions [13, 14]. The Streamline Upwind Petrov-Galerkin (SUPG) stabilization method [15–21] has also been widely studied in very general settings. A first approach to the VEM-SUPG stabilization is discussed in [22], in a non-consistent formulation.

Another issue related to advection-diffusion problems is the derivation of robust a posteriori error estimates. In such context, the term *robustness* refers to the property of obtaining a relation between the error and the error estimator with constants which are independent of the Péclet number [23–28]. An a posteriori analysis for the reaction-convection-diffusion problem with the VEM is provided in [29], not addressing robustness aspects and the SUPG-like stabilization issues.

The aim of this work is to devise a consistent SUPG formulation compatible with the VEM. A key aspect of the VEM is that the basis functions of the discrete functional space are not known explicitly, but only through their degrees of freedom. As a consequence, computability of discrete operators requires special care and, in particular, the consistent VEM-SUPG formulation devised in the present work requires the introduction of a second-order term in the weak formulation of the problem, computed by resorting to polynomial projections of the virtual element basis functions.

An *a priori* error estimate for the stabilized VEM discrete solution is also proven, showing that the order of convergence is not affected by the stabilizing perturbation added to the problem. Numerical tests proposed in the paper confirm the theoretical results on triangular and polygonal meshes in both the convection-dominated regime and the diffusion-dominated regime.

The paper is organised as follows. In Section 2 we state the model problem, define some useful notations and make some standard hypothesis on the model parameters. In Section 3 we introduce the spatial discretization and the Virtual Element functional space based on it. In particular, the VEM-SUPG formulation of the problem is presented in Subsection 3.1, equations (13), (19) and (20). In Section 4 the *a priori* error estimate for the stabilized VEM discrete solution is derived, the main result being stated in Theorem 2. Finally, in Section 5 we propose some numerical tests aimed at confirming the theoretical results.

2. The model problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let us consider the following convection-diffusion problem:

$$\begin{cases} -\nabla\cdot\left(\mathsf{K}\nabla u\right) + \beta\cdot\nabla u = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $\mathsf{K} \in \mathrm{L}^\infty\left(\Omega\right)$ is a positive function satisfying $\mathsf{K}(x) \geq \mathsf{K}_0\ \forall x \in \Omega$ for a given $\mathsf{K}_0 > 0$, and $\beta \in \left[\mathrm{L}^\infty\left(\Omega\right)\right]^2$, $\nabla\cdot\beta \in \mathrm{L}^2\left(\Omega\right)$. We additionally assume $\nabla\cdot\beta = 0$.

The notation throughout the paper is as follows: $\left(\cdot,\cdot\right)$ and $\left\|\cdot\right\|$ denote the $\mathrm{L}^2\left(\Omega\right)$ scalar products and norms; $\left(\cdot,\cdot\right)_\omega$ and $\left\|\cdot\right\|_\omega$ denote the $\mathrm{L}^2\left(\omega\right)$ scalar products and norms, for any $\omega \subseteq \Omega$; moreover, $\left\|\cdot\right\|_\alpha$ and $\left|\cdot\right|_\alpha$ denote the $\mathrm{H}^\alpha\left(\Omega\right)$ norm and semi-norm; $\left\|\cdot\right\|_{\alpha,\omega}$ and $\left|\cdot\right|_{\alpha,\omega}$ denote the $\mathrm{H}^\alpha\left(\omega\right)$ norm and semi-norm; whereas $\left\|\cdot\right\|_{\mathrm{W}^q_p(\omega)}$ and $\left|\cdot\right|_{\mathrm{W}^q_p(\omega)}$ denote the $\mathrm{W}^q_p(\omega)$ norm and semi-norm, where p is the Lebesgue regularity and q is the order of the Sobolev space.

For future reference, we recall the classical weak formulation of the problem. Defining $B\colon \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ and $F\colon \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ such that

$$B(w,v) := (\mathsf{K}\nabla w, \nabla v) + (\beta \cdot \nabla w, v) \quad \forall w, v \in \mathrm{H}^1_0(\Omega),$$

and

$$F(v) := (f, v) \quad \forall v \in \mathrm{H}^1_0(\Omega),$$

the variational form of (1) is

$$B(u,v) = F(v) \quad \forall v \in \mathrm{H}^1_0(\Omega). \tag{2}$$

We remark that, for the sake of improving readability, here we limit ourself to formulation (2). More general boundary conditions can be considered as well. Furthermore, K can be taken as a symmetric positive definite tensor, with minor changes in some definitions.

3. VEM discretization

Let $\mathcal{T}_h$ be a set of open polygons partitioning Ω, h being the maximum diameter of these elements. For VEM-based discretizations these polygons can have a different number of edges from one to another and also nodes can be placed between edges forming a flat angle, thus allowing for hanging-node-like configurations. As usually done for VEM discretizations (see [2]), we ask that every polygon is star-shaped with respect to a ball whose radius is greater or equal than γh_E, being h_E the element diameter and γ a global constant. Finally, for each $E \in \mathcal{T}_h$, we set

$$\mathsf{K}_E := \sup_{x\in E} \mathsf{K}(x), \qquad \mathsf{K}_E^{\vee} := \inf_{x\in E} \mathsf{K}(x),$$
$$\beta_E := \sup_{x\in E} \|\beta(x)\|^2_{\mathbb{R}^2}.$$

To define the Virtual Element space of order $k > 0$, for some $k \in \mathbb{N}$, we denote by $\mathbb{P}_k(\mathcal{T}_h)$ the space of possibly discontinuous functions which are polynomials of degree less than or equal to k on each polygon and we introduce the piecewise polynomial oblique projection $\Pi^{\nabla}_k\colon \mathrm{H}^1(E) \to \mathbb{P}_k(\mathcal{T}_h)$ such that, $\forall E \in \mathcal{T}_h$,

$$\left(\nabla\left(v - \Pi^{\nabla}_k v\right), \nabla p\right)_E = 0 \quad \forall p \in \mathbb{P}_k(E),$$
$$\text{and} \quad \left(\Pi^{\nabla}_{k,E} v, 1\right)_{\partial E} = (v, 1)_{\partial E}.$$

The local Virtual space of order k is defined as follows: $\forall E \in \mathcal{T}_h$,

$$V_h^E := \Big\{ v_h \in \mathrm{H}^1(E) : \Delta v_h \in \mathbb{P}_k(E),\, v_h \in \mathbb{P}_k(e)\ \forall e \subset \partial E,$$
$$(v_h, p)_E = \left(\Pi^{\nabla}_{k,E} v_h, p\right)_E\ \forall p \in \mathbb{P}_k(E) \setminus \mathbb{P}_{k-2}(E) \Big\},$$

and, asking for global continuity we obtain the global space $V_h \subset \mathrm{H}^1_0(\Omega)$:

$$V_h := \left\{ v_h \in C^0(\Omega) : v_h \in V_h^E \quad \forall E \in \mathcal{T}_h \right\}.$$

A function belonging to such space is uniquely identified by its polynomial expression on each edge of the discretization and by its moments against polynomials of degree $\leq k-2$ (see [1]). As in [1], we choose the following set of degrees of freedom (see also [30] for more details):

1. the values at the vertices of each polygon;
2. if $k \geq 2$, for each edge $e \subset \partial E$, the values at $k-1$ internal points of e. For practical purposes, we may choose these points to be the internal Gauss-Lobatto quadrature nodes;
3. if $k \geq 2$, for each $v_h \in V_h$ the moments $(v_h, m_{\boldsymbol{\alpha}})_E$, for each $E \in \mathcal{T}_h$ and all the monomials $m_{\boldsymbol{\alpha}} \in \mathcal{M}_{k-2}(E)$, defined as

$$m_{\boldsymbol{\alpha}}(x,y) := \frac{(x-x_E)^{\alpha_1}(y-y_E)^{\alpha_2}}{h_E^{\alpha_1+\alpha_2}} \quad \forall (x,y) \in E\,,$$

with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 \leq k-2$.

3.1. VEM-SUPG formulation

It is well known that discretizing the variational formulation (2) leads to instabilities when the convective term $(\beta \cdot \nabla w, v)$ is dominant with respect to the diffusive term $(\mathsf{K}\nabla w, \nabla v)$. In such situations a stabilized form of the problem is required in order to prevent spurious oscillations that can completely alter the numerical solution. In the following we recast the classical *Streamline Upwind Petrov Galerkin* (SUPG) approach [17] in the framework of the VEM, showing that the optimal order of convergence can be preserved.

We define the bilinear form $B_{\text{supg}} \colon \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ such that

$$B_{\text{supg}}(w,v) := a(w,v) + b(w,v) + d(w,v)\,, \tag{3}$$

being

$$a(w,v) := (\mathsf{K}\nabla w, \nabla v) + \sum_{E \in \mathcal{T}_h} \tau_E\,(\beta \cdot \nabla w, \beta \cdot \nabla v)\,, \tag{4}$$

$$b(w,v) := (\beta \cdot \nabla w, v)\,, \tag{5}$$

$$d(w,v) := -\sum_{E \in \mathcal{T}_h} \tau_E\,(\nabla \cdot (\mathsf{K}\nabla w), \beta \cdot \nabla v)_E\,. \tag{6}$$

The stability parameter τ_E is defined as usual, $\forall E \in \mathcal{T}_h$, by

$$\tau_E := \frac{h_E}{2\beta_E} \min\{\mathrm{Pe}_E, 1\}\,, \tag{7}$$

where Pe_E is the *mesh Péclet number* of E, given by

$$\mathrm{Pe}_E := m_k^E \frac{\beta_E h_E}{2\mathsf{K}_E}\,, \tag{8}$$

and

$$m_k^E := \begin{cases} \frac{1}{3} & \text{if } \nabla \cdot (\mathsf{K}\nabla v_h) = 0 \;\forall v_h \in V_h^E, \\ 2\tilde{C}_k^E & \text{otherwise,} \end{cases}$$

having set $\tilde{C}_k^E$ to be the largest constant satisfying the following inverse inequality:

$$\tilde{C}_k^E h_E^2 \left\|\nabla \cdot (\mathsf{K}\nabla v_h)\right\|_E^2 \leq \left\|\mathsf{K}\nabla v_h\right\|_E^2 \quad \forall v_h \in V_h^E\,. \tag{9}$$

Remark 1. We point out that if $u \in \mathrm{H}^2(\Omega)$, we have that

$$B_{\text{supg}}(u, v) = F_{\text{supg}}(v) := (f, v) + \sum_{E \in \mathcal{T}_h} \tau_E (f, \beta \cdot \nabla v) \quad \forall v \in \mathrm{H}^1_0(\Omega) . \tag{10}$$

Remark 2. From the definition of τ_E we have the following two estimates, that will be used in the following:

$$\tau_E \leq \frac{\tilde{C}_k^E h_E^2}{2\mathsf{K}_E} \quad \text{if } \nabla \cdot (\mathsf{K} \nabla v_h) \neq 0 \text{ for some } v_h \in V_h^E , \tag{11}$$

$$\tau_E \leq \frac{h_E}{2\beta_E} . \tag{12}$$

The Finite Element discretization of the bilinear form (3) has been widely studied, for example in [15, 17], in which optimal orders of convergence were proved. In order to write a *computable* VEM discretization of problem (2), we define the discrete bilinear form $B_{\text{supg},h} \colon V_h \times V_h \to \mathbb{R}$ such that

$$B_{\text{supg},h}(w_h, v_h) := a_h(w_h, v_h) + b_h(w_h, v_h) + d_h(w_h, v_h) \quad \forall w_h, v_h \in V_h , \tag{13}$$

where

$$\begin{aligned} a_h(w_h, v_h) :=& \left(\mathsf{K} \Pi^0_{k-1} \nabla w_h, \Pi^0_{k-1} \nabla v_h\right) \\ &+ \sum_{E \in \mathcal{T}_h} \tau_E \left(\beta \cdot \Pi^0_{k-1} \nabla w_h, \beta \cdot \Pi^0_{k-1} \nabla v_h\right)_E \\ &+ \left(\mathsf{K}_E + \tau_E \beta_E^2\right) S^E \left(\left(I - \Pi^\nabla_k\right) w_h, \left(I - \Pi^\nabla_k\right) v_h\right) , \end{aligned} \tag{14}$$

$$b_h(w_h, v_h) := \left(\beta \cdot \Pi^0_{k-1} \nabla w_h, \Pi^0_{k-1} v\right) , \tag{15}$$

$$d_h(w_h, v_h) := - \sum_{E \in \mathcal{T}_h} \tau_E \left(\nabla \cdot \left(\mathsf{K} \Pi^0_{k-1} \nabla w_h\right), \beta \cdot \Pi^0_{k-1} \nabla v_h\right) , \tag{16}$$

where Π^0_r is the element-wise orthogonal L^2 projection on the space of polynomials of degree less than or equal to r, as used in [2]. The stabilization form $S^E \colon V_h \times V_h \to \mathbb{R}$ in (14) must statisfy the following property:

$$S^E(v_h, v_h) \sim \|\nabla v_h\|_E^2 \quad \forall v_h \in \ker \Pi^\nabla_k . \tag{17}$$

A possible choice for S^E is

$$S^E(v_h, w_h) = \sum_{i=1}^{N_E} \chi_i(v_h) \chi_i(w_h), \tag{18}$$

where N_E is the number of degrees of freedom on the element E and χ_i is the operator that selects the i-th degree of freedom.

With the above definitions we can state a SUPG-stabilized discrete formulation of (2) as: find $u_h \in V_h$ such that

$$B_{\text{supg},h}(u_h, v_h) = F_{\text{supg},h}(v_h) \quad \forall v_h \in V_h , \tag{19}$$

having defined the discrete right-hand-side as

$$F_{\mathrm{supg},h}\left(v_h\right) = \left(f, \Pi^0_{k-1} v_h\right) + \sum_{E\in\mathcal{T}_h} \tau_E \left(f, \beta\cdot\Pi^0_{k-1}\nabla v_h\right)_E \,. \tag{20}$$

Finally, in order to provide an estimation of the constant $\tilde{C}^E_k$ for each polygon, we can make use of classical theoretical results on triangles [31] thanks to the following proposition.

Proposition 1. *Given a regular polygon $E \in \mathcal{T}_h$, let $\mathcal{T}_{h,E}$ be a triangulation of E composed by triangular elements with an edge on the boundary of E and one vertex in the centre of the ball with respect to which E is star-shaped. Let $\tilde{C}^E_k$ be the constant of inequality* (9)*. Then,*

$$\tilde{C}^E_k \geq \tilde{C}_k \left(\frac{\min_{t\in\mathcal{T}_{h,E}} h_t}{h_E}\right)^2 \,,$$

where $\tilde{C}_k$ is such that, $\forall v_h \in V^E_h$,

$$\tilde{C}_k h_t^2 \left\|\nabla\cdot\left(\mathsf{K}\nabla v_h\right)\right\|_t^2 \leq \left\|\mathsf{K}\nabla v_h\right\|_t^2 \quad \forall t \in \mathcal{T}_{h,E} \,.$$

Proof. Summing up the inequalities on internal triangles we have

$$\tilde{C}_k \sum_{t\in\mathcal{T}_{h,E}} h_t^2 \left\|\nabla\cdot\left(\mathsf{K}\nabla v_h\right)\right\|_t^2 \leq \sum_{t\in\mathcal{T}_{h,E}} \left\|\mathsf{K}\nabla v_h\right\|_t^2 \,,$$

from which it follows

$$\tilde{C}_k \left(\min_{t\in\mathcal{T}_{h,E}} h_t^2\right) \left\|\nabla\cdot\left(\mathsf{K}\nabla v_h\right)\right\|_E^2 \leq \left\|\mathsf{K}\nabla v_h\right\|_E^2 \,,$$

and therefore

$$\tilde{C}_k \left(\frac{\min_{t\in\mathcal{T}_{h,E}} h_t}{h_E}\right)^2 h_E^2 \left\|\nabla\cdot\left(\mathsf{K}\nabla v_h\right)\right\|_E^2 \leq \left\|\mathsf{K}\nabla v_h\right\|_E^2 \,,$$

which proves the thesis. □

4. Error Analysis

Let $h := \max_{E\in\mathcal{T}_h} h_E$ and define the following norm:

$$|||v||| := \left\{ \left\|\sqrt{\mathsf{K}}\nabla v\right\|^2 + \sum_{E\in\mathcal{T}_h} \tau_E \left\|\beta\cdot\nabla v\right\|_E^2 \right\}^{\frac{1}{2}} \quad \forall v \in \mathrm{H}^1_0\left(\Omega\right) \,.$$

In the following, we will use the symbol $\lesssim$ for inequalities which are satisfied up to a multiplicative constant independent of the meshsize and the problem data K and β, and the symbol $\precsim$ for inequalities satisfied up to a multiplicative constant independent of the meshsize only. All the constants may depend on the regularity of the VEM mesh.

4.1. Discretization errors

The following Lemmas are devoted to estimate the error of approximation of the bilinear forms defined by (4), (5) and (6) with the discrete ones defined by (14), (15) and (16), respectively. The results are based on the following approximation results for polynomial projections (see [2, Lemma 5.1]): $\forall E \in \mathcal{T}_h$,

$$\left\| v - \Pi^0_{k-1} v \right\|_{m,E} \lesssim h_E^{s-m} \left| v \right|_{s,E} \quad \forall v \in \mathrm{H}^s\left(E\right),\, m \leq s \leq k\,, \tag{21}$$

$$\left\| v - \Pi^\nabla_{k} v \right\|_{m,E} \lesssim h_E^{s-m} \left| v \right|_{s,E} \quad \forall v \in \mathrm{H}^s\left(E\right),\, m \leq s \leq k+1, s \geq 1\,. \tag{22}$$

Lemma 1. *For any sufficiently regular function w and $\forall v_h \in V_h$,*

$$b_h\left(w, v_h\right) \lesssim \max_{E \in \mathcal{T}_h} \frac{\beta_E}{\sqrt{\mathsf{K}_E^\vee}} \left\| \sqrt{\mathsf{K}} \nabla w \right\| \left\| v_h \right\| \,. \tag{23}$$

Moreover, if $\beta \in \left[\mathrm{W}^s_\infty(\Omega)\right]^2$ for some $s \in \{0, \dots, k\}$, then

$$\left| b\left(w, v_h\right) - b_h\left(w, v_h\right) \right| \lesssim \max_{E \in \mathcal{T}_h} \left\| \beta \right\|_{\mathrm{W}^s_\infty(E)} h^{s+1} \left\| w \right\|_{s+1} \left\| v_h \right\|_1 \,. \tag{24}$$

Proof. Regarding (23), by the Cauchy-Schwarz inequality and the continuity of Π^0_{k-1} and Π^0_k we have, $\forall E \in \mathcal{T}_h$,

$$\left(\beta \cdot \Pi^0_{k-1} \nabla w, \Pi^0_{k-1} v_h\right)_E \leq \beta_E \left\| \Pi^0_{k-1} \nabla w \right\|_E \left\| \Pi^0_{k-1} v_h \right\|_E \lesssim \frac{\beta_E}{\sqrt{\mathsf{K}_E^\vee}} \left\| \sqrt{\mathsf{K}} \nabla w \right\|_E \left\| v_h \right\|_E \,,$$

from which (23) readily follows.

Concerning (24), let $E \in \mathcal{T}_h$ be fixed. By adding and subtracting $\left(\beta \cdot \Pi^0_{k-1} \nabla w, v_h\right)_E$ in the left-hand side and using the triangle inequality,

$$\begin{aligned}
&\left| \left(\beta \cdot \nabla w, v_h\right)_E - \left(\beta \cdot \Pi^0_{k-1} \nabla w, \Pi^0_{k-1} v_h\right)_E \right| = \\
&\quad = \left| \left(\beta \cdot \left(\nabla w - \Pi^0_{k-1} \nabla w\right), v_h\right)_E + \left(\beta \cdot \Pi^0_{k-1} \nabla w, v_h - \Pi^0_{k-1} v_h\right)_E \right| \leq \\
&\qquad \leq \left| \left(\beta \cdot \left(\nabla w - \Pi^0_{k-1} \nabla w\right), v_h\right)_E \right| + \left| \left(\beta \cdot \Pi^0_{k-1} \nabla w, v_h - \Pi^0_{k-1} v_h\right)_E \right| \,.
\end{aligned}$$

We consider the two terms in the sum separately. The first one can be written as

$$\left(\beta \cdot \left(\nabla w - \Pi^0_{k-1} \nabla w\right), v_h\right)_E = \sum_{i=1}^{2} \left(\frac{\partial w}{\partial x_i} - \Pi^0_{k-1} \frac{\partial w}{\partial x_i}, \beta_i v_h \right)_E \,.$$

Estimating each term in the right-hand side we have, $\forall i \in \{1, 2\}$,

$$\begin{aligned}
&\left(\frac{\partial w}{\partial x_i} - \Pi^0_{k-1} \frac{\partial w}{\partial x_i}, \beta_i v_h \right)_E = \left(\frac{\partial w}{\partial x_i} - \Pi^0_{k-1} \frac{\partial w}{\partial x_i}, \beta_i v_h - \Pi^0_{k-1}(\beta_i v_h) \right)_E \leq \\
&\quad \leq \left\| \frac{\partial w}{\partial x_i} - \Pi^0_{k-1} \frac{\partial w}{\partial x_i} \right\|_E \left\| \beta_i v_h - \Pi^0_{k-1}(\beta_i v_h) \right\|_E \lesssim h_E^s \left| w \right|_{s+1,E} \cdot h_E \left| \beta_i v_h \right|_{1,E} \leq \\
&\qquad \leq h_E^{s+1} \left\| \beta_i \right\|_{\mathrm{W}^1_\infty(E)} \left| w \right|_{s+1,E} \left\| v_h \right\|_{1,E} \,.
\end{aligned}$$

Concerning the second term, we have that

$$\left(\beta \cdot \Pi^0_{k-1} \nabla w, v_h - \Pi^0_{k-1} v_h\right)_E = \sum_{i=1}^{2} \left(\beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i}, v_h - \Pi^0_{k-1} v_h\right)_E .$$

Thus, using the properties of projectors to add polynomials of degree less or equal than $k-1$, we have

$$\begin{aligned}
&\left(\beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i}, v_h - \Pi^0_{k-1} v_h\right)_E = \\
&\quad = \left(\beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i} - \Pi^0_{k-1}\left(\beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i}\right), v_h - \Pi^0_{k-1} v_h\right)_E \leq \\
&\quad \leq \left\| \beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i} - \Pi^0_{k-1}\left(\beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i}\right)\right\|_E \left\| v_h - \Pi^0_{k-1} v_h \right\|_E \leq \\
&\quad \leq h_E \left\| \beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i} - \Pi^0_{k-1}\left(\beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i}\right)\right\|_E \left\| \nabla v_h \right\|_E ,
\end{aligned}$$

and the proof ends using the best approximation property of the projection, the triangle inequality and (21):

$$\begin{aligned}
&\left\| \beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i} - \Pi^0_{k-1}\left(\beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i}\right)\right\|_E \leq \\
&\quad \leq \left\| \beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i} - \Pi^0_{k-1}\left(\beta_i \frac{\partial w}{\partial x_i}\right)\right\|_E \leq \left\| \beta_i \Pi^0_{k-1} \frac{\partial w}{\partial x_i} - \beta_i \frac{\partial w}{\partial x_i}\right\|_E \\
&\quad + \left\| \beta_i \frac{\partial w}{\partial x_i} - \Pi^0_{k-1}\left(\beta_i \frac{\partial w}{\partial x_i}\right)\right\|_E \leq h_E^s \beta_E \left| w \right|_{s+1,E} + h_E^s \left| \beta_i \frac{\partial w}{\partial x_i} \right|_{s,E} \leq \\
&\quad \leq h_E^s \left(\beta_E \left| w \right|_{s+1,E} + \left\| \beta_i \right\|_{\mathrm{W}^s_\infty(E)} \left\| w \right\|_{s+1,E} \right) .
\end{aligned}$$

□

Lemma 2. *For any sufficiently regular function w and $\forall v_h \in V_h$,*

$$d_h\left(w, v_h\right) \lesssim \max_{E \in \mathcal{T}_h} \frac{\mathsf{K}_E}{\mathsf{K}_E^{\vee}} \left\| \sqrt{\mathsf{K}} \nabla w \right\| \left\| \sqrt{\mathsf{K}} \nabla v_h \right\| . \tag{25}$$

Moreover, if $\mathsf{K} \in \mathrm{W}^s_\infty(\Omega)$ and $\beta \in \left[\mathrm{W}^{s+1}_\infty(\Omega)\right]^2$ for some $s \in \{0, \dots, k\}$, then

$$\begin{aligned}
\left| d\left(w, v_h\right) - d_h\left(w, v_h\right) \right| \lesssim \max_{E \in \mathcal{T}_h} \frac{\left\| \beta \right\|_{\mathrm{W}^{s+1}_\infty(E)} \left\| \mathsf{K} \right\|_{\mathrm{W}^s_\infty(E)} \left(\mathsf{K}_E + \beta_E\right)}{\mathsf{K}_E \beta_E \sqrt{\mathsf{K}_E^{\vee}}} h^{s+1} \left\| w \right\|_{s+1} \times \\
\times \left\| \sqrt{\mathsf{K}} \nabla v_h \right\| . \quad (26)
\end{aligned}$$

Proof. To prove (25), we assume $\nabla \cdot \left(\mathsf{K} \nabla w\right) \neq 0$, since otherwhise the inequality is obviously true. We use (12), the Cauchy-Schwarz inequality, the continuity of Π^0_{k-1} and

(9): $\forall E \in \mathcal{T}_h$,

$$\tau_E \left(\nabla\cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right), \beta \cdot \Pi^0_{k-1}\nabla v_h\right)_E \leq \beta_E \frac{h_E}{2\beta_E} \left\|\nabla\cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right)\right\|_E \times$$
$$\times \left\|\Pi^0_{k-1}\nabla v_h\right\|_E \lesssim \frac{1}{2\sqrt{\tilde{C}_E}} \left\|\mathsf{K}\Pi^0_{k-1}\nabla w\right\|_E \left\|\nabla v_h\right\|_E \lesssim \frac{\mathsf{K}_E}{2\sqrt{\tilde{C}_E}} \times$$
$$\times \left\|\Pi^0_{k-1}\nabla w\right\|_E \left\|\nabla v_h\right\|_E \lesssim \frac{\mathsf{K}_E}{2\mathsf{K}^\vee_E\sqrt{\tilde{C}_E}} \left\|\sqrt{\mathsf{K}}\nabla w\right\|_E \left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E .$$

Regarding (26), by applying the triangle inequality we have:

$$\left|\sum_{E\in\mathcal{T}_h} \tau_E \left(\nabla\cdot (\mathsf{K}\nabla w), \beta\cdot\nabla v_h\right)_E - \tau_E \left(\nabla\cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right), \beta \cdot \Pi^0_{k-1}\nabla v_h\right)_E\right| \leq$$
$$\leq \sum_{E\in\mathcal{T}_h} \tau_E \left|\left(\nabla\cdot (\mathsf{K}\nabla w - \mathsf{K}\Pi^0_{k-1}\nabla w), \beta\cdot\nabla v_h\right)_E\right|$$
$$+ \tau_E \left|\left(\nabla\cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right), \beta \cdot \left(\nabla v_h - \Pi^0_{k-1}\nabla v_h\right)\right)_E\right| . \quad (27)$$

To estimate the first term of the right-hand-side of (27), we suppose $\nabla\cdot(\mathsf{K}\nabla w - \mathsf{K}\Pi^0_{k-1}\nabla w) \neq 0$, we use the Cauchy-Schwarz inequality, (11), (9) and (21):

$$\tau_E \left|\left(\nabla\cdot (\mathsf{K}\nabla w - \mathsf{K}\Pi^0_{k-1}\nabla w), \beta\cdot\nabla v_h\right)_E\right| \leq \frac{\tilde{C}_E h_E^2 \beta_E}{2\mathsf{K}_E} \left\|\nabla\cdot \left(\mathsf{K}\nabla w - \mathsf{K}\Pi^0_{k-1}\nabla w\right)\right\|_E \times$$
$$\times \left\|\nabla v_h\right\|_E \leq \frac{\sqrt{\tilde{C}_E} h_E \beta_E}{2\mathsf{K}_E} \left\|\mathsf{K}\nabla w - \mathsf{K}\Pi^0_{k-1}\nabla w\right\|_E \left\|\nabla v_h\right\|_E \leq$$
$$\leq \frac{\sqrt{\tilde{C}_E}\beta_E}{2\sqrt{\mathsf{K}^\vee_E}} h_E \left\|\nabla w - \Pi^0_{k-1}\nabla w\right\|_E \left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E \lesssim \frac{\sqrt{\tilde{C}_E}\beta_E}{2\sqrt{\mathsf{K}^\vee_E}} h_E^{s+1} \left|w\right|_{s+1,E} \left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E .$$

Concerning the second term of (27), we have

$$\tau_E \left(\nabla\cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right), \beta \cdot \left(\nabla v_h - \Pi^0_{k-1}\nabla v_h\right)\right)_E =$$
$$= \tau_E \sum_{i=1}^{2} \left(\beta_i \nabla\cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right), \frac{\partial v_h}{\partial x_i} - \Pi^0_{k-1}\left(\frac{\partial v_h}{\partial x_i}\right)\right)_E ,$$

and we can bound each term of the sum by using the properties of the projection, the

Cauchy-Schwarz inequality and the triangle inequality:

$$
\begin{aligned}
\tau_E \left(\beta_i \nabla\cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right), \frac{\partial v_h}{\partial x_i} - \Pi^0_{k-1}\frac{\partial v_h}{\partial x_i}\right)_E &= \\
\tau_E \left(\nabla\cdot \left(\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w\right), \frac{\partial v_h}{\partial x_i} - \Pi^0_{k-1}\frac{\partial v_h}{\partial x_i}\right)_E & \\
+ \tau_E \left(-\nabla\beta_i \cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right), \frac{\partial v_h}{\partial x_i} - \Pi^0_{k-1}\frac{\partial v_h}{\partial x_i}\right)_E &= \\
= \tau_E \left(\nabla\cdot \left(\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w\right) - \nabla\cdot\left(\Pi^0_{k-1}\big(\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w\big)\right), \frac{\partial v_h}{\partial x_i} - \Pi^0_{k-1}\frac{\partial v_h}{\partial x_i}\right)_E & \\
+ \tau_E \left(\Pi^0_{k-1}\big(\nabla\beta_i \cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right)\big) - \nabla\beta_i \cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right), \frac{\partial v_h}{\partial x_i} - \Pi^0_{k-1}\frac{\partial v_h}{\partial x_i}\right)_E &\lesssim \\
\lesssim \frac{\tau_E}{\sqrt{\mathsf{K}^{\vee}_E}} \Big(\left\|\nabla\cdot \left(\beta_i\mathsf{K}\Pi^0_{k-1}(\nabla w) - \Pi^0_{k-1}\big(\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w\big)\right)\right\|_E & \\
+ \left\|\Pi^0_{k-1}\big(\nabla\beta_i \cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right)\big) - \nabla\beta_i \cdot \left(\mathsf{K}\Pi^0_{k-1}\nabla w\right)\right\|_E \Big) \left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E &.
\end{aligned}
$$

We consider the two terms inside the parentheses separately. To estimate the first one, we first use the fact that Π^0_{k-1} is the best $\mathrm{L}^2(E)$ approximation in $\mathbb{P}_{k-1}(E)$, then inequalities (11) and (9), and finally (21):

$$
\begin{aligned}
\tau_E \left\|\nabla\cdot \left(\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w - \Pi^0_{k-1}\big(\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w\big)\right)\right\|_E &\leq \\
\leq \frac{\tilde{C}_E h_E^2}{2\mathsf{K}_E} \left\|\nabla\cdot \left(\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w - \Pi^0_{k-1}\big(\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w\big)\right)\right\|_E &\leq \\
\leq \frac{\sqrt{\tilde{C}_E} h_E}{2\mathsf{K}_E} \left\|\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w - \Pi^0_{k-1}\big(\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w\big)\right\|_E &\leq \\
\leq \frac{\sqrt{\tilde{C}_E} h_E}{2\mathsf{K}_E} \left\|\beta_i\mathsf{K}\Pi^0_{k-1}\nabla w - \Pi^0_{k-1}(\beta_i\mathsf{K}\nabla w)\right\|_E &\leq \\
\leq \frac{\sqrt{\tilde{C}_E} h_E}{2\mathsf{K}_E} \left(\left\|\beta_i\mathsf{K}\left(\Pi^0_{k-1}\nabla w - \nabla w\right)\right\|_E + \left\|\beta_i\mathsf{K}\nabla w - \Pi^0_{k-1}(\beta_i\mathsf{K}\nabla w)\right\|_E \right) &\lesssim \\
\lesssim \frac{\sqrt{\tilde{C}_E} h_E}{2\mathsf{K}_E} \left(h_E^s \beta_E \mathsf{K}_E \left|w\right|_{s+1,E} + h_E^s \left|\beta_i\mathsf{K}\nabla w\right|_{s,E} \right) &\lesssim \\
\lesssim \frac{\sqrt{\tilde{C}_E} h_E^{s+1}}{2\mathsf{K}_E} \left(\beta_E \mathsf{K}_E \left|w\right|_{s+1,E} + \left\|\beta\right\|_{\mathrm{W}^s_\infty(E)} \left\|\mathsf{K}\right\|_{\mathrm{W}^s_\infty(E)} \left\|w\right\|_{s+1,E} \right) &.
\end{aligned}
$$

To estimate the second term we use the fact that Π^0_{k-1} is the best approximation in

$\mathbb{P}_{k-1}(E)$, the triangle inequality, inequality (12) and the estimate (21):

$$\tau_E \left\| \Pi^0_{k-1}\big(\nabla\beta_i \cdot \big(\mathsf{K}\Pi^0_{k-1}\nabla w\big)\big) - \nabla\beta_i \cdot \big(\mathsf{K}\Pi^0_{k-1}\nabla w\big) \right\|_E \leq$$

$$\leq \tau_E \left\| \Pi^0_{k-1}(\nabla\beta_i \cdot \mathsf{K}\nabla w) - \nabla\beta_i \cdot \mathsf{K}\Pi^0_{k-1}\nabla w \right\|_E \leq \frac{h_E}{2\beta_E} \times$$

$$\times \left(\left\| \Pi^0_{k-1}(\nabla\beta_i \cdot \mathsf{K}\nabla w) - \nabla\beta_i \cdot \mathsf{K}\nabla w \right\|_E + \left\| \nabla\beta_i \cdot \mathsf{K}\left(\nabla w - \Pi^0_{k-1}\nabla w\right) \right\|_E \right) \lesssim$$

$$\lesssim \frac{h_E}{2\beta_E} \left(h_E^s \left| \nabla\beta_i \cdot \mathsf{K}\nabla w \right|_{s,E} + h_E^s \mathsf{K}_E \left\| \beta \right\|_{\mathrm{W}^1_\infty(E)} \left| w \right|_{s+1,E} \right) \lesssim$$

$$\lesssim \frac{h_E^{s+1}}{2\beta_E} \left(\left\| \mathsf{K} \right\|_{\mathrm{W}^s_\infty(E)} \left\| \beta \right\|_{\mathrm{W}^{s+1}_\infty(E)} \left\| w \right\|_{s+1,E} + \mathsf{K}_E \left\| \beta \right\|_{\mathrm{W}^1_\infty(E)} \left| w \right|_{s+1,E} \right) .$$

□

Lemma 3. *For any sufficiently regular function w and $\forall v_h \in V_h$,*

$$a_h(w, v_h) \lesssim \max_{E \in \mathcal{T}_h} \frac{\mathsf{K}_E + \tau_E \beta_E^2}{\mathsf{K}_E^\vee} \left\| \sqrt{\mathsf{K}}\nabla w \right\| \left\| \sqrt{\mathsf{K}}\nabla v_h \right\| . \tag{28}$$

Moreover, if $\mathsf{K} \in \mathrm{W}^s_\infty(\Omega)$ and $\beta \in [\mathrm{W}^s_\infty(\Omega)]$ for some $s \in \{0, \dots, k\}$, then

$$\left| a(w, v_h) - a_h(w, v_h) \right| \lesssim \left(\max_{E \in \mathcal{T}_h} \frac{\left\| \mathsf{K} \right\|_{\mathrm{W}^s_\infty(E)} + \frac{\left\| \beta \right\|^2_{\mathrm{W}^s_\infty(E)}}{\beta_E}}{\sqrt{\mathsf{K}_E^\vee}} \right) h^s \left\| w \right\|_{s+1} \times$$

$$\times \left\| \sqrt{\mathsf{K}}\nabla v_h \right\| . \tag{29}$$

Proof. Let $v_h, w \in V_h$. We first prove (28) considering $E \in \mathcal{T}_h$. Regarding the terms involving the VEM stabilization, we first point out that, as a consequence of (17), we have

$$S^E\left(\left(I - \Pi^\nabla_k\right) w, \left(I - \Pi^\nabla_k\right) v_h\right) \lesssim \left\| \nabla\left(w - \Pi^\nabla_k w\right) \right\|_E \left\| \nabla\left(v_h - \Pi^\nabla_k v_h\right) \right\|_E . \tag{30}$$

Applying the Cauchy-Schwarz inequality, (30) and the continuity of projectors,

$$a_h^E(w, v_h) = \left(\mathsf{K}\Pi^0_{k-1}\nabla w, \Pi^0_{k-1}\nabla v_h\right)_E + \tau_E\left(\beta \cdot \Pi^0_{k-1}\nabla w, \beta \cdot \Pi^0_{k-1}\nabla v_h\right)_E$$

$$+ \left(\mathsf{K}_E + \tau_E\beta_E^2\right) S^E\left(\left(I - \Pi^\nabla_k\right) w, \left(I - \Pi^\nabla_k\right) v_h\right) \lesssim \left(\mathsf{K}_E + \tau_E\beta_E^2\right) \times$$

$$\times \left(\left\| \Pi^0_{k-1}\nabla w \right\|_E \left\| \Pi^0_{k-1}\nabla v_h \right\|_E + \left\| \left(I - \Pi^\nabla_k\right) w \right\|_E \left\| \left(I - \Pi^\nabla_k\right) v_h \right\|_E \right) \lesssim$$

$$\lesssim \frac{\mathsf{K}_E + \tau_E\beta_E^2}{\mathsf{K}_E^\vee} \left\| \sqrt{\mathsf{K}}\nabla w \right\|_E \left\| \sqrt{\mathsf{K}}\nabla v_h \right\|_E .$$

Concerning (29), by adding and subtracting $\left(\mathsf{K}\nabla w, \Pi^0_{k-1}\nabla v_h\right)_E = \left(\Pi^0_{k-1}(\mathsf{K}\nabla w), \nabla v_h\right)_E$ and $\left(\beta\beta^\intercal\nabla w, \Pi^0_{k-1}\nabla v_h\right)_E = \left(\Pi^0_{k-1}(\beta\beta^\intercal\nabla w), \nabla v_h\right)_E$ and exploiting the triangle inequality we have, $\forall E \in \mathcal{T}_h$,

$$\left| a^E(w, v_h) - a_h^E(w, v_h) \right| \leq \left| \left(\mathsf{K}\nabla w - \mathsf{K}\Pi^0_{k-1}\nabla w, \Pi^0_{k-1}\nabla v_h\right)_E \right|$$

$$+ \left| \left(\mathsf{K}\nabla w - \Pi^0_{k-1}(\mathsf{K}\nabla w), \nabla v_h\right)_E \right| + \left(\mathsf{K}_E + \tau_E\beta_E^2\right) \times$$

$$\times \left| S^E\left(\left(I - \Pi^\nabla_k\right) w, \left(I - \Pi^\nabla_k\right) v_h\right) \right| + \tau_E \left| \left(\beta\beta^\intercal\nabla w - \beta\beta^\intercal\Pi^0_{k-1}\nabla w, \Pi^0_{k-1}\nabla v_h\right)_E \right|$$

$$+ \tau_E \left| \left(\beta\beta^\intercal\nabla w - \Pi^0_{k-1}(\beta\beta^\intercal\nabla w), \nabla v_h\right)_E \right| .$$

The first term is bounded as follows, exploiting the definition of projection, its continuity and (21):

$$\left(\mathsf{K}\left(\nabla w-\Pi^0_{k-1}\nabla w\right),\Pi^0_{k-1}\nabla v_h\right)_E \lesssim \sqrt{\mathsf{K}_E}\left\|\nabla w-\Pi^0_{k-1}\nabla w\right\|_E \left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E \lesssim$$
$$\lesssim \sqrt{\mathsf{K}_E} h_E^s \left|w\right|_{s+1,E}\left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E .$$

The second term is bounded by the Cauchy-Schwarz inequality and (21):

$$\left(\mathsf{K}\nabla w-\Pi^0_{k-1}(\mathsf{K}\nabla w),\nabla v_h\right)_E \leq \left\|\mathsf{K}\nabla w-\Pi^0_{k-1}(\mathsf{K}\nabla w)\right\|_E \left\|\nabla v_h\right\|_E \lesssim$$
$$\lesssim h_E^s\left|\mathsf{K}\nabla w\right|_{s,E} \leq h_E^s \frac{\left\|\mathsf{K}\right\|_{\mathrm{W}^s_\infty(E)}}{\sqrt{\mathsf{K}_E^\vee}}\left\|w\right\|_{s+1,E}\left\|\sqrt{\mathsf{K}}\nabla v_h\right\| .$$

The third term is estimated using (12), the Cauchy-Schwarz inequality, the continuity of Π^0_{k-1} and (21):

$$\tau_E\left(\beta\beta^\intercal\left(\nabla w-\Pi^0_{k-1}\nabla w\right),\Pi^0_{k-1}\nabla v_h\right)_E \lesssim \frac{\beta_E}{2}h_E\left\|\nabla w-\Pi^0_{k-1}\nabla w\right\|_E\left\|\nabla v_h\right\|_E \lesssim$$
$$\lesssim \frac{\beta_E}{2\sqrt{\mathsf{K}_E^\vee}}h_E^{s+1}\left|w\right|_{s+1,E}\left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E \lesssim \frac{\left\|\beta\right\|^2_{\mathrm{W}^s_\infty(E)}}{\beta_E\sqrt{\mathsf{K}_E^\vee}}h_E^{s+1}\left|w\right|_{s+1,E}\left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E .$$

The fourth term can be estimated similarly:

$$\tau_E\left(\beta\beta^\intercal\nabla w-\Pi^0_{k-1}(\beta\beta^\intercal\nabla w),\nabla v_h\right)_E \leq \tau_E\left\|\beta\beta^\intercal\nabla w-\Pi^0_{k-1}(\beta\beta^\intercal\nabla w)\right\|_E\left\|\nabla v_h\right\|$$
$$\lesssim \tau_E\frac{h_E^s}{\sqrt{\mathsf{K}_E^\vee}}\left|\beta\beta^\intercal\nabla w\right|_{s,E}\left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E \lesssim \frac{\left\|\beta\right\|^2_{\mathrm{W}^s_\infty(E)}}{\beta_E\sqrt{\mathsf{K}_E^\vee}}h_E^{s+1}\left\|w\right\|_{s+1,E}\left\|\sqrt{\mathsf{K}}\nabla v_h\right\| .$$

Finally, we consider the terms involving the VEM stabilization and, applying again (30), we are left to estimate projection errors. Proceeding as above, exploiting the continuity of Π^∇_k, (22) and the estimate on τ_E in (12) we obtain

$$\mathsf{K}_E\left\|\nabla\left(w-\Pi^\nabla_k w\right)\right\|_E\left\|\nabla\left(v_h-\Pi^\nabla_k v_h\right)\right\|_E \leq \frac{\mathsf{K}_E}{\sqrt{\mathsf{K}_E^\vee}}h_E^s\left|w\right|_{s+1,E}\left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E ,$$
$$\tau_E\beta_E^2\left\|\nabla\left(w-\Pi^\nabla_k w\right)\right\|_E\left\|\nabla\left(v_h-\Pi^\nabla_k v_h\right)\right\|_E \leq \frac{\beta_E}{2\sqrt{\mathsf{K}_E^\vee}}h_E^{s+1}\left|w\right|_{s+1,E}\left\|\sqrt{\mathsf{K}}\nabla v_h\right\|_E .$$

□

4.2. Well-posedness of the discrete problem

In this subsection we prove, in Theorem 1, an inf-sup condition for the discrete bilinear form defined by (13), which ensures the well-posedness of problem (19).

Lemma 4. *There exist a constant $\alpha > 0$ such that*

$$a_h\left(v_h,v_h\right) \geq \alpha \left|\!\left|\!\left| v_h \right|\!\right|\!\right|^2 \quad \forall v_h \in V_h . \tag{31}$$

Proof. Let $v_h \in V_h$ and fix $E \in \mathcal{T}_h$. From the definition of a_h in (14) we have

$$
\begin{aligned}
a_h^E\left(v_h, v_h\right) := & \left\|\sqrt{\mathsf{K}}\Pi_{k-1}^0 \nabla v_h\right\|_E^2 + \tau_E \left\|\beta \cdot \Pi_{k-1}^0 \nabla v_h\right\|_E^2 \\
& + \left(\mathsf{K}_E + \tau_E \beta_E^2\right) S^E\left(\left(I - \Pi_k^\nabla\right) v_h, \left(I - \Pi_k^\nabla\right) v_h\right) .
\end{aligned}
$$

From (17) and the properties of the orthogonal projection, we have that there exists $c^* > 0$ such that, $\forall E \in \mathcal{T}_h$,

$$
S^E\left(\left(I - \Pi_k^\nabla\right) v_h, \left(I - \Pi_k^\nabla\right) v_h\right) \geq c^* \left\|\nabla\left(v_h - \Pi_k^\nabla v_h\right)\right\|_E^2 \geq c^* \left\|\nabla v_h - \Pi_{k-1}^0 \nabla v_h\right\|_E^2 ,
$$

and then

$$
\begin{aligned}
& \left(\mathsf{K}_E + \tau_E \beta_E^2\right) S^E\left(\left(I - \Pi_k^\nabla\right) v_h, \left(I - \Pi_k^\nabla\right) v_h\right) \geq \\
& \qquad \geq c^* \left(\mathsf{K}_E + \tau_E \beta_E^2\right) \left\|\nabla v_h - \Pi_{k-1}^0 \nabla v_h\right\|_E^2 \geq \\
& \qquad \geq c^* \left(\left\|\sqrt{\mathsf{K}}\left(\nabla v_h - \Pi_{k-1}^0 \nabla v_h\right)\right\|_E^2 + \tau_E \left\|\beta \cdot \left(\nabla v_h - \Pi_{k-1}^0 \nabla v_h\right)\right\|_E^2\right) .
\end{aligned}
$$

The thesis is thus proven choosing $\alpha = \min\left\{c^*, 1\right\}$:

$$
\begin{aligned}
a_h^E\left(v_h, v_h\right) \geq & \left\|\sqrt{\mathsf{K}}\Pi_{k-1}^0 \nabla v_h\right\|_E^2 + \sum_{E \in \mathcal{T}_h} \tau_E \left\|\beta \cdot \Pi_{k-1}^0 \nabla v_h\right\|_E^2 \\
& + c^* \left(\left\|\sqrt{\mathsf{K}}\left(\nabla v_h - \Pi_{k-1}^0 \nabla v_h\right)\right\|_E^2 + \tau_E \left\|\beta \cdot \left(\nabla v_h - \Pi_{k-1}^0 \nabla v_h\right)\right\|_E^2\right) \geq \\
& \min\left\{c^*, 1\right\} \left(\left\|\sqrt{\mathsf{K}}\Pi_{k-1}^0 \nabla v_h\right\|^2 + \left\|\sqrt{\mathsf{K}}\left(\nabla v_h - \Pi_{k-1}^0 \nabla v_h\right)\right\|^2\right. \\
& \left. + \tau_E \left\|\beta \cdot \Pi_{k-1}^0 \nabla v_h\right\|_E^2 + \tau_E \left\|\beta \cdot \left(\nabla v_h - \Pi_{k-1}^0 \nabla v_h\right)\right\|_E^2\right) \geq \\
& \geq \left\{c^*, 1\right\} \left(\left\|\sqrt{\mathsf{K}}\nabla v_h\right\|^2 + \tau_E \left\|\beta \cdot \nabla v_h\right\|_E^2\right) .
\end{aligned}
$$

□

Lemma 5. *Let $q \in \mathrm{H}_0^1(\Omega)$. Then there exists $q^* \in V_h$ such that*

$$
a_h\left(q^*, v_h\right) = a\left(q, v_h\right) \quad \forall v_h \in V_h .
$$

Moreover,

$$
|||q^*||| \leq \frac{1}{\alpha} |||q||| , \tag{32}
$$

$$
\left\|q - q^*\right\| \precsim h |||q||| , \tag{33}
$$

being α the coercivity constant in (31).

Proof. The proof is formally the same as the one for [2, Lemma 5.6].

□

Lemma 6. *For any $v_h \in V_h$,*

$$B_{\text{supg}}\left(v_h, v_h\right) \geq \frac{1}{2} \left\| v_h \right\| \, . \tag{34}$$

Proof. Let $v_h \in V_h$. Since we have homogeneous Dirichlet boundary conditions and $\nabla \cdot \beta = 0$, it holds

$$\left(\beta \cdot \nabla v_h, v_h\right) = -\frac{1}{2} \left(\nabla \cdot \beta, v_h^2\right) = 0 \, .$$

We have, using the definition of B_{supg}, and the Cauchy-Schwarz and Young inequalities and the estimate (9),

$$\begin{aligned}
B_{\text{supg}}\left(v_h, v_h\right) &= \left\| \sqrt{\mathsf{K}} \nabla v_h \right\|^2 + \sum_{E \in \mathcal{T}_h} \tau_E \left\| \beta \cdot \nabla v_h \right\|_E^2 - \tau_E \left(\nabla \cdot \left(\sqrt{\mathsf{K}} \nabla v_h \right), \beta \cdot \nabla v_h \right)_E \geq \\
&\geq \left\| \sqrt{\mathsf{K}} \nabla v_h \right\|^2 + \sum_{E \in \mathcal{T}_h} \tau_E \left\| \beta \cdot \nabla v_h \right\|_E^2 - \tau_E \left\| \nabla \cdot \left(\sqrt{\mathsf{K}} \nabla v_h \right) \right\|_E \left\| \beta \cdot \nabla v_h \right\|_E \geq \left\| \sqrt{\mathsf{K}} \nabla v_h \right\|_E^2 \\
&+ \sum_{E \in \mathcal{T}_h} \frac{1}{2} \tau_E \left\| \beta \cdot \nabla v_h \right\|_E^2 - \frac{1}{2} \tau_E \left\| \nabla \cdot \left(\mathsf{K} \nabla v_h \right) \right\|_E^2 \geq \left\| \sqrt{\mathsf{K}} \nabla v_h \right\|^2 - \sum_{E \in \mathcal{T}_h} \frac{1}{4} \left\| \sqrt{\mathsf{K}} \nabla v_h \right\|_E^2 \\
&+ \frac{1}{2} \tau_E \left\| \beta \cdot \nabla v_h \right\|_E^2 \geq \frac{3}{4} \left\| \sqrt{\mathsf{K}} \nabla v_h \right\|^2 + \sum_{E \in \mathcal{T}_h} \frac{1}{2} \tau_E \left\| \beta \cdot \nabla v_h \right\|_E^2 \geq \frac{1}{2} \left\| v_h \right\|^2 \, .
\end{aligned}$$

□

Theorem 1. *Suppose $\mathsf{K} \in \mathrm{L}^\infty(\Omega)$ and $\beta \in \left[\mathrm{W}^1_\infty(\Omega)\right]^2$. Then, $\forall v_h \in V_h$ and for h sufficiently small,*

$$\sup_{w_h \in V_h} \frac{B_{\text{supg},h}\left(v_h, w_h\right)}{\left\| w_h \right\|} \gtrsim \left\| v_h \right\| \, . \tag{35}$$

Proof. Let $v_h \in V_h$ be fixed and let $v_h^* \in V_h$ be the function, whose existence is guaranteed by Lemma 5, such that $a_h\left(v_h^*, w_h\right) = a\left(v_h, w_h\right)$, $\forall w_h \in V_h$. By definitions (3) and (13), since a_h is symmetric, we have, by (34),

$$\begin{aligned}
B_{\text{supg},h}\left(v_h, v_h^*\right) &= a_h\left(v_h, v_h^*\right) + b_h\left(v_h, v_h^*\right) + d_h\left(v_h, v_h^*\right) = a\left(v_h, v_h\right) + b_h\left(v_h, v_h^*\right) \\
&+ d_h\left(v_h, v_h^*\right) = B_{\text{supg}}\left(v_h, v_h\right) + r\left(v_h, v_h^*\right) \geq \frac{1}{2} \left\| v_h \right\|^2 + r\left(v_h, v_h^*\right) \, ,
\end{aligned}$$

where

$$\begin{aligned}
r\left(v_h, v_h^*\right) &= b_h\left(v_h, v_h^*\right) - b\left(v_h, v_h^*\right) + b\left(v_h, v_h^* - v_h\right) \\
&+ d_h\left(v_h, v_h^*\right) - d\left(v_h, v_h^*\right) + d\left(v_h, v_h^* - v_h\right) \, .
\end{aligned}$$

By Lemmas 1 and 2, the continuity of b and d, that can be proven as for (23) and (25), and by (32) and (33), there exists a constant $C_r > 0$ depending on $\left\| \mathsf{K} \right\|_{\mathrm{L}^\infty(\mathsf{K})}$, $\left\| \beta \right\|_{\mathrm{W}^1_\infty(\Omega)}$ and on the approximation constants in (21) and (22), such that

$$\left| r\left(v_h, v_h^*\right) \right| \leq C_r h \left\| \sqrt{\mathsf{K}} \nabla v_h \right\| \left\| \sqrt{\mathsf{K}} \nabla v_h^* \right\| \leq C_r h \left\| v_h \right\| \left\| v_h^* \right\| \leq C_r h \left\| v_h \right\|^2 \, . \tag{36}$$

Then, by (32) and (36) the following lower bound holds:

$$B_{\mathrm{supg},h}(v_h, v_h) \geq \frac{1}{2} |||v_h|||^2 + r(v_h, v_h^*) \geq \left(\frac{\alpha}{2} - C_r h\right) |||v_h||| \, |||v_h^*||| \ ,$$

which yields the thesis for

$$h < \frac{\alpha}{2C_r} .$$

□

4.3. A priori error estimate

To derive an a priori estimate that shows optimality of the rate of convergence of this SUPG approach, we will use the following estimate on the VEM interpolator (see [2, Lemma 5.1]):

$$\forall E \in \mathcal{T}_h, \forall \varphi \in \mathrm{H}^s(E), \, \|\varphi - \varphi_I\|_{m,E} \lesssim h_E^{s-m} |\varphi|_{s,E} : \forall s, m \in \mathbb{N}, \, m \leq s \leq k+1, \, s \geq 2 . \tag{37}$$

We are now ready to prove the following result.

Theorem 2. *Suppose* $u \in \mathrm{H}^{s+1}(\Omega)$, $\mathsf{K} \in \mathrm{W}^s_\infty(\Omega)$, $\beta \in \left[\mathrm{W}^{s+1}_\infty(\Omega)\right]^2$ *for some* $s \in \{1, \ldots, k\}$. *Then, for h sufficiently small,*

$$|||u - u_h||| \lesssim h^s \left(\|u\|_{s+1} + \|f\|_s\right) . \tag{38}$$

Proof. First, by the triangle inequality we have

$$|||u - u_h|||^2 \leq |||u - u_I|||^2 + |||u_h - u_I|||^2 \ ,$$

and, by (37),

$$\begin{aligned} |||u - u_I|||^2 &= \sum_{E \in \mathcal{T}_h} \left\|\sqrt{\mathsf{K}} \nabla (u - u_I)\right\|_E^2 + \|\beta \cdot \nabla (u - u_I)\|_E^2 \leq \\ &\leq \sum_{E \in \mathcal{T}_h} \left(\mathsf{K}_E + \beta_E^2\right) \|\nabla (u - u_I)\|_E^2 \lesssim \sum_{E \in \mathcal{T}_h} \left(\mathsf{K}_E + \beta_E^2\right) h_E^{2s} |u|_{s+1,E}^2 . \end{aligned}$$

We are left to estimate the norm of $e_h := u_h - u_I$. Since $e_h \in V_h$, by (35) there exists $w_h \in V_h$ such that

$$B_{\mathrm{supg},h}(e_h, w_h) \gtrsim |||e_h||| \, |||w_h||| \ .$$

Using the exact and discrete problems (10) and (19),

$$\begin{aligned} |||e_h||| \, |||w_h||| &\lesssim B_{\mathrm{supg},h}(u_h - u_I, w_h) = F_{\mathrm{supg},h}(w_h) - B_{\mathrm{supg},h}(u_I, w_h) = \\ &= F_{\mathrm{supg},h}(w_h) - F_{\mathrm{supg}}(w_h) + B_{\mathrm{supg}}(u, w_h) - B_{\mathrm{supg},h}(u_I, w_h) = F_{\mathrm{supg},h}(w_h) \\ &\quad - F_{\mathrm{supg}}(w_h) + B_{\mathrm{supg},h}(u - u_I, w_h) + B_{\mathrm{supg}}(u, w_h) - B_{\mathrm{supg},h}(u, w_h) . \end{aligned} \tag{39}$$

Note that for our choice of the degrees of freedom and stabilization (defined in (18)), it makes sense to compute $B_{\mathrm{supg},h}(u, w_h)$ as in (13)-(16), because $u \in \mathrm{H}^2(\Omega) \subset C^0(\Omega)$ for $\Omega \subset \mathbb{R}^2$. If the solution u does not have the regularity for pointwise evaluation, definition (18) for the VEM-stabilization function has to be properly modified.

The first difference in (39) can be written as:

$$F_{\text{supg},h}\left(w_h\right) - F_{\text{supg}}\left(w_h\right) = \sum_{E\in\mathcal{T}_h} \left(f, \left(\Pi^0_{k-1} - I\right) w_h + \beta \cdot \left(\Pi^0_{k-1} - I\right) \nabla w_h\right)_E \,. \tag{40}$$

The first term of the sum in (40) is bounded as follows:

$$\begin{aligned}\left(f, \left(\Pi^0_{k-1} - I\right) w_h\right) = \left(\left(I - \Pi^0_{k-1}\right) f, \left(\Pi^0_{k-1} - I\right) w_h\right) \leq \left\|f - \Pi^0_{k-1} f\right\|_E \times \\ \times \left\|w_h - \Pi^0_{k-1} w_h\right\|_E \lesssim h_E^{s-1} \left|f\right|_{s-1,E} h_E \left\|\nabla w_h\right\| \leq \\ \leq \frac{h_E^s}{\sqrt{\mathsf{K}_E^\vee}} \left|f\right|_{s-1,E} \left\|\sqrt{\mathsf{K}} \nabla w_h\right\|_E \leq \frac{h_E^s}{\sqrt{\mathsf{K}_E^\vee}} \left|f\right|_{s-1,E} \left\|\left|w_h\right|\right\|_E \,.\end{aligned}$$

The second term of the sum in (40) can be treated as follows:

$$\begin{aligned}\left(f, \beta \cdot \left(\Pi^0_{k-1} - I\right) \nabla w_h\right)_E = \sum_{i=1}^{2} \left(\left(\Pi^0_{k-1} - I\right)\left(\beta_i f\right), \frac{\partial w_h}{\partial x_i}\right)_E \leq \\ \leq \sum_{i=1}^{2} \left\|\left(I - \Pi^0_{k-1}\right)\left(\beta_i f\right)\right\|_E \left\|\frac{\partial w_h}{\partial x_i}\right\|_E \lesssim \frac{h^s}{\sqrt{\mathsf{K}_E^\vee}} \sum_{i=1}^{2} \left|\beta_i f\right|_{s,E} \left\|\sqrt{\mathsf{K}} \nabla w_h\right\|_E \leq \\ \leq \frac{\left\|\beta\right\|_{\mathrm{W}^s_\infty(E)}}{\sqrt{\mathsf{K}_E^\vee}} h_E^s \left\|f\right\|_{s,E} \left\|\left|w_h\right|\right\|_E \,.\end{aligned}$$

Going back to (39), we estimate the continuity of $B_{\text{supg},h}$, given by (23), (25) and (28), and the estimate on the VEM interpolator in (37):

$$B_{\text{supg},h}\left(u - u_I, w_h\right) \precsim \left\|u - u_I\right\|_1 \left\|w_h\right\|_1 \lesssim h^s \left\|u\right\|_{s+1} \left\|\left|w_h\right|\right\| \,.$$

The estimate of the last difference in (39) is obtained by applying (24), (26) and (29):

$$\begin{aligned}\left|B_{\text{supg}}\left(u, w_h\right) - B_{\text{supg},h}\left(u, w_h\right)\right| \leq \left|a\left(u, w_h\right) - a_h\left(u, w_h\right)\right| \\ + \left|b\left(u, w_h\right) - b_h\left(u, w_h\right)\right| + \left|d\left(u, w_h\right) - d_h\left(u, w_h\right)\right| \precsim h^s \left\|u\right\|_{s+1} \left\|w_h\right\|_1 \,.\end{aligned}$$

□

5. Numerical Results

In this section we will consider two benchmark problems in the domain $\Omega = (0,1) \times (0,1)$ in order to numerically evaluate the rates of convergence of the discussed VEM-SUPG stabilization both in the convection-dominated regime and the diffusion-dominated regime. VEM orders from one to three are used.

5.1. Test 1

As a first test we consider problem (1) with constant K and β. In particular the transport velocity field is

$$\beta(x,y) = \left(\tfrac{1}{2}, -\tfrac{1}{3}\right),$$

and we perform two sets of simulations corresponding to two different values of K: in a first set of simulations we use $\mathsf{K} = 10^{-3}$, whereas $\mathsf{K} = 10^{-9}$ is used for a second set of simulations. The meshsize range is chosen in such a way that for all the values of the VEM order k the mesh Péclet number is both greater and lower than one for $\mathsf{K} = 10^{-3}$, whereas it is much greater than one for $\mathsf{K} = 10^{-9}$.

The exact solution for this problem is given by

$$u(x, y) = \frac{65536}{729} x^3 (1 - x) y^3 (1 - y) .$$

In Figures 1a–1f we show the convergence curves obtained with $\mathsf{K} = 10^{-3}$ (left) and $\mathsf{K} = 10^{-9}$ (right). The error reported is based on the difference between the exact solution and the projection of the discrete solution on the space of polynomials of degree k, accordingly to the VEM order k varying from 1 to 3. The error is measured in the $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^1(\Omega)$-norms and is plotted with respect to the number of degrees of freedom (Ndof). For each mesh we also report the values of the minimum and maximum mesh Péclet numbers. Note that the left y-axes scales refer to the mesh Péclet numbers, whereas the right ones refer to the error measure. The very good agreement between the numerical behaviour and the expected rates of convergence in (38) is evident.

5.2. Test 2

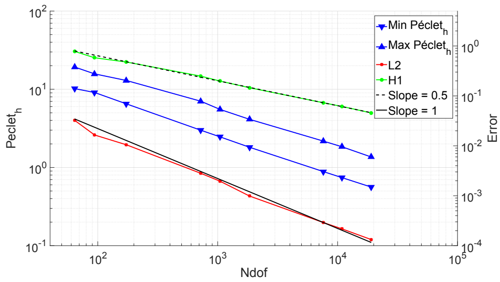
For the second test, non-constant coefficients are used and the flow regime is transport dominated in all the simulations performed. We have set:

$$\mathsf{K}(x, y) = 10^{-7} \begin{pmatrix} 1 + x^2 & xy \\ xy & 1 + y^2 \end{pmatrix} ,$$
$$\beta(x, y) = \left(\tfrac{1}{3} + 10y(x + y^2)^4, -\tfrac{1}{2} - 5(x + y^2)^4 \right) ,$$

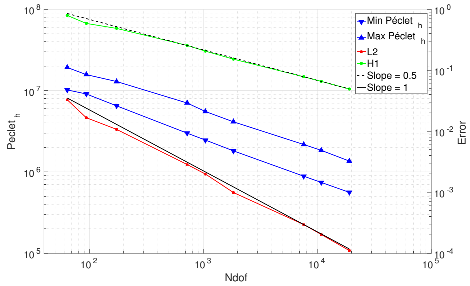
and the exact solution in this case is:

$$u(x, y) = 600xy(1 - x)(1 - y) \left(x - \frac{1}{5} \right) \left(y - \frac{2}{5} \right) \left(y - \frac{3}{5} \right) .$$

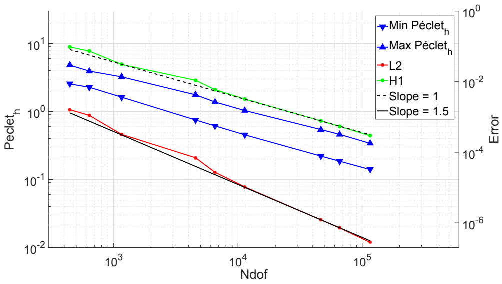
We now compare the solution obtained with the VEM-SUPG method described in the present work on a family of polygonal Voronoi meshes generated by `PolyMesher` [32], made up of polygons with four to eight edges (see Figure 2a), with the solution obtained on standard triangular meshes. Figures 2c and 2d show a comparison between the unstabilized solution and the one obtained using the SUPG stabilization for second order VEM, showing a very good agreement with the exact solution (Figure 2b) for a given polygonal mesh. Convergence curves were obtained for VEM formulations of order from 1 to 3 and are reported in Figure 3. The error was obtained by comparing the exact solution to the polynomial projections of the discrete solutions. On each plot we also report the maximum and minimum mesh Péclet number for each considered meshsize. Also in this case, the left y-axes refer to the mesh Péclet numbers, whereas the right ones refer to the error measure. Note that for all orders and meshes, this problem is always convection-dominant ($\min_{E \in \mathcal{T}_h} \mathrm{Pe}_E \gg 1$ for all meshes). Again, the plots show a very good agreement between the experimental orders of convergence and the ones provided by Theorem 2, independently of the mesh used.



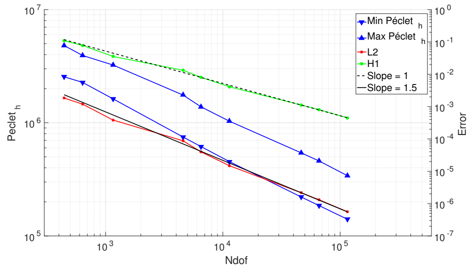
(a) Order 1, $\mathsf{K} = 10^{-3}$



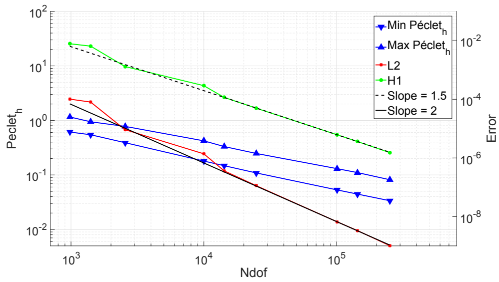
(b) Order 1, $\mathsf{K} = 10^{-9}$



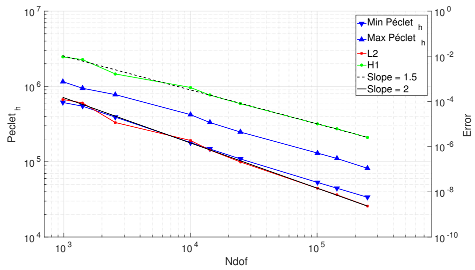
(c) Order 2, $\mathsf{K} = 10^{-3}$



(d) Order 2, $\mathsf{K} = 10^{-9}$

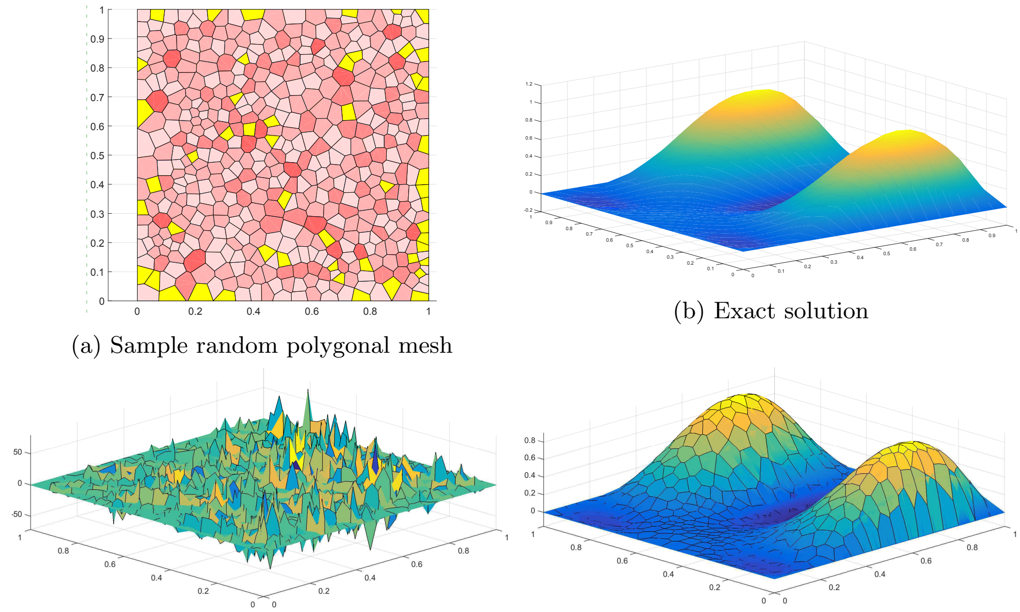


(e) Order 3, $\mathsf{K} = 10^{-3}$



(f) Order 3, $\mathsf{K} = 10^{-9}$

Figure 1: Test 1. Convergence curves



(a) Sample random polygonal mesh

(b) Exact solution

(c) Unstabilized solution

(d) Stabilized solution

Figure 2: Test 2. Sample mesh, exact, unstabilized and stabilized solutions

6. Conclusions

In this paper we have considered the advection-diffusion problem with a VEM based approach. The stabilization considered is a natural extension to the VEM of the classical SUPG stabilization for the standard FEM. It is known from the VEM literature that VEM discretizations require the introduction of a stabilization term to ensure coercivity of the discrete operators. A VEM stabilization of the SUPG stabilization is therefore needed. Under sufficient regularity assumptions of the data and of the exact solution, we have shown that both the advective-SUPG stabilization and the corresponding VEM stabilization for coercivity (stabilization of a stabilization) do not pollute the rates of convergence of the VEM discretization.

Numerical results confirm the proven theoretical behaviour. Moreover, stable good discrete solutions are obtained also for very large Péclet numbers in the order of 10^9 and mesh Péclet numbers in the order of 10^7. The numerical results also show a reliable stabilizing effect for the proposed formulation of the SUPG stabilization without the introduction of an excessive diffusive effect.

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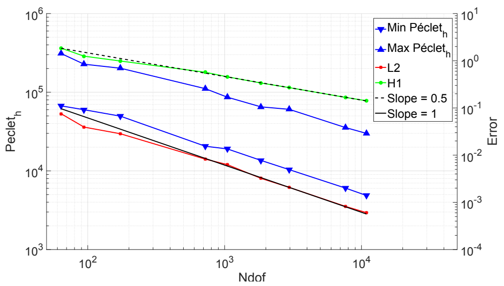
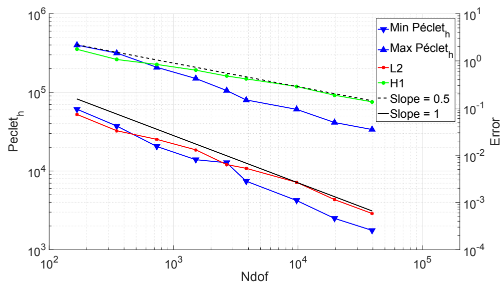
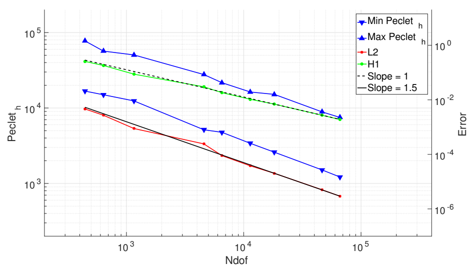


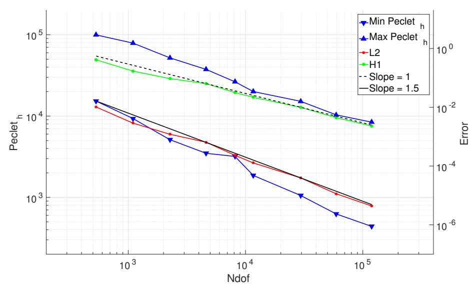
(a) Order 1, triangular mesh



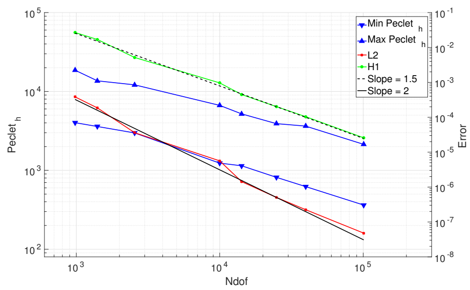
(b) Order 1, polygonal mesh



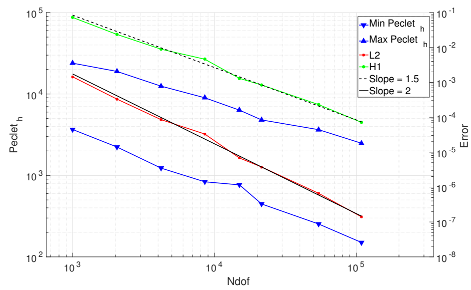
(c) Order 2, triangular mesh



(d) Order 2, polygonal mesh



(e) Order 3, triangular mesh



(f) Order 3, polygonal mesh

Figure 3: Test 2. Convergence curves

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