

A note on integer polynomials with small integrals. II

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# A NOTE ON INTEGER POLYNOMIALS WITH SMALL INTEGRALS II

DANILO BAZZANELLA

ABSTRACT. The smart method of Gelfond–Shnirelman–Nair allows one to obtain in elementary way a lower bound for the prime counting function  $\pi(x)$  in terms of integrals of suitable integer polynomials. In this paper we carry on the study of the properties of the sets of integer polynomials relevant for the method.

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## 1. INTRODUCTION

In 1851, Chebyshev [7] made the first step towards the Prime Number Theorem by proving that, given  $\varepsilon > 0$ ,

$$(c_1 - \varepsilon) \frac{N}{\log N} \leq \pi(N) \leq (c_2 + \varepsilon) \frac{N}{\log N}$$

where  $c_1 = \log(2^{1/2}3^{1/3}5^{1/5}/30^{1/30})$ ,  $c_2 = 6c_1/5$  and  $N$  is sufficiently large. This result was proved using elementary approaches, i.e. without use of complex analysis and the Riemann zeta function. A survey of elementary methods in the study of the distribution of prime numbers may be found in Diamond [8].

In 1936 Gelfond and Shnirelman, see Gelfond's editorial remarks in the 1944 edition of Chebyshev's Collected Works [7, pag. 287-288], proposed a new elementary and clever method for deriving a lower bound for the prime counting functions  $\pi(x)$  and  $\psi(x)$ . In 1982 the Gelfond–Shnirelman method was rediscovered and developed by Nair, see [10] and [11]. The method of Gelfond–Shnirelman–Nair allows one to obtain in elementary way a lower bound for  $\pi(x)$  in terms of integrals of suitable integer polynomials and it is based on the fact that the least common multiple of the integers not greater than  $N$ , say  $d_N$ , satisfied

$$d_N \leq \prod_{p \leq N} p^{\log N / \log p},$$

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where  $p$  belongs to the set of prime numbers, which implies

$$(1) \quad \pi(N) \geq \frac{\log d_N}{\log N}.$$

Considering a polynomial of degree  $N - 1$  with integral coefficients

$$P(x) = \sum_{n=0}^{N-1} a_n x^n$$

and letting

$$I(P) = \int_0^1 P(x) dx = \sum_{n=0}^{N-1} \frac{a_n}{n+1},$$

we note that  $I(P)d_N$  is an integer, and hence if  $I(P) \neq 0$  we have

$$d_N |I(P)| \geq 1$$

and then

$$d_N \geq \frac{1}{|I(P)|}.$$

From the above and (1) we get

$$(2) \quad \pi(N) \geq \frac{\log(1/|I(P)|)}{\log N}.$$

By the definition of  $I(P)$ , it follows that the small positive value of  $|I(P)|$  is  $1/d_N$  and it is reached if

$$\sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n = \pm 1.$$

Since the integer coefficients  $d_N, d_N/2, \dots, d_N/N$  are relatively prime, we have that for all  $N$  there exists a polynomial of degree less than  $N$  such that  $I(P) = 1/d_N$ . This leads to define the following sets of polynomials.

**Definition.** Let  $N \geq 2$ . We define

$$\begin{aligned} Z_N &= \{P(x) \in \mathbb{Z}[x], \deg(P) < N\}, \\ R_N &= \{P(x) \in \mathbb{Z}[x], \deg(P) < N, I(P) = 0\} \end{aligned}$$

and

$$S_N = \{P(x) \in \mathbb{Z}[x], \deg(P) < N, I(P) = 1/d_N\},$$

where  $d_N$  denotes the least common multiple of the integers  $1, 2, \dots, N$ .

It is simple to verify that, for every  $N$ ,  $Z_N$  is a free  $\mathbb{Z}$ -module and  $R_N$  is a submodule of  $Z_N$  and then it is also free.  $S_N$  is the affine space of the integer polynomials with positive and minimal integral on  $[0, 1]$ .

In the precedent paper [3] we proved some results about the roots of polynomials of the sets  $S_N$ . In the present paper we carry on the study of the properties of the sets of integer polynomials relevant for the method.

2. PROPERTIES OF THE SETS  $\overline{R}_N$ 

We start giving a theorem about the structure of the modules  $R_N$ .

**Theorem 1.** *A basis  $B_N$  of the module  $R_N$  can be constructed by adding to a basis  $B_{N-1}$  of the module  $R_{N-1}$  a suitable polynomial  $q(x) \in R_N$ . More precisely*

- (1) *if  $N$  is a prime:  $q(x) = 1 - Nx^{N-1}$ ;*
- (2) *if  $N$  is a power of a prime:  $q(x) = x^{n-1} - px^{N-1}$ , where  $N = p^k$  and  $n = p^{k-1}$ ;*
- (3) *otherwise:  $q(x) = a_1x^{n_1-1} + a_2x^{n_2-1} - x^{N-1}$ , where  $p_1$  and  $p_2$  are primes dividing  $N$ ,  $a_1$  and  $a_2$  are such that  $a_1p_1 + a_2p_2 = 1$ ,  $n_1 = N/p_1$  and  $n_2 = N/p_2$ .*

*Proof.* Let  $N$  prime and  $p(x) \in R_N$ . Then we can write

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}$$

with

$$a_0d_N + a_1\frac{d_N}{2} + a_2\frac{d_N}{3} + \cdots + a_{N-2}\frac{d_N}{N-1} + a_{N-1}\frac{d_N}{N} = 1.$$

Since  $N$  is a prime number we have that  $d_N = Nd_{N-1}$  and then

$$N/d_N, N/\frac{d_N}{2}, N/\frac{d_N}{3}, \dots, N/\frac{d_N}{N-1}$$

and  $N$  does not divide  $d_N/N$ . From this it follows that  $a_{N-1}/N$  is an integer.

Now we define

$$r(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{N-2}x^{N-2} + \frac{a_{N-1}}{N},$$

which implies

$$p(x) = r(x) + a_{N-1}x^{N-1} - \frac{a_{N-1}}{N} = r(x) - \frac{a_{N-1}}{N} (1 - Nx^{N-1}).$$

Then (1) is proved, since  $r(x) \in R_{N-1}$ .

To prove (2) we let  $N = p^k$ . In this case  $d_N = p d_{N-1}$  and more precisely

$$d_N = \prod_{q \leq N} q^{[\ln q / \ln N]} = p^k \prod_{q \leq N, q \neq p} q^{[\ln q / \ln N]} = Nm,$$

where  $(m, p) = 1$  and  $q$  runs over primes. From this follows that

$$p/d_N, p/\frac{d_N}{2}, p/\frac{d_N}{3}, \dots, p/\frac{d_N}{N-1}$$

and  $p$  does not divide  $d_N/N$ , hence  $a_{N-1}/p$  is an integer.

Now we define  $n = p^{k-1}$  and

$$r(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{N-2}x^{N-2} + \frac{a_{N-1}}{p}x^{n-1},$$

which implies

$$p(x) = r(x) + a_{N-1}x^{N-1} - \frac{a_{N-1}}{p}x^{n-1} = r(x) - \frac{a_{N-1}}{p} (x^{n-1} - px^{N-1}).$$

Then also (2) is proved, since  $r(x) \in R_{N-1}$ .

To prove (3) we observe that if  $N$  is neither prime nor power of a prime then there exist two primes  $p_1 \neq p_2$  both dividing  $N$ . Let  $a_1$  and  $a_2$  integers such that  $a_1 p_1 + a_2 p_2 = 1$ , we define  $n_1 = N/p_1$ ,  $n_2 = N/p_2$  and

$$r(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-2} x^{N-2} + a_1 a_{N-1} x^{n_1-1} + a_2 a_{N-1} x^{n_2-1}.$$

We conclude that

$$p(x) = r(x) - a_{N-1} (a_1 x^{n_1-1} + a_2 x^{n_2-1} - x^{N-1}).$$

and then the proof of the theorem is complete, since  $r(x) \in R_{N-1}$ .  $\square$

Using Theorem 1 we can fully describe the sets  $R_N$ . By the definition we have

$$R_2 = \{p(x) \in \mathbb{Z}[x], p(x) = a_0 + a_1 x, 2a_0 + a_1 = 0\} = \{p(x) \in \mathbb{Z}[x], p(x) = a_0(1 - 2x), a_0 \in \mathbb{Z}\}.$$

Then a basis  $B_2$  of the set  $R_2$  is

$$B_2 = \{1 - 2x\}.$$

Using several times Theorem 1 we can get a basis  $B_N$  of the set  $R_N$  for many values of  $N$ :

$$\begin{aligned} B_3 &= \{1 - 2x, 1 - 3x^2\}, \\ B_4 &= \{1 - 2x, 1 - 3x^2, -x + 2x^3\}, \\ B_5 &= \{1 - 2x, 1 - 3x^2, -x + 2x^3, 1 - 5x^4\}, \\ B_6 &= \{1 - 2x, 1 - 3x^2, -x + 2x^3, 1 - 5x^4, x(1 - x - x^4)\}, \\ B_7 &= \{1 - 2x, 1 - 3x^2, -x + 2x^3, 1 - 5x^4, x(1 - x - x^4), 1 - 7x^6\}, \\ B_8 &= \{1 - 2x, 1 - 3x^2, -x + 2x^3, 1 - 5x^4, x(1 - x - x^4), 1 - 7x^6, x^3 - 2x^7\}, \\ B_9 &= \{1 - 2x, 1 - 3x^2, -x + 2x^3, 1 - 5x^4, x(1 - x - x^4), 1 - 7x^6, x^3 - 2x^7, x^2 - 3x^8\}, \dots \end{aligned}$$

### 3. PROPERTIES OF THE SETS $S_N$

To describe the sets  $S_N$  is much more complicated. Since  $S_N$  are affine spaces, we can write

$$S_N = \{\bar{p}(x) + r(x) : r(x) \in R_N\},$$

where  $\bar{p}(x)$  is a fixed polynomial of  $S_N$ . For small values of  $N$  it is simple to find such a suitable polynomial

$N$	$\bar{p}(x)$	$N$	$\bar{p}(x)$
3	$x(1-x)$	14	$x^7(1-x)^4(2x-1)(-3+4x)$
4	$x^2(1-x)$	15	$x^7(1-x)^6(2x-1)$
5	$x^2(1-x)(2x-1)$	16	$x^8(1-x)^6(4-7x)$
6	$x^3(1-x)^2$	17	$x^8(1-x)^6(2x-1)(4-5x)$
7	$x^3(1-x)^2(2x-1)$	18	$x^9(1-x)^6(2x-1)(3-4x)$
8	$x^4(1-x)^2(2-3x)$	19	$x^9(1-x)^6(2x-1)^2(53-77x)$
9	$x^4(1-x)^3(2x-1)$	20	$x^{10}(1-x)^6(2x-1)^2(42-59x)$
10	$x^4(1-x)^3(2x-1)$	21	$x^{10}(1-x)^7(2x-1)^2(-2+3x)$
11	$x^5(1-x)^3(2x-1)(-4+5x)$	22	$x^{12}(1-x)^6(2x-1)^2(17-23x)$
12	$x^6(1-x)^3(2x-1)(-3+4x)$	23	$x^{12}(1-x)^7(2x-1)^2(-62+87x)$
13	$x^6(1-x)^4(2x-1)(-4+5x)$	24	$x^{12}(1-x)^7(2x-1)^3(-3+4x)$

Unfortunately it is very difficult to find out such a polynomial for a generic value of  $N$ . However we may provide some theorems about their factorization.

**Theorem 2.** *For every  $N \geq 3$  there exists a polynomial  $p(x) \in S_N$  such that  $p(0) = p(1) = 0$ , namely  $p(x) = x(1-x)q(x)$  with  $q(x) \in \mathbb{Z}[x]$ .*

*Proof.* The list of polynomials given before shows that the theorem is true for  $3 \leq N \leq 7$ . Then we let  $N \geq 8$  and  $p(x) \in S_N$ , that is

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_{N-1}x^{N-1}$$

and

$$(3) \quad a_0d_N + a_1\frac{d_N}{2} + a_2\frac{d_N}{3} + a_3\frac{d_N}{4} + \dots + a_{N-2}\frac{d_N}{N-1} + a_{N-1}\frac{d_N}{N} = 1.$$

The Diophantine equation

$$a_3\frac{d_N}{4} + a_4\frac{d_N}{5} \dots + a_{N-2}\frac{d_N}{N-1} + a_{N-1}\frac{d_N}{N} = 1$$

has an integer solution  $(a_3, a_4, a_{N-1})$ , since for  $N \geq 8$  we have

$$\left(\frac{d_N}{4}, \frac{d_N}{5}, \dots, \frac{d_N}{N-1}, \frac{d_N}{N}\right) = 1.$$

Setting  $a_0 = 0$ ,  $a_1 = 2(a_3 + a_4 + \dots + a_{N-1})$  and  $a_2 = -3(a_3 + a_4 + \dots + a_{N-1})$  we have that  $(a_0, a_1, a_2, \dots, a_{N-1})$  is a solution of (3) and verify  $p(0) = a_0 = 0$  and

$$p(1) = a_0 + a_1 + a_2 + \dots + a_{N-1} = 0,$$

which concludes the proof of the theorem. □

At the cost of some complications we can prove a similar result also including the factor  $(2x-1)$ .

**Theorem 3.** *Let  $N \geq 4$ .*

- (1) *If  $N$  is not a power of 2, then there exists a polynomial  $p(x) \in S_N$  such that  $p(0) = p(1) = p(1/2) = 0$ , namely such that  $p(x) = x(1-x)(2x-1)q(x)$  with  $q(x) \in \mathbb{Z}[x]$ ;*
- (2) *If  $N$  is a power of 2, then there not exists a polynomial  $p(x) \in S_N$  such that  $(2x-1)/p(x)$ .*

*Proof.* Let  $p(x) = (2x-1)(b_0 + b_1x + b_2x^2 + \dots + a_{N-2}x^{N-2})$ . The condition

$$\int_0^1 p(x) dx = \frac{1}{d_N}$$

is equivalent to

$$(4) \quad \sum_{k=1}^{N-2} \frac{d_N k}{(k+1)(k+2)} b_k = 1.$$

If  $N$  is a power of 2, then all the coefficients

$$\frac{d_N k}{(k+1)(k+2)}$$

are even and thus the equation (4) has no solutions, therefore there not exists a polynomial  $p(x) \in S_N$  such that  $(2x-1)/p(x)$ .

The list of polynomials given before shows that (1) is true for  $4 \leq N \leq 24$  and then we need only to consider the case  $N \geq 25$ . If  $N$  is not a power of 2, then we are able to prove that the coefficients

$$\frac{d_N k}{(k+1)(k+2)}$$

are relatively prime. In order to prove the coprimality, we suppose on the contrary that there exists a prime  $p$  dividing

$$\frac{d_N k}{(k+1)(k+2)}$$

for every  $k = 1, 2, \dots, N-2$ . Let  $H = p^j$ , with  $j = \max\{i : p^i \leq N\}$  and observe that  $p$  does not divide  $d_N/H$ . Then at least one of the two coefficients

$$\frac{d_N (H-1)}{H(H+1)} \quad \text{and} \quad \frac{d_N (H-2)}{(H-1)H},$$

is not divisible by  $p$ , a contradiction. By the coprimality of the coefficients of the Diophantine equation (4) follows that there exists  $p(x) \in S_N$  such that  $(2x-1)/p(x)$ .

To have also the factors  $x$  and  $(1-x)$  it is sufficient to note that the integer  $H$  defined above is greater than 7, since  $N \geq 25$ , and then there exists a solution  $(b_4, b_5, \dots, b_{N-2})$  of the Diophantine equation

$$(5) \quad \sum_{k=4}^{N-2} \frac{d_N k}{(k+1)(k+2)} b_k = 1.$$

We conclude the proof as above by setting  $b_0 = b_1 = 0$ ,  $b_2 = 9(b_4 + b_5 + \cdots + b_{N-2})$  and  $b_3 = -10(b_4 + b_5 + \cdots + b_{N-2})$ .  $\square$

Applying similar ideas we can prove the following theorem.

**Theorem 4.** *Let  $N \geq 4$  and let  $0 < m < n$  natural numbers such that  $(n, m) = 1$ .*

- (1) *If  $N$  is not a power of a prime, then there exists  $p(x) \in S_N$  such that  $(nx - m)/p(x)$ ;*
- (2) *If  $N$  is a power of a prime  $p$ , then there exists  $p(x) \in S_N$  such that  $(nx - m)/p(x)$  if and only if  $(p, n) = 1$ .*

*Proof.* Let  $N \geq 4$  and  $p(x) = (nx - m)(b_0 + b_1x + b_2x^2 + \cdots + a_{N-2}x^{N-2})$ . The condition

$$\int_0^1 p(x) dx = \frac{1}{d_N}$$

is equivalent to

$$(6) \quad \sum_{k=0}^{N-2} d_N \frac{(k+1)(n-m) - m}{(k+1)(k+2)} b_k = 1.$$

If  $N$  is not a power of a prime, then we are able to prove that the coefficients of the Diophantine equation (6) are relatively prime. In order to prove the coprimality, we suppose that there exists a prime  $q$  dividing

$$d_N \frac{(k+1)(n-m) - m}{(k+1)(k+2)}$$

for every  $k = 1, 2, \dots, N-2$ , with the goal of obtaining a contradiction. Let  $H = q^j$ , with  $j = \max\{i : q^i \leq N\}$  and consider the coefficient

$$(7) \quad d_N \frac{(H-1)(n-m) - m}{H(H-1)} = \frac{d_N}{H} n - \frac{d_N}{H-1} m,$$

which arise from  $k = H-2$ . By the definition of  $H$ ,  $q$  does not divide  $d_N/H$  and divides  $d_N/(H-1)$ . If  $q$  does not divide  $n$  then  $q$  does not divide (7) and we reach the desired contradiction. If instead  $q$  divides  $n$ , and then does not divide  $m$ , therefore  $q$  does not divide the coefficient

$$d_N \frac{H(n-m) - m}{H(H+1)} = d_N \frac{n}{H+1} - \frac{d_N}{H} m,$$

which arise from  $k = H-1$ , and this leads again to contradiction.

If  $N$  is a power of a prime, namely  $N = p^k$  with  $k \geq 1$ , and  $(n, p) > 1$  this implies that  $p$  divides all the coefficients

$$d_N \frac{(k+1)(n-m) - m}{(k+1)(k+2)}$$

and then the equation (6) has no solutions.

Finally if  $N = p^k$  and  $(n, p) = 1$  then we suppose that a prime  $q$  divides all the coefficients of the equation (6) and find as above that one of such coefficient is not divisible by  $q$ , a contradiction.  $\square$



4. INTEGER POLYNOMIALS IN  $S_N$  NON-NEGATIVE IN  $[0, 1]$ 

In the first paper of the series we proposed the following conjecture:

**Conjecture.** *For every  $N$ , or at least for infinitely many values of  $N$ , there exists an integer polynomial  $p(x) \in S_N$  such that  $p(x) \geq 0$  in the interval  $[0, 1]$ .*

A straightforward way to obtain a negative conclusion about the existence of integer polynomials of  $S_N$  non-negative in  $[0, 1]$  is to consider  $0 \leq x_1 < x_2 < x_3 \cdots < x_n \leq 1$  and a generic polynomial  $p(x) \in S_N$  in the form

$$p(x) = \sum_{k=0}^{N-1} a_k x^k.$$

Since  $p(x) \in S_N$ , we have

$$\int_0^1 p(x) dx = \frac{1}{d_N},$$

that is

$$\sum_{k=0}^{N-1} \frac{d_N}{k+1} a_k = 1$$

and consider the following linear Diophantine system composed of an equality and  $n$  inequalities

$$(8) \quad \begin{cases} \sum_{k=0}^{N-1} \frac{d_N}{k+1} a_k = 1 \\ p(x_1) \geq 0 \\ p(x_2) \geq 0 \\ \dots \\ p(x_n) \geq 0. \end{cases}$$

If we are able to prove that, for a fixed value of  $N$ , the above linear system have no integer solutions  $a_1, a_2 \dots a_{N-1}$ , we obtain that there not exists an integer polynomial  $p(x) \in S_N$  such that  $p(x) \geq 0$  in the interval  $[0, 1]$ .

By the branch and cut algorithm, used in many mathematical software systems, we can verify in deterministic way that for  $N = 5$  and  $x_k = k/4$ , with  $k = 0, 1, \dots, 4$ , the system (8) has no integer solutions, although it has infinitely many real solutions, which implies that there are no integer polynomials  $p(x) \in S_5$  such that  $p(x) \geq 0$  in the interval  $[0, 1]$ . Hence we disproved the strong form of the conjecture.

For  $N = 6$  there exists the polynomial  $p(x) = x^3(1-x)^2 \in S_6$ , non-negative for all values of  $x \in [0, 1]$ . Then the case  $N = 5$  might appears as an exceptional case. Instead we can verify that for many values of  $N$  there not exists a polynomial in  $S_N$  non-negative in  $[0, 1]$ . More precisely we can verify that there not exists an integer polynomial  $p(x) \in S_N$  such that  $p(x) \geq 0$  in the interval  $[0, 1]$  for all  $7 \leq N \leq 20$ , with the only exclusion of the case  $N = 10$ , for which we have the polynomial  $p(x) = x^3(1-x)^4(2x-1)^2$ .

To find out any others non-negative polynomials it might be difficult because the calculations involved, but one can prove that such polynomials cannot exist for large values of  $N$ . A Nikolskii-type inequality gives that there is a constant  $C > 0$  such that

$$\max_{x \in [0,1]} |p(x)| \leq CN^2 \int_0^1 |p(x)| dx$$

for any polynomial  $p(x)$  of degree  $N - 1$ , see e.g. [16, Corollary 13.3.3]. If we suppose that there exists a sequence of non-negative polynomials  $p_N(x) \in S_N$ , we have

$$\frac{1}{d_N} = \int_0^1 p_N(x) dx = \int_0^1 |p_N(x)| dx \geq \frac{1}{CN^2} \max_{x \in [0,1]} |p_N(x)|$$

and hence

$$\lim_{N \rightarrow +\infty} \left( \max_{x \in [0,1]} |p_N(x)| \right)^{1/N} \leq \lim_{N \rightarrow +\infty} d_N^{-1/N}.$$

It follows from the Prime Number Theorem that

$$\lim_{N \rightarrow +\infty} d_N^{-1/N} = e,$$

see [12, page 180]. On the other hand, Gorshkov's bound [12, page 187] gives that

$$\lim_{N \rightarrow +\infty} \left( \max_{x \in [0,1]} |p_N(x)| \right)^{1/N} \geq 0.42,$$

which is a contradiction. This implies that there are only finitely many values of  $N$  for which there exists a non-negative polynomial in  $S_N$  and then we have also disproved the weak form of the Conjecture.

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