Minimal helix submanifolds and minimal Riemannian foliations

MINIMAL HELIX SUBMANIFOLDS AND MINIMAL RIEMANNIAN FOLIATIONS

ANTONIO J. DI SCALA AND GABRIEL RUIZ-HERNÁNDEZ

Abstract. We investigate minimal helix submanifolds of any dimension and codimension immersed in Euclidean space. Our main result proves that a ruled minimal helix submanifold is a cylinder. As an application we classify complex helix submanifolds of $\mathbb{C}^n$: They are extrinsic products with a complex line as a factor. The key tool is Corollary 1.3 which allows us to classify Riemannian foliations of open subsets of the Euclidean space with minimal leaves. Finally, we consider the case of a helix hypersurface with constant mean curvature and prove that it is either a cylinder or an open part of a hyperplane.

1. Introduction

A submanifold $M \subset \mathbb{R}^n$ is called a *helix* with respect to $\overrightarrow{\mathrm{d}} \in \mathbb{R}^n$ if the angle

$$\theta(p) := \angle(T_pM, \overrightarrow{\mathrm{d}})$$

between the tangent space T_pM and a fixed direction $\overrightarrow{\mathrm{d}} \in \mathbb{R}^n$ is constant, i.e. $\theta(p)$ does not depend upon $p \in M$. Observe that the angle $\theta(p)$ is related to the splitting $\overrightarrow{\mathrm{d}} = \overrightarrow{\mathrm{d}}^{\top} + \overrightarrow{\mathrm{d}}^{\perp}$ according to the tangent and normal components of $\overrightarrow{\mathrm{d}}$ at $p \in M$. Indeed, the norm $\|\overrightarrow{\mathrm{d}}^{\top}\|$ at $p \in M$ is given by $\|\overrightarrow{\mathrm{d}}\|\cos(\theta(p))$. Then $M \subset \mathbb{R}^n$ is a helix with respect to $\overrightarrow{\mathrm{d}}$ if and only if the norm $\|\overrightarrow{\mathrm{d}}^{\top}\|$ is constant along M. Observe that $\overrightarrow{\mathrm{d}}^{\top}$ is the gradient of the height function $h_{\overrightarrow{\mathrm{d}}}(x) := \langle x, \overrightarrow{\mathrm{d}}\rangle$ by [13, Proposition 4.1.1, page 65]. So M is a helix with respect to $\overrightarrow{\mathrm{d}}$ if and only if the height function $h_{\overrightarrow{\mathrm{d}}}$ is a so called eikonal function i.e. the

Date: October 6, 2015.

1991 *Mathematics Subject Classification.* Primary 53C40, 53C42.

Key words and phrases. Helix submanifolds, constant angle submanifolds, minimal submanifolds.

This research was partially supported by Ministero degli Affari Esteri from Italy and CONACYT from Mexico. The third named author was partially supported by DGAPA-UNAM-PAPIIT, under Project IN100414.

norm of its gradient $\nabla_M h_{\vec{\mathrm{d}}}$ is constant on M.

In this paper we are interested in the local geometry of the helix M i.e. all the claims are of local nature unless otherwise specified. Important examples of helix submanifolds are totally geodesic submanifolds of shadow boundaries. We refer to [9] and [14] for details. Helix submanifolds are also called constant angle submanifolds and had been studied in other ambient spaces, see for example [6] and [10].

The integral curves of $T := \frac{\vec{\mathrm{d}}^{\top}}{\|\vec{\mathrm{d}}^{\top}\|}$ are geodesics of a helix M. When such integral curves are also geodesics in the Euclidean space the helix is called ruled. See [5, page 194, Definition 2.3] for details.

Here is the main result of this paper.

Theorem 1.1. *If $M \subset \mathbb{R}^n$ is a full minimal ruled helix with respect to $\vec{\mathrm{d}} \in \mathbb{R}^n$ then $\vec{\mathrm{d}}$ is tangent to M. That is to say, the helix angle θ is zero and M is a cylinder over a minimal submanifold contained in a hyperplane H orthogonal to $\vec{\mathrm{d}}$.*

We do not know if the hypothesis of being **ruled** can be omitted in the above statement.

Then we obtain the classification of complex helix submanifolds of $\mathbb{C}^n$.

Theorem 1.2. *Let $M^m \subset \mathbb{C}^n$ be a full complex submanifold of complex dimension m. Assume that M is a helix of angle θ with respect to a direction $\vec{\mathrm{d}} \in \mathbb{C}^n$. Then $\theta = 0$ and so M is locally an extrinsic product*

$$M = \mathbb{C} \times N \subset \mathbb{C} \times \mathbb{C}^{n-1},$$

where $N \subset \mathbb{C}^{n-1}$ is a complex submanifold.

It is important to notice that the above theorem is not a direct consequence of Theorem 1.1 since we do not assume the complex helix submanifold to be ruled.

The main tool to prove the above theorems is Lemma 2.5 which we think is interesting in itself. Indeed, in Submanifold Geometry [1] it is well-known that if the parallel manifolds $M_{t\xi} := M + t\xi \subset \mathbb{R}^n$ in the direction of a normal parallel vector field ξ are minimal submanifolds for small values of t then ξ is constant in $\mathbb{R}^n$. We show that this is still true just assuming that ξ has constant length (i.e. the hypothesis on ξ of being normal parallel is not necessary). Namely, we have the following corollary of Lemma 2.5.

Corollary 1.3. *Let $M \subset \mathbb{R}^n$ be a submanifold and let $\xi \in \Gamma(\nu(M))$ be a normal vector field of constant length i.e. $\|\xi\| = constant$. If the submanifolds $M_{t\xi} := M + t\xi \subset \mathbb{R}^n$ are minimal submanifolds for small values of t then ξ is constant in $\mathbb{R}^n$, i.e. ξ is parallel with respect the normal connection and $A_\xi \equiv 0$, where A_ξ is the shape operator of M in direction ξ.*

The above corollary have the following interesting application to Riemannian foliations of the Euclidean space. In [11, page 450] the author wrote

... it is easy to construct non-trivial examples of regular complex Riemannian foliations in $\mathbb{C}^n$ of all codimensions. (sic)

Indeed, the totally geodesic foliation given by the family of parallel affine subspaces $\{\mathbf{V} + p\}$, $p \in \mathbf{V}^\perp$ to a fixed vector subspace $\mathbf{V} \subset \mathbb{C}^n$ give such examples. The following theorem shows that they are (even locally) the unique examples.

Theorem 1.4. *Let $\mathcal{F}$ be a Riemannian foliation of an open subset U of $\mathbb{R}^n$ with minimal leaves i.e. any leave of $\mathcal{F}$ is a minimal submanifold of $\mathbb{R}^n$. Then $\mathcal{F}$ is totally geodesic. More precisely, for each $p \in U$ there is a neighborhood G of p such that the leaves of the restriction $\mathcal{F}|_G$ are open subsets of a foliation of $\mathbb{R}^n$ by parallel affine subspaces. In particular, any complex Riemannian foliation of an open subset of $\mathbb{C}^n$ is totally geodesic.*

In section 5 we give general results and discuss some interesting examples about (non necessarily ruled) minimal helices and its intrinsic geometry.

Finally we give the following generalization of a result in [7].

Theorem 1.5. *A helix hypersurface $M^n \subset \mathbb{R}^{n+1}$ with constant mean curvature is either a cylinder $M = \mathbb{R} \times N \subset \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$ over a hypersurface $N \subset \mathbb{R}^n$ with constant mean curvature or an open subset of a hyperplane i.e. M is a totally geodesic hypersurface of $\mathbb{R}^{n+1}$.*

The above theorem is a special case of [8, Theorem 15] where a similar result valid for hypersurfaces of products $\mathbb{R} \times N$ is obtained by using Bochner's formula. Instead our proof is based in Ruh-Vilms's theorem [15] and a maximum principle for harmonic maps due to Sampson [16, Theorem 2]. We also explain why our proof can not be extended to the case of higher codimensional minimal helix submanifolds.

2. Minimal ruled helices

Let us briefly recall the two methods to study helix submanifolds that were developed in [4], [5]. Namely, the **projection method** and the **slice method**.

The **projection method** considers the helix M as the graph of a function f defined on the projection B of M to an hyperplane H orthogonal to $\overrightarrow{\mathrm{d}}$. More precisely, let $M \subset \mathbb{R}^n$ be a helix submanifold of angle $\theta \notin \{0, \frac{\pi}{2}\}$ with respect to the unit vector $\overrightarrow{\mathrm{d}}$. Let $\pi : \mathbb{R}^n \to H$ be the orthogonal projection to an hyperplane H orthogonal to $\overrightarrow{\mathrm{d}}$. The restriction of π to M is an immersion and $B = \pi(M)$ is called the base of the helix M. Then M looks locally as the graph of a function $f : U \subset B \to \mathbb{R}$ which is an eikonal function with respect to the intrinsic geometry of B. That is to say, M is locally the image of the map $\phi : B \to \mathbb{R}^n = H \times \mathbb{R}$ defined as

$$\phi(p) := (i(p), f(p)) \tag{1}$$

where i is the canonical inclusion of $\pi(M)$.

Conversely we can start from a submanifold $B \subset H$ and an eikonal function $f \in C^\infty(B)$ and construct $M \subset \mathbb{R}^n$ as the graph of f (see Theorem 2.1).

The **slice method** can be used when the helix is ruled, i.e. the integral curves of $T := \frac{\overrightarrow{\mathrm{d}}^\top}{\|\overrightarrow{\mathrm{d}}^\top\|}$ are geodesics in the Euclidean space [5, page 194, Definition 2.3]. Let us briefly recall the local structure of a ruled helix, for more details see [5, Theorem 4.6, page 202]. Let $L = M \bigcap H$ be a slice of M where H is a hyperplane perpendicular to $\overrightarrow{\mathrm{d}}$. Observe that T is a normal vector field of $L = M \bigcap H$.

If the helix is ruled then M is the union of the parallel manifolds L_{sT} to L in the T-direction. Namely, M is the image of the map

$$e : L \times (-\epsilon, \epsilon) \to \mathbb{R}^n$$

defined as

$$e(p, s) := p + sT(p)\,.$$

The projection and slice methods are related via the height function $h_{\overrightarrow{\mathrm{d}}}$ in the following way. Let M be the helix submanifold, $B \subset H$ be its base and $f : B \to \mathbb{R}$ be the eikonal function as explained above. Then $h_{\overrightarrow{\mathrm{d}}} = f \circ \pi$ hence $\nabla_M h_{\overrightarrow{\mathrm{d}}}$ is parallel to the vector field T. Therefore the slices of M with hyperplanes orthogonal to $\overrightarrow{\mathrm{d}}$ are the parallel submanifolds L_{sT}.

The following result proved in [4, Theorem 3.4, page 211] is going to play a key role along this paper.

Theorem 2.1. [4, Theorem 3.4, page 211] *In the above notation, the submanifold M is a helix if and only if f is an eikonal function of B, i.e. $\|\nabla_B f\|$ is constant on B. Here $\nabla_B f$ is the gradient of f with respect to the induced metric on B from $H \subset \mathbb{R}^n$.*

Let L be an immersed $l-$dimensional submanifold in $\mathbb{R}^n$, let η be a normal vector field to L of constant length. The shape operator A_η of L in direction η is given by

$$A_\eta(X) = -(D_X\eta)^\top \ ,$$

where D is the directional derivative of $\mathbb{R}^n$. Let L_η be the parallel submanifold given by the immersion $t_\eta(p) = p + \eta(p)$ where $p \in L$ (cf. [1] page 117). We also assume that 1 is not an eigenvalue of A_η.

Lemma 2.2. *Let E_i, $1 \le i \le l$ be an orthonormal local frame of TL such that $A_\eta(E_i) = \lambda_i E_i$, i.e. this frame diagonalize the shape operator A_η of L in direction η. Let ξ_j, $1 \le j \le n-l$ be a local orthonormal frame of νL the normal bundle of L. Then the corresponding tangent X_i and normal $\tilde{\xi_j}$ frames of L_η are given by*

$$\begin{cases} X_i = (1-\lambda_i)E_i + \nabla^\perp_{E_i}\eta \,, \\ \tilde{\xi_j} = \xi_j - \sum_{k=1}^{l} \frac{\langle \nabla^\perp_{E_k}\eta, \xi_j \rangle}{1-\lambda_k} E_k \,. \end{cases}$$

In particular, the metric $G = (G_{rs})$ of L_η with respect to the frame X_i's is given by

$$G_{rs} = (1-\lambda_r)(1-\lambda_s)\delta_{rs} + \langle \nabla^\perp_{E_r}\eta, \nabla^\perp_{E_s}\eta \rangle. \tag{2}$$

Proof. The vectors fields X_i's are tangent to L_η because

$$X_i = (t_\eta)_*(E_i) = D_{E_i}(p + \eta(p)) = E_i + D_{E_i}\eta = E_i + \nabla^\perp_{E_i}\eta - A_\eta(E_i).$$

Let us see that the vectors fields $\tilde{\xi_j}$ are orthogonal to the X_j's:

$$\langle \tilde{\xi_j}, X_i \rangle = \langle \nabla^\perp_{E_i}\eta, \xi_j \rangle - \sum_{k=1}^{l} \frac{1-\lambda_i}{1-\lambda_k} \delta_{ik} \langle \nabla^\perp_{E_k}\eta, \xi_j \rangle = 0.$$

□

Let $M \subset \mathbb{R}^n$ be helix with respect to the direction $\overrightarrow{\mathrm{d}} \in \mathbb{R}^n$. Let $\pi : \mathbb{R}^n \to H$ be the projection to a normal hyperplane H to $\overrightarrow{\mathrm{d}}$.

Proposition 2.3. *Let $M \subset \mathbb{R}^n$ be a full minimal ruled helix and let $B = \pi(M) \subset H$ be its base. Let $L = M \cap H \subset \pi(M) = B$ be a slice. Let $\eta := T$ be the restriction of T to the slice L. Then either M is a cylinder over a submanifold of H or it is the union of the η-parallel manifolds $L_{s\eta}$ to L which are minimal submanifolds of hyperplanes parallel to H.*

Proof. Let us assume that M is not a cylinder over a submanifold of H. That is to say the helix constant angle θ between its tangent spaces and $\overrightarrow{\mathrm{d}}$ is not zero. We already explained, at the beginning of section 2, that the η-parallel manifolds $L_{s\eta}$ to L are the slices of M. So by [5, Theorem 7.1, page 208] we get that the η-parallel manifolds $L_{s\eta}$ are minimal submanifolds. □

Lemma 2.4. *Under the above assumptions, the trace of the shape operator A^s_η of $L_{s\eta}$ in direction η is given by*

$$\mathrm{Tr}(A^s_\eta) = \mathrm{Tr}((\mathrm{D} - s\mathrm{D}^2 - s\mathrm{N})[\mathbf{1} - 2s\mathrm{D} + s^2(\mathrm{D}^2 + \mathrm{N})]^{-1}),$$

where D, N *are the matrices:* $\mathrm{D}_{ij} = \lambda_i \delta_{ij}$ *and* $\mathrm{N}_{ij} = \langle \nabla^\perp_{E_i}\eta, \nabla^\perp_{E_j}\eta\rangle$.

Proof. As explained at the beginning of section 2, $\eta = T$ is orthogonal to the slices $L_{s\eta}$. We will denote by A_η the shape operator of L in direction η. Let $E_1, \cdots, E_{\dim(L)}$ be the frame of L and let $X^s_1, \cdots, X^s_{\dim(L)}$ be the frame of $L_{s\eta}$ introduced in Lemma 2.2. The following computation follows the same ideas as in the classical "tube formula" (cf. [1, page 121]):

$$\begin{aligned}
\langle A^s_\eta(X^s_i), X^s_j\rangle &= -\langle D_{E_i}\eta, X^s_j\rangle = \langle \eta, D_{E_i}X^s_j\rangle \\
&= \langle \eta, D_{E_i}((1 - s\lambda_j)E_j + s\nabla^\perp_{E_j}\eta)\rangle \\
&= (1 - s\lambda_j)\langle \eta, \alpha(E_i, E_j)\rangle + s\langle \eta, \nabla^\perp_{E_i}\nabla^\perp_{E_j}\eta\rangle \\
&= (1 - s\lambda_j)\langle A_\eta(E_i), E_j\rangle - s\langle \nabla^\perp_{E_i}\eta, \nabla^\perp_{E_j}\eta\rangle \\
&= (1 - s\lambda_j)\lambda_i\delta_{ij} - s\langle \nabla^\perp_{E_i}\eta, \nabla^\perp_{E_j}\eta\rangle \\
&= \lambda_i\delta_{ij} - s\lambda_i\lambda_j\delta ij - s\langle \nabla^\perp_{E_i}\eta, \nabla^\perp_{E_j}\eta\rangle
\end{aligned}$$

Therefore, we have that

$$\langle A^s_\eta(X^s_i), X^s_j\rangle = \mathrm{D}_{ij} - s\mathrm{D}^2_{ij} - s\mathrm{N}_{ij}.$$

Now equation (2) in Lemma 2.2 give us the metric G_{ij} of $L_{s\eta}$ with respect to the frame $X^s_1, \cdots, X^s_{\dim(L)}$:

$$G_{ij} = \delta_{ij} - s\delta_{ij}(\lambda_i + \lambda_j) + s^2\lambda_i\lambda_j\delta_{ij} + s^2\langle \nabla^\perp_{E_i}\eta, \nabla^\perp_{E_j}\eta\rangle$$

So,

$$G = \mathbf{1} - 2s\mathrm{D} + s^2(\mathrm{D}^2 + \mathrm{N}).$$

Then, we have that

$$\mathrm{Tr}(A^s_\eta) = \mathrm{Tr}((\mathrm{D} - s\mathrm{D}^2 - s\mathrm{N})[\mathbf{1} - 2s\mathrm{D} + s^2(\mathrm{D}^2 + \mathrm{N})]^{-1}).$$

□

For the proof of Theorem 1.1 we will need the following lemma.

Lemma 2.5. *Let* N, D *be symmetric square matrices with* N *positive semi-definite. Set* $\mathrm{H} := \mathrm{D}^2 + \mathrm{N}$ *and let* $\epsilon > 0$ *be such that the matrix* $\mathbf{1} - 2s\mathrm{D} + s^2\mathrm{H}$ *is invertible for all* $s \in (0, \epsilon)$. *If*

$$\mathrm{Tr}\left((\mathrm{D} - s\mathrm{H})(\mathbf{1} - 2s\mathrm{D} + s^2\mathrm{H})^{-1}\right) = 0$$

for all $s \in (0, \epsilon)$ *then*

$$\mathrm{D} = \mathrm{N} = \mathrm{H} = \mathbf{0}.$$

Proof. The inverse G^{-1} of an invertible matrix G can be computed by means of its *adjoint* matrix $\mathrm{adj}(G)$. Namely,

$$G^{-1} = \frac{\mathrm{adj}(G)}{\det(G)}\,.$$

Then for $s \in (0, \epsilon)$ we have

$$\begin{aligned} \mathrm{Tr}\left((\mathrm{D} - s\mathrm{H})(\mathbf{1} - 2s\mathrm{D} + s^2\mathrm{H})^{-1}\right) &= \mathrm{Tr}\left((\mathrm{D} - s\mathrm{H})\tfrac{\mathrm{adj}(\mathbf{1} - 2s\mathrm{D} + s^2\mathrm{H})}{\det(\mathbf{1} - 2s\mathrm{D} + s^2\mathrm{H})}\right) \\ &= 0\,. \end{aligned}$$

Since the polynomial $P(s) := \det(\mathbf{1} - 2s\mathrm{D} + s^2\mathrm{H})$ has a finite number of zeros we get that

$$\mathrm{Tr}\left((\mathrm{D} - s\mathrm{H})(\mathbf{1} - 2s\mathrm{D} + s^2\mathrm{H})^{-1}\right) = 0$$

for all real numbers $s \in \mathbb{R}$ up to the finite number of zeroes of $P(s)$. Changing $s = \frac{1}{t}$ we get

$$\mathrm{Tr}\left((t\mathrm{D} - \mathrm{H})(t^2\mathbf{1} - 2t\mathrm{D} + \mathrm{H})^{-1}\right) = 0 \tag{3}$$

for all $t \in \mathbb{R}$ up to a finite number of exceptions.

Let $\overrightarrow{v} \in \ker(\mathrm{H})$ be a vector in the kernel of H then

$$\mathrm{H}\overrightarrow{v} = \mathrm{D}^2\overrightarrow{v} + \mathrm{N}\overrightarrow{v} = 0\,.$$

So

$$\mathrm{D}\overrightarrow{v}.\mathrm{D}\overrightarrow{v} = -\mathrm{N}\overrightarrow{v}.\overrightarrow{v}$$

hence $\mathrm{D}\overrightarrow{v} = \mathrm{N}\overrightarrow{v} = \mathrm{H}\overrightarrow{v} = 0$ since N is positive semi-definite. Then $\ker(\mathrm{H}) \subset \ker(\mathrm{D})$ and $\ker(\mathrm{H}) \subset \ker(\mathrm{N})$. Since D and N are symmetric matrices they preserve $\ker(\mathrm{H})^\perp$ and we get the following block decomposition with respect to the splitting $\ker(\mathrm{H}) \oplus \ker(\mathrm{H})^\perp$:

$$\mathrm{D} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{D}_1 \end{pmatrix}, \ \mathrm{N} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{N}_1 \end{pmatrix}, \text{ and } \mathrm{H} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{H}_1 \end{pmatrix}.$$

Now equation (3) reduce to

$$\mathrm{Tr}\left((t\mathrm{D}_1 - \mathrm{H}_1)(t^2\mathbf{1} - 2t\mathrm{D}_1 + \mathrm{H}_1)^{-1}\right) = 0\,.$$

Letting $t \to 0$ we get

$$\mathrm{Tr}\left((-\mathrm{H}_1)(\mathrm{H}_1)^{-1}\right) = \mathrm{Tr}\,(-\mathbf{1}) = 0$$

which is a contradiction unless $\mathrm{H}_1 = 0$. So $\mathrm{H}_1 = 0$ hence $\mathrm{H} = 0$ and also $\mathrm{D} = \mathrm{N} = 0$ since $\ker(\mathrm{H})^{\perp} = \{0\}$. □

2.1. **Proof of Theorem 1.1.** Let M be a ruled minimal helix submanifold of $\mathbb{R}^n$ with constant angle $\theta \neq 0$. We are going to show that M is not full, that is to say M is contained in a hyperplane.
By Proposition 2.3, the helix M is a union of parallel submanifolds $L_{s\eta}$, where L is a slice and $\eta = T$ is a normal vector field of L of constant length.
By Lemma 2.4 and since $L_{s\eta}$ is minimal for small values of s,

$$0 = \mathrm{Tr}(A^s_\eta) = \mathrm{Tr}((\mathrm{D} - s\mathrm{D}^2 - s\mathrm{N})[\mathbf{1} - 2s\mathrm{D} + s^2(\mathrm{D}^2 + \mathrm{N})]^{-1}),$$

where D, N are the matrices: $\mathrm{D}_{ij} = \lambda_i \delta_{ij}$ and $\mathrm{N}_{ij} = \langle \nabla^{\perp}_{E_i}\eta, \nabla^{\perp}_{E_j}\eta \rangle$.
Now, by Lemma 2.5, $\mathrm{D} = 0$ and $\mathrm{N} = 0$, that is to say the vector field η is parallel with respect to the normal connection and its shape operator $A_\eta = 0$. Hence η is constant in the ambient space along L. This implies that T is a constant vector along M in the ambient space $\mathbb{R}^n$ hence $\overrightarrow{\mathrm{d}}^{\top}$ is constant along M in the ambient space $\mathbb{R}^n$. Therefore, $\overrightarrow{\mathrm{d}}^{\perp}$ is a constant vector field along M in the ambient space $\mathbb{R}^n$. Since we assumed that $\theta \neq 0$ we get that M is contained in a hyperplane orthogonal to $\overrightarrow{\mathrm{d}}^{\perp} \neq 0$ i.e. M is not full. □

2.2. **Proof of Corollary 1.3.** The corollary follows by applying Lemma 2.2, Lemma 2.4 and Lemma 2.5 to $L = M$ and $\eta = \xi$.

3. Complex helix submanifolds: Proof of Theorem 1.2.

It is well-known that a complex submanifolds of $\mathbb{C}^n$ is also a minimal submanifold. We notice that Theorem 1.2 is not an immediate corollary of Theorem 1.1 since we do not assume the complex submanifold $M \subset \mathbb{C}^m$ to be a **ruled** helix.

We need the following lemma.

Lemma 3.1. *Let $N^2 \subset \mathbb{R}^n$ be a minimal helix surface (not necessarily ruled). Then N^2 is a totally geodesic submanifold (hence ruled).*

Proof. Under the hypothesis the induced metric on N^2 is flat. Indeed, this is obvious if the helix angle θ is zero. If $\theta \neq 0$ then N^2 carries an harmonic eikonal function, hence two perpendicular totally geodesic foliations, which implies flatness. Now it is a well-known fact that the Gauss equation implies that a minimal and Ricci-flat submanifold of $\mathbb{R}^n$ is totally geodesic. □

Proof of Theorem 1.2. We will show that $M^m \subset \mathbb{C}^n$ is a ruled helix submanifold. Let $\overrightarrow{\mathrm{d}} = \cos(\theta)\mathrm{T} + \sin(\theta)\xi$ be the decomposition of $\overrightarrow{\mathrm{d}}$ in its tangent and normal components. Let J be the complex structure of $\mathbb{C}^n$ regarded as an automorphism of $\mathbb{C}^n$. Then M is also a helix with respect to the direction $\mathrm{J}\overrightarrow{\mathrm{d}}$. So both T and JT are geodesic vector fields of M^m. Let $\mathcal{T} = \operatorname{span}\{\mathrm{T}, \mathrm{JT}\}$ be the 2-dimensional distribution generated by T and JT. We claim that $\mathcal{T}$ is involutive. Indeed, by computing the bracket we have

$$\begin{aligned}
\mathrm{J}[\mathrm{T}, \mathrm{JT}] &= \mathrm{J}\left(\nabla_{\mathrm{T}}\mathrm{JT} - \nabla_{\mathrm{JT}}\mathrm{T}\right)\\
&= \mathrm{J}\nabla_{\mathrm{T}}\mathrm{JT} - \mathrm{J}\nabla_{\mathrm{JT}}\mathrm{T}\\
&= -\nabla_{\mathrm{T}}\mathrm{T} - \nabla_{\mathrm{JT}}\mathrm{JT}\\
&= 0 - 0
\end{aligned}$$

and so $[\mathrm{T}, \mathrm{JT}] = 0$ showing that $\mathcal{T}$ is involutive. Notice that the leaves of $\mathcal{T}$ are complex surfaces which are helix with respect to both $\overrightarrow{\mathrm{d}}$ and $\mathrm{J}\overrightarrow{\mathrm{d}}$. Then by the above lemma it follows that the leaves of $\mathcal{T}$ are complex totally geodesic surfaces of $\mathbb{C}^n$. Therefore the flow lines of both vector fields T and JT are straight lines of $\mathbb{C}^n$. So M is a minimal ruled helix and we can apply Theorem 1.1 to get that M splits as required. □

Now, we will extend Theorem 1.2 to the case when the isometric immersion of a Kähler manifold is not necessarily a holomorphic isometric immersion. The next statement was taken from [3] but it is a result of Dajczer and Gromoll.

Theorem 3.2. ([2]) *Let M be a simply connected Kähler manifold (not necessarily complete) and let $f : M \longrightarrow \mathbb{R}^n$ be a minimal isometric immersion. Then there exists a minimal isometric immersion $g : M \longrightarrow \mathbb{R}^n$ such that $\overline{f} : M \longrightarrow \mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n$ given by $\overline{f}(p) = (\frac{f(p)}{\sqrt{2}}, \frac{g(p)}{\sqrt{2}})$ is isometric and holomorphic with respect to the complex structure $J(u, v) = (-v, u)$ on $\mathbb{R}^n \times \mathbb{R}^n$.*

We are ready to give the extension of Theorem 1.2.

Corollary 3.3. *Let M^m be a simply connected Kähler manifold (not necessarily complete) and let $f : M \longrightarrow \mathbb{R}^n$ be a minimal isometric immersion. If under this immersion M is a helix submanifold then M is a cylinder.*

Proof. We can assume that $f(M)$ is a helix submanifold with respect to the direction induced by the factor $\mathbb{R}$ in $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. Let us observe that in Theorem 3.2, we are identifying $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$ with the map $I : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}^n$ given by

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \mapsto (x_1 + iy_1, x_2 + iy_2, \cdots, x_n + iy_n).$$

By Theorem 3.2, $I \circ \overline{f} : M \longrightarrow \mathbb{C}^n$ is a holomorphic isometric immersion, i.e. M is a Kähler submanifold of $\mathbb{C}^n$. Therefore, Theorem 1.2, implies that $I \circ \overline{f}(M) = \left\{ I(\frac{f(p)}{\sqrt{2}}, \frac{g(p)}{\sqrt{2}}) | p \in M \right\}$ is an extrinsic product $\mathbb{C} \times N \subset \mathbb{C} \times \mathbb{C}^{n-1}$. This proves that the original immersed submanifold $f(M)$ is an extrinsic product in $\mathbb{R} \times \mathbb{R}^{n-1}$, i.e. it is a cylinder.

□

4. Minimal Riemannian foliations: Proof of Theorem 1.4.

Let $p \in U$ and let F_p be the leave of $\mathcal{F}$ through p. Let $m = \dim(F_p)$ be the dimension of F_p and let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a parametrization of F_p near p i.e. $f(0) = p$ and $f(W)$ is an open subset of F_p for a neighborhood W of 0.

Due to the fact that $\mathcal{F}$ is a Riemannian foliation we have that for $q \in U$ near to p the leave F_q is obtained from F_p and a normal vector field $\xi \in \Gamma(\nu(F_p))$ of constant length. Namely, $f_{t\xi}(x) := f(x) + t\xi(x)$ is parametrization of a neighborhood of $q \in F_{p+t\xi(p)}$ for small fixed t. Then Corollary 1.3 implies that ξ is constant in $\mathbb{R}^n$ along $f(W) \subset F_p$. That is to say $f(W)$ is contained in the affine hyperplane

$$H_\xi := \{x \in \mathbb{R}^n : \langle \xi(p), x \rangle = \langle \xi(p), p \rangle\}.$$

Since $\mathcal{F}$ is a foliation of U we get that for each normal direction $\xi \in \nu_p(F_p)$ $f(W)$ is contained in the hyperplane H_ξ. So F_p is near p an open subset of an affine subspace and the Riemannian foliation $\mathcal{F}$ consist of the parallel affine subspaces as we wanted to show.

Since complex submanifolds of $\mathbb{C}^n$ are minimal submanifolds the last claim of Theorem 1.4 follows from the first part.

5. The geometry of the helix submanifolds

In this section we investigate some relations between the extrinsic geometry of the the helix M and the intrinsic geometry of its base $B = \pi(M) \subset \mathbb{R}^n$. Our analysis is based on the eikonal function of the projection method. The notation α_B and $\mathbf{H}_B$ means respectively the second fundamental form of the submanifold $B \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$ and its mean curvature vector field. The gradient $\nabla_B f$ and the Laplacian $\Delta_B f$ of the function f are computed with respect the Riemannian metric on B induced by the inclusion $B \subset \mathbb{R}^n$.

Theorem 5.1. *Let B be the base of the helix M and let $f \in C^\infty(B)$ be the associated eikonal function. Then M is a minimal submanifold of $\mathbb{R}^n$ if and only if the following holds:*

$$\begin{cases} \mathbf{H}_B = \frac{\alpha_B(\nabla_B f, \nabla_B f)}{1+\|\nabla_B f\|^2}, \\ \Delta_B f = 0\,. \end{cases}$$

Proof. Let $\xi_1, \cdots, \xi_r \in \Gamma(\nu(B))$ be a (local) normal frame of $B \subset \mathbb{R}^{n-1}$. Then the vectors $\xi_1(p), \cdots, \xi_r(p)$ are also normal to M at the point $\phi(p) = (p, f(p)) \in M$. The vector field

$$N = \frac{(\nabla_B f, -1)}{\sqrt{1 + \|\nabla_B f\|^2}}$$

is normal to M so $N, \xi_1, \cdots, \xi_r$ is a normal frame of M.
Let $E_1, \cdots, E_{\dim(B)}$ be an orthonormal local frame of B with $E_1 := \frac{\nabla_B f}{\|\nabla_B f\|}$. Then the vector fields $X_1, \cdots, X_{\dim(M)}$ defined by

$$X_i := (E_i, \mathrm{d}f(E_i)) \in \mathbb{R}^{n-1} \times \mathbb{R}$$

give us a frame of M.

In terms of this frame the second fundamental form α_M of M is given by

$$\langle \alpha_M(X_i, X_j), \xi_k \rangle = \langle D_{E_i} E_j + E_i(\mathrm{d}f(E_j)) \overrightarrow{\mathrm{d}}, \xi_k \rangle = \langle \alpha_B(E_i, E_j), \xi_k \rangle$$

$$\begin{aligned} \langle \alpha_M(X_i, X_j), N \rangle &= \langle D_{E_i} E_j + E_i(\mathrm{d}f(E_j)) \overrightarrow{\mathrm{d}}, \frac{(\nabla_B f, -1)}{\sqrt{1 + \|\nabla_B f\|^2}} \rangle \\ &= \langle \nabla_{E_i} E_j, \frac{\nabla_B f}{\sqrt{1 + \|\nabla_B f\|^2}} \rangle - \frac{E_i(\mathrm{d}f(E_j))}{\sqrt{1 + \|\nabla_B f\|^2}} \\ &= \langle \nabla_{E_i} E_j, \frac{\nabla_B f}{\sqrt{1 + \|\nabla_B f\|^2}} \rangle \end{aligned}$$

Let $G = (G_{ij} = \langle X_i, X_j \rangle)$ be the matrix of the metric of M with respect to the frame $X_1, \cdots, X_{\dim(M)}$. Then the matrix of the shape operators A_N, A_ξ with respect to the frame $X_1, \cdots, X_{\dim(M)}$ are:

$$A_{\xi_k} = A_k G^{-1} \ , \ A_N = RG^{-1}$$

where $(A_k)_{i,j} := \langle \alpha_M(X_i, X_j), \xi_k \rangle = \langle \alpha_B(E_i, E_j), \xi_k \rangle$ and $(R_{ij}) := \langle \alpha_M(X_i, X_j), N \rangle$. Observe that G is the diagonal matrix $G = \mathrm{diag}(1 + \|\nabla_B f\|^2, 1, \cdots, 1)$ since $df(E_1) = \|\nabla_B f\|$.

Then for all ξ_k we have

$$\begin{aligned} \mathrm{trace}(A_{\xi_k}) = \mathrm{trace}(A_k G^{-1}) = & \\ &= \frac{\langle \alpha_B(E_1, E_1), \xi_k \rangle}{1 + \|\nabla_B f\|^2} + \langle \alpha_B(E_2, E_2), \xi_k \rangle + \cdots \\ &+ \langle \alpha_B(E_{\dim(B)}, E_{\dim(B)}), \xi_k \rangle \\ &= \frac{\langle \alpha_B(E_1, E_1), \xi_k \rangle}{1 + \|\nabla_B f\|^2} - \langle \alpha_B(E_1, E_1), \xi_k \rangle + \langle \mathbf{H}_B, \xi_k \rangle \, . \end{aligned}$$

So $\mathrm{trace}(A_{\xi_k}) = 0$ for all k if and only if

$$\mathbf{H}_B = \frac{\|\nabla_B f\|^2}{1 + \|\nabla_B f\|^2} \alpha_B(E_1, E_1) = \frac{\alpha_B(\nabla_B f, \nabla_B f)}{1 + \|\nabla_B f\|^2}$$

and we get the first identity. We also have

$$\begin{aligned} \mathrm{trace}(A_N) = \mathrm{trace}(RG^{-1}) = & \\ &= \frac{\langle \nabla_{E_1} E_1, \frac{\nabla_B f}{\sqrt{1 + \|\nabla_B f\|^2}} \rangle}{1 + \|\nabla_B f\|^2} + \sum_{j=2}^{\dim(B)} \langle \nabla_{E_j} E_j, \frac{\nabla_B f}{\sqrt{1 + \|\nabla_B f\|^2}} \rangle = \\ &= \frac{\langle \nabla_{E_1} E_1, \frac{\nabla_B f}{\sqrt{1 + \|\nabla_B f\|^2}} \rangle}{1 + \|\nabla_B f\|^2} + \frac{\langle \nabla_{E_1} \nabla_B f, E_1 \rangle}{\sqrt{1 + \|\nabla_B f\|^2}} - \frac{\Delta_B f}{\sqrt{1 + \|\nabla_B f\|^2}} \\ &= - \frac{\Delta_B f}{\sqrt{1 + \|\nabla_B f\|^2}} \end{aligned}$$

the last equation follows from the fact that $E_1 = \frac{\nabla_B f}{\|\nabla_B f\|}$ and $\|\nabla_B f\|$ is a constant. So $\mathrm{trace}(A_N) = 0$ if and only if $\Delta_B f = 0$. $\square$

An interesting application of the above result is given in Theorem 5.15 below.

5.1. **The intrinsic geometry of helix submanifolds.** As we recall in section 2, any helix submanifold M is locally constructed with the projection method where we used a Riemannian manifold $B := \pi(M) \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$ called the basis. Here we study the relations between the geometries of M and B.

So if we want to construct a helix M in $\mathbb{R}^n$, we can consider a Riemannian manifold (B,g) of dimension m with an immersion of (B,g) in $\mathbb{R}^n$ given by $\phi(p)=(i(p),f(p))$ where $i:B\longrightarrow\mathbb{R}^{n-1}$ is an isometric immersion and where $f:B\longrightarrow\mathbb{R}$ is an non constant eikonal function on B. By Theorem 2.1, $M=\phi(B)$ is a helix submanifold of $\mathbb{R}^n$ with its induced metric H. Then we have an isometry between (M,H) and $(B,h:=\phi^*H)$. First, let us observe that the relation between the metrics of (B,g) and (B,h) is given by

$$h(X,Y):=(\phi^*H)(X,Y)=H(\phi_*(X),\phi_*(Y))=g(X,Y)+df(X)df(Y).$$

So, in this subsection we will compare (B,g) with (B,h) and $f:(B,g)\longrightarrow\mathbb{R}$ will be a non constant C^∞ eikonal function.
Let $E_1=\frac{\nabla_g f}{\|\nabla_g f\|},E_2,\cdots,E_m$ be a local frame orthonormal of (B,g). Since $h(E_1,E_1)=1+\|\nabla_g f\|^2$, we can consider the following orthonormal local frame of (B,h): $\tilde{E}_1=\frac{1}{\sqrt{1+\|\nabla_g f\|^2}}E_1,E_2,\cdots,E_m$.

Let us observe that in the basis $E_1=\frac{\nabla_g f}{\|\nabla_g f\|},E_2,\cdots,E_m$, the relation between the metrics looks like

$$h(E_i,E_j)=\begin{cases} g(E_i,E_j)=\delta_{ij}, & \text{if either } i>1 \text{ or } j>1,\\ (1+\|\nabla_g f\|^2)g(E_1,E_1), & \text{if } i=j=1.\end{cases} \tag{4}$$

Remark 5.2. Under ϕ the local vector field $\tilde{E}_1$ is identified with $T=\overrightarrow{\mathrm{d}}^\top/\|\overrightarrow{\mathrm{d}}^\top\|$ the unit tangent component of the helix direction $\overrightarrow{\mathrm{d}}$. Indeed,

$$\begin{aligned}\phi_*(\tilde{E}_1)&=\frac{1}{\|\nabla_g f\|\sqrt{1+\|\nabla_g f\|^2}}\phi_*(\nabla_g f)\\ &=\frac{1}{\|\nabla_g f\|\sqrt{1+\|\nabla_g f\|^2}}(\nabla_g f+\|\nabla_g f\|^2\overrightarrow{\mathrm{d}})=T\,.\end{aligned}$$

Notice that the function f regarded as a function of M is given by the height function $f(x)=\langle x,\overrightarrow{\mathrm{d}}\rangle$ with $x\in M$. So the gradient in M of f is $\overrightarrow{\mathrm{d}}^\top$ and the unitary projection η of $\overrightarrow{\mathrm{d}}^\top$ in B is a constant multiple of the gradient of $\nabla_g f$ when we regard f as a function of B.

In the next Proposition 5.3, we give the relation between the volume forms of the metrics h and g.

Proposition 5.3. *Let ω_g and ω_h be the volume forms of (B,g) and (B,h), respectively. Then*

$$\omega_h=\sqrt{1+\|\nabla_g f\|^2}\;\omega_g.$$

Proof. Let $E_1 = \frac{\nabla_g f}{\|\nabla_g f\|}, E_2, \cdots, E_m$ be the basis defined above. The volume forms are given by $\omega_g(E_1, \cdots, E_m) = \sqrt{\det(g(E_i, E_j))} = 1$ because the basis is orthonormal with the metric g. In the case of metric h we have: $\omega_h(E_1, \cdots, E_m) = \sqrt{\det(h(E_i, E_j))} = \sqrt{1 + \|\nabla_g f\|^2}$. $\square$

Proposition 5.4. *Let $\nabla_g f$ and $\nabla_h f$ be the gradients of f in (B, g) and (B, h), respectively. Then*

$$\nabla_h f = \frac{1}{1 + \|\nabla_g f\|^2} \nabla_g f. \tag{5}$$

Proof. For every j, we have the relation:

$$h(\nabla_h f, E_j) = df(E_j) = g(\nabla_g f, E_j)$$

and in particular we have for $j > 2$:
$h(\nabla_h f, E_j) = g(\nabla_g f, E_j) = 0$. When $j = 1$: $E_1 = \frac{\nabla_g f}{\|\nabla_g f\|}$,

$$h(\nabla_h f, E_1) = g(\nabla_g f, E_1) = g(\nabla_g f, \frac{\nabla_g f}{\|\nabla_g f\|}) = \|\nabla_g f\|.$$

We can calculate $\nabla_h f$ as

$$\nabla_h f \quad = \quad \frac{1}{1 + \|\nabla_g f\|^2} h(\nabla_h f, E_1) E_1 = \frac{1}{1 + \|\nabla_g f\|^2} \nabla_g f.$$

$\square$

Proposition 5.5. *Let $\nabla_g f$ be the gradient of f in (B, g). Then the Levi-Civita connection ∇^h of (B, h) is given by*

$$\nabla^h_X Y = \nabla^g_X Y + \frac{Hess_g f(X, Y)}{1 + \|\nabla_g f\|^2} \nabla_g f. \tag{6}$$

Proof. Let us recall Koszul's formula:

$$\begin{aligned} 2g(\nabla^g_X Y, Z) \quad &= \quad Xg(Y, Z) - Zg(X, Y) + Yg(Z, X) \\ &- \quad g(X, [Y, Z]) + g(Z, [X, Y]) + g(Y, [Z, X]). \end{aligned}$$

To prove the relation (6), we only have to check it for X and Y in a local frame. Let $E_1 = \frac{\nabla_g f}{\|\nabla_g f\|}, E_2 \cdots, E_m$ be a local frame orthonormal of (B, g). Since $h(E_1, E_1) = 1 + \|\nabla_g f\|^2$, we can consider the following orthonormal local frame of (B, h): $\tilde{E}_1 = \frac{1}{\sqrt{1+\|\nabla_g f\|^2}} E_1, E_2, \cdots, E_m$.

Using Koszul's formula: $i,\ j,\ k > 1$,

$$\begin{aligned}
2g(\nabla^g_{E_j} E_i, E_1) &= -g(E_j, [E_i, E_1]) + g(E_1, [E_j, E_i]) + g(E_i, [E_1, E_j]) \\
&= -g(E_j, [E_i, E_1]) + g(E_i, [E_1, E_j]). \\
2g(\nabla^g_{E_j} E_i, E_k) &= -g(E_j, [E_i, E_k]) + g(E_k, [E_j, E_i]) + g(E_i, [E_k, E_j]). \\
2g(\nabla^g_{E_1} E_i, E_k) &= -g(E_1, [E_i, E_k]) + g(E_k, [E_1, E_i]) + g(E_i, [E_k, E_1]), \\
&= g(E_k, [E_1, E_i]) + g(E_i, [E_k, E_1]).
\end{aligned}$$

A similar calculus and the properties
$h(E_1, [E_i, E_j]) = 0,\ h(E_j, [E_i, E_k]) = g(E_j, [E_i, E_k]),\ h(E_j, [E_i, E_1]) = g(E_j, [E_i, E_1])$ (see (4)) proves that:

$$\begin{aligned}
h(\nabla^h_{E_j} E_i, E_1) &= g(\nabla^g_{E_j} E_i, E_1), \\
h(\nabla^h_{E_j} E_i, E_k) &= g(\nabla^g_{E_j} E_i, E_k), \\
h(\nabla^h_{E_1} E_i, E_k) &= g(\nabla^g_{E_1} E_i, E_k).
\end{aligned}$$

Thus we can calculate for $i,\ j > 1$,

$$\begin{aligned}
\nabla^h_{E_j} E_i &= h(\nabla^h_{E_j} E_i, \tilde{E}_1)\tilde{E}_1 + \sum_{k>1} h(\nabla^h_{E_j} E_i, E_k) E_k \\
&= \frac{1}{1 + \|\nabla_g f\|^2} h(\nabla^h_{E_j} E_i, E_1) E_1 + \sum_{k>1} h(\nabla^h_{E_j} E_i, E_k) E_k \\
&= \nabla^g_{E_j} E_i - \frac{\|\nabla_g f\|^2}{1 + \|\nabla_g f\|^2} g(\nabla^g_{E_j} E_i, E_1) E_1.
\end{aligned}$$

Let us analyse the last term:

$$\begin{aligned}
-g(\nabla^g_{E_j} E_i, E_1) &= g(E_i, \nabla^g_{E_j} E_1) = \frac{1}{\|\nabla_g f\|} g(E_i, \nabla^g_{E_j}(\nabla_g f)) \\
&= \frac{1}{\|\nabla_g f\|} Hess_g f(E_i, E_j).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla^h_{E_j} E_i &= \nabla^g_{E_j} E_i + \frac{\|\nabla_g f\|}{1 + \|\nabla_g f\|^2} Hess_g f(E_i, E_j) E_1 \\
&= \nabla^g_{E_j} E_i + \frac{Hess_g f(E_i, E_j)}{1 + \|\nabla_g f\|^2} \nabla_g f.
\end{aligned}$$

When $i = 1$ or $j = 1$, $\nabla^h_{E_j} E_i = \nabla^g_{E_j} E_i$. Since f is eikonal in (B, g) and by Proposition 5.4, we deduce that f is eikonal in (B, h). Therefore, $\nabla^h_{E_1} E_1 = \nabla^g_{E_1} E_1 = 0$. Finally, other consequence is that for every $X \in TB$, $Hess_g f(E_1, X) = 0$. $\square$

Remark 5.6. Let us observe that Equations (5) and (6) implies that if $\nabla_g f$ is parallel in (B, g) then $\nabla_h f$ is a parallel vector field in (B, h). Also it is true that the integral lines of $\nabla_g f$ are geodesics in (B, g) if and only if the integral lines of $\nabla_h f$ are geodesics in (B, h), i.e. $\nabla^h_{\nabla_h f} \nabla_h f = 0$ if and only if $\nabla^g_{\nabla_g f} \nabla_g f = 0$.

Proposition 5.7. *Let $\nabla_g f$ and $Hess_g f$ be the gradient and the Hessian respectively, of f in (B, g). Then*

$$Hess_h f = \frac{1}{1 + \|\nabla_g f\|^2} Hess_g f. \tag{7}$$

Proof. If $i,\ j > 1$ we have that,

$$\begin{aligned}
& Hess_h f(E_i, E_j) = \\
= \ & h(\nabla^h_{E_i}(\nabla_h f), Ej) = \frac{1}{1 + \|\nabla_g f\|^2} h(\nabla^h_{E_i}(\nabla_g f), Ej) \\
= \ & \frac{\|\nabla_g f\|}{1 + \|\nabla_g f\|^2} h(\nabla^h_{E_i} E_1, Ej) = -\frac{\|\nabla_g f\|}{1 + \|\nabla_g f\|^2} h(E_1, \nabla^h_{E_i} Ej) \\
= \ & -\frac{\|\nabla_g f\|}{1 + \|\nabla_g f\|^2} g(E_1, \nabla^g_{E_i} Ej) = \frac{\|\nabla_g f\|}{1 + \|\nabla_g f\|^2} g(\nabla^g_{E_i} E_1, Ej) \\
= \ & \frac{1}{1 + \|\nabla_g f\|^2} Hess_g f(E_i, E_j).
\end{aligned}$$

Finally,

$$Hess_h f(E_1, E_j) = h(\nabla^h_{E_1}(\nabla_h f), E_j) \ = \ 0 \tag{8}$$

$$Hess_g f(E_1, E_j) = g(\nabla^g_{E_1}(\nabla_g f), E_j) \ = \ 0, \tag{9}$$

because $\nabla^h_{E_1} E_1 = 0$, $\nabla^g_{E_1} E_1 = 0$. The property that f is eikonal both in (B, g) and (B, h) implies the latter two equalities. □

Corollary 5.8. The relation between the Laplacians is given by

$$\triangle_h f = \frac{1}{1 + \|\nabla_g f\|^2} \triangle_g f,$$

where $\triangle_h f$ and $\triangle_g f$ are the Laplacians of f in (B, h) and (B, g), respectively.

Proof. It follows by taking the trace in both sides of formula (7) and applying (8) and (9). □

As an application we obtain a different proof of the second part of Theorem 5.1.

Let us observe that we have applied two notations $\triangle_g f$ and $\triangle_B f$ which are the same: The Laplacian for the isometric immersion of

(B, g) in $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ where $B = \pi(M)$ is the projection of the helix M. Moreover, the metric of the helix M is (M, H) which is isometric to (B, h).

Corollary 5.9. *Let M be a helix submanifold. Let f be the associated eikonal function $f : B = \pi(M) \to \mathbb{R}$. If M is minimal then $\triangle_B f = 0$, in particular f is an isoparametric function.*

Proof. Since M is a helix submanifold, locally $M = \{(x, f(x))\}$ where $f : B \longrightarrow \mathbb{R}$ is a height function. It is well known that the height functions of M are harmonic with the metric of M because M is minimal. Therefore $\triangle_h f = 0$. Therefore by Corollary 5.8, $\triangle_g f = \triangle_h f = 0$. So, $\triangle_B f = \triangle_g f = 0$. ☐

Remark 5.10. Let us recall that a height function on M, $f : M \longrightarrow \mathbb{R}$ given by $f(x) = \langle x, \overrightarrow{\mathrm{d}} \rangle$ is harmonic when the submanifold is minimal. Here $\overrightarrow{\mathrm{d}}$ is a unit direction in $\mathbb{R}^n$. In our case of helix submanifolds, there is other way to calculate the Laplacian of a height function:
According to [5, page 194] for any helix submanifold we have the structure equation

$$\nabla_X T = \tan(\theta) A^\xi(X)$$

with A^ξ the shape operator of the immersion $M \subset \mathbb{R}^n$ with respect to the vector $\xi = \overrightarrow{\mathrm{d}}^{\perp}/\| \overrightarrow{\mathrm{d}}^{\perp}\|$. Taking an orthonormal basis of TM we can do the sum over the basis to obtain that

$$\triangle_M f = \sum_{i=1}^{m} \langle \nabla_{X_i}(\nabla_M f), X_i\rangle = \cos(\theta) \sum_{i=1}^{m} \langle \nabla_{X_i} T, X_i\rangle = \sin(\theta)\langle H, \xi\rangle,$$

where $\cos(\theta) = \langle T, \overrightarrow{\mathrm{d}} \rangle = \| \overrightarrow{\mathrm{d}}^{\top}\|$, $\sin(\theta) = \langle \xi, \overrightarrow{\mathrm{d}} \rangle$, $\overrightarrow{\mathrm{d}}^{\top} = \nabla_M f$ and $T = \overrightarrow{\mathrm{d}}^{\top}/\| \overrightarrow{\mathrm{d}}^{\top}\| = \nabla_M f / \cos(\theta)$. So, it is clear that if M is minimal then f is harmonic in M. In general, it is well known the formula for Euclidean immersed submanifolds $\triangle_M f = \langle H, \overrightarrow{\mathrm{d}} \rangle$. The two relations for the Laplacian are compatible because $\overrightarrow{\mathrm{d}} = \cos(\theta) T + \sin(\theta)\xi$.

Now we are going to find a relation between the Ricci curvature Ric_g of (B, g) and Ric_h of (B, h).
The Riemannian tensor of curvature is given by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

and the Ricci curvature

$$Ric(X, Y) = \sum_{i=1}^{m} \langle R(X, X_j)Y, X_j\rangle,$$

where $X_1, \ldots, X_m$ is an orthonormal basis of TB.

Proposition 5.11. *The Ricci curvature Ric_h in direction $\nabla_h f$ is related to the Ricci curvature Ric_g in direction $\nabla_g f$ by the formula*

$$Ric_h(\nabla_h f, \nabla_h f) = \frac{1}{(1+\|\nabla_g f\|^2)^2} Ric_g(\nabla_g f, \nabla_g f). \tag{10}$$

Proof. Let $E_1 = \frac{\nabla_g f}{\|\nabla_g f\|}, \dots, E_m$ and $\tilde{E}_1 = \frac{1}{\sqrt{1+\|\nabla_g f\|^2}} E_1, E_2 \dots, E_m$ be the local orthonormal frames defined in the beginning of Subsection 5.1. Let us observe that for every $Y \in TB$,

$$Hess_g f(\nabla_g f, Y) = \langle \nabla^g_{\nabla_g f} \nabla_g f, Y \rangle = 0$$

because the integral lines of $\nabla_g f$ are geodesics of (B, g). It follows from formula (6) that for every $X \in TB$,

$$\nabla^h_X(\nabla_g f) = \nabla^g_X(\nabla_g f), \quad \nabla^h_{\nabla_g f} X = \nabla^g_{\nabla_g f} X.$$

We deduce by substitution that

$$\nabla^h_{\nabla_g f} \nabla^h_Y(\nabla_g f) = \nabla^h_{\nabla_g f} \nabla^g_Y(\nabla_g f) = \nabla^g_{\nabla_g f} \nabla^g_Y(\nabla_g f).$$

Analogously,

$$\nabla^h_{[\nabla_g f, Y]}(\nabla_g f) = \nabla^g_{[\nabla_g f, Y]}(\nabla_g f).$$

Since the integral curves of $\nabla_h f$ and $\nabla_g f$ are geodesics in (B, g) and (B, h) respectively,

$$\nabla^h_Y \nabla^h_{\nabla_g f}(\nabla_g f) = 0 = \nabla^g_Y \nabla^g_{\nabla_g f}(\nabla_g f).$$

By definition,
$R^g(\nabla_g f, Y)\nabla_g f = -\nabla^g_{\nabla_g f} \nabla^g_Y \nabla_g f + \nabla^g_Y \nabla_{\nabla_g f} \nabla_g f + \nabla^g_{[\nabla_g f, Y]} \nabla_g f$ and a similarly formula for R^h. Then

$$R^h(\nabla_g f, Y)\nabla_g f = R^g(\nabla_g f, Y)\nabla_g f.$$

Therefore,

$$\begin{array}{rcc} Ric^h(\nabla_g f, \nabla_g f) & = & h(R^h(\nabla_g f, \tilde{E}_1)\nabla_g f, \tilde{E}_1) \\ & + & \sum_{i=2}^m h(R^h(\nabla_g f, E_j)\nabla_g f, E_j) \\ & = & \sum_{i=2}^m h(R^h(\nabla_g f, E_j)\nabla_g f, E_j) \\ & = & \sum_{i=2}^m g(R^g(\nabla_g f, E_j)\nabla_g f, E_j) = Ric^g(\nabla_g f, \nabla_g f). \end{array}$$

To obtain formula (10), we have to use equation (5) which is the relation between the gradients $\nabla_g f$ and $\nabla_h f$. □

Corollary 5.12. *Let M be an immersed helix hypersurface in $\mathbb{R}^{n+1}$ with respect to an unitary direction $\overrightarrow{\mathrm{d}} \in \mathbb{R}^{n+1}$. Then the Ricci curvature of M in direction of the tangent component of $\overrightarrow{\mathrm{d}}$ is zero:*

$$Ricc_M(T, T) = 0,$$

where $T = \overrightarrow{\mathrm{d}}^{\top} / \|d^{\top}\|$.

Proof. If M is a cylinder with direction d, we are ready. Otherwise, let B be as before: The orthogonal projection of M into an open part of a hyperplane $span\{d^\perp\}$ orthogonal to d. So, B is Ricci-flat because it is an open part of a Euclidean space and in particular $Ricc_g(\eta,\eta) = 0$, where η is the unitary projection of T into the hyperplane $span\{d^\perp\}$. Let us observe that η is a constant multiple of $\nabla_g f$: By Remark 5.2, $T = \frac{1}{\|\nabla_g f\|\sqrt{1+\|\nabla_g f\|^2}}(\nabla_g f + \|\nabla_g f\|^2 \overrightarrow{\mathrm{d}})$ and so η is a constant multiple of $T - \frac{1}{\|\nabla_g f\|\sqrt{1+\|\nabla_g f\|^2}}\|\nabla_g f\|^2 \overrightarrow{\mathrm{d}} = \frac{1}{\|\nabla_g f\|\sqrt{1+\|\nabla_g f\|^2}}\nabla_g f$. In fact since we are looking for η to be unitary in (B,g) we deduce that $\eta = E_1 = \nabla_g f/\|\nabla_g f\|$. Since (M,H) and (B,h) are isometric, $Ric_M(T,T) = Ric_h(\tilde{E}_1,\tilde{E}_1)$. By Equation (10), $Ric_h(\tilde{E}_1,\tilde{E}_1)$ is a constant multiple of $Ricc_g(\eta,\eta)$, see Remark 5.2. This relations prove that $Ric_M(T,T) = 0$. ☐

Remark 5.13. Another proof of the above corollary is as follows. Notice that if M is helix hypersurface then the vector field T is in the relative nullity distribution i.e. the kernel of the shape operator. So by Gauss equation the curvature tensor of M vanish when contracted with T hence $Ricc_M(T,T) = 0$.

Example 5.14. Let us consider the Sol geometry: $(\mathbb{R}^3, g_{Sol})$, where the metric is $g_{Sol} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$. The function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $f(x,y,z) = z$ is harmonic, see Corollary 4.3 in [12]. This function is also eikonal, its gradient $\nabla f = \partial_z$ has constant length, it satisfies that $\|\nabla f\| = 1$. We should remark that the level hypersurfaces are minimal submanifolds but not totally geodesic, because the latter condition is equivalent to the parallelism of the gradient vector field $\nabla f = \partial_z$. We can see using the formula of Koszul that this vector field satisfies that $\nabla_{\partial_x}\partial_z = \partial_x$, i.e. ∂_z is not a parallel vector field. Similarly, we have the following relations

$$\begin{aligned}
\nabla_{\partial_x}\partial_x &= -e^{2z}\partial_z,\ \nabla_{\partial_x}\partial_y = 0, \nabla_{\partial_x}\partial_z = \partial_x\\
\nabla_{\partial_y}\partial_y &= e^{-2z}\partial_z,\ \nabla_{\partial_y}\partial_z = -\partial_y\\
\nabla_{\partial_z}\partial_z &= 0.
\end{aligned}$$

Now, we are ready for the calculus of the Riemannian curvature tensor, for example

$$R(\partial_x,\partial_y)\partial_x = e^{2z}\partial_y,\ R(\partial_x,\partial_z)\partial_x = -e^{2z}\partial_z.$$

Therefore,

$$\langle R(\partial_x,\partial_y)\partial_x,\partial_y\rangle = 1,\ \langle R(\partial_x,\partial_z)\partial_x,\partial_z\rangle = -e^{2z}.$$

Finally, a direct calculus show that the Ricci curvature of this Sol geometry is

$$\begin{aligned} Ric(\partial_x,\partial_x) &= 0,\ Ric(\partial_y,\partial_y)=0,\ Ric(\partial_z,\partial_z)=-2\\ Ric(\partial_x,\partial_y) &= 0,\ Ric(\partial_x,\partial_z)=0,\ Ric(\partial_y,\partial_z)=0. \end{aligned}$$

From this we conclude that $(\mathbb{R}^3, g_{Sol})$ can not be isometrically immersed as a minimal submanifold (even locally) in any euclidean space $\mathbb{R}^n$ of any dimension. Indeed, assume that such isometric immersion do exists. Then from Gauss equation we get that the kernel of the Ricci tensor $\ker(Ric) = span\{\partial_x, \partial_y\}$ is the kernel of the second fundamental form of the immersion, i.e. the so called relative nullity distribution. Since this distribution has dimension 2 we get that $(\mathbb{R}^3, g_{Sol})$ is a flat Riemannian manifold. This contradicts $Ric(\partial_z, \partial_z) = -2$ and prove our claim.

5.2. **Other results about minimal helices.** Let $M \subset \mathbb{R}^n$ be helix with respect to the direction $\overrightarrow{\mathrm{d}} \in \mathbb{R}^n$. Let $\pi : \mathbb{R}^n \to span\{\overrightarrow{\mathrm{d}}^{\perp}\}$ be the projection to a normal hyperplane $\overrightarrow{\mathrm{d}}^{\perp}$ to $\overrightarrow{\mathrm{d}}$. Since we work locally we can assume that $\pi(M)$ is a submanifold of $\mathbb{R}^{n-1} \cong span\{\overrightarrow{\mathrm{d}}^{\perp}\}$.

Theorem 5.15. *Let $M \subset \mathbb{R}^n$ be a full minimal helix of any codimension with respect to the direction $\overrightarrow{\mathrm{d}} \in \mathbb{R}^n$. If the Ricci curvature of the submanifold $B := \pi(M)$ is non-negative then M is a totally geodesic submanifold of $\mathbb{R}^n$.*

Proof. If M is a cylinder, then B is minimal with non-negative Ricci curvature and therefore $(B = \pi(M), g)$ is totally geodesic. It is a consequence that a cylinder over a totally geodesic submanifold is also totally geodesic. Otherwise, we can apply the projection method where is important the condition $\theta \neq 0$. By [4, Theorem] we have that locally the immersion $M \subset \mathbb{R}^n$ is given as

$$\phi(p) = (p, f(p))$$

where $M = \phi(B)$ locally. Notice that $\phi : (B, h) \to M \subset \mathbb{R}^n$ is a isometry. The function f is eikonal either in $(B = \pi(M), g)$ or (B, g) and Theorem 5.1 implies $\Delta_B f := \Delta_g f = 0$. Bochner's formula for functions together with the hypothesis that (B, g) has non-negative Ricci curvature implies that the $\nabla_B f := \nabla_g f$ is a parallel vector field of (B, g) and therefore, $\nabla_h f$ is parallel in (B, h). Since ϕ is a isometry and by Remark 5.2, $\phi_*(\tilde{E}_1) = T$ we deduce that T is parallel in M. In particular $\mathrm{Ric}_B(\nabla_g f) = 0$. By using Gauss equation we have

$$\mathrm{Ric}_B(\nabla_B f) = \langle A_{\mathrm{H}_B}(\nabla_B f), \nabla_B f\rangle - \sum_{i=1}^{\dim(B)} \|\alpha(\nabla_B f, E_i)\|^2 .$$

Then from Theorem 5.1 we get

$$0 = \langle \frac{\alpha_B(\nabla_B f, \nabla_B f)}{1 + \|\nabla_B f\|^2}, \alpha(\nabla_B f, \nabla_B f)\rangle - \sum_{i=1}^{\dim(B)} \|\alpha(\nabla_B f, E_i)\|^2 .$$

Setting $E_1 := \frac{\nabla_B f}{\|\nabla_B f\|}$ we get

$$0 = \frac{\|\alpha(\nabla_B f, \nabla_B f)\|^2}{1 + \|\nabla_B f\|^2} - \frac{\|\alpha(\nabla_B f, \nabla_B f)\|^2}{\|\nabla_B f\|^2} - \sum_{i=2}^{\dim(B)} \|\alpha(\nabla_B f, E_i)\|^2$$

and so

$$0 = \frac{-\|\alpha(\nabla_B f, \nabla_B f)\|^2}{(1 + \|\nabla_B f\|^2)\|\nabla_B f\|^2} - \sum_{i=2}^{\dim(B)} \|\alpha(\nabla_B f, E_i)\|^2 .$$

Thus, $\alpha_B(\nabla_B f, \nabla_B f) = \alpha_B(\nabla_B f, E_i) = 0$ for $i = 2, \cdots, \dim(B)$. Then $\nabla_B f$ is in the nullity of the second fundamental form. By Theorem 5.1, (B, g) is minimal. Then B is a minimal submanifold with non-negative Ricci tensor. It follows that B is a totally geodesic submanifold. Since, f is eikonal and harmonic in (B, g) with B an Euclidean space we have that f is a linear function and so its graph over B is other Euclidean space, i.e. $M = \phi(B)$ is a totally geodesic submanifold. $\square$

6. Helix hypersurfaces with constant mean curvature

In this section we give a proof of the following theorem which generalize Corollary 4.2 in [7]. For the proof we need the following corollary of the maximum principle for harmonic maps in [16, Theorem 2].

Lemma 6.1. *Let $f : M \to N$ be a harmonic map between the Riemannian manifolds M, N. Assume that $f(M)$ is contained in the hypersurface $H \subset N$. If the shape operator of H is definite then f is a constant map.*

Proof of Theorem 1.5 . If the helix angle is zero then it is clear that the hypersurface is a cylinder. So assume that the constant angle is different from zero. So a normal vector is not perpendicular the constant direction $\overrightarrow{\mathrm{d}}$. Observe that the subset H of the sphere consisting of vectors whose angle with a fix vector $\overrightarrow{\mathrm{d}}$ is constant different from

$\frac{\pi}{2}$ is a totally umbilical non-totally geodesic submanifold. Hence the shape operator of H is definite. Now by Ruh-Vilms' theorem [15] the Gauss map of our helix surface is harmonic. By the previous observation the image of such Gauss map is contained in the hypersurface H. Then by the above lemma the Gauss map is constant. Hence the helix hypersurface is an open subset of some hyperplane. $\Box$

Unfortunately the above idea does not work for higher codimensional helix submanifolds. Let us explain where is the problem. Let $\mathrm{G}(n,r)$ be the Grassmanian of r-planes in $\mathbb{R}^n$. For $\overrightarrow{\mathrm{d}} \in \mathbb{R}^n$ define $H(\overrightarrow{\mathrm{d}},\theta) \subset \mathrm{G}(n,r)$ as the subset of r-planes whose angle with $\overrightarrow{\mathrm{d}}$ is θ. Notice that for $\theta \neq 0$ the subset $H(\overrightarrow{\mathrm{d}},\theta) \subset \mathrm{G}(n,r)$ is a smooth hypersurface. It is not difficult to see that $H(\overrightarrow{\mathrm{d}},\theta) \subset \mathrm{G}(n,r)$ is an orbit of the natural action of the subgroup $SO(n)_{\overrightarrow{\mathrm{d}}}$ of $SO(n)$ which leaves $\overrightarrow{\mathrm{d}}$ fixed, i.e. the isotropy subgroup of $\overrightarrow{\mathrm{d}}$. The subgroup $SO(n)_{\overrightarrow{\mathrm{d}}}$ is symmetric in $SO(n)$. Indeed, it is the fixed subgroup associated to the involution σ of $SO(n)$ induced by the symmetry with respect to the hyperplane $\langle \overrightarrow{\mathrm{d}}, \cdot \rangle = 0$ in $\mathbb{R}^n$. The principal curvatures of the orbits $H(\overrightarrow{\mathrm{d}},\theta) \subset \mathrm{G}(n,r)$ were computed in [17, p.65, Proposition 6]. So we see that unless the Grassmanian $\mathrm{G}(n,r)$ is a projective space the shape operator of the hypersurfaces $H(\overrightarrow{\mathrm{d}},\theta) \subset \mathrm{G}(n,r)$ is never definite. Notice that the dimension of $H(\overrightarrow{\mathrm{d}},0)$ is $(r-1)(n-r)$ so if the codimension $n-r$ is greater than one $H(\overrightarrow{\mathrm{d}},0)$ is not a hypersurface of $\mathrm{G}(n,r)$. Finally, in codimension one the hypersurface $H(\overrightarrow{\mathrm{d}},0)$ is totally geodesic hence its shape operator is non-definite. So this explains the existence of non-totally geodesic cylinders over hypersurfaces with constant mean curvature.

Acknowledgements

We would like to thank Francisco Vittone for several useful comments. The second author thanks the hospitality of DISMA at Politecnico di Torino where this work was started.

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