The relative nullity of complex submanifolds and the Gauss map

THE RELATIVE NULLITY OF COMPLEX SUBMANIFOLDS AND THE GAUSS MAP

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1. Introduction

In this paper we denote by $\mathbb{CP}^n$ the complex projective space endowed with its Fubini-Study metric g_{FS} of constant holomorphic sectional curvature. By a complete submanifold $M \subset \mathbb{CP}^n$ we mean that the induced Riemannian metric g_{FS} on M is complete.

Let α be the second fundamental form of $M \subset \mathbb{CP}^n$. The index of relative nullity

$$\bar{\mu}(p) = \dim_{\mathbb{C}}(RN_p)$$

at $p \in M$ is the dimension of the relative nullity subspace RN_p:

$$RN_p := \{X \in T_pM : \alpha(X, Y) = 0 \text{ for all } Y \in T_pM\}.$$

The minimum μ of $\{\bar{\mu}(p) : p \in M\}$ is called the index of relative nullity of M.

The following theorem was first stated in [2, Corollary 5].

Theorem 1.1. *Let $M^m \subset \mathbb{CP}^n$ be a m-dimensional complex submanifold. If M is complete then the index of relative nullity μ of M is either zero or M is a totally geodesic complex submanifold of $\mathbb{CP}^n$.*

The proof of Abe [2] uses the so called conullity operator (also called splitting tensor [5, p. 761]). The idea of the Riccati type differential equation for the conullity operator comes back to Ferus [6].

If the submanifold $M \subset \mathbb{CP}^n$ is compact, then it is algebraic due to a well known theorem of W-L. Chow. In this case the above theorem is phrased, in algebraic geometry, by means of the Gauss map [8, p. 393, (2.29)] and [12, Corollary 2.8]. Indeed, the second fundamental form α is the differential of the Gauss map [8, p. 379] which is degenerated, in the sense of [8], if and only if $\mu > 0$.

One of the goals of this paper is to give a short and geometric proof of Theorem 1.1 based on Jacobi fields.

For complex submanifolds of the complex Euclidean space $\mathbb{C}^n$ Theorem 1.1 does not hold. Indeed, cylinders in $\mathbb{C}^3$ over complete curves of $\mathbb{C}^2$ are examples showing that μ can be different from zero. A more interesting example is the hypersurface H of $\mathbb{C}^4$ defined by the equation $w = xz^2 - yz$, where (x, y, z, w) are the standard complex coordinates of $\mathbb{C}^4$. The hypersurface H is not a cylinder but it is complete and its index of relative nullity $\mu = 1$. So in order to generalize Theorem 1.1 to complex submanifolds of $\mathbb{C}^n$ one needs further assumptions. In this direction, Abe [2, Theorem 7] proved a splitting result under a non vanishing assumption of the holomorphic sectional curvatures.

By imposing conditions on the Ricci curvatures we have the following splitting theorem.

Theorem 1.2. *Let $M^m \subset \mathbb{C}^n$ be a m-dimensional complex submanifold and let $\mathbf{Ric}_M$ be its Ricci tensor w.r.t. the induced Riemannian metric. Let U be the open subset of M where $\bar{\mu}(x) = \mu$. Assume that there is a point $q \in U$ such that for any sequence of unit tangent vectors $X_{p_j} \in RN_{p_j}^{\perp}$, where $p_j \in U$ is an unbounded sequence of points, the following holds:*

$$\limsup_{j \to \infty} |\mathbf{Ric}_M(X_{p_j}, X_{p_j})| \mathrm{dist}^2(p_j, q) = \infty \tag{1}$$

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If M is complete then M splits as an extrinsic product of a leaf of the relative nullity distribution by a complex complete submanifold, i.e. M is cylindrical.

In the special case that the Ricci tensor has a pinching we have the following corollary.

Corollary 1.3. *Let $M^m \subset \mathbb{C}^n$ be a m-dimensional complex submanifold and let $\mathbf{Ric}_M$ be its Ricci tensor w.r.t. the induced Riemannian metric. Let U be the open subset of M where $\bar{\mu}(x) = \mu$. Assume that there is a constant $c < 0$ such that for all $X_p \in T_pU$ perpendicular to RN_p the following holds:*

$$\mathbf{Ric}_M(X_p, X_p) \leq c\|X_p\|^2 \,.$$

If M is complete then M splits as an extrinsic product of a leaf of the relative nullity distribution by a complex complete submanifold M' with $\mathbf{Ric}_{M'} \leq c < 0$ (in particular, M is cylindrical).

Observe that any complex submanifold of $\mathbb{C}^n$ is minimal and so its Ricci curvatures are non-positive. Moreover, there are examples of complete complex submanifolds $M' \subset \mathbb{C}^n$ with $\mathbf{Ric}_{M'} \leq c < 0$ (see Section 5.1).

2. Preliminaries

We will follow the usual notation of submanifold geometry in [3]. The Riemannian metric will be denoted by $\langle\, , \rangle$. The symbols ∇ and $\mathbf{R}$ will denote the Levi-Civita connection and its curvature tensor of either the Fubini-Study metric of $\mathbb{CP}^n$ or the flat standard metric of $\mathbb{C}^n$. The curvature tensor $\mathbf{R}$ is explicitly given by (see [9, Proposition 7.3]):

$$\mathbf{R}_{X,Y}Z = \frac{c}{4}\left((X \wedge Y)Z + (JX \wedge JY)Z - 2\langle JX, Y\rangle JZ\right) \tag{2}$$

where J is the complex structure and $c \geq 0$ is the holomorphic constant curvature. If $c > 0$, then the Jacobi operator $J_\xi = \mathbf{R}_{\cdot,\xi}\xi$, when restricted to the orthogonal complement $\mathbb{W}$ of $\mathbb{R}\xi$, has two different eigenvalues λ_1, λ_2. Namely, $\lambda_1 = c\|\xi\|^2$, associated to the (real) eigenspace $\mathbb{R}J\xi$ and $\lambda_2 = \frac{c}{4}\|\xi\|^2$, associated to the eigenspace $\mathbb{V} = (\mathbb{C}\xi)^\perp$.

Let $\gamma(t) = \exp(t\xi)$ be a geodesic of $\mathbb{CP}^n$. A Jacobi vector field $X(t)$ along $\gamma(t)$ is a solution of the so called Jacobi equation (for details see [9, Chapter VIII]):

$$X''(t) + \mathbf{R}_{X(t),\gamma'(t)}\gamma'(t) = 0\,,$$

where $X''(t) = (\frac{D}{\mathrm{d}t})^2 X(t)$ and $\frac{D}{\mathrm{d}t}$ is the covariant derivative along $\gamma(t)$ associated to ∇.

If both initial conditions $X(0), X'(0)$ of a Jacobi vector field $X(t)$ are in $\mathbb{V}$, then

$$X(t) = \cos(\omega t)\mathbf{a} + \sin(\omega t)\mathbf{b} \tag{3}$$

where $\omega^2 = \frac{c\|\xi\|^2}{4}$ and $\mathbf{a}, \mathbf{b}$ are parallel vector fields along $\gamma(t)$ with initial conditions $X(0) = \mathbf{a}(0), X'(0) = \omega\mathbf{b}(0)$.

Let M, N be Riemannian manifolds and let $f : M \to N$ be an smooth map of constant rank. We use the language of fiber bundles, although f need not be a fiber bundle. In particular, $\mathcal{V} = \ker(\mathrm{d}f)$ is the vertical distribution and $\mathcal{H} = \mathcal{V}^\perp$ the so called horizontal distribution. A vector field $\mathbf{X}$ of M (resp. $\mathbf{Y}$) is called *vertical* (resp. *horizontal*) if it is tangent to $\mathcal{V}$ (resp. if it is tangent to $\mathcal{H}$). A vector field $\mathbf{X}$ of M is *basic* if it is horizontal and locally it is f-related to a vector field $\mathbf{v}$ (locally) defined on the image of f.

For us f will be the well-known Gauss map of a submanifold M. For the convenience of the reader we explain it for $\mathbb{CP}^n$. Let $M \subset \mathbb{CP}^n$ be a complex submanifold of $\mathbb{CP}^n$. The Gauss map

$$f : M \to \mathbb{G}(m, n)$$

where $\mathbb{G}(m, n)$ is the Grassmannian of all totally geodesic $\mathbb{CP}^m$ in $\mathbb{CP}^n$ [8, p. 363] f is the map

$$p \to f(p) := \exp(T_pM)$$

where $\exp(T_pM)$ is the totally geodesic projective subspace of $\mathbb{CP}^n$ tangent to T_pM at $p \in \mathbb{CP}^n$. The Grassmannian $\mathbb{G}(m, n)$ is identified with the usual Grassmannian $\mathrm{G}(m+1, n+1)$ of $(m+1)$-dimensional complex subspaces of $\mathbb{C}^{n+1}$. Let U be the open subset of M where $\bar{\mu}(x) = \mu$. Observe that for $p \in U$ the vertical distribution $\mathcal{V}_p$ is the relative nullity distribution RN_p defined in the introduction.

The following result is well-known.

Theorem 2.1. [6] *The nullity distribution RN in U is autoparallel. If M is complete then the (totally geodesic) fibers of $f_{|U}$ are also complete submanifolds.*

The first part of the above theorem is a simple consequence of the Codazzi equations (see e.g. [1, Proposition 5]). The second part was first proved in [6] by using the conullity operator. Recently, in [11] C. Olmos and F. Vittone gave a conceptual proof without using the conullity operator. It is interesting to notice that in complex or algebraic geometry the above theorem is stated and proved by using quit different language and methods (e.g. [8, p. 388, (2.19)], [12, c), Theorem 2.3]).

3. Proof of Theorem 1.1

The proof is by contradiction. So assume that M is not a totally geodesic submanifold of $\mathbb{CP}^n$ and that $\mu > 0$. We will show that this is not possible hence either $\mu = 0$ or M is a totally geodesic submanifold i.e. $\mu = \dim_{\mathbb{C}}(M)$.

Fix $p_0 \in U$ and let $\mathcal{H}_{p_0} := (RN_{p_0})^{\perp} \subset T_{p_0}M$ be the horizontal distribution at p_0. Let, for $\xi \in RN_{p_0} \setminus \{0\}$, $\gamma_\xi(t) := \exp_{p_0}(t.\xi)$ be the geodesic starting at p_0 in the ξ-direction. For $v \in \mathcal{H}_{p_0}$ let $X(t)$ be the Jacobi vector field along $\gamma_\xi(t)$ obtained by the horizontal lift, along $\gamma_\xi(t)$, of the vector $\mathrm{d}f(v) \in T_{f(p_0)}f(U)$ (see [11, p. 91]). We define a map $\mathrm{T} : RN_{p_0} \times \mathcal{H}_{p_0} \to \mathcal{H}_{p_0}$ as follows:

$$\mathrm{T}(\xi, v) := X'(0) = \left.\frac{\mathbf{D}}{\mathrm{d}\,t}\right|_{t=0} X(t)\,.$$

That $\mathrm{T}(\xi, v)$ lands in $\mathcal{H}_{p_0}$ is due to the the first part of Lemma 2.1 (cf. with the example of the Segre embedding of the last section where $X'(0)$ is not horizontal w.r.t. the projection f to a factor).

Observe that $\mathrm{T}(\xi, v)$ can be defined as

$$\mathrm{T}(\xi, v) := (\nabla_\xi X)_{p_0} \tag{4}$$

where X the horizontal lift of $\mathrm{d}f(v) \in T_{f(p_0)}f(U)$ along the whole fiber $f^{-1}(f(p_0))$.

Lemma 3.1. *The map $\mathrm{T} : RN_{p_0} \times \mathcal{H}_{p_0} \to \mathcal{H}_{p_0}$ is $\mathbb{C}$-bilinear.*

Proof. The $\mathbb{C}$-linearity in v is clear due to the facts that the Riemannian metric of $\mathbb{CP}^n$ is Kähler and f is holomorphic. The $\mathbb{R}$-linearity in ξ follows from Equation (4). Let $\mathbf{v}$ be vector field defined around $f(p_0) \in f(U)$ such that $\mathbf{v}_{f(p_0)} = \mathrm{d}f(v)$. Let $\mathbf{X}$ be the horizontal lift of $\mathbf{v}$ and let $\bar{\xi}$ be a vertical local extension of ξ around $p_0 \in M$. Let J be the complex structure of $\mathbb{CP}^n$. Then a direct computation shows

$$\nabla_{J\bar{\xi}}\mathbf{X} = J\nabla_{\bar{\xi}}\mathbf{X} + J[\mathbf{X}, \bar{\xi}] + [J\bar{\xi}, \mathbf{X}]\,.$$

Since $\mathrm{T}(\xi, v)$ is horizontal we conclude that

$$(\nabla_{J\bar{\xi}}\mathbf{X}) = J\nabla_{\bar{\xi}}\mathbf{X} \quad \text{and} \quad J[\mathbf{X}, \bar{\xi}] + [J\bar{\xi}, \mathbf{X}] = 0$$

(we have used that the Lie bracket of basic vector field by a vertical one is vertical).

This shows that $\mathrm{T}(\xi, v)$ is also $\mathbb{C}$-linear in ξ. □

For fixed $\xi \in RN_{p_0} \setminus \{0\}$ let us denote by $\mathrm{T}_\xi := \mathrm{T}(\xi, \cdot\,)$. The proof of Theorem 1.1 is based on the following lemma:

Lemma 3.2. *Assume that $0 < \mu < \dim M$. Then, for any $0 \neq \xi \in RN_{p_0} \setminus \{0\}$, the linear map T_ξ has no real eigenvalues.*

Proof. Assume that there is $\lambda \in \mathbb{R}$ and $v \in \mathcal{H} \setminus \{0\}$ such that

$$T_\xi(v) = \lambda v\,.$$

Let $X(t)$ be the Jacobi vector field along $\gamma_\xi(t)$ obtained by the restriction to $\gamma_\xi(t)$, of the horizontal lift of the vector $\mathrm{d}f(v) \in T_{f(p_0)}f(U)$. Observe that both initial conditions of $X(t)$ are proportional to v and that $X(t)$ is a Jacobi vector field of $\mathbb{CP}^n$ along $\gamma_\xi(t)$. Then from Equation (3) we get

$$X(t) = \cos(\omega t)\mathbf{v} + \sin(\omega t)\frac{\lambda}{\omega}\mathbf{v} = \left(\cos(\omega t) + \sin(\omega t)\frac{\lambda}{\omega}\right)\mathbf{v}\,.$$

Since the fibers of the Gauss map are complete the vector field $X(t)$ is defined for all values of $t \in \mathbb{R}$. So if we take $t_0 \in \mathbb{R}$ such that $\cos(\omega t_0) + \sin(\omega t_0)\frac{\lambda}{\omega} = 0$ then $X(t_0) = 0$. But this contradicts the construction of $X(t)$ as the horizontal lift of a non-zero vector $\mathrm{d}f(v) \in T_{f(p_0)}f(U)$. $\square$

Let us now prove Theorem 1.1. Choose any $\xi \in RN_{p_0} \setminus \{0\}$. Let λ be a complex eigenvalue of $\mathbf{T}_\xi$. So there is $v \neq 0$ such that $\mathrm{T}(\xi, v) = \lambda v$. From Lemma 2.1 $\lambda \notin \mathbb{R}$ and so $\lambda \neq 0$. Then $\mathrm{T}(\lambda^{-1}\xi, v) = v$. Hence 1 is an eigenvalue of $\mathrm{T}_{\lambda^{-1}\xi}$ contradicting Lemma 2.1. This completes our proof of Theorem 1.1. $\square$

4. Proof of Theorem 1.2

Let $M \subset \mathbb{C}^n$ be a complex submanifold and let $f : M \to \mathrm{G}(m,n)$ be its Gauss map. Let g_{G} be the standard Riemannian metric of $\mathrm{G}(m,n)$. The following identity is crucial for the proof:

$$\mathbf{Ric}_M = -f^* g_{\mathrm{G}} \tag{5}$$

where $\mathbf{Ric}_M$ is the Ricci tensor of M. That is, the Ricci tensor is minus the pull-back, via the Gauss map, of the canonical metric of the Grassmannian.

Then if $X(t)$ is the basic Jacobi vector field along the geodesic $\gamma_\xi(t) := \exp_{p_0}(t.\xi)$, defined in the proof of Theorem 1.1, we get

$$-\mathbf{Ric}_M(X(t), X(t)) = f^* g_{\mathrm{G}}(X(t), X(t)) = g_{\mathrm{G}}(\mathrm{d}f(X(t)), \mathrm{d}f(X(t))) = g_{\mathrm{G}}(\mathrm{d}f(v), \mathrm{d}f(v)))\,.$$

So $g_{\mathrm{G}}(\mathrm{d}f(v), \mathrm{d}f(v)))$ is a constant hence $\mathbf{Ric}_M(X(t), X(t))$ does not depends on t.

On the other hand, the Jacobi vector field $X(t)$ along the geodesic $\gamma_\xi(t)$ of $\mathbb{C}^n$ has the form

$$X(t) = X(0) + tX'(0)\,.$$

So if $X'(0) \neq 0$ we get that $\|X(t)\|^2 = o(\mathrm{dist}^2(\gamma_\xi(t), q))$ as $t \to \infty$. Then

$$\begin{aligned} |\mathbf{Ric}_M(X(t), X(t))| &= \|X(t)\|^2 |\mathbf{Ric}_M(\frac{X(t)}{\|X(t)\|}, \frac{X(t)}{\|X(t)\|})| \\ &= o(\mathrm{dist}^2(\gamma_\xi(t), q)) |\mathbf{Ric}_M(\frac{X(t)}{\|X(t)\|}, \frac{X(t)}{\|X(t)\|})| \end{aligned}$$

as $t \to \infty$. If we now assume equation (1), then $|\mathbf{Ric}_M(X(t), X(t))|$ goes to infinity. A contradiction, since it is constant as explained above. Thus, we must have $X'(0) = 0$ for any basic Jacobi vector field constructed as in the proof of Theorem 1.1. So $\nabla_\xi X = 0$ for any ξ vertical and X basic vector field. This implies that M is a local product $\mathbb{C}^\mu \times M'$ around each point of the open subset $U \subset M$. Let $\widetilde{M}$ be the universal cover of M endowed with the pull-back Riemannian metric. Since the Riemannian metric is real analytic the local Riemannian product (of an open subset) implies that also the global holonomy group of $\widetilde{M}$ has a flat factor $\mathbb{C}^\mu$. Then by De Rham's splitting Theorem $\widetilde{M}$ splits as $\mathbb{C}^\mu \times M'$. Then also the submanifold M is an extrinsic product as it follows by using either Moore's Lemma [1, Theorem 16] or Calabi's rigidity theorem [4, Theorem 2]. $\square$

5. Examples

In the first part we construct an example of a complete complex submanifolds $M' \subset \mathbb{C}^n$ with $\mathbf{Ric}_{M'} \leq c < 0$. In the second part, by using the Segre embedding $\mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathbb{CP}^3$ we will see the importance of various ingredients of our proof of Theorem 1.1.

5.1. **Example of complete complex submanifold M with $\mathbf{Ric}_{M'} \leq c < 0$.** Let S be a non hyperelliptic compact Riemannian surface of genus $g \geq 3$. That S is non hyperelliptic means that any non constant holomorphic map $f : S \to \mathbb{CP}^1$ has degree greater than 2. Here we follow the notation in [7, p. 228]. Let $\mathcal{J}(S) := \mathbb{C}^g/\Lambda$ be the Jacobian variety of S. Then there is a natural embedding

$$\tau : S \to \mathcal{J}(S)\,.$$

The map τ is usually called the Abel-Jacobi map. Then the flat metric of the torus $\mathcal{J}(S)$ induces a metric ρ_0 on S called the Bergmann metric in [10, p. 317] (ρ_0 is also called the *theta* metric by other authors). Since S is non hyperelliptic the theorem in [10, p. 317] implies that the Gauss curvature $\kappa(p)$ of the metric ρ_0 is different from zero at all points $p \in S$. Thus, since S is compact, there are constants $a < b < 0$ such that

$$a < \kappa(p) < b < 0$$

for all points $p \in S$.

Let $\widetilde{S}$ be the universal cover of S. Then, by a basic property of universal covers concerning the lifting of mappings, we have a map $\widetilde{\tau} : \widetilde{S} \to \mathbb{C}^g$ making the following diagram commutative:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{\tau}} & \mathbb{C}^g \\ \big\downarrow \pi & & \big\downarrow \pi \\ S & \xrightarrow{\tau} & \mathcal{J}(S) \end{array}$$

Then $M' := \widetilde{\tau}(\widetilde{S})$ is a complex submanifold of $\mathbb{C}^g$ such that the induced metric is complete and $\mathbf{Ric}_{M'} = 2\kappa < 2b = c < 0$ on M'.

Remark 5.1. The immersion $\widetilde{\tau} : \widetilde{S} \to M'$ is not injective. Indeed, the fundamental group of M' is the commutator group $[\pi_1(S), \pi_1(S)]$, i.e. M' is the Galois covering of S associated to the commutator group $[\pi_1(S), \pi_1(S)]$.

5.2. **The Segre embedding.** The Segre embedding $\mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathbb{CP}^3$ is defined in homogeneous coordinates as

$$([x, y], [u, v]) \to [xu : xv : yu : yv]\,.$$

It is standard to check that the above map is isometric w.r.t. the Fubini-Study metrics. Let $f : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ be the projection to the first factor. We can regard as vertical distribution the tangent spaces to the fibers of f i.e. the tangent spaces to the second factor. The horizontal distribution coincides with the distribution associated to the first factor. Observe that both horizontal and vertical distributions are totally geodesic. Then any tangent vector v_p, where p is a point of the first factor, can be horizontally lifted to Jacobi vector field X along the fiber $f^{-1}(p)$. In this case the initial condition $X'(0)$ can not be horizontal for all such Jacobi vector fields. Indeed, if for all horizontal lifts X the initial condition $X'(0)$ is horizontal then the shape operator of $\mathbb{CP}^1 \times \mathbb{CP}^1$ preserves the factors. But this is not possible as proved in [1, Theorem 17].

Then it is crucial, in Theorem 1.1, that the totally geodesic fibers are given by the Gauss map.

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