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HELIX SURFACES IN EUCLIDEAN SPACES

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Abstract. We study helix surfaces with parallel mean curvature vector in Euclidean space from a local point of view. Our main result says that they are either part of a cylinder of revolution or a plane. One way to prove this is with the generalization we found about the Laplacian of a support function of a hypersurface. This allows us to study the constant mean curvature surfaces in space forms which have constant angle with respect to a closed and conformal vector field. The result we find says that these surfaces are totally umbilic.

1. Introduction

Helix or constant angle submanifolds are those whose angle function between its tangent spaces and a vector field in the ambient is constant. In 2012 Dillen and his collaborators investigated in [2] the surfaces in the ambient $\mathbb{S}^3 \times \mathbb{R}$ which have constant angle with respect to the parallel vector field induced by the $\mathbb{R}$-direction. They obtained the classification of such surfaces with parallel mean curvature vector. In [10, page 66, Theorem 5.1] the third author showed that a complete surface of $\mathbb{R}^n$ with parallel mean curvature whose tangent space made a constant angle with respect to a fixed vector is either totally geodesic or a revolution cylinder. His proof is based on Chen-Yau's classification of surfaces with parallel mean curvature [3, page 660], [13, page 358, Theorem 4]. Both Chen and Yau proofs are related to ideas of H. Hopf and uses isothermal coordinates and a holomorphicity argument. The first goal of the present paper is to show that the completeness assumption in [10, page 66, Theorem 5.1] can be removed. Namely, we have the following local result.

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Theorem 1. *A surface of $\mathbb{R}^n$ with parallel mean curvature whose tangent space made a constant angle $\theta \neq \frac{\pi}{2}$ with respect to a fixed vector is either totally geodesic or an open subset of a revolution cylinder.*

We point out that our first proof does not use Chen-Yau's classification and is based in elementary facts related to the Gauss-Codazzi-Ricci equations from submanifold geometry [1].

In Section 6, we develop some tools to give a second proof of our main result Theorem 1 with the help of the mentioned Chen-Yau's result. In particular, we generalize in our Proposition 6.7 the formula in Proposition 1.3.5 given in [12] about the Laplacian of a support function of a hypersurface. One condition to obtain this more general formula is to consider a closed conformal vector field in a space form in instead of a constant vector field or parallel in Euclidean space.
As other application of this technique we investigated constant angle surfaces of three dimensional space forms with constant mean curvature. In Theorem 6.18, we prove that constant mean curvature surfaces in space forms which have constant angle with respect to a closed and conformal vector field are totally umbilic.

2. Preliminaries and basic properties

Let us recall how to use an eikonal function to construct a helix submanifold, as in [8] we will call it the **projection method**. Let $M \subset \mathbb{R}^n$ be a helix submanifold of angle $\theta \notin \{0, \frac{\pi}{2}\}$ with respect to the unit vector $\overrightarrow{\mathrm{d}}$. So we have

$$\overrightarrow{\mathrm{d}} = \cos(\theta)T + \sin(\theta)\xi \tag{1}$$

where T tangent to M and ξ normal to M. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the orthogonal projection to the orthogonal complement of $\overrightarrow{\mathrm{d}}$ identified with $\mathbb{R}^{n-1}$. The restriction of π to M is an immersion. Then M looks locally as the graph of a function $f : U \subset \pi(M) \to \mathbb{R}$. That is to say, M is locally the image of the map $\phi : \pi(M) \to \mathbb{R}^{n-1} \times \mathbb{R}$ define as

$$\phi(p) := (i(p), f(p)) \tag{2}$$

where i is the canonical inclusion of $\pi(M)$. In simple words we can start with a Riemannian submanifold $B \subset \mathbb{R}^{n-1}$ and a function $f \in C^\infty(B)$ and construct $M \subset \mathbb{R}^n$ has the graph of f. In [6] we proved that M is an helix if f is a so called Eikonal function of B. Moreover, any helix comes locally from such a construction. This is what we mean by the **projection method**. We will call the manifold B the **base** of the helix M.

We know from [7] that a helix submanifold of arbitrary codimension satisfies the equations: For every $X \in TM$,

$$\cos(\theta)\nabla_X T - \sin(\theta)A^{\xi}(X) = 0, \tag{3}$$

$$\cos(\theta)\alpha(X,T) + \sin(\theta)\nabla^{\perp}_X \xi = 0, \tag{4}$$

where ∇, α, A^{ξ} is the Levi-Civita connection, the second fundamental form and the shape operator, respectively, of the isometric immersion of M in $\mathbb{R}^n$.

3. Helix surfaces with constant curvature or with parallel normal direction

The following proposition shows how to construct helix surfaces with constant Gauss curvature via the projection method.

Proposition 3.1. *An immersed helix surface in $\mathbb{R}^n$ has constant curvature if and only if either it is a cylinder over a curve or its base has the same signed constant curvature .*

Proof. Let B the base of the helix surface M and let f be the eikonal function on B given by the projection method. Let us observe that the function f induces a function on M call also f. In our previous work [8] in Proposition 2.11 we proved that $Ricc_M(\nabla f, \nabla f)$ and $Ricc_B(\nabla f, \nabla f)$, calculated in the corresponding Riemannian structures of M and B respectively, are related by multiplying one of them by a positive constant to obtain the other one.
But for the case of Riemannian surfaces, as M and B, the Ricci curvature becomes the sectional curvature. So, we are done. □

We recall that a submanifold $M \subset \mathbb{R}^n$ is said to be full if it is not contained in a hyperplane of $\mathbb{R}^n$ (see [1, page 22]). The following proposition gives the description of helix surfaces with parallel normal direction introduced in [7, Section 6].

Proposition 3.2. *Let M be an immersed full helix surface in $\mathbb{R}^n$. Then ξ is normal parallel if and only if M is flat and ruled helix.*

Proof. Since M is full the vector field T is well-defined. Indeed, T is not defined only in the case that $\overrightarrow{\mathrm{d}}$ is perpendicular to TM i.e. when the helix angle $\theta = \frac{\pi}{2}$. Since M has dimension two ξ is parallel if and only if $\nabla^{\perp}_T \xi = 0$ and $\nabla^{\perp}_W \xi = 0$, where W is a unit vector field perpendicular to T. We know by Equation (4) that the first condition is equivalent to $\alpha(T,T) = 0$ which is equivalent for M to be a ruled helix. Similarly,

the condition $\nabla^{\perp}_{W}\xi = 0$ is equivalent to $\alpha(T, W) = 0$.
Therefore, ξ is normal parallel if and only if

$$\alpha(T, T) = 0 \text{ and } \alpha(T, W) = 0.$$

By Gauss equation, M is flat if and only if

$$\langle \alpha(T, T), \alpha(W, W)\rangle - |\alpha(T, W)|^2 = 0.$$

Therefore, if ξ is normal parallel it implies that M is flat and ruled.
Finally, if M is flat and ruled then $|\alpha(T, W)|^2 = 0$, i.e. $\alpha(T, W) = 0$. $\square$

4. Minimal helix in spheres

Let $M^2 \to \mathbb{R}^n$ be a helix surface of angle $0 < \theta < \frac{\pi}{2}$ with helix direction $\overrightarrow{\mathrm{d}} = e_1$. Then the vector field T is (locally) the gradient of an eikonal function $t : M^2 \to \mathbb{R}$. So around any point there exists a local chart (t, x) where the metric of M has the form

$$\mathrm{d}s^2 = dt^2 + \frac{\mathrm{d}x^2}{B^2},$$

where $B(t, x)$ is smooth. Then with respect to this local chart the immersion $\phi : M^2 \to \mathbb{R}^n$ is given by

$$\phi(t, x) = (\cos(\theta)t, F(t, x)).$$

Now, if M^2 is minimal in a sphere of radius r then according to Beltrami's formula $\Delta\phi = -2r^{-2}\phi$. This implies

$$\Delta t = -2r^{-2}t$$

where Δ is the Laplacian of $\mathrm{d}s^2$. Here is the Laplacian of $\mathrm{d}s^2$:

$$\Delta = B\frac{\partial}{\partial t}\left(B^{-1}\frac{\partial}{\partial t}\right) + B\frac{\partial}{\partial x}\left(B\frac{\partial}{\partial x}\right).$$

So

$$-2r^{-2}t = \Delta t = B\frac{\partial}{\partial t}\left(B^{-1}\right) = -\frac{\partial \log(B)}{\partial t}$$

and we get

$$B(t, x) = \mathrm{e}^{r^{-2}t^2 + a(x)}.$$

Lemma 4.1. *Let $M^2 \to \mathbb{R}^n$ be a helix surface of angle $0 < \theta < \frac{\pi}{2}$. Assume that $M^2 \to \mathbb{R}^n$ is a minimal surface of a sphere of radius r. Then around each point $p \in M^2$ there exists a coordinate system (t, y) (centered at p, i.e. $(0, 0)$ corresponds to p) such that*

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{e}^{-2r^{-2}t^2}\mathrm{d}y^2 \tag{5}$$

Proof. According to the previous discussion we have a coordinate system of the form

$$\mathrm{d}s^2 = dt^2 + \frac{\mathrm{d}x^2}{B^2},$$

where $B(t,x) = \mathrm{e}^{r^{-2}t^2 + a(x)}$. Note that $B(t,x) = \mathrm{e}^{r^{-2}t^2}.\mathrm{e}^{a(x)}$. Let $y(x)$ such that $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-a(x)}$. Then

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{e}^{-2r^{-2}t^2} dy^2$$

as we claimed. □

By using the formula for the Gaussian curvature

$$\kappa = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial u} \frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v} \frac{E_v}{\sqrt{EG}} \right)$$

we get that the Gauss curvature of the metric $\mathrm{d}s^2$ is

$$2r^{-2}(1 - 2r^{-2}t^2) \ .$$

So we have the following result.

Proposition 4.2. *A helix surface $M^2 \to \mathbb{R}^n$ with helix angle $0 \le \theta < \frac{\pi}{2}$ can not be a minimal surface of a sphere.*

Proof. Assume by contradiction that such a minimal helix M^2 of a sphere do exists. Then, by the above lemma any point p of M^2 has a local chart (t, y) centered on p on which the metric has the same expression i.e. given by equation (5). So the metric of M^2 must have constant Gauss curvature. Indeed, the Gaussian curvature at point p is computed through the local expression of the metric in coordinates (t, y). But the metric $\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{e}^{-2r^{-2}t^2}\mathrm{d}y^2$ has not constant curvature as a direct computation shows. This contradiction proves that the helix M^2 cannot be a minimal surface of a sphere. □

5. Helix surfaces with parallel mean curvature (PMC)

Lemma 5.1. *Let $M^2 \subset \mathbb{R}^n$ be a PMC helix surface with helix angle $\theta < \frac{\pi}{2}$. Assume that $\alpha(T, W) \equiv 0$. Then M^2 is either totally geodesic or (locally) a cylinder of revolution contained in an affine 3-dimensional subspace of $\mathbb{R}^n$.*

Proof. The condition $\alpha(T, W) \equiv 0$ implies $R^\perp \equiv 0$ so by [5, page 57, Theorem 4.4] we can assume $n = 4$. The condition to be PMC implies that the following expression is tangent to M

$$D_W H = D_W \left(D_T T + D_W W - \langle D_W W, T \rangle T \right)$$

where H is the mean curvature of M. Then

$$\begin{aligned}
D_W H &\equiv D_W D_T T + D_W D_W W - \langle D_W W, T\rangle D_W T \\
&\equiv D_W D_T T + D_W D_W W - \langle D_W W, T\rangle (\nabla_W T + \alpha(T,W)) \\
&\equiv D_W D_T T + D_W D_W W \\
&\equiv D_T D_W T + D_{[W,T]} T + D_W D_W W \\
&\equiv D_T D_W T + \alpha([W,T],T) + D_W D_W W \\
&\equiv D_T D_W T + \alpha(\nabla_W T, T) + D_W D_W W \\
&\equiv D_T D_W T + D_W D_W W \\
&\equiv D_T \nabla_W T + D_W D_W W \\
&\equiv \alpha(T, \nabla_W T) + D_W D_W W \\
&\equiv D_W D_W W
\end{aligned}$$

where the symbol $\equiv$ means an equality module $\Gamma(TM)$. Now by ([7]) the integral curve of W is contained in a hyperplane perpendicular to $\overrightarrow{\mathrm{d}}$. By using the Frenet-Serret frame in the hyperplane containing the integral curve of W we get

$$D_W D_W W \equiv k_1' N_1 - k_1^2 W + k_1 k_2 N_2 \, .$$

So if $D_W D_W W$ is tangent to M then either T is tangent to the hyperplane or $k_1' = 0, k_1 k_2 = 0$. Since we assumed the helix angle to be different from $\frac{\pi}{2}$ the vector field T cannot be tangent to the hyperplane. Then we have to cases:

(a) $k_1 \equiv 0$. Then the integral curve of W is an straight line in $\mathbb{R}^4$ and the surface must be an extrinsic product. Then the surface M^2 is an extrinsic product $\mathbb{R} \times \gamma$ where $\gamma \subset \mathbb{R}^3$ the integral curve of T is a helix curve with PMC hence it is either a piece of a circle or a line in a plane and we are done.

(b) $k_1 \neq 0$. Then $k_1 \equiv c \neq 0$ and $k_2 \equiv 0$ hence the integral curve S of W is a circle in a 2-plane. To complete the proof we are going to show that this case also gives a totally geodesic or (locally) a cylinder of revolution contained in an affine 3-dimensional subspace of $\mathbb{R}^4$.

Notice that an integral curve $\gamma(t)$ of T starting at a point p of S is contained in the normal spaces $\nu_p(S)$ of S as submanifold of $\mathbb{R}^4$. So as in [9, page 92, Lemma 4.7] we have that the surface M^2 is constructed by normal parallel transport of $\gamma(t) \subset \nu_p(S)$ along S. The parallel transport of the normal connection of a circle S in a 2-plane $\Pi \subset \mathbb{R}^4$ is given by the rotations around the center of S. Thus we get a local parametrization of $M^2 \subset \mathbb{R}^4$ as follows: Put coordinates in $\mathbb{R}^4$ such that

- $\Pi =$ linear span$\{\mathrm{e}_1, \mathrm{e}_2\}$,
- the center of S is the origin of Π,
- the point p is $r\mathrm{e}_1$,
- the helix direction $\overrightarrow{\mathrm{d}} = \mathrm{e}_4$,

so $\gamma(t) \subset p +$ linear span$\{\mathrm{e}_1, \mathrm{e}_3, \mathrm{e}_4\}$. Since $\gamma(t)$ is a helix w.r.t. $\overrightarrow{\mathrm{d}}$ we have the following parametrization of $\gamma(t)$:

$$\gamma(t) = \begin{pmatrix} x(t) \\ 0 \\ z(t) \\ t\cos(\theta) \end{pmatrix}$$

Then the local parametrization of $M^2 \subset \mathbb{R}^4$ is as follows:

$$\begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) & 0 & 0 \\ \sin(\vartheta) & \cos(\vartheta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ 0 \\ z(t) \\ t\cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\vartheta)x(t) \\ \sin(\vartheta)x(t) \\ z(t) \\ t\cos(\theta) \end{pmatrix}$$

In terms of this parametrization the mean curvature vector field H is given by:

$$\begin{aligned} H &= D_T T + D_W W - \langle D_W W, T\rangle T \\ &= \begin{pmatrix} \cos(\vartheta)x''(t) \\ \sin(\vartheta)x''(t) \\ z''(t) \\ 0 \end{pmatrix} + \frac{1}{x(t)} \begin{pmatrix} -\cos(\vartheta) \\ -\sin(\vartheta) \\ 0 \\ 0 \end{pmatrix} + \frac{x'(t)}{x(t)} \begin{pmatrix} \cos(\vartheta)x'(t) \\ \sin(\vartheta)x'(t) \\ z'(t) \\ \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\vartheta)x''(t) - \frac{\cos(\vartheta)}{x(t)} + \frac{x'(t)^2\cos(\vartheta)}{x(t)} \\ \sin(\vartheta)x''(t) - \frac{\sin(\vartheta)}{x(t)} + \frac{x'(t)^2\sin(\vartheta)}{x(t)} \\ z''(t) + \frac{x'(t)z'(t)}{x(t)} \\ \frac{x'(t)\cos(\theta)}{x(t)} \end{pmatrix} \end{aligned}$$

The condition $\nabla^{\perp}_T H = 0$ implies $D_T H = mT$ for a smooth function m. So we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \cos(\vartheta)x''(t) - \frac{\cos(\vartheta)}{x(t)} + \frac{x'(t)^2\cos(\vartheta)}{x(t)} \\ \sin(\vartheta)x''(t) - \frac{\sin(\vartheta)}{x(t)} + \frac{x'(t)^2\sin(\vartheta)}{x(t)} \\ z''(t) + \frac{x'(t)z'(t)}{x(t)} \\ \frac{x'(t)\cos(\theta)}{x(t)} \end{pmatrix} = m \begin{pmatrix} \cos(\vartheta)x'(t) \\ \sin(\vartheta)x'(t) \\ z'(t) \\ \cos(\theta) \end{pmatrix}$$

which is equivalent to the following system

$$\begin{cases} \Big(x''(t) - \frac{1}{x(t)} + \log(x(t))'.x'(t)\Big)' = \log(x(t))''.x'(t)\,, \\ \\ (z(t)'' + \log(x(t))'z'(t))' = \log(x(t))''.z'(t)\,. \end{cases}$$

Then we get

$$\begin{cases} x'''(t) + \log(x(t))'.x''(t) = (\frac{1}{x(t)})'\,, \\ \\ z(t)''' + \log(x(t))'z''(t) = 0\,. \end{cases}$$

The second equation implies $x(t)z''(t) = c_2$ for $c_2 \in \mathbb{R}$. The first equation is equivalent to

$$\frac{x(t)x'''(t) + x'(t)x''(t)}{x(t)} = \frac{(x(t)x''(t))'}{x(t)} = (\frac{1}{x(t)})'\ .$$

Then we have

$$\begin{cases} x(t)z''(t) = c_2\,, \\ x(t)x''(t) = -\log(x(t)) + c_1\,. \end{cases}$$

Since $\gamma(t)$ is parametrized by the natural parameter we get the following over-determined system

$$\begin{cases} x(t)z''(t) = c_2\,, \\ x(t)x''(t) = -\log(x(t)) + c_1\,, \\ x'(t)^2 + z'^2(t) = \sin^2(\theta)\,. \end{cases}$$

We claim that the above system have no solutions for $c_2 \neq 0$. Indeed, set $v(t) := x'(t)$ and $w(t) := z'(t)$ and $f(x) := \frac{\log(x) - c_1}{x}$. Then derivation of $v^2 + w^2 = \sin^2(\theta)$ yields to

$$0 = v.v' + w.w' = vf(x) + w\frac{c_2}{x} = 0\,,$$

so

$$w = -\frac{vxf(x)}{c_2}$$

Another derivation gives

$$0 = v^2 A + B \tag{6}$$

where

$$\begin{cases} A = f(x) + xf'(x)\,, \\ B = xf^2(x) + (\frac{c_2}{x})^2\,. \end{cases}$$

Another derivation of equation (6) gives

$$v(v^2A' + B' + 2f(x)A) = 0\,.$$

So if $v \equiv 0$ we get $x(t) \equiv 1$ and this implies that M^2 is contained in an hyperplane and we are done. If $v \neq 0$ elimination of v^2 yields the following ODE for $f(x)$

$$2f(x) + (\frac{B}{A})' = 0\,.$$

A straightforward calculation shows that

$$2f(x) + (\frac{B}{A})' = \frac{c_2^2 + 4c_1 x - 4x\log(x)}{x^2}$$

and this is not possible since $\log(x)$ is not a rational function.
This complete the proof of Lemma 5.1. □ □

5.1. **Proof of Theorem 1.** Here we give the proof of Theorem 1.

Proof. Let H be the mean curvature vector field of the surface. If $H \equiv 0$ then the surface is a totally geodesic submanifold as follows from [10, page 58, Theorem 3.2]. So we can assume $H \neq 0$. If the shape operator A^H is a multiple of the identity then the surface is a minimal surface of some sphere. This is not possible by Proposition 4.2. So we have that the shape operator A^H has two different eigenvalues. Then the Ricci equation $\langle R^{\perp}_{X,Y}H, \eta\rangle = \langle [A^H, A^{\eta}]X, Y\rangle$ implies that the surface has flat normal bundle i.e. $R^{\perp} \equiv 0$. Since we assumed $\theta \neq \frac{\pi}{2}$ the tangent vector T in equation (1) is not zero. Since T is in the kernel of A^{ξ} (see [7, Proposition 2.4., page 195]) we get that either $\alpha(T, W) \equiv 0$ or $A^{\xi} \equiv 0$. In the first case the theorem follows from Lemma 5.1. In the second case ξ is perpendicular to the first normal space. Then by [5, page 57, Theorem 4.4] that the helix direction $\overrightarrow{\mathrm{d}}$ is either tangent or normal to M. Since $\overrightarrow{\mathrm{d}}$ can not be normal we get that M is (locally) a cylinder i.e. $\theta = 0$. □ □

6. Surfaces in space forms

This section is motivated by the following two lemmas which say that if a helix surface in $M \subset \mathbb{R}^4$ with parallel mean curvature vector is contained in $\mathbb{S}^3$ then $M \subset \mathbb{S}^3$ is a surface with constant mean curvature and satisfies the condition $\langle Z, T\rangle$ is constant. Then we will investigate this class of surfaces in $\mathbb{S}^3$. As application we will give a second proof of Theorem 1 above about helix surfaces in $\mathbb{R}^n$ with parallel mean curvature vector. Let us give the details.

Lemma 6.1. *Let $M \subset \mathbb{R}^4$ be an isometric immersion of the surface M. Let us assume that M is contained in the unitary sphere $\mathbb{S}^3$ of $\mathbb{R}^4$. If M has parallel mean curvature vector in $\mathbb{R}^4$ then M has constant mean curvature in $\mathbb{S}^3$.*

Proof. Given the isometric immersions $M \subset \mathbb{S}^3 \subset \mathbb{R}^4$ we will denote by $\alpha_M, \alpha_S, \alpha$ the second fundamental forms of $M \subset \mathbb{R}^4$, $\mathbb{S}^3 \subset \mathbb{R}^4$ and $M \subset \mathbb{R}^4$, respectively. Similarly, we will use H_M, H_S and H to denote the mean curvature vector of the immersions. Let us observe that H_M is constant by hypothesis.
So we want to prove that $\langle H, H\rangle$ is constant in M:
Let $x \in M \subset \mathbb{S}^3 \subset \mathbb{R}^4$ be any point and $X, Y \in T_pM$ two arbitrary tangent vectors. It well known that $\alpha_S(X,Y) = -\langle X, Y\rangle x$ and that $\alpha_M(X,Y) = \alpha(X,Y) + \alpha_S(X,Y)$.
Therefore $\alpha_M(X,Y) = \alpha(X,Y) - \langle X, Y\rangle x$. This relation implies that the next one about the mean curvature vector $H_M = H - 2x$. In particular $\langle H_M(x), x\rangle = \langle H(x), x\rangle - 2 = -2$ because $H(x)$ is tangent to $\mathbb{S}^3$ and x is orthogonal to $\mathbb{S}^3$. Now the last calculus:

$$\begin{aligned} \langle H(x), H(x)\rangle &= \langle H_M(x), H_M(x)\rangle + 4 + 2\langle H_M(x), 2x\rangle \\ &= \langle H_M(x), H_M(x)\rangle - 8. \end{aligned}$$

So, H has constant length because H_M is constant by hypothesis. □

Lemma 6.2. *Let M be an immersed surface in $\mathbb{R}^4$ and let $\vec{\mathrm{d}}$ be a constant vector field of unitary length. Let us assume that M is contained in $\mathbb{S}^3 \subset \mathbb{R}^4$. Then M is a helix in $\mathbb{R}^4$ with respect to $\vec{\mathrm{d}}$ if and only if the function $\langle Z, T\rangle$ is constant, where Z is the tangent part of $\vec{\mathrm{d}}$ in $\mathbb{S}^3$ and T is the unitary tangent part of Z in M.*

Proof. It is an exercise that $Z : \mathbb{S}^3 \longrightarrow T\mathbb{S}^3$ is given by $Z(x) := \vec{\mathrm{d}}(x) - \langle \vec{\mathrm{d}}(x), x\rangle x$. Let us observe that for $x \in M$, $T(x) = \lambda(p)(Z(x) - \langle Z(x), \xi(x)\rangle \xi(x))$ where ξ is a unitary orthogonal vector field along M and λ is defined by $\langle T, T\rangle = 1$. By substitution $T = \lambda(\vec{\mathrm{d}} - \langle \vec{\mathrm{d}}, x\rangle x - \langle \vec{\mathrm{d}}, \xi\rangle \xi)$. This proves that T is also the unitary tangent part of $\vec{\mathrm{d}}$ in M. Moreover, $\langle Z, T\rangle = \langle \vec{\mathrm{d}}, T\rangle$ because $\langle x, T\rangle = 0$. Finally, by hypothesis M is a helix with respect to $\vec{\mathrm{d}}$, i.e. $\langle \vec{\mathrm{d}}, T\rangle$ is constant along M. This finish the proof. □

6.1. **Closed conformal vector fields.** In the next results we will see that Z in Lemma 6.2, is a so called closed conformal vector field in $\mathbb{S}^3$. So, in order to investigate helix surfaces in $\mathbb{R}^4$ with parallel mean curvature vector we will study surfaces in $\mathbb{S}^3$ with constant mean curvature and constant angle with respect to a closed conformal vector field Z.

Definition 6.3. A vector field Z in a Riemannian manifold (N, g) is called *closed conformal* vector field if there exist a smooth function

$\varphi : N \longrightarrow \mathbb{R}$, such that for every vector field X in N,

$$\bar{\nabla}_X Z = \varphi X,$$

where $\bar{\nabla}$ is the Levi-Civita connection of (N, g).

Remark 6.4. We should observe that the flow of Z is given by conformal transformations and that the 1-form $\beta(\cdot) := \langle Z, \cdot \rangle$ is closed.

We are going to study surfaces of codimension one in the connected, simply connected complete space forms $\mathbb{Q}^n_c$ with constant sectional curvature $c = 1,\ -1$. First, let us describe the closed conformal vector fields in $\mathbb{Q}^n_c$ that we will use.

For this we have to consider isometric immersions of $\mathbb{Q}^n_1 := \mathbb{S}^n$ into the euclidean space $\mathbb{R}^{n+1}_c := \mathbb{R}^{n+1}$ if $c = 1$ and an isometric immersion of the hyperbolic space $\mathbb{Q}^n_{-1} := \mathbb{H}^n$ in the Minkowski space $\mathbb{R}^{n+1}_c := \mathbb{L}^{n+1}$ of dimension four if $c = -1$. Let us recall that $\mathbb{R}^{n+1}_c$ has the standard Riemannian (if $c = 1$) or Lorentzian metric (if $c = -1$) given by $\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots + cu_{n+1}v_{n+1}$. The immersion of $\mathbb{Q}^n_c$ in $\mathbb{R}^{n+1}_c$ is given by the inclusion of the points $x = (x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}_c$ such that $\langle x, x\rangle = c$. When $c = -1$ we also need the condition $x_{n+1} > 0$ to define the immersion of $\mathbb{H}^n$ in the Minkowski space $\mathbb{R}^{n+1}_c$.
Let $D, \bar{\nabla}$ the Levi-Civita connection of $\mathbb{R}^{n+1}_c$ and $\mathbb{Q}^n_c$, respectively. Finally, let us denote by $\overrightarrow{\mathrm{d}}$ a constant vector field in $\mathbb{R}^{n+1}_c$. We will assume that this parallel vector field satisfies the condition $\langle \overrightarrow{\mathrm{d}}, \overrightarrow{\mathrm{d}}\rangle = c$. With the above immersion, let us define the vector field Z on $\mathbb{Q}^n_c$ as the tangent component of $\overrightarrow{\mathrm{d}}$ into $\mathbb{Q}^n_c$, i.e.

$$Z(x) := \overrightarrow{\mathrm{d}}(x) - c\langle \overrightarrow{\mathrm{d}}(x), x\rangle x.$$

Let us observe that the position vector x is orthogonal to the described immersion of $\mathbb{Q}^n_c$ and $\langle x, x\rangle = c$.

Proposition 6.5. *Under the above construction, Z is a closed conformal vector field in $\mathbb{Q}^n_c$ satisfying $\bar{\nabla}_X Z = \varphi X$, for every vector field X on $\mathbb{Q}^n_c$ and where $\varphi(x) := -\langle \overrightarrow{\mathrm{d}}(x), x\rangle$ is a function on $\mathbb{Q}^n_c$. Moreover $c|Z|^2 + \varphi^2 = 1$ and in particular,*

$$X \cdot \varphi = -c\langle X, Z\rangle. \tag{7}$$

Proof.

$$\begin{aligned} \bar{\nabla}_X Z &= (D_X Z)^T = -(D_X(\langle \overrightarrow{\mathrm{d}}(x), x\rangle x))^T \\ &= -(\langle \overrightarrow{\mathrm{d}}(x), X\rangle x - \langle \overrightarrow{\mathrm{d}}(x), x\rangle X)^T = -\langle \overrightarrow{\mathrm{d}}(x), x\rangle X = \varphi X. \end{aligned}$$

Finally,

$$\begin{array}{rcl} |Z|^2 & = & \langle \overrightarrow{\mathrm{d}}(x), \overrightarrow{\mathrm{d}}(x)\rangle + 2c\varphi\langle \overrightarrow{\mathrm{d}}(x), x\rangle + \varphi^2\langle x, x\rangle \\ & = & c - 2c\varphi^2 + c\varphi^2 = c - c\varphi^2. \end{array}$$

Finally, we take the derivative in direction X in this relation: $2c\varphi\langle X, Z\rangle + 2\varphi(X \cdot \varphi) = 0$. This concludes the proof. □

Remark 6.6. A consequence of the equations $c|Z|^2 + \varphi^2 = 1$ and $X \cdot \varphi = -c\langle X, Z\rangle$ is that φ and then $|Z|$ are constant along orthogonal directions to Z. Moreover, if φ is a constant function then $|Z|$ is a constant function which implies that Z would be parallel, i.e. $\varphi = 0$. Finally, since c is either 1 or -1, then $\mathbb{Q}^n_c$ do not has parallel vector fields. This implies that φ is not constant along the Z direction.

The following result and its proof is a generalization of the Proposition 1.3.5 in [12], page 14. Morover, a particular case of this formula was applied by James Simons in his famous work [11] in pages 88-89, to prove an extrinsic rigidity theorem for closed minimal hypersurfaces in a round sphere.

Proposition 6.7. *Let M be a isometrically immersed oriented hypersurface in $\mathbb{Q}^{n+1}_c$ with an unit vector field $\xi \in \Gamma(TM^\perp)$ and let $Z \in \Gamma(T\mathbb{Q}^{n+1}_c)$ be a closed conformal vector field. If M has constant mean curvature then*

$$\triangle_M\langle Z, \xi\rangle + |\alpha|^2\langle Z, \xi\rangle + \varphi\langle H, \xi\rangle = 0, \tag{8}$$

where α is the second fundamental form of the immersion with squared norm $|\alpha|^2 = \sum_{i,j=1}^n |\alpha(X_i, X_j)|^2$ and $H = \sum_{i=1}^n \alpha(X_i, X_i)$ its mean curvature vector (X_i's base orthonormal of TM).

Proof. We will prove the relation pointwise. Let $p \in M$. Let $e_1, \dots, e_n$ be a local frame in M around p such that $\nabla_{e_i} e_{j|p} = 0$ for every $i,\ j$. The Laplacian is given by

$$\begin{array}{rcl} \triangle_M\langle Z, \xi\rangle & = & \sum_{i=1}^n e_i \cdot e_i \cdot \langle Z, \xi\rangle = \sum_{i=1}^n e_i \cdot \langle Z, \bar{\nabla}_{e_i}\xi\rangle \\ & = & -\sum_{i=1}^n e_i \cdot \langle Z, A^\xi(e_i)\rangle \\ & = & -\varphi\sum_{i=1}^n\langle e_i, A^\xi(e_i)\rangle - \sum_{i=1}^n\langle Z, \bar{\nabla}_{e_i}A^\xi(e_i)\rangle \\ & = & -\varphi\langle H, \xi\rangle - \sum_{i=1}^n\langle Z, \nabla_{e_i}A^\xi(e_i)\rangle \\ & & -\sum_{i=1}^n\langle Z, \alpha(e_i, A^\xi(e_i))\rangle \\ & = & -\varphi\langle H, \xi\rangle - \langle Z, \sum_{i=1}^n \nabla_{e_i}A^\xi(e_i)\rangle - |\alpha|^2\langle Z, \xi\rangle. \end{array}$$

We applied the next calculation

$$\begin{array}{rcl} \sum_{i=1}^n\langle Z, \alpha(e_i, A^\xi(e_i))\rangle & = & \sum_{i=1}^n\langle \alpha(e_i, A^\xi(e_i)), \xi\rangle\langle Z, \xi\rangle \\ & = & \sum_{i=1}^n\langle A^\xi(e_i), A^\xi(e_i)\rangle\langle Z, \xi\rangle \\ & = & |A|^2\langle Z, \xi\rangle = |\alpha|^2\langle Z, \xi\rangle. \end{array}$$

Now let us analyse the next term at point p

$$\begin{array}{rcc} \nabla_{e_i} A^\xi(e_i) & = & \sum_{j=1}^n \nabla_{e_i}(\langle A^\xi(e_i), e_j\rangle e_j) = \sum_{j=1}^n (e_i \cdot \langle \alpha(e_i, e_j), \xi\rangle) e_j \\ & = & \sum_{j=1}^n (e_i \cdot \langle \bar\nabla_{e_j} e_i, \xi\rangle) e_j = \sum_{j=1}^n \langle \bar\nabla_{e_i} \bar\nabla_{e_j} e_i, \xi\rangle e_j \\ & = & \sum_{j=1}^n \langle \bar\nabla_{e_j} \bar\nabla_{e_i} e_i, \xi\rangle e_j \\ & = & \sum_{j=1}^n \langle \bar\nabla_{e_j} \nabla_{e_i} e_i, \xi\rangle e_j + \sum_{j=1}^n \langle \bar\nabla_{e_j} \alpha(e_i, e_i), \xi\rangle e_j \\ & = & \sum_{j=1}^n \langle \alpha(e_j, \nabla_{e_i} e_i), \xi\rangle e_j + \sum_{j=1}^n \langle \nabla^\perp_{e_j} \alpha(e_i, e_i), \xi\rangle e_j \\ & = & \sum_{j=1}^n \langle \nabla^\perp_{e_j} \alpha(e_i, e_i), \xi\rangle e_j \end{array}$$

Taking the sum over i we have that,

$$\sum_{i=1}^n \nabla_{e_i} A^\xi(e_i) = \sum_{j=1}^n \langle \nabla^\perp_{e_j} H, \xi\rangle e_j = 0.$$

In the latter equality we applied that M has constant mean curvature and that $\nabla_{e_i} e_{i|p} = 0$.
Let us observe that in the above equalities we applied that for a space form holds the relation $\langle \bar\nabla_{e_i} \bar\nabla e_j e_k, \xi\rangle = \langle \bar\nabla_{e_j} \bar\nabla e_i e_k, \xi\rangle$ when e_i, e_j, e_k are orthogonal to ξ and $[e_i, e_j] = 0$ as in our case at point p. By substitution we conclude that

$$\triangle_M \langle Z, \xi\rangle = -\varphi\langle H, \xi\rangle - |\alpha|^2 \langle Z, \xi\rangle.$$

$\square$

We will need the following result in the next subsections.

Proposition 6.8. *Let M be an immersed connected surface in $\mathbb{Q}^3_c$ with constant mean curvature. Let $Z \in \Gamma(T\mathbb{Q}^3_c)$ be a closed conformal vector field on $\mathbb{Q}^3_c$. If Z is tangent to M, then M is totally geodesic.*

Proof. Let S be a surface orthogonal to Z and let us denote $\gamma := M \cap S \neq \emptyset$ and let u be a unit tangent vector to γ. In particular, $u, Z/|Z|$ is an orthonormal basis of TM.
Since $\langle Z, \xi\rangle = 0$ along M by hypothesis, the Equation (8) in the Proposition 6.7, becomes $\varphi\langle H, \xi\rangle = 0$. In our case we are assuming that φ is nowhere zero along M. This implies that $H = 0$, i.e. M is a minimal surface in $\mathbb{Q}^3_c$:

$$\alpha_{M,Q}(u, u) + \alpha_{M,Q}(Z/|Z|, Z/|Z|) = 0.$$

Now, the Gauss formula for the immersion $M \subset \mathbb{Q}^3_c$ says:

$$0 = \bar\nabla_{Z/|Z|}(Z/|Z|) = \nabla_{Z/|Z|}(Z/|Z|) + \alpha_{M,Q}(Z/|Z|, Z/|Z|),$$

where ∇ is the connection of M. Therefore, $\alpha_{M,Q}(Z/|Z|, Z/|Z|) = 0$.
So, we have that $\alpha_{M,Q}(u, u) = 0$.
Finally, $\alpha_{M,Q}(u, Z/|Z|) = \nabla_u(\bar{Z}/|Z|) - \nabla_u(Z/|Z|)$. Let us observe that Z is also closed conformal in M. This implies that $\bar\nabla_u(Z/|Z|) =$

$u \cdot (1/|Z|)Z + (1/|Z|)\varphi u$ and $\nabla_u(Z/|Z|) = u \cdot (1/|Z|)Z + (1/|Z|)\varphi u$. Hence,

$$\alpha_{M,Q}(u, Z/|Z|) = 0.$$

We conclude that M is totally geodesic. □

6.2. **Surfaces in space forms with $\langle Z, T\rangle$ constant.** Let M be an oriented surface isometrically immersed in $\mathbb{Q}^3_c$ with the property that

$$\langle Z, T\rangle = b, \tag{9}$$

is a constant function on M. Here Z is a closed conformal vector field and T is the unit tangent component of Z relative to M, i.e.

$$Z = \langle Z, T\rangle T + \langle Z, \xi\rangle \xi, \tag{10}$$

where ξ is a vector field orthogonal to M of unit constant length.
From now, in this subsection if we do not say otherwise, both $\langle Z, T\rangle \neq 0$ and $\langle Z, \xi\rangle \neq 0$.
Finally, W is a local vector field of M such that $\{T, W\}$ is a positive oriented orthonormal basis on TM.

Proposition 6.9. *Under the above conditions we have that the connection ∇, the shape operator A^ξ and the second fundamental form α of M satisfy the following equalities*

$$\begin{array}{cc} \alpha(T,T) = -\frac{\varphi}{\langle Z,\xi\rangle}\xi, & \alpha(T,W) = 0. \\ A^\xi(T) = -\frac{\varphi}{\langle Z,\xi\rangle}T & \nabla_W T = \frac{\varphi W + \langle Z,\xi\rangle A^\xi(W)}{\langle Z,T\rangle} \\ \nabla_T T = 0 & \nabla_T W = 0. \end{array}$$

In particular, T and W are principal directions of the immersion and the integral curves of T are geodesics of M.

Proof. Taking the derivative of Equation (9) in direction $X \in TM$, we have that
$\varphi\langle X, T\rangle + \langle Z, \bar{\nabla}_X T\rangle = \varphi\langle X, T\rangle + \langle Z, \nabla_X T\rangle + \langle Z, \alpha(X, T)\rangle = 0$. Using Equation (10), we deduce that

$$\alpha(X, T) = -\frac{\varphi\langle X, T\rangle}{\langle Z, \xi\rangle}\xi. \tag{11}$$

In particular, we have that $\alpha(T, W) = 0$, $\alpha(T, T) = -\frac{\varphi}{\langle Z,\xi\rangle}\xi$ and $A^\xi(T) = -\frac{\varphi}{\langle Z,\xi\rangle}T$. This says that T and W are principal directions.

Finally, we take the derivative of Equation (10):

$$\begin{aligned} \varphi X &= \langle Z, T\rangle \bar{\nabla}_X T + (X\langle Z, \xi\rangle)\xi + \langle Z, \xi\rangle \bar{\nabla}_X \xi \\ &= \langle Z, T\rangle(\nabla_X T + \alpha(X, T)) + (X\langle Z, \xi\rangle)\xi - \langle Z, \xi\rangle A^\xi(X) \end{aligned}$$

The tangent part of this equation is $\varphi X = \langle Z, T\rangle \nabla_X T - \langle Z, \xi\rangle A^{\xi}(X)$. Since T, W are principal directions, this implies that

$$\nabla_T T = 0, \quad \nabla_W T = \frac{\varphi W + \langle Z, \xi\rangle A^{\xi}(W)}{\langle Z, T\rangle}. \tag{12}$$

□

Now, let us assume that M has constant mean curvature in $\mathbb{Q}^3_c$.

Proposition 6.10. *If M satisfies that $\langle Z, T\rangle$ is constant and has constant mean curvature with mean curvature vector given by $H = \alpha(T,T) + \alpha(W,W) = a\xi$, then*

$$A^{\xi}(W) = (a + \frac{\varphi}{\langle Z, \xi\rangle})W, \quad \alpha(W,W) = (a + \frac{\varphi}{\langle Z, \xi\rangle})\xi, \tag{13}$$

$$\nabla_W W = -\frac{2\varphi + a\langle Z, \xi\rangle}{\langle Z, T\rangle} T. \tag{14}$$

Proof. Then $\langle A^{\xi}(W), W\rangle = \langle \alpha(W,W), \xi\rangle = a - \langle \alpha(T,T), \xi\rangle = a + \frac{\varphi}{\langle Z, \xi\rangle}$. We are ready for the next calculus:

$$\nabla_W W = \langle \nabla_W W, T\rangle T = -\langle W, \nabla_W T\rangle T = -\frac{2\varphi + a\langle Z, \xi\rangle}{\langle Z, T\rangle} T.$$

□

Now we are ready to calculate the Laplacian in M of $\langle Z, \xi\rangle$. We will do the calculus in two ways, first with the extrinsic geometry of M and second with the intrinsic information. We will assume that the closed conformal vector field Z satisfies as before that $c|Z|^2 + \varphi^2 = 1$. Let us assume also that M has constant mean curvature with mean curvature vector given as in Corollary 6.10.

Corollary 6.11. *(Extrinsic Laplacian) If $M \subset \mathbb{Q}^3_c$ is an isometrically immersed surface with $\langle Z, T\rangle$ constant and constant mean curvature, then*

$$\triangle_M \langle Z, \xi\rangle = \frac{-2\varphi^2 - 3a\varphi\langle Z, \xi\rangle - a^2\langle Z, \xi\rangle^2}{\langle Z, \xi\rangle}. \tag{15}$$

Proof. By Equation (8) we have to calculate

$$\triangle_M \langle Z, \xi\rangle = -|\alpha|^2 \langle Z, \xi\rangle - \varphi\langle H, \xi\rangle.$$

Since $H = \alpha(T,T) + \alpha(W,W) = a\xi$, $-\varphi\langle H, \xi\rangle = -a\varphi$. Finally,

$$\begin{array}{rcl} |\alpha|^2 & = & |\alpha(T,T)|^2 + |\alpha(W,W)|^2 + 2|\alpha(T,W)|^2 \\ & = & |\alpha(T,T)|^2 + |\alpha(W,W)|^2 = \frac{\varphi^2}{\langle Z,\xi\rangle^2} + (a + \frac{\varphi}{\langle Z,\xi\rangle})^2 \\ -|\alpha|^2\langle Z, \xi\rangle & = & \frac{-2\varphi^2 - 2a\varphi\langle Z,\xi\rangle - a^2\langle Z,\xi\rangle^2}{\langle Z,\xi\rangle}. \end{array}$$

Therefore, $\triangle_M\langle Z,\xi\rangle = \frac{-2\varphi^2-3a\varphi\langle Z,\xi\rangle-a^2\langle Z,\xi\rangle^2}{\langle Z,\xi\rangle}$.

□

Corollary 6.12. *(Intrinsic Laplacian) If $M \subset \mathbb{Q}_c^3$ is an isometrically immersed surface with $\langle Z,T\rangle$ constant and constant mean curvature, then*

$$\triangle_M\langle Z,\xi\rangle = \frac{(c\langle Z,T\rangle^2-1)\langle Z,T\rangle^2}{\langle Z,\xi\rangle^3} + \frac{2\varphi^2+a\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle}. \tag{16}$$

Proof. The intrinsic formula for the Laplacian is $\triangle_M\langle Z,\xi\rangle = TT\langle Z,\xi\rangle + WW\langle Z,\xi\rangle - \nabla_T T\langle Z,\xi\rangle - \nabla_W W\langle Z,\xi\rangle$. In our situation we have that $W\langle Z,\xi\rangle = 0$ and $\nabla_T T = 0$. So, the calculus we have to do is $\triangle_M\langle Z,\xi\rangle = TT\langle Z,\xi\rangle - \nabla_W W\langle Z,\xi\rangle$.
Let us start: $T\langle Z,\xi\rangle = -\langle Z, A_\xi T\rangle = -\langle Z,T\rangle\langle T, A_\xi T\rangle = \frac{\langle Z,T\rangle}{\langle Z,\xi\rangle}\varphi$. Now,

$$\begin{array}{rcl} TT\langle Z,\xi\rangle & = & \langle Z,T\rangle\frac{\langle Z,\xi\rangle(T\cdot\varphi)-\varphi^2\frac{\langle Z,T\rangle}{\langle Z,\xi\rangle}}{\langle Z,\xi\rangle^2} = \langle Z,T\rangle\frac{\langle Z,\xi\rangle^2(T\cdot\varphi)-\varphi^2\langle Z,T\rangle}{\langle Z,\xi\rangle^3} \\ -\nabla_W W\langle Z,\xi\rangle & = & \frac{2\varphi+a\langle Z,\xi\rangle}{\langle Z,T\rangle}T\cdot\langle Z,\xi\rangle = \frac{2\varphi+a\langle Z,\xi\rangle}{\langle Z,\xi\rangle}\varphi = \frac{2\varphi^2+a\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle}. \end{array}$$

Therefore, $\triangle_M\langle Z,\xi\rangle = \langle Z,T\rangle\frac{\langle Z,\xi\rangle^2(T\cdot\varphi)-\varphi^2\langle Z,T\rangle}{\langle Z,\xi\rangle^3} + \frac{2\varphi^2+a\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle}$.
By Proposition 6.5, $T\cdot\varphi = -c\langle Z,T\rangle$ and $c|Z|^2+\varphi^2 = 1$. By substitution:

$$\begin{array}{rcl} \triangle_M\langle Z,\xi\rangle & = & \frac{-c\langle Z,\xi\rangle^2-\varphi^2}{\langle Z,\xi\rangle^3}\langle Z,T\rangle^2 + \frac{2\varphi^2+a\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle} \\ & = & \frac{(c\langle Z,T\rangle^2-1)\langle Z,T\rangle^2}{\langle Z,\xi\rangle^3} + \frac{2\varphi^2+a\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle}. \end{array}$$

□

Theorem 6.13. *If $M \subset \mathbb{Q}_c^3$ is an isometrically immersed connected surface with $\langle Z,T\rangle$ constant and constant mean curvature, then M is a totally umbilical surface of $\mathbb{Q}_c^3$ orthogonal to Z.*

Proof. First case: We will prove that with the above hypothesis do not exist surfaces $M \subset \mathbb{Q}_c^3$ with $\langle Z,T\rangle \neq 0$ and such that $\langle Z,\xi\rangle \neq 0$ on M. Under these conditions, we can apply (16) and (15). We obtain that

$$\frac{(c\langle Z,T\rangle^2-1)\langle Z,T\rangle^2}{\langle Z,\xi\rangle^3} + \frac{4\varphi^2+a^2\langle Z,\xi\rangle^2}{\langle Z,\xi\rangle} = \frac{-4a\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle}.$$

Multiplying by $\langle Z,\xi\rangle^3$ we get that

$$(c\langle Z,T\rangle^2-1)\langle Z,T\rangle^2 + (4\varphi^2+a^2\langle Z,\xi\rangle^2)\langle Z,\xi\rangle^2 = -4a\varphi\langle Z,\xi\rangle^3.$$

Let us observe that $\varphi^2 = 1 - c\langle Z,T\rangle^2 - c\langle Z,\xi\rangle^2$ where we applied the equality $|Z|^2 = \langle Z,T\rangle^2 + \langle Z,\xi\rangle^2$.
To obtain a polynomial, denoted by $P(\langle Z,\xi\rangle)$, in the variable $\langle Z,\xi\rangle$ we have to square the latter equation. The constant term in $P(\langle Z,\xi\rangle)$ is $a_0 = (c\langle Z,T\rangle^2-1)^2\langle Z,T\rangle^4$.

When $c = -1$ this constant a_0 is non zero if and only if $\langle Z, T\rangle \neq 0$. In the case that $c = 1$, the constant $a_0 \neq 0$ if and only if $\langle Z, T\rangle \neq 0, 1$. We are working in the case $\langle Z, T\rangle \neq 0$. If we had $\langle Z, T\rangle = 1$ along M, would imply that $Z = T + \langle Z, \xi\rangle\xi$. This is a contradiction: We are assuming $c = 1$ and so $\mathbb{Q}^3_c$ is a round sphere of radius one. By construction Z is the tangent part of a constant vector field $\overrightarrow{\mathrm{d}}$ in $\mathbb{R}^4$ of unit length. Therefore the length of Z is at most one. The above formula for Z would imply that $\langle Z, \xi\rangle = 0$ and we are working in the case $\langle Z, \xi\rangle \neq 0$.
This implies that some term of degree great or equal that one of $P(\langle Z, \xi\rangle)$ is not zero. Otherwise $(c\langle Z, T\rangle^2 - 1)^2\langle Z, T\rangle^4$ would be zero. This proves that $\langle Z, \xi\rangle$ satisfies a non zero polynomial of degree greater or equal than one. So $\langle Z, \xi\rangle$ is constant and thus $|Z|$ is constant in M because $\langle Z, T\rangle$ is constant and $|Z|^2 = \langle Z, T\rangle^2 + \langle Z, \xi\rangle^2$. Since Z is closed conformal we deduce by Remark 6.6, $|Z|$ is constant only along the directions orthogonal to Z. For example $W \cdot |Z| = 0$. Therefore, we can conclude that Z is orthogonal to M and in particular $\langle Z, T\rangle = 0$. This is a contradiction because our conditions say that $\langle Z, T\rangle \neq 0$. This proves that there are not CMC surfaces such that $\langle Z, T\rangle$ is constant, $\langle Z, T\rangle \neq 0$ and $\langle Z, \xi\rangle \neq 0$.
Second case: $\langle Z, T\rangle = 0$ implies that M is orthogonal to Z and therefore M is totally umbilical orthogonal to Z.
Last case: If $\langle Z, \xi\rangle = 0$ at some point, then $\langle Z, \xi\rangle = 0$ along M. Let us prove this assertion: If there were a point with $\langle Z, \xi\rangle \neq 0$, then there is an open $U \subset M$ with this property. Now, let us observe that we can assume that $\langle Z, T\rangle \neq 0$ along M. We have a contradiction because by the first case in this proof, a surface like U can not exist. Therefore, $\langle Z, \xi\rangle = 0$ along M. The latter property means that Z is tangent to M. By Equation (10), this implies that $|Z| = |\langle Z, T\rangle|$ is constant along M. But this is not possible: It only happens if M is orthogonal to Z but we are in the case that Z is tangent to M. □

Now, we will apply Theorem 6.13 to the study of helix surfaces in $\mathbb{R}^4$ with parallel mean curvature vector.

Second Proof of Theorem 1

Proof. In this second proof we will use Chen [3] and Yau's [13] classification: A surface M of $\mathbb{R}^n$ has parallel mean curvature vector if and only if M is one of the following surfaces

- M is a minimal surface in $\mathbb{R}^n$,
- a minimal surface of a hypersphere of $\mathbb{R}^n$,

- a surface in $\mathbb{R}^3$ with constant mean curvature,
- a surface of a 3-sphere in $\mathbb{R}^4$ with constant mean curvature.

This sentence was taken from [4] page 11.
Such classification give us four possibilities. We will apply that M is also a helix surface in $\mathbb{R}^n$. If M is a minimal surface in $\mathbb{R}^n$, we conclude that it is totally geodesic and so M is an open part of a plane. See Theorem 3.1 in [10] for details. If M is a minimal surface of a hypersphere of $\mathbb{R}^n$, we can apply Proposition 4.2, to see that this does not happen. Finally, M either is a CMC surface contained in $\mathbb{R}^3$ or contained in $\mathbb{S}^3$ with constant mean curvature. In the first situation M should be a plane or a cylinder of revolution. See proof of Theorem 5.1 in [10] for details. In the second case, Lemma 6.2, M should be a surface in $\mathbb{S}^3$ with $\langle Z, T\rangle$ constant. Here Z is the tangent part of the helix direction $\overrightarrow{\mathrm{d}}$ of M. By Proposition 6.5, Z is closed conformal in $\mathbb{S}^3$. So, by Lemma 6.1, M is a CMC surface in $\mathbb{S}^3$ with $\langle Z, T\rangle$ constant, where T is the unit tangent component of Z in M. Therefore, we can apply the techniques of this subsection to study M.
Finally, by Theorem 6.13, M is a totally umbilical surface in $\mathbb{S}^3$ and orthogonal to Z. Then M is the intersection of a hyperplane $\mathbb{R}^3$, orthogonal to $\overrightarrow{\mathrm{d}}$, in the $\mathbb{R}^4$ that contains $\mathbb{S}^3$ with $\mathbb{S}^3$. Then M is a CMC surface contained in such $\mathbb{R}^3$. But our hypothesis says that $\theta \neq \pi/2$. So this case is not possible. □

6.3. **Constant angle surfaces in space forms.** Let us consider a isometrically immersed oriented surface M in a space form $\mathbb{Q}_c$ which has constant angle with respect to a closed conformal vector field Z.
Let T and ξ as before, then the condition to have constant angle means that $\langle Z/|Z|, T\rangle$ is constant.

$$Z = |Z|\langle Z/|Z|, T\rangle T + \langle Z, \xi\rangle \xi, \tag{17}$$

equivalently

$$Z/|Z| = \cos(\theta)T + \sin(\theta)\xi, \tag{18}$$

i.e. $\cos(\theta) = \langle Z/|Z|, T\rangle$, $\sin(\theta) = \langle Z/|Z|, \xi\rangle$. Here θ is the angle between T_pM and $Z(p)$ for every $p \in M$.
From now, in this subsection if we do not say otherwise, θ will be different from $0, \pi/2$. This is equivalent to say that both $\langle Z, T\rangle \neq 0$ and $\langle Z, \xi\rangle \neq 0$.
Finally, W is a local vector field of M such that $\{T, W\}$ is a positive oriented orthonormal basis on TM.

Proposition 6.14. *Under the above conditions we have that the connection ∇, the shape operator A^ξ and the second fundamental form α*

of M satisfy the following equalities

$$\begin{array}{cc} \alpha(T,T) = \frac{\varphi(\langle Z/|Z|,T\rangle^2-1)}{\langle Z,\xi\rangle}\xi, & \alpha(T,W)=0. \\ A^\xi(T) = \frac{\varphi(\langle Z/|Z|,T\rangle^2-1)}{\langle Z,\xi\rangle}T & \nabla_W T = \frac{\varphi W+\langle Z,\xi\rangle A^\xi(W)}{\langle Z,T\rangle} \\ \nabla_T T = 0 & \nabla_T W = 0. \end{array}$$

In particular, T and W are principal directions of the immersion and the integral curves of T are geodesics of M.

Proof. We take the derivative of Equation (17): The tangent part of the resulting equation is

$$\varphi X = (X\cdot|Z|)\langle Z/|Z|,T\rangle T + \langle Z,T\rangle \nabla_X T - \langle Z,\xi\rangle A^\xi(X). \tag{19}$$

We choose $X = W$ in Equation (19) to obtain

$$\begin{array}{rcl} \varphi W &=& (W\cdot|Z|)\langle Z/|Z|,T\rangle T + \langle Z,T\rangle\nabla_W T - \langle Z,\xi\rangle A^\xi(W) \\ &=& \langle Z,T\rangle\nabla_W T - \langle Z,\xi\rangle A^\xi(W) \end{array}$$

Thus $\nabla_W T = \frac{\langle Z,\xi\rangle A^\xi(W)+\varphi W}{\langle Z,T\rangle}$. Since $dim M = 2$ and we are assuming that $\langle Z,T\rangle \neq 0 \neq \langle Z,\xi\rangle$ we deduce that W and then T are principal directions. We also have that $\alpha(T,W)=0$ because we have codimension one and $\langle\alpha(T,W),\xi\rangle = \langle A^\xi(T),W\rangle$.
Now, we take $X = T$:

$$\varphi T = (T\cdot|Z|)\langle Z/|Z|,T\rangle T + \langle Z,T\rangle\nabla_T T - \langle Z,\xi\rangle A^\xi(T).$$

This equation implies that $\nabla_T T = 0$ which is equivalent in dimension two to $\nabla_W T = 0$. Thus
$(T\cdot|Z|)\langle Z/|Z|,T\rangle T - \langle Z,\xi\rangle A^\xi(T) = \varphi T$. We need the following calculus: $2|Z|(T\cdot|Z|) = T\cdot\langle Z,Z\rangle = 2\varphi\langle T,Z\rangle$. By substitution in the above equation we have $\varphi\langle T,Z/|Z|\rangle\langle Z/|Z|,T\rangle T - \langle Z,\xi\rangle A^\xi(T) = \varphi T$. This is equivalent to $A^\xi(T) = \frac{\langle Z/|Z|,T\rangle^2-1}{\langle Z,\xi\rangle}\varphi T$. □

Corollary 6.15. *If M has constant angle and has constant mean curvature with mean curvature vector given by $H = \alpha(T,T)+\alpha(W,W) = a\xi$, then*

$$\begin{array}{rcl} A^\xi(W) &=& \frac{a\langle Z,\xi\rangle-\varphi(\langle Z/|Z|,T\rangle^2-1)}{\langle Z,\xi\rangle}W, \\ \alpha(W,W) &=& \frac{a\langle Z,\xi\rangle-\varphi(\langle Z/|Z|,T\rangle^2-1)}{\langle Z,\xi\rangle}\xi, \\ \nabla_W W &=& \frac{\varphi\langle Z/|Z|,T\rangle^2-a\langle Z,\xi\rangle-2\varphi}{\langle Z,T\rangle}T. \end{array}$$

Proof. It follows from the hypothesis that

$$\alpha(W,W) = a\xi - \alpha(T,T) = a\xi - \frac{\varphi(\langle Z/|Z|,T\rangle^2-1)}{\langle Z,\xi\rangle}\xi.$$

Therefore

$$\begin{aligned} A^{\xi}(W) &= \langle A^{\xi}(W), W\rangle W = \langle \alpha(W,W), \xi\rangle W \\ &= \tfrac{a\langle Z,\xi\rangle - \varphi(\langle Z/|Z|,T\rangle^2-1)}{\langle Z,\xi\rangle} W. \end{aligned}$$

By Proposition 6.14 and the above equation we deduce that

$$\langle \nabla_W T, W\rangle = \langle \frac{\varphi W + \langle Z,\xi\rangle A^{\xi}(W)}{\langle Z,T\rangle}, W\rangle = \frac{2\varphi + a\langle Z,\xi\rangle - \varphi\langle Z/|Z|,T\rangle^2}{\langle Z,T\rangle}$$

Thus

$$\begin{aligned} \nabla_W W &= \langle \nabla_W W, T\rangle T = -\langle W, \nabla_W T\rangle T \\ &= \tfrac{\varphi\langle Z/|Z|,T\rangle^2 - a\langle Z,\xi\rangle - 2\varphi}{\langle Z,T\rangle} T. \end{aligned}$$

□

Corollary 6.16. *(Extrinsic Laplacian II) If $M \subset \mathbb{Q}^3_c$ is an isometrically immersed surface with constant angle and constant mean curvature, then*

$$\triangle_M \langle Z,\xi\rangle = \frac{-2\sin^4(\theta)\varphi^2 - a^2\langle Z,\xi\rangle^2 - a(1+2\sin^2(\theta))\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle}. \tag{20}$$

Proof. By Equation (8) we have to calculate

$$\triangle_M\langle Z,\xi\rangle = -|\alpha|^2\langle Z,\xi\rangle - \varphi\langle H,\xi\rangle.$$

Since $H = \alpha(T,T) + \alpha(W,W) = a\xi$, $-\varphi\langle H,\xi\rangle = -a\varphi$. Finally,

$$\begin{aligned} |\alpha|^2 &= |\alpha(T,T)|^2 + |\alpha(W,W)|^2 + 2|\alpha(T,W)|^2 \\ &= |\alpha(T,T)|^2 + |\alpha(W,W)|^2 \\ &= \tfrac{\varphi^2(\langle Z/|Z|,T\rangle^2-1)^2}{\langle Z,\xi\rangle^2} + \tfrac{(a\langle Z,\xi\rangle - \varphi(\langle Z/|Z|,T\rangle^2-1))^2}{\langle Z,\xi\rangle^2} \\ &= \tfrac{\varphi^2\sin^4(\theta)}{\langle Z,\xi\rangle^2} + \tfrac{(a\langle Z,\xi\rangle + \varphi\sin^2(\theta))^2}{\langle Z,\xi\rangle^2} \\ -|\alpha|^2\langle Z,\xi\rangle &= \tfrac{-2\sin^4(\theta)\varphi^2 - a^2\langle Z,\xi\rangle^2 - 2a\sin^2(\theta)\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle}. \end{aligned}$$

In the above calculus we applied that $\cos(\theta) = \langle Z/|Z|, T\rangle$, see (18). Therefore, $\triangle_M\langle Z,\xi\rangle = \frac{-2\sin^4(\theta)\varphi^2 - a^2\langle Z,\xi\rangle^2 - a(1+2\sin^2(\theta))\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle}$.

□

Corollary 6.17. *(Intrinsic Laplacian II) If $M \subset \mathbb{Q}^3_c$ is an isometrically immersed surface with constant angle and constant mean curvature, then*

$$\triangle_M\langle Z,\xi\rangle = \frac{(\sin^4(\theta) + \sin^2(\theta))\varphi^2 + a\sin^2(\theta)\varphi\langle Z,\xi\rangle - c\cos^2(\theta)\langle Z,\xi\rangle^2}{\langle Z,\xi\rangle}. \tag{21}$$

Proof. The intrinsic formula for the Laplacian is $\triangle_M\langle Z,\xi\rangle = TT\langle Z,\xi\rangle + WW\langle Z,\xi\rangle - \nabla_T T\langle Z,\xi\rangle - \nabla_W W\langle Z,\xi\rangle$. In our situation we have that $W\langle Z,\xi\rangle = 0$ and $\nabla_T T = 0$. So, the calculus we have to do is $\triangle_M\langle Z,\xi\rangle = TT\langle Z,\xi\rangle - \nabla_W W\langle Z,\xi\rangle$.
Let us start:

$$\begin{array}{rcl} T\langle Z,\xi\rangle & = & -\langle Z, A_\xi T\rangle = -\langle Z,T\rangle\langle T, A_\xi T\rangle \\ & = & -\langle Z,T\rangle \frac{\varphi(\langle Z/|Z|,T\rangle^2-1)}{\langle Z,\xi\rangle} = \sin(\theta)\cos(\theta)\varphi. \end{array}$$

By Proposition 6.5, $T\cdot\varphi = -c\langle Z,T\rangle$. Then,

$$\begin{array}{rcl} TT\langle Z,\xi\rangle & = & \sin(\theta)\cos(\theta)(T\cdot\varphi) = -c\sin(\theta)\cos(\theta)\langle Z,T\rangle \\ & = & -c\sin(\theta)\cos(\theta)\cot(\theta)\langle Z,\xi\rangle = -c\cos^2(\theta)\langle Z,\xi\rangle. \\ -\nabla_W W\langle Z,\xi\rangle & = & -\frac{\varphi\langle Z/|Z|,T\rangle^2 - a\langle Z,\xi\rangle - 2\varphi}{\langle Z,T\rangle} T\cdot\langle Z,\xi\rangle \\ & = & -\sin(\theta)\cos(\theta)\varphi\frac{\varphi\langle Z/|Z|,T\rangle^2 - a\langle Z,\xi\rangle - 2\varphi}{\langle Z,T\rangle} \\ & = & -\sin^2(\theta)\frac{(\cos^2(\theta)-2)\varphi^2 - a\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle}. \end{array}$$

Here we applied that $\langle Z,T\rangle = \langle Z,\xi\rangle\cot(\theta)$. Therefore,

$$\begin{array}{rcl} \triangle_M\langle Z,\xi\rangle & = & -\sin^2(\theta)\frac{(\cos^2(\theta)-2)\varphi^2 - a\varphi\langle Z,\xi\rangle}{\langle Z,\xi\rangle} - \frac{c\cos^2(\theta)\langle Z,\xi\rangle^2}{\langle Z,\xi\rangle} \\ & = & \frac{-\sin^2(\theta)(-\sin^2(\theta)-1)\varphi^2 + a\sin^2(\theta)\varphi\langle Z,\xi\rangle - c\cos^2(\theta)\langle Z,\xi\rangle^2}{\langle Z,\xi\rangle} \end{array}$$

□

Theorem 6.18. *If $M \subset \mathbb{Q}^3_c$ is an isometrically immersed connected surface with constant angle and constant mean curvature, then either M is totally umbilical or a totally geodesic surface in $\mathbb{Q}^3_c$.*

Proof. First case: We will prove that there are not surfaces with the above hypothesis with $\langle Z,T\rangle \neq 0$ and $\langle Z,\xi\rangle \neq 0$.
Applying Corollaries 6.17 and 6.16, we have the equation

$$(3\sin^4(\theta)+\sin^2(\theta))\varphi^2 + a(1+3\sin^2(\theta))\varphi\langle Z,\xi\rangle + (a^2 - c\cos^2(\theta))\langle Z,\xi\rangle^2 = 0.$$

This is a polynomial in the variable $\langle Z,\xi\rangle$, denoted by $Q(\langle Z,\xi\rangle) = 0$. Let us see that the constant term of this polynomial Q is non zero. First, let us recall that

$$\begin{array}{rcl} \varphi^2 & = & 1 - c\langle Z,T\rangle^2 - c\langle Z,\xi\rangle^2 = 1 - c\langle Z,\xi\rangle^2\cot^2(\theta) - c\langle Z,\xi\rangle^2 \\ & = & 1 - c\csc^2(\theta)\langle Z,\xi\rangle^2. \end{array}$$

We can conclude that the constant term a_0 of Q is $a_0 = (3\sin^4(\theta) + \sin^2(\theta))^2$ because we have to square the above formula $Q(\langle Z,\xi\rangle) = 0$ to be able to substitute φ. Now it is clear that $a_0 \neq 0$ if and only if $\sin\theta \neq 0$. This implies that $\langle Z,\xi\rangle$ is constant.

Then $|Z|^2 = |Z|^2\cos^2\theta + \langle Z,\xi\rangle^2$ is constant in M. But $|Z|$ is constant only in the orthogonal surfaces to Z which are totally umbilic because

Z is closed conformal.
Second case: $\langle Z, T\rangle = 0$. This implies that M is orthogonal to Z and so a totally umbilical surface.
Last case: $\langle Z, \xi\rangle = 0$. This means that Z is tangent to M and by Proposition 6.8, we can conclude that M is a totally geodesic surface in $\mathbb{Q}^3_c$. □

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