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# SNR Gap Between MIMO Linear Receivers: Characterization and Applications

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**Abstract**—This paper presents a statistical characterization of the SNR gap between MIMO Zero-Forcing (ZF) and Minimum Mean Squared Error (MMSE) equalizers, beyond the Rayleigh assumption for the interfering streams amplitude fading. Results are valid for arbitrary transmit SNR values and number of transmit/receive antennas. Specifically, we provide the exact closed-form distribution of the random variable representing the difference between the output SNR on a generic receive filter branch, under MMSE and ZF equalization. Analytical results turn particularly useful for the study of heterogeneous cellular networks.

## I. INTRODUCTION

Zero Forcing (ZF) and Minimum Mean Squared Error (MMSE) are the most popular linear receivers, due to their satisfactory trade-off between implementation complexity and achievable performance. Their supposed performance equivalence in the high Signal to Noise Ratio (SNR) regime was contradicted in [1]. Therein, it was shown that, given the transmission of  $n_t$  signal streams, with common value of the normalized transmit SNR, the output SNR for independent stream decoding,  $\gamma_k^{\text{mmse}}$ , measured in correspondence of an arbitrary branch of the MMSE receiving filter, can be expressed as [1], [2]

$$\gamma_k^{\text{mmse}} = \gamma_k^{\text{zf}} + \eta_k \quad (1)$$

In (1),  $\gamma_k^{\text{zf}}$  represents the output SNR, of the ZF equalizer on branch  $k$  and  $\eta_k$  denotes a non-decreasing function of the SNR, accounting for the energy nulled out by the ZF but not by the MMSE receiver. A gap between the output SNR values strongly impacts on both the outage (see [1, Eq. (53)]) as well as the error probability (EP), as outlined, e.g., in [1, Eq. (50)]. Moreover, knowledge of the statistics of  $\eta_k$  can serve as a high-SNR upper bound to the Interference to Noise Ratio at the output of an MMSE filter [1, Lemma III.2]. Due to its relevance as a performance index for linear receivers, hereinafter we investigate the statistics of  $\eta_k$ .

*Related work.* To date, the probability density function (pdf) of  $\eta_k$  has been derived in closed form only for asymptotically large transmit power. In [1, Thm. III.1], it was shown that the pdf of a properly scaled  $\eta_k$ , as the SNR grows large, converges to the  $\mathcal{F}_{2(n_t-2), 2(n_r-n_t+2)}$  distribution<sup>1</sup>, under the assumption that both the intended, as well as the interfering streams, are

subject to uncorrelated Rayleigh fading. The case of a system with a large number of transmit and receive antennas, impaired by transmit-correlated Rayleigh fading, has been investigated in [2]. Irrespectively of the presence of spatial correlation, under Rayleigh fading,  $\gamma_k^{\text{zf}}$  is Gamma distributed [3].

Very recent works [4, and references therein] deal with the distribution of  $\gamma_k^{\text{zf}}$  in the Line-of-Sight (LOS) environment, i.e., where either the desired or the interfering streams [5] experience Rician fading, while remaining streams experience Rayleigh fading. No results on either  $\gamma_k^{\text{zf}}$  or  $\eta_k$  are available for other channel models. The exact characterization of  $\gamma_k^{\text{mmse}}$ , in turn, is a more challenging task<sup>2</sup>, calling for the separate investigation of  $\gamma_k^{\text{zf}}$  and  $\eta_k$ , which are easier to handle.

*Aim of the work.* We capitalize on the fact that  $\eta_k$ , in [1, Formula (26)], is cast in terms of an Hermitian quadratic form, where the desired signal stream only appears in the vector, while the interfering streams are confined to the kernel matrix. This algebraic structure strongly favors its explicit statistical characterization, which can be carried out by means of standard analytical tools of multivariate analysis in several fading cases.

We focus on the case of Rayleigh-faded intended stream. Under this assumption, we first provide a non-asymptotic result complementing the fundamental one of [1, Thm. III.1], by giving the gap distribution for Rayleigh-faded interfering streams but arbitrary rather than arbitrarily large SNR values. The case of transmit-correlated Rayleigh fading, possibly corresponding to spatially-clustered interferers, is addressed in the same vein. Then, we derive the corresponding statistics in the case of Rician-faded interfering streams, for arbitrary rank and eigenvalues multiplicity of the LOS matrix. Such a scenario is representative of a worst-case cellular transmission, where the useful stream comes, e.g., from the cell edge and dominant interferers have LOS path toward the base station. Finally, we evaluate the pdf of  $\eta_k$  for the case where interfering streams undergo multiple Rayleigh scattering<sup>3</sup>. This last scenario is of particular interest in foreseen small-cells networks. Indeed, it

<sup>2</sup>To the best of the authors' knowledge, the pdf of  $\gamma_k^{\text{mmse}}$  is available when both desired and interfering signals are Rayleigh faded [6], while a closed-form expression for its moments has been derived also in the case of Rice-faded intended stream/Rayleigh faded interference [7].

<sup>3</sup>See [10] for the statistical analysis of a multiple scattering system, and [11], [12], [13, and references therein] for practical justification and information-theoretic performance of such channels, as the number of scattering stages vary.

<sup>1</sup>I.e., the pdf of  $\mathcal{F}_{m,n}$  is:  $f_x(x) = \frac{\Gamma(\frac{n+m}{2})n^{n/2}m^{m/2}x^{m/2-1}}{\Gamma(m/2)\Gamma(n/2)(n+mx)^{\frac{n+m}{2}}}$ .

adequately represents, with no approximation in the channel statistics, the case where interfering signals come from non colocated single-antenna equipped users, whose signals may undergo multiple scattering phenomena.

As an instance of application of our results, we evaluate the key statistics of  $\eta_k$ , relating the values of the Bit Error Probability (BER) as well as those of the Outage Probability (OP) of an MMSE receiver with the corresponding performance indices in the case of ZF.

## II. NOTATION

Boldface uppercase and lowercase letters denote matrices and vectors, respectively.  $\mathbf{I}$  is the identity matrix. The determinant and the conjugate transpose of the generic matrix  $\mathbf{A}$  are denoted by  $|\mathbf{A}|$  and  $\mathbf{A}^H$ , respectively, while  $\mathbf{A}_{i,j}$  is the  $(i, j)$ -th element of  $\mathbf{A}$ . Moreover,  $\mathbb{E}_a[\cdot]$  represents the average operator with respect to the random variable  $a$ . For any  $m \times m$  Hermitian matrix  $\mathbf{A}$  with eigenvalues  $a_1, \dots, a_m$ , the Vandermonde determinant is defined as:  $\mathcal{V}(\mathbf{A}) = \prod_{1 \leq k < \ell \leq m} (a_k - a_\ell)$ .  $G_{a,b}^{c,d}(\cdot)$ , with integer parameters  $a, b, c, d$ , denotes the Meijer-G function [14, Ch. 8]. The Gauss hypergeometric function is denoted by  ${}_2F_1(a, b; c; x)$ . The complex multivariate Gamma function is defined as [15]:  $\Gamma_p(q) = \pi_p \prod_{\ell=1}^p (q - \ell)!$ , with  $p$  and  $q$  non-negative integers such that  $p \leq q$ , and  $\pi_p = \pi^{p(p-1)/2}$ . We also denote by  $(a)_k = \Gamma(a+k)/\Gamma(a)$  the Pochhammer symbol of order  $k$ .  $f_a(a)$  denotes the pdf of the scalar random variable  $a$  (for random matrices pdf we skip the subscript). If two variables  $a$  and  $b$  share the same distribution, we write  $a \sim b$ .

## III. SYSTEM MODEL

Consider a linear system:

$$\mathbf{y} = \sqrt{\rho} \mathbf{H} \mathbf{x} + \mathbf{n} \quad (2)$$

where  $\mathbf{y}$  is a vector of size  $n_r$ ,  $\mathbf{H}$  is the  $n_r \times n_t$  random channel matrix,  $\mathbf{x}$  is a random vector of size  $n_t$  with covariance  $\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathcal{E}_s/n_t \mathbf{I}$ , and  $\mathbf{n}$  represents Gaussian noise with covariance  $\mathbb{E}[\mathbf{n}\mathbf{n}^H] = \mathcal{N}_0 \mathbf{I}$ . For simplicity, we assume  $n_t \leq n_r$ . The channel matrix is assumed to be normalized such that  $\rho = n_r n_t / \text{Tr}\{\mathbb{E}[\mathbf{H}\mathbf{H}^H]\}$ .

The expression of the instantaneous received SNR on the  $k$ -th receive filter branch ( $k = 1, \dots, n_t$ ) depends on the adopted equalizer. In terms of the channel matrix and system parameters, we have [16, Ch. 6]:

$$\gamma_k^{\text{mmse}} = \frac{1}{\left[ (\mathbf{I} + \delta \mathbf{H}^H \mathbf{H})^{-1} \right]_{k,k}} - 1, \quad \gamma_k^{\text{zf}} = \frac{\delta}{\left[ (\mathbf{H}^H \mathbf{H})^{-1} \right]_{k,k}}, \quad (3)$$

where  $\delta = \frac{\mathcal{E}_s}{\mathcal{N}_0} \frac{\rho}{n_t}$  and [1, Eq. (26)]. We denote by  $\mathbf{h}_k$  the  $k$ -th column of  $\mathbf{H}$  and by  $\mathbf{H}_k$  the matrix obtained by removing  $\mathbf{h}_k$  from  $\mathbf{H}$ . The SNR gap  $\gamma_k^{\text{mmse}} - \gamma_k^{\text{zf}}$  is then given by [1]

$$\eta_k = \mathbf{h}_k^H \mathbf{U}_k (\delta^{-1} \mathbf{I}_{n_t-1} + \mathbf{\Lambda}_k)^{-1} \mathbf{U}_k^H \mathbf{h}_k, \quad (4)$$

where  $\mathbf{H}_k = \mathbf{U}_k \mathbf{\Lambda}_k^{1/2} \mathbf{V}_k^H$  is the singular value decomposition of  $\mathbf{H}_k$ ,  $\mathbf{U}_k$  has size  $n_r \times (n_t - 1)$  and both  $\mathbf{\Lambda}_k$  and  $\mathbf{V}_k$  are square matrices of size  $(n_t - 1)$ .

## IV. PROBLEM FORMULATION

Our work focuses on the statistical characterization of (4), under different assumptions on the statistics of the diagonal matrix  $\mathbf{\Lambda}_k$  of size  $n_t - 1$ , while  $\mathbf{h}_k \sim \mathcal{CN}(0, \mathbf{I}_{n_r})$ , i.e.  $\mathbf{h}_k$  is a length- $n_r$  vector of zero-mean uncorrelated Gaussian random variable with unit variance. This assumption on the intended stream allows us to state the following lemma on the gap distribution, conditionally on the interfering streams fading law.

*Lemma 4.1: For independent stream decoding, the SNR gap  $\eta_k$ , as defined in (4), is a random variable whose conditional law w.r.t. the interfering streams fading distribution can be expressed as*

$$f_{\eta_k | \mathbf{\Lambda}_k}(y) = \frac{|\Delta|}{\mathcal{V}(\mathbf{\Lambda}_k)} \prod_{\ell=1}^{n_t-1} \left( \frac{1}{\delta} + \lambda_\ell \right)^{n_t-2}. \quad (5)$$

$\Delta$  is the determinant of a size- $(n_t - 1)$  matrix, with generic element

$$\Delta_{i,j} = \begin{cases} \ell_i^{j-1}, & 1 \leq i \leq n_t - 1, 1 \leq j \leq n_t - 2 \\ \ell_i^{n_t-3} e^{-y/\ell_i}, & 1 \leq i \leq n_t - 1, j = n_t - 1 \end{cases}$$

$$\ell_i = \left( \frac{1}{\delta} + \lambda_i \right)^{-1}.$$

*Proof:* Since  $\mathbf{h}_k \sim \mathcal{CN}(0, \mathbf{I}_{n_r})$ ,  $\mathbf{U}_k^H \mathbf{h}_k \sim \mathcal{CN}(0, \mathbf{I}_{n_t-1})$ , by the invariance of the complex Gaussian distribution to linear transformations. Henceforth, the conditional distribution of  $\eta_k | \mathbf{\Lambda}_k$  can be written directly as per [17, Eq. (8)]. ■

Equipped with (5), we derive its unconditional expression for each listed fading scenario. For sake of formula's compactness, we define  $\zeta = \frac{1+\delta}{\delta}$  and  $\tau = n_r - n_t + 1$ . Due to the lack of space, the proofs of the following Propositions are omitted, and relegated to [20].

### A. Rayleigh fading

Under the uncorrelated Rayleigh assumption for the interferers,  $\mathbf{H}_k$  is modeled as a matrix whose entries are zero-mean independent complex Gaussian random variables with unit variance. As  $\delta$  takes on arbitrary but finite values,  $\eta_k$ 's law can be expressed as per the following proposition.

*Proposition 4.1: The distribution of (4), under the assumption of Rayleigh-faded interfering streams, can be written as*

$$f_{\eta_k}(y) = \mathcal{K}_{n_t-1, n_r} |\mathbf{Z}|, \quad (6)$$

where

$$\mathcal{K}_{n_t-1, n_r} = \frac{\pi_{n_t-1}^2}{\Gamma_{n_t-1}(n_t-1) \Gamma_{n_t-1}(n_r)},$$

$$\mathbf{Z}_{i,j} = \begin{cases} \beta! \sum_{\ell=0}^{\beta} \frac{(n_t-j-\ell)_{n_r-i}}{\delta^\ell \ell!} & 1 \leq i \leq n_t-1 \\ & 1 \leq j \leq n_t-2 \\ \frac{e^{-y/\delta} (n_r-i)!}{\delta (y+1)^{n_r-i+1}} \left[ 1 + \frac{\delta (n_r-i+1)}{y+1} \right] & 1 \leq i \leq n_t-1 \\ & j = n_t-1 \end{cases}$$

and, hereinafter,  $\beta = n_t - j - 1$ .

*Remark 1:* Notice that (6) can be also cast as

$$f_{\eta_k}(y) = \mathcal{K}_{n_t-1, n_r} \sum_{i=1}^{n_t-1} \frac{\mathcal{D}_i e^{-y/\delta} (n_r-i)!}{\delta (y+1)^{n_r-i+1}} \left[ 1 + \frac{\delta (n_r-i+1)}{y+1} \right], \quad (7)$$

by virtue of Laplace expansion of  $|\mathbf{Z}|$  w.r.t. its last column, the only one depending on  $\eta_k$ . In (7),  $\mathcal{D}_i$  is the  $(i, n_t - 1)$ -th cofactor of the matrix  $\mathbf{Z}$ . Such an expression holds for each of the following newly derived pdfs. Though less compact than that appearing in the Proposition's statements, (7) turns out to be effective in averaging w.r.t.  $\eta_k$ , under any fading scenario.

*Corollary 4.1:* Under the assumptions of Proposition 4.1, if the rows of  $\mathbf{H}_k$  are correlated with common covariance matrix  $\Sigma_k$ , with distinct<sup>4</sup> eigenvalues  $\sigma_i$ 's,  $i = 1, \dots, n_t - 1$

$$f_{\eta_k}(y) = \frac{\pi_{n_t-1} |\mathbf{Z}|}{\Gamma_{n_t-1}(n_r) \mathcal{V}(\Sigma_k) |\Sigma_k|^{\tau+1}}, \quad (8)$$

$$\mathbf{Z}_{i,j} = \begin{cases} \sum_{\ell=0}^{\beta} \beta! \sigma_i^{n_r-j+1} (n_r-j-\ell)! & 1 \leq i \leq n_t-1, \\ (\delta \sigma_i)^\ell \ell! (\beta-\ell)! & 1 \leq j \leq n_t-2 \\ \frac{e^{-y/\delta} \tau!}{\delta \left(\frac{1}{\sigma_i} + y\right)^{\tau+1}} \left[ 1 + \frac{\delta(\tau+1)}{\left(\frac{1}{\sigma_i} + y\right)} \right] & \begin{matrix} 1 \leq i \leq n_t-1, \\ j = n_t-1 \end{matrix} \end{cases}$$

### B. Rice fading

Assuming that all interfering streams undergo Rice fading is tantamount to say that

$$\mathbf{H}_k = \bar{\mathbf{H}}_k + \tilde{\mathbf{H}}_k. \quad (9)$$

In the above expression  $\bar{\mathbf{H}}_k$  is a deterministic matrix, representing the LOS signal component between each interfering transmitter and each receiver antenna. The entries of  $\tilde{\mathbf{H}}_k$  are independent, zero-mean complex Gaussian random variables. We provide hereinafter the gap characterization for arbitrary geometric behavior of the LOS component, i.e., we assume the matrix  $\Omega = \bar{\mathbf{H}}_k^H \bar{\mathbf{H}}_k$  to have rank  $L \leq n_t - 1$ . We denote by  $\{\omega_1, \dots, \omega_L\}$  its  $L$  non-zero eigenvalues.

*Proposition 4.2:* The distribution of the Hermitian quadratic form in (4), under the assumption of Rice-faded interfering streams, with rank- $L$  LOS matrix, can be written as

$$f_{\eta_k}(y) = \mathcal{K}_\Omega |\mathbf{Z}|, \quad (10)$$

where

$$\mathcal{K}_\Omega = \frac{e^{-\text{Tr}\{\Omega\}} \prod_{\ell=1}^L \omega_\ell^{L+1-n_t}}{\Gamma_{n_t-L-1}(n_t-L-1) \mathcal{V}(\Omega)},$$

$\mathcal{V}(\Omega) = \prod_{\ell < k}^L (\omega_\ell - \omega_k)$  and

$$\mathbf{Z}_{i,j} = \begin{cases} \sum_{\ell=0}^{\beta} \frac{(n_r-j-\ell)!}{e^{-\omega_j} \ell! \delta^\ell} & 1 \leq i \leq L \\ \times \sum_{q=0}^{\beta-\ell} \frac{(n_t-j-\ell-q)_{t+q}}{\omega_j^{-q} q! (\tau+q)!} & \begin{matrix} 1 \leq i \leq L \\ 1 \leq j \leq n_t-2 \end{matrix} \\ \frac{\beta!}{(n_r-i)!} \sum_{\ell=0}^{\beta} \frac{(n_t-j-\ell)_{n_r-i}}{\ell! \delta^\ell} & \begin{matrix} L+1 \leq i \leq n_t-1 \\ 1 \leq j \leq n_t-2 \end{matrix} \\ \frac{e^{-y/\delta + \omega_i}}{\delta(1+y)^{\tau+1}} \left[ 1 + \frac{\delta(\tau+1 + \omega_i)}{y+1} \right] & \begin{matrix} 1 \leq i \leq L \\ j = n_t-1 \end{matrix} \\ \frac{e^{-y/\delta}}{\delta(y+1)^{n_r-i+1}} \left[ 1 + \frac{\delta(n_r-i+1)}{1+y} \right] & \begin{matrix} L+1 \leq i \leq n_t-1 \\ j = n_t-1 \end{matrix} \end{cases}$$

<sup>4</sup>The case of multiple eigenvalues can be addressed resorting to classical limiting procedures, which we do not report here for lack of space.

*Corollary 4.2:* Under the assumption of Proposition 4.2, if  $L = n_t - 1$  and  $\omega_i = \alpha n_r$ ,  $i = 1, \dots, n_t - 1$ ,  $\alpha > 0$ ,

$$f_{\eta_k}(y) = \frac{e^{-\alpha n_r (n_t-1)} |\mathbf{Z}|}{\Gamma_{n_t-1}(n_t-1)}, \quad (11)$$

where

$$\mathbf{Z}_{i,j} = \begin{cases} e^{\alpha n_r} \sum_{\ell=0}^{\beta} \frac{(n_r+n_t-i-j-\ell-1)!}{\ell! \delta^\ell} & \\ \times \sum_{q=0}^{\beta-\ell} \frac{(n_t-j)_{\ell+q} (\alpha n_r)^q}{q! (n_r-i+q)!} & \begin{matrix} 1 \leq i \leq n_t-1 \\ 1 \leq j \leq n_t-2 \end{matrix} \\ \frac{e^{-y/\delta + \alpha n_r / (1+y)}}{\delta(1+y)^{n_r-i+1}} & \\ \times \left[ 1 + \frac{\delta(n_r-i+1 + \frac{\alpha n_r}{1+y})}{y+1} \right] & \begin{matrix} 1 \leq i \leq n_t-1 \\ j = n_t-1 \end{matrix} \end{cases}$$

### C. Small-cells and multiple scattering

Consider a multiple-scattering channel with  $N - 1$  clusters of  $n_i$  independent scatterers each. In this case the matrix  $\mathbf{H}_k$  can be represented by the product of  $N$  matrices,  $\mathbf{S}_i$ , of size  $n_i \times n_{i-1}$ ,  $i = 1, \dots, N$ , with  $n_0 = n_t - 1$  and  $n_N = n_r$ . The entries of  $\mathbf{S}_i$  are zero-mean unit variance complex Gaussian independent random variables. For sake of compactness, we define the set of auxiliary variables  $\nu_i = n_i - n_0$ ,  $i = 1, \dots, N$ . In the following we assume  $n_t - 1 \leq n_1 \leq \dots \leq n_r$ , thus such variables are non-negative integers<sup>5</sup>.

*Proposition 4.3:* The distribution of (4), under the assumption of multiple Rayleigh scattering affecting the interfering streams, can be written as

$$f_{\eta_k}(y) = \frac{|\mathbf{Z}|}{\prod_{i=1}^{n_0} \prod_{\ell=0}^N \Gamma(i + \nu_\ell)} \quad (12)$$

where

$$\mathbf{Z}_{i,j} = \begin{cases} \sum_{\ell=0}^{\beta} \binom{\beta}{\ell} \frac{\Gamma(n_1+i-j-\ell)}{\delta^\ell} & \\ \times \prod_{q=2}^N (n_q-j-\ell)!, & \begin{matrix} 1 \leq i \leq n_t-1 \\ 1 \leq j \leq n_t-2 \end{matrix} \\ \frac{e^{-y/\delta}}{\delta} \left[ G_{N,1}^{1,N} \left( \begin{matrix} -\nu_i \\ 1 \end{matrix} \middle| y \right) \right. & \\ \left. + \delta G_{N,1}^{1,N} \left( \begin{matrix} -\nu_i-1 \\ 1 \end{matrix} \middle| y \right) \right] & \begin{matrix} 1 \leq i \leq n_t-1 \\ j = n_t-1 \end{matrix} \end{cases}$$

and  $\nu_i = [\nu_N, \dots, \nu_2, \nu_1 + i - 1]$ .

## V. APPLICATIONS

Here we present two main applications of the SNR gap statistics evaluation, focusing on the BER and the OP.

<sup>5</sup>This assumption can be relaxed based on the observations in [10].

### A. BER of ZF and MMSE via SNR gap statistics

In presence of a BPSK-modulated input signal, ZF and MMSE uncoded BER are linked as per [1, Eq. (50)]. In a non-asymptotic setting, with reference to  $M$ -PSK<sup>6</sup>,

$$P_b^{\text{mmse}} \approx \mathbb{E}_{\eta_k} [e^{-\eta_k \sin^2(\pi/M)}] P_b^{\text{zf}}. \quad (13)$$

With reference to the above equation and setting  $M = 2$  for simplicity, we can write the required expectation in the uncorrelated Rayleigh case as

$$\mathbb{E}_{\eta_k} [e^{-\eta_k}] = \mathcal{K}_{n_t-1, n_r} \sum_{i=1}^{n_t-1} \frac{\mathcal{D}_i}{\delta} [\mathcal{I}_{n_r-i, \zeta} + \delta \mathcal{I}_{n_r-i+1, \zeta}], \quad (14)$$

and, in the rank- $L$  Rician case, as

$$\mathbb{E}_{\eta_k} [e^{-\eta_k}] = \mathcal{K}_{\Omega} \left[ \sum_{j=1}^L \frac{e^{\mu_j} \mathcal{D}_j}{\tau! \delta} \left( \mathcal{I}_{\tau, \zeta} + \frac{\delta(\tau + \mu_j + 1) \mathcal{I}_{\tau+1, \zeta}}{\tau + 1} \right) \sum_{j=1+L}^{n_t-1} \frac{\mathcal{D}_j}{\delta} (\mathcal{I}_{n_r-j+1, \zeta} + \delta \mathcal{I}_{n_r-j+2, \zeta}) \right], \quad (15)$$

where

$$\mathcal{I}_{m, \zeta} = (-\zeta)^m \left[ \sum_{\ell=1}^m \frac{(\ell-1)!}{(-\zeta)^\ell} - e^\zeta \text{Ei}(-\zeta) \right].$$

Finally, in presence of multiple scattering,

$$\mathbb{E}_{\eta_k} [e^{-\eta_k}] = \sum_{i=1}^{n_t-1} \frac{\mathcal{D}_i}{\mathcal{Z}_{n_t-1} \delta} \left[ G_{1, N+1}^{N+1, 1} \left( \begin{matrix} -1 \\ 0, \bar{\nu}_i \end{matrix} \middle| \zeta \right) + \delta G_{1, N+1}^{N+1, 1} \left( \begin{matrix} -1 \\ 0, \bar{\nu}_i + 1 \end{matrix} \middle| \zeta \right) \right] \quad (16)$$

where  $\bar{\nu}_i = [\nu_N, \dots, \nu_1 + i]$ .

*Remark III:* Notice that (14), together with [1, Eq. (39)], provides an analytical approximation of  $P_b^{\text{mmse}}$  under Rayleigh fading, under the assumptions of Gaussian approximation for the interference and non rank-deficient channel<sup>7</sup>. Numerical results, not reported herein due to lack of space, confirm also, for moderate values of  $\delta$ , the presence of an offset between ZF and MMSE per-stream BER, as observed in [1, Fig. 3]. Extension to different fading laws and/or signal constellation is subject of ongoing work.

### B. Outage Probability evaluation

Employing independent codes of rate  $R$  over each of the transmit antennas, the MMSE  $k$ -th stream OP can be evaluated from<sup>8</sup> [1, Eq. (52)], namely

$$P_{\text{out}, k}^{\text{mmse}}(R) = \int_0^{2^R-1} F_{\gamma_k^{\text{zf}}} \left( \frac{2^R-1-x}{\delta} \right) f_{\eta_k}(x) dx, \quad (17)$$

which once again only requires the cumulative distribution of  $\gamma_k^{\text{zf}}$  and the density of  $\eta_k$ . In the case of Rayleigh-faded

<sup>6</sup>Our results can be applied to  $M$ -QAM constellations, too, but we do not report this case here due to space limitations.

<sup>7</sup>See again [1] for a detailed discussion on those assumptions' validity.

<sup>8</sup>We put in the OP expression the generic CDF of  $\gamma_k^{\text{zf}}$ , instead that the one used in [1, Eq. (52)], whose expression only refers to Rayleigh fading.

interfering streams, (17) can be expressed in closed form by substituting into (17) the expression for  $f_{\eta_k}(x)$  given in (7) and the expression for  $F_{\gamma_k^{\text{zf}}}(x)$  given in [1], i.e.,  $F_{\gamma_k^{\text{zf}}}(x) = 1 - e^{-x} \sum_{\ell=0}^{n_r-n_t} \frac{x^\ell}{\ell!}$ . Indeed, (17) can be further cast as

$$P_{\text{out}, k}^{\text{mmse}}(R) = F_{\eta_k}(2^R-1) - e^{-(2^R-1)/\delta} \sum_{\ell=0}^{n_r-n_t} \int_0^{2^R-1} \frac{e^{x/\delta} (2^R-1-x)^\ell}{\delta^\ell \ell!} f_{\eta_k}(x) dx, \quad (18)$$

whose first term can be evaluated from (7) by virtue of [14, 3.353.1], while the second requires the exploitation of [14, 3.196.1] to be expressed in closed form. As a consequence, we obtain

$$P_{\text{out}, k}^{\text{mmse}}(R) = 1 - e^{-(2^R-1)/\delta} \mathcal{K}_{n_t-1, n_r} \sum_{i=0}^{n_t-1} \frac{\mathcal{D}_i (n_r-i)!}{2^{R(n_r-i)}} \times \left\{ \frac{1}{2^R} + \sum_{\ell=0}^{n_r-n_t} \frac{(2^R-1)^{\ell+1}}{2^{R\ell} (\ell+1)! \delta^{\ell+1}} [\delta (n_r-i+1) \times {}_2F_1(\ell+1, \xi+1; \ell+2; 1-2^R) - {}_2F_1(\ell+1, \xi; \ell+2; 1-2^R)] \right\} \quad (19)$$

where  $\xi = \ell - n_r + i + 1$ , while  $\mathcal{K}_{n_t-1, n_r}$  and  $\mathcal{D}_i$  are defined in Proposition 4.1. Ongoing numerical investigation aims to identify the values of  $\delta$  beyond which the high-SNR MMSE OP approximation [1, Formula (53)] and our exact result (19) closely match.

The corresponding expression for rank 1 and full rank Rice-faded interferers, can be obtained by plugging [4, Formula (51)] and [4, Formula (53)], respectively, in (17), while the derivation of an equivalent formula for the case of arbitrary Rician rank is subject of ongoing work. Unfortunately, no closed-form expression for  $P_{\text{out}, k}^{\text{mmse}}(R)$  is available in the multiple-scattering case [13], due to the absence of any expression for  $F_{\gamma_k^{\text{zf}}}$ .

## VI. NUMERICAL RESULTS

We now validate our analytical expressions for the pdf of  $\eta_k$ , against numerical (i.e., Monte Carlo) simulations.

Figure 1 shows analytical and numerical results, represented by markers and solid line, respectively. Results have been obtained for  $n_t = 3$ ,  $n_r = 5$  and different (finite) values of normalized SNR  $\delta$ . The dashed line corresponds instead to the asymptotic expression of the pdf as  $\text{SNR} \rightarrow \infty$ , which is given in [1]. The plot highlights the excellent match between analysis and simulation, thus validating our derivations. Furthermore, we observe that the asymptotic expression provides a very good approximation of the pdf for values of normalized SNR greater than 10 dB, but it is far from providing an accurate representation for lower values of SNR.

In Figure 2 we show the pdf of the SNR gap  $\eta_k$  for the multiscattering channel and for a different numbers of scatterers and scattering stages. Again, the match between analysis and simulation is very tight. We observe that, as the number of scattering stages increases, the probability mass of  $f_{\eta_k}$  moves toward smaller values. Thus, the ZF performance approaches that of the MMSE receiver.

## VII. CONCLUSION

The output SNR gap between linear MIMO MMSE and ZF has been investigated for arbitrary finite values of transmit power and finite number of antennas. Its pdf is provided in closed form, under the assumption of Rayleigh-faded intended stream and independent stream decoding. Interfering streams are, in turn, assumed to undergo Rayleigh, or Rician (with arbitrary rank and eigenvalues multiplicity) fading. The case of multiple Rayleigh scattering on the interfering streams is considered, too, thus providing a comprehensive analysis of fading scenarios, arising in current and foreseen wireless cellular settings. The extension beyond Rayleigh fading for the desired signal stream is currently under study.

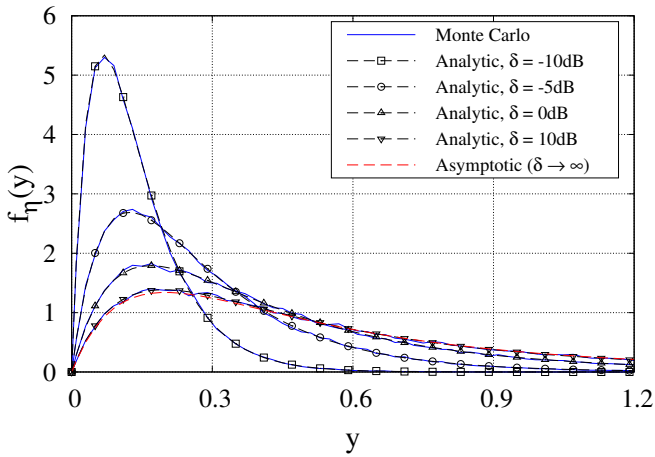


Fig. 1. Pdf of the SNR gap  $\eta_k$  for Rayleigh channel: comparison between analytical and numerical results, for  $n_t = 3$ ,  $n_r = 5$  and different values of normalized SNR  $\delta$ . Results are also compared against the asymptotic expression [1].

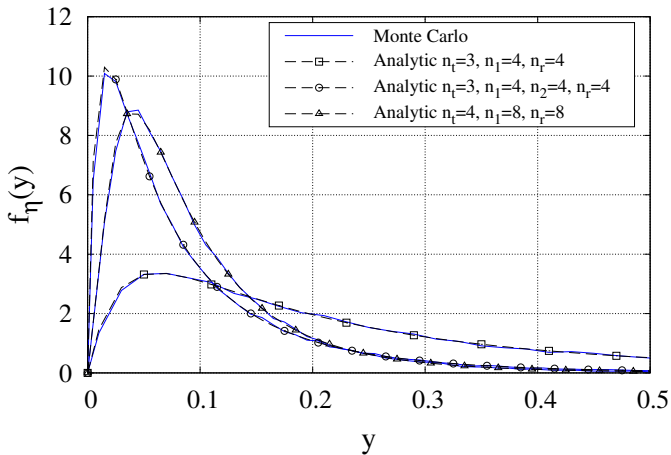


Fig. 2. Pdf of the SNR gap  $\eta_k$  for multiscattering channel: comparison between analytical and numerical results, for  $n_t = 3$ ,  $n_1 = 4$ ,  $n_r = 4$ ;  $n_t = 3$ ,  $n_1 = 4$ ,  $n_2 = 4$ ,  $n_r = 4$ ; and  $n_t = 4$ ,  $n_1 = 8$ ,  $n_r = 8$ .

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