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Geometrical Formulation of Supergravity Theories

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Per la mia cara Evelyn

Contents

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Publications

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Abstract

This thesis deals with a geometrical formulation of diverse Supergravity theories. In particular, the construction of Supergravity actions in four and three dimensions are considered in different frameworks with interesting physical implications.

Before approaching supersymmetry, we briefly review some gravity theories in the Cartan formalism. The formalism used in the introductory chapter is crucial in order to understand the development of the present thesis. Some interesting results are presented in chapter 2 using the semigroup expansion method in the Chern-Simons (CS) and Born-Infeld (BI) gravity theories. Subsequently, a brief introduction of supersymmetry and some supergravity models are considered in chapter 3.

Chapters 4, 5, 6 and 7 contain the main results of this thesis which are based on five articles written during the cotutelle research process.

Initially, we present a family of superalgebras using the semigroup expansion of the Anti-de Sitter superalgebra. In the MacDowell-Mansouri approach, we study the construction of diverse four-dimensional supergravity theories for different superalgebras. Interestingly, we show that the pure supergravity action can be obtained as a MacDowell-Mansouri like action using the Maxwell symmetries. Additionally, a generalized supersymmetric cosmological constant term can be included to a supergravity theory using a particular supersymmetry, called AdS-Lorentz. Furthermore, we present a supergravity model in three dimensions using the CS formalism and the Maxwell superalgebras.

Subsequently, the thesis is focused on a supergravity model with partial breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ supersymmetry which, in the low energy limit, gives rise to a $\mathcal{N}=1$ supersymmetric theory.

Eventually, the thesis ends with some comments about possible developments.

Introduction

The observers are crucial in order to describe Physics. They can measure and define mathematical objects in order to represent physical concepts. However, following the Copernican principle, there are not privileged observers of the universe. Thus, the Physics has to be observer-independent. Interestingly, the diverse results obtained by different observers can be related by symmetry transformations. In particular, the symmetry of a physical system is a feature which remains unchanged under some transformation. Ones of the symmetries of nature are the space-time symmetries successfully described by the General Relativity (GR) theory. On the other hand, the internal symmetries are understood through gauge theories described by the Standard Model.

Nowadays, three of the four forces of nature are successfully described by the Standard Model as Yang-Mills theories. They are elegantly related to gauge symmetries allowing renormalizability and ensuring a viable quantum theory. On the other hand, gravity described by General Relativity, resists to the quantization. In spite of the huge success of the General Relativity theory, there is not a consistent quantum description of gravity which prevents a possible unification of gravity to the other interactions.

The fundamental interactions of nature and their coupling to matter are based on the invariance under local transformations of some gauge group. In particular, the local symmetry is achieved if matter is coupled to bosonic gauge fields which are the mediators of an interaction. The coupling of matter to the electromagnetic fields described by the Quantum-electrodynamics (QED) is seen as a $U(1)$-gauge theory. While the Weak and electromagnetic interactions are unified in the Standard model as local $SU(2) \times U(1)$-gauge theory. On the other hand, the $SU(3)$ gauge group describe the strong interactions (QCD). In the same way, General Relativity can be seen as the

”gauge” theory of the Poincaré group whose gauge boson is associated with the local space-time translation generators P_a.

In order to unify gravity with the other interactions in a unique theory, it is necessary to put together the internal symmetries with the space-time symmetries. A good candidate for this purpose is the supersymmetry. Supersymmetric theories are remarkable theories since they unify space-time with internal symmetries relating bosonic and fermionic particles in an elegant way. Indeed, the particles of different spin can be associated in a bigger group called the supersymmetry group or supergroup. This allows to introduce a new algebraic structure known as the Lie superalgebra. In particular, the supersymmetric generalization of the Poincaré algebra can be obtained introducing in addition to the bosonic generators, the fermionic generators Q which satisfy the Poincaré (anti)commutation relations.

The supersymmetric extension of gravity, described by General Relativity, corresponds to the supergravity theory. Thus, the simplest supergravity theory can be viewed as the “gauge” theory of the Poincaré superalgebra where the fermionic generators Q are gauged by the superpartner of the graviton (spin-2), which corresponds to a spin-3/2 field ψ called the gravitino. There is a particular interest in superalgebras going beyond the superPoincaré one, in order to study richer supergravity theories. Furthermore, there are several models depending on the amount of supersymmetry charges $\mathcal{N}$ and on the choice of the space-time dimension D. The larger $\mathcal{N}$ and the larger D, more constraints are presents in the theory. It is known that increase $\mathcal{N}$ beyond 8 or the dimension D beyond 11 makes difficult a consistent coupling to gravity.

Interestingly, other features can be incorporated to supergravity theories like matter couplings and the presence of cosmological constant. The inclusion of matter in a supergravity theory has important consequences in the geometrical structure leading to a vast variety of supergravity theories with diverse physical implications. In particular, pure supergravity models can be coupled to matter multiplets in order to obtain more realistic theories. The models of particular relevance are the supergravity theories in ten and eleven dimensions since they describe the low-energy dynamics of superstring and M-theory, on flat space-time, respectively. Besides with the well celebrated duality between superstring theory realized on an AdS space-time and the conformal field theory on its boundary (AdS/CFT duality) made supergravity a useful tool for studying nonperturbative properties of gauge theories.

The purpose of the present thesis to study diverse features of supergravity models using different geometric formalisms. First, we shall approach enlarged supersymmetries using a Lie algebra expansion (S-expansion) method in order to analyze and construct four-dimensional supergravity theories. We shall see that the pure supergravity action can be obtained as a MacDowell-Mansouri like action using the Maxwell symmetries. Additionally, we shall present an alternative way to introduce a generalized cosmological term to a supergravity action à la MacDowell-Mansouri using the AdS-Lorentz superalgebra.

Subsequently, we will study, in the Chern-Simons formalism, the construction of a three-dimensional supergravity action using a minimal Maxwell superalgebra. In particular, a supersymmetric theory for the usual Maxwell superalgebra can be obtained combining the expansion and contraction procedures. Eventually, we shall present the multi-vector generalization of the partially broken $\mathcal{N} = 2$ rigid supersymmetric theory as a rigid limit of a $\mathcal{N} = 2$ supergravity theory. Our purpose is to elucidate the supergravity origin of the multifield Born-Infeld supersymmetric theory and to understand the origin of the electric and magnetic Fayet-Iliopoulos terms.

Chapter 1

Differential geometry and gravity

1.1 Introduction

It is well known that gravity described by General Relativity theory can be formulated from a variational principle,

$$S_g = \int \mathcal{L}\ d^4x = \kappa \int \sqrt{-g} L_g\ d^4x. \tag{1.1}$$

The scalar L_g can be obtained considering that the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{1.2}$$

are second order in the metric $g_{\mu\nu}$. This restricts the scalar L_g to contain only the metric and their first derivatives through the connection

$$\Gamma^{\gamma}{}_{\mu\nu} = \frac{1}{2} g^{\gamma\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right). \tag{1.3}$$

However there is no way to construct an invariant scalar only with these ingredients.

The problem was elegantly solved in 1915 by D. Hilbert proposing that L_g must also contain second order derivative of the metric through the Riemann curvature tensor

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}. \tag{1.4}$$

In four dimensions, there are 14 invariant scalars which can be constructed from the metric coefficients, theirs first and second derivatives. Nevertheless, only the curvature scalar $R = g^{\mu\nu}R_{\mu\nu}$ is lineal in the second derivatives of $g_{\mu\nu}$. Then the Einstein field equations can be derived from the Einstein-Hilbert (EH) action

$$S = \kappa \int \sqrt{-g} R \, d^4x. \tag{1.5}$$

It remains an open problem to find an action which describes the unification of gravity with the others interactions. Along the thesis we will try to approach this problem generalizing the Einstein theory to diverse gravity theories.

1.2 First order formulation for gravity

The differential forms are an useful tool in order to describe a gravity theory beyond General Relativity. Before to study (super)gravity in this formalism it is necessary to introduce some useful concepts for the understanding of the thesis.

Let us consider the space-time as a four-dimensional differential manifold M. For each point P of the manifold we can define a tangent space built from all the tangent vectors defined on P. Let x^μ be a coordinate system defined on the tangent space which contains P. Then the vectors $\partial_i = \partial_\mu (P)$ define a coordinate basis of the tangent space in P. This basis is not necessary orthonormal but rather

$$\partial_i \cdot \partial_k = g_{ik}, \tag{1.6}$$

where $g_{ik} = g_{\mu\nu}(P)$ corresponds to the metric components in the coordinate basis. However, an orthonormal basis can be defined using a tetrad (also known as vierbein) $e_a = e_a^i \partial_i$,

$$e_a \cdot e_b = e_a^i e_b^k g_{ik} = \eta_{ab}, \tag{1.7}$$

where η_{ab} is the Minkowski metric. The inverse matrix e_i^a allows to relate the Minkowski metric to the metric g_{ik},

$$g_{ik} = e_i^a e_k^b \eta_{ab}. \tag{1.8}$$

Thus the space-time metric can be directly derived if we know the local orthonormal frame e_i^a. Nevertheless, the choice of the e_i^a is not unique since they transform as a contravariant vector under local Lorentz $SO(3,1)$ rotations

$$e_i^a \to e_i^{\bar{a}} = \Lambda^{\bar{a}}{}_b e_i^b, \tag{1.9}$$

where the matrices Λ form the Lorentz group and satisfy

$$\Lambda^a_{\ c}\Lambda^b_{\ d}\eta_{ab} = \eta_{cd}. \tag{1.10}$$

Although the vierbein e^a behave as a vector under local Lorentz transformations, the exterior derivative de^a does not. A covariant exterior derivative D has to be introduced such that De^a transforms covariantly under local Lorentz rotations,

$$De^a \to De^{\bar{a}} = \Lambda^{\bar{a}}_{\ b}De^b. \tag{1.11}$$

The covariant exterior derivative D required the presence of a gauge field ω

$$De^a \equiv de^a + \omega^a_{\ b}e^b, \tag{1.12}$$

where ω^{ab} is known as the one-form spin connection and obeys the following transformation law

$$\omega^a_{\ b} = \Lambda^d_{\ b}\Lambda^a_{\ c}\omega^c_{\ d} - \Lambda^c_{\ b}d\Lambda^a_{\ c}. \tag{1.13}$$

Analogously to the Yang-Mills theory, a field strength can be associated to the gauge potential ω,

$$R^a_{\ bik} = \partial_i\omega^a_{\ bk} - \partial_k\omega^a_{\ bi} + \omega^a_{\ ci}\omega^c_{\ bk} - \omega^a_{\ ck}\omega^c_{\ bi}, \tag{1.14}$$

which allows to define the curvature 2-form

$$R^a_{\ b} = d\omega^a_{\ b} + \omega^a_{\ c}\omega^c_{\ b} = \frac{1}{2}R^a_{\ bik}dx^i \wedge dx^k. \tag{1.15}$$

On the other hand, we introduce the torsion 2-form as

$$T^a = De^a = \frac{1}{2}T^a_{\ ik}dx^i \wedge dx^k, \tag{1.16}$$

with

$$T^a_{\ ik} = \partial_i e^a_{\ k} - \partial_k e^a_{\ i} + \omega^a_{\ bi}e^b_{\ k} - \omega^a_{\ bk}e^b_{\ i}. \tag{1.17}$$

The equations $(1.15) - (1.16)$ are the structure equations and describe the geometrical structure of the manifold. These 2-forms satisfy the first and the second Bianchi identity

$$DT^a = R^a_{\ b}e^b, \tag{1.18}$$

$$DR^a_{\ b} = 0. \tag{1.19}$$

In the Cartan formalism, the Einstein-Hilbert action (without cosmological constant) can be written in terms of the vierbein e^a, the spin connection ω^{ab} and their respective fields strengths $\left(T^a,\ R^{ab}\right)$ as

$$S_{EH} = \int \epsilon_{abcd} R^{ab} e^c e^d. \tag{1.20}$$

The field equations can be obtained in this formalism varying the action with respect to the vierbein and the spin connection

$$\delta S_{EH} = \int \epsilon_{abcd} \left(\delta R^{ab} e^c e^d + 2R^{ab} e^c \delta e^d\right). \tag{1.21}$$

Then $\delta S = 0$ requires the following field equations

$$\epsilon_{abcd} R^{ab} e^c = 0, \tag{1.22}$$
$$\epsilon_{abcd} T^c e^d = 0. \tag{1.23}$$

which correspond to the Einstein field equations in the Cartan formalism. Let us note that the second equation express the vanishing of the torsion. This allows to write ω^{ab} in terms of e^a from a variational equation and does not correspond to a priori constraint.

1.3 Poincaré symmetries

A gauge symmetry is a crucial ingredient in order to have a Yang-Mills theory and ensure renormalization. However, gravity described by General Relativity have only a diffeomorphism invariance which makes difficult the unification with the other three interactions.

In this section we will briefly discuss the invariance of the EH action under the Poincaré symmetries. This discussion will be fundamental for a correct understanding of the thesis and will be generalized to other (super)symmetries.

One of the simplest gauge symmetries in order to describe gravity corresponds to the Poincaré group $ISO(3,1)$. The generators of the Poincaré Lie algebra are given by

$$T_A = \left(P_a, J_{ab}\right), \tag{1.24}$$

where J_{ab} are the Lorentz transformations and P_a correspond to the four-dimensionañ translational generators. These generators satisfy the following commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{1.25}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{1.26}$$

$$[P_a, P_b] = 0. \tag{1.27}$$

The corresponding gauge fields are the one-form vierbein e^a and the one-form spin connection ω^{ab}. These gauge fields can be viewed as a single multiplet in the adjoint representation of the Poincaré group. Then the one-form gauge connection A can be written as

$$A = A^A T_A = \frac{1}{l} e^a P_a + \frac{1}{2} \omega^{ab} J_{ab}. \tag{1.28}$$

The introduction of the length scale l is necessary in order to interpret the gauge field as the vierbein. In fact, one can always choose the generators T_A to be dimensionless so that the one-form connection A must also be dimensionless. However, the vierbein $e^a = e_i^a dx^i$ must have length dimensions since it is related to the spacetime metric $g_{ik} = e_i^a e_k^b \eta_{ab}$. This means that that the true gauge field must be of the form e^a/l.

The field strength $F = dA + A^2$ is defined as

$$F = F^A T_A = \frac{1}{l} T^a P_a + \frac{1}{2} R^{ab} J_{ab},$$

where the corresponding Poincaré Lie algebra-valued curvature 2-forms are

$$R^{ab} = d\omega^{ab} + \omega^a_{\ c} \omega^{cb}, \tag{1.29}$$

$$T^a = de^a + \omega^a_{\ b} e^b \equiv De^a. \tag{1.30}$$

Thus the Lorentz curvature R^{ab} corresponds to the field strength of the the spin connection while the torsion T^a is the field strength of the vierbein. The formalism used here shows explicitly the relation between the algebraic structure of a symmetry group and the geometrical structure of a manifold.

In order to have a true gauge theory of gravity the action has to be invariant under the whole gauge algebra. Nevertheless the Einstein-Hilbert action (1.20) is not a Yang-Mills action so that the EH action is not invariant under the Poincaré algebra $\mathfrak{iso}(3,1)$, but only under the Lorentz subalgebra $\mathfrak{so}(3,1)$.

The non invariance of the EH action (1.20) under the Poincaré algebra can be viewed using the Poincaré gauge transformations

$$\delta_{gauge} A^A = \nabla \lambda^A, \tag{1.31}$$

where λ^A is the gauge parameter $\lambda^a \equiv \left(\rho^a, \kappa^{ab}\right)$ and ∇ is the Poincaré covariant derivative. Then the transformation laws of the gauge fields are

$$\delta e^a = D\rho^a + e_c \kappa^{ca}, \tag{1.32}$$

$$\delta \omega^{ab} = D\kappa^{ab}. \tag{1.33}$$

In particular, the translational transformations correspond to

$$\delta e^a = D\rho^a, \tag{1.34}$$

$$\delta \omega^{ab} = 0. \tag{1.35}$$

It is straightforward to see that the EH action is not invariant under (1.34) – (1.35). In fact, if we consider the variation of the action (1.20) under local transformations, we find

$$\delta S_{EH} = 2 \int \epsilon_{abcd} R^{ab} e^c \delta e^d = 2 \int \epsilon_{abcd} R^{ab} T^c \rho^d \neq 0. \tag{1.36}$$

Beside, the constraint $T^a = 0$ is not invariant under Poincaré transformations,

$$\delta T^a = R^{ab} \rho_b \neq 0. \tag{1.37}$$

The non invariance of the EH action seems strange since a translation can be though as a coordinate transformation. However, a coordinate transformation is not a gauge translation but a Lie derivative. Then we said that the EH action is invariant under diffeomorphisms.

The situation is completely different in odd dimensions where the $D = 2n - 1$ EH action is truly invariant under the Poincaré algebra. Interestingly the inclusion of the cosmological constant in the EH action leads to an anti-de Sitter (AdS) invariance. The generalization of General Relativity to higher dimensions and to other symmetries will be discussed along the thesis.

Chapter 2

Beyond General Relativity

2.1 The Lanczos-Lovelock theory

It is an accepted assumption in Physics that the spacetime may have more than four dimensions. This requires a generalization of General Relativity theory of gravity that includes general covariance and second order field equations for the metric. Although the Einstein-Hilbert action can be generalized to higher dimensions, it is interesting to analyze a more general gravity theory.

The most general metric theory of gravity satisfying the criteria of general covariance and second order field equations is a polynomial of degree $[D/2]$ in the curvature known as the Lanczos-Lovelock gravity theory (LL) [1, 2]. The LL action can be written as the most general D-form invariant under local Lorentz transformations, constructed out of the vielbein e^a, the spin connection ω^{ab} and their exterior derivatives without using the Hodge dual [3, 4],

$$S = \int \sum_{p=0}^{[D/2]} \alpha_p L^{(p)}, \tag{2.1}$$

where α_p are arbitrary constants and they are not fixed from first principles, and

$$L^{(p)} = \epsilon_{a_1 a_2 \cdots a_D} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_D}. \tag{2.2}$$

Although the EH action is contained in the LL action, the action with higher powers of curvature are dynamically different from General Relativity and are not perturbatively related.

It was shown in ref. [5] that requiring the LL theory to have the maximum possible number of degrees of freedom, fixes the α_p's coefficients in terms of the gravitational and the cosmological constants.

In odd dimensions, the parameters are given by

$$\alpha_p = \alpha_0 \frac{(2n-1)(2\gamma)^p}{(2n-2p-1)} \binom{n-1}{p}, \tag{2.3}$$

with

$$\alpha_0 = \frac{\kappa}{Dl^D}, \tag{2.4}$$

$$\gamma = -sgn(\Lambda)\frac{l^2}{2}, \tag{2.5}$$

where l is a length parameter related to the cosmological constant by

$$\Lambda = \pm \frac{(D-1)(D-2)}{2l^2}, \tag{2.6}$$

and the gravitational constant G is related to κ through

$$\kappa^{-1} = 2(D-2)!\Omega_{D-2}G. \tag{2.7}$$

With these coefficients, the LL Lagrangian is a Chern-Simons (CS) $(2n-1)$-form

$$L_{CS} = \kappa\epsilon_{a_1a_2\cdots a_{2n-1}} \sum_{p=0}^{n} \frac{l^{2(p-n)+1}}{2(n-p)-1} \binom{n-1}{p} R^{a_1a_2}\cdots R^{a_{2p-1}a_{2p}} e^{a_{2p+1}}\cdots e^{a_{2n-1}}. \tag{2.8}$$

The Lagrangian (2.8) is invariant not only under local Lorentz rotation, but also under a local AdS boost,

$$\delta e^a = -D\rho^a, \tag{2.9}$$

$$\delta\omega^{ab} = \frac{1}{l^2}\left(\rho^a e^b - \rho^b e^a\right). \tag{2.10}$$

Meanwhile in even dimensions, the coefficient are given by

$$\alpha_p = \alpha_0 (2\gamma)^p \binom{n}{p}, \tag{2.11}$$

With these coefficients the LL Lagrangian takes the form [5]

$$L = \frac{\kappa}{2n}\epsilon_{a_1a_2\cdots a_{2n}} \bar{R}^{a_1a_2}\cdots\bar{R}^{a_{2n-1}a_{2n}}, \tag{2.12}$$

which is the Pfaffian of the 2-form $\bar{R}^{ab} = R^{ab} + \frac{1}{l^2} e^a e^b$ and can be written as a Born-Infeld like form [5, 6],

$$L = 2^{n-1}(n-1)!\kappa\sqrt{\det\left(R^{ab} + \frac{1}{l^2}e^a e^b\right)}. \tag{2.13}$$

The corresponding Born-Infeld (BI) gravity Lagrangian is given by

$$L_{BI} = \kappa\epsilon_{a_1 a_2 \cdots a_{2n}} \sum_{p=0}^{n} \frac{l^{2p-2n}}{2n} \binom{n}{p} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_{2n}}, \tag{2.14}$$

which is off-shell invariant under the Lorentz Lie algebra $\mathfrak{so}(2n-1,1)$.

The Levi-Civita symbol $\epsilon_{a_1 a_2 \cdots a_{2n}}$ in (2.14) can be viewed as the only non-vanishing component of the $SO(2n-1,1)$ invariant tensor of rank n, namely

$$\left\langle J_{a_1 a_2} \cdots J_{a_{2n-1} a_{2n}} \right\rangle = \frac{2^{n-1}}{n} \epsilon_{a_1 a_2 \cdots a_{2n}}. \tag{2.15}$$

Let us note that this choice of the invariant tensor breaks the AdS group to their Lorentz subgroup. If $\langle T_A \cdots T_B \rangle$ is an invariant tensor for the $\mathfrak{so}(D-1,2)$ algebra then the Lagrangian corresponds to a topological invariant.

If the Lovelock gravity theory is the appropriate theory to provide a framework for the gravitational interaction, then it must satisfy the correspondence principle, namely it must be related to General Relativity theory. Nevertheless, from the Lovelock action, it is apparent that neither the $l \to 0$ nor the $l \to \infty$ limit allows to recover the Einstein-Hilbert term. In the following sections, we will discuss a particular choice of symmetry that permits to establish a relation between General Relativity and the Lovelock gravity theory.

2.2 Maxwell symmetries and General Relativity

It is known that the Maxwell algebra[1] $\mathcal{M}$ corresponds to a modification of the Poincaré algebra, where a constant electromagnetic field background is added to the Minkowski space [7, 8]. In four dimensions, this algebra is obtained by adding to

[1]Also known as $\mathfrak{B}_4$ algebra.

the Poincaré generators (J_{ab}, P_a) the tensorial central charges Z_{ab}. This enlarges the Poincaré algebra and modifies the commutation relations as follows

$$\begin{aligned}
[P_a, P_b] &= Z_{ab}, \qquad [J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, && (2.16)\\
[J_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, && (2.17)\\
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, && (2.18)\\
[Z_{ab}, Z_{cd}] &= 0, \qquad [Z_{ab}, P_c] = 0. && (2.19)
\end{aligned}$$

Recently, it was shown in ref. [9] an alternative way of introducing the cosmological constant term using the Maxwell symmetries. In particular, the Maxwell type algebras [2] allow to recover the Einstein equations from a Lovelock gravity theory in a certain limit of a coupling constant [10, 11, 12, 13]. In the next section we will briefly review the relation between General Relativity and the Maxwell algebras type using the abelian semigroup expansion approach.

2.2.1 Standard General Relativity from Chern-Simons gravity

In this section, following ref. [10] we discuss how to recover General Relativity from a Chern-Simons gravity theory using a Lie algebra expansion procedure.

A Generalization of the odd-dimensional General Relativity theory corresponds to the *AdS* Chern-Simons gravity theory. The CS theory has the advantage to describe a gauge gravity theory in a odd-dimensional spacetime.

In $(4+1)$ dimensions, the general expression of the Chern-Simons Lagrangian is given by [14, 15]

$$L_{CS}^{(4+1)} = \kappa \left\langle A\,(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5 \right\rangle, \tag{2.20}$$

where $\langle \cdots \rangle$ denotes a symmetric invariant tensor. Then in order to write down a CS Lagrangian for the *AdS* algebra, we start from the *AdS*-valued one-form gauge connection

$$A = \frac{1}{2}\omega^{ab}\tilde{J}_{ab} + \frac{1}{l}e^a\tilde{P}_a, \tag{2.21}$$

[2]Also known as generalized Poincaré algebra.

where the $\mathfrak{so}(4,2)$ generators satisfy the following commutation relations

$$\left[\tilde{J}_{ab},\tilde{J}_{cd}\right]=\eta_{bc}\tilde{J}_{ad}-\eta_{ac}\tilde{J}_{bd}-\eta_{bd}\tilde{J}_{ac}+\eta_{ad}\tilde{J}_{bc}, \tag{2.22}$$

$$\left[\tilde{J}_{ab},\tilde{P}_{c}\right]=\eta_{bc}\tilde{P}_{a}-\eta_{ac}\tilde{P}_{b}, \tag{2.23}$$

$$\left[\tilde{P}_{a},\tilde{P}_{b}\right]=\tilde{J}_{ab}. \tag{2.24}$$

The Levi-Civita symbol ϵ_{abcde} corresponds to the only non-vanishing component of the $\mathfrak{so}(4,2)$-invariant tensor. Therefore, the $D=5$ Chern-Simons Lagrangian invariant under the AdS algebra can be written as

$$L_{AdS}^{(5)}=\kappa\epsilon_{abcde}\left(\frac{1}{5l^{5}}e^{a}e^{b}e^{c}e^{d}e^{e}+\frac{2}{3l^{3}}R^{ab}e^{c}e^{d}e^{e}+\frac{1}{l}R^{ab}R^{cd}e^{e}\right). \tag{2.25}$$

One can see that neither the $l\to\infty$ nor the $l\to 0$ limit allows to recover the EH Lagrangian alone. Nevertheless, the AdS Lie algebra is not the only possible choice in order to describe a gravity theory. In particular, a family of Maxwell type algebras $\mathcal{M}_{2m+1}$ can be defined using the abelian semigroup expansion procedure.

The S-expansion method is a powerful tool in order to derive new lie (super)algebras and construct new (super)gravity theories [See Appendix A]. Basically it consists on combining the multiplication law of a semigroup S with the structure constants of a Lie (super)algebra $\mathfrak{g}$ [16].

Following ref. [10], let $S_{E}^{(3)}=\{\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ be the relevant finite abelian semigroup with the following multiplication law

$$\lambda_{\alpha}\lambda_{\beta}=\begin{cases}\lambda_{\alpha+\beta}, & \text{when } \alpha+\beta\leq 4,\\ \lambda_{4}, & \text{when } \alpha+\beta>4.\end{cases} \tag{2.26}$$

Here λ_4 plays the role of the zero element of the semigroup $S_{E}^{(3)}$, so we have for each $\lambda_{\alpha}\in S_{E}^{(3)}$, $\lambda_4\lambda_{\alpha}=\lambda_4=0_s$. Let us consider the $S_{E}^{(3)}$-expansion of the $\mathfrak{so}(4,2)$ Lie algebra. The Maxwell type algebra[3] $\mathcal{M}_5$ is obtained after extracting a resonant subalgebra and performing its 0_s-reduction [10]. The new algebra is generated by $\{J_{ab},\ P_a,\ Z_{ab},\ Z_a\}$

[3]Also known as $\mathfrak{B}_5$ algebra.

which are related to the $\mathfrak{so}(4,2)$ generators through

$$J_{ab} = \lambda_0 \otimes \tilde{J}_{ab}, \tag{2.27}$$

$$Z_{ab} = \lambda_2 \otimes \tilde{J}_{ab}, \tag{2.28}$$

$$P_a = \lambda_1 \otimes \tilde{P}_a, \tag{2.29}$$

$$Z_a = \lambda_3 \otimes \tilde{P}_a. \tag{2.30}$$

The $\mathcal{M}_5$ generators satisfy the following commutation relations

$$[P_a, P_b] = Z_{ab}, \qquad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{2.31}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{2.32}$$

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{2.33}$$

$$[Z_{ab}, P_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \tag{2.34}$$

$$[J_{ab}, Z_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \tag{2.35}$$

$$[Z_{ab}, Z_{cd}] = [Z_{ab}, Z_c] = [P_a, Z_b] = [Z_a, Z_b] = 0. \tag{2.36}$$

Let us note that the P_a generators are no longer AdS boost, nevertheless the vielbein e^a still transforms as vector under Lorentz transformations.

A very useful advantage of the S-expansion method is that it provides with an invariant tensor for the S-expanded (super)algebra $\mathfrak{G} = S \times \mathfrak{g}$ in terms of an invariant tensor for the original (super)algebra $\mathfrak{g}$. Using Theorem VII.2 from ref. [16], one can see that the only non-vanishing components of a $\mathcal{M}_5$ invariant tensor are given by

$$\begin{aligned} \langle J_{ab} J_{cd} P_e \rangle_{\mathcal{M}_5} &= \alpha_1 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \tilde{P}_e \right\rangle_{AdS} \\ &= \frac{4}{3} l^3 \alpha_1 \epsilon_{abcde}, \end{aligned} \tag{2.37}$$

$$\begin{aligned} \langle J_{ab} J_{cd} Z_e \rangle_{\mathcal{M}_5} &= \alpha_3 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \tilde{P}_e \right\rangle_{AdS} \\ &= \frac{4}{3} l^3 \alpha_3 \epsilon_{abcde}, \end{aligned} \tag{2.38}$$

$$\begin{aligned} \langle J_{ab} Z_{cd} P_e \rangle_{\mathcal{M}_5} &= \alpha_3 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \tilde{P}_e \right\rangle_{AdS} \\ &= \frac{4}{3} l^3 \alpha_3 \epsilon_{abcde}, \end{aligned} \tag{2.39}$$

where α_1 and α_3 are arbitrary constants of dimension $[\text{length}]^{-3}$.

The one-form gauge connection for the Maxwell type algebra $\mathcal{M}_5$ is

$$A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{l}e^a P_a + \frac{1}{2}k^{ab}Z_{ab} + \frac{1}{l}h^a Z_a. \tag{2.40}$$

Inserting the one-form connection (2.40) into the general expression of the CS action (2.20) and using the invariant tensor $(2.37-2.39)$, we can write explicitly the Chern-Simons gravity action for the Maxwell algebra type $\mathcal{M}_5$ [10],

$$L_{CS}^{(5)} = \alpha_1 l^2 \epsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \epsilon_{abcde} \left(\frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right). \tag{2.41}$$

The Lagrangian (2.41) is split into two independent pieces proportional to α_1 and α_3. The first term corresponds to the CS Lagrangian invariant under the Poincaré algebra $\mathfrak{iso}(4,1)$. Meanwhile the term proportional to α_3 contains the EH Lagrangian and the coupling between the curvature and the new Maxwell fields k^{ab} and h^a.

Interestingly, when the coupling constant l equals zero, we obtain only the EH term,

$$L_{l\rightarrow 0}^{(5)} = \frac{2}{3}\alpha_3 \epsilon_{abcde} R^{ab} e^c e^d e^e. \tag{2.42}$$

Analogously, the limit $l \rightarrow 0$ in the variation of the Lagrangian leads to the Einstein equations in vacuum,

$$\delta L_{CS}^{(5)} = 2\alpha_3 \epsilon_{abcde} R^{ab} e^c e^d \delta e^e + 2\alpha_3 \epsilon_{abcde} \delta\omega^{ab} e^c e^d T^e. \tag{2.43}$$

This result can be generalized to every odd dimension using a bigger semigroup leading to a bigger Maxwell algebra type. However we have pointed out in Theorem 4 of ref. [12] that only some Maxwell type algebras allow to recover General Relativity from a Chern-Simons gravity theory.

Theorem 1 *Let $\mathcal{M}_{2m+1}$ be the Maxwell type algebra, which is obtained from the AdS algebra by a resonant reduced $S_E^{(2m-1)}$-expansion. If $L_{CS}^{(2p+1)}$ is a $(2p+1)$-dimensional Chern-Simons Lagrangian invariant under the $\mathcal{M}_{2m+1}$ algebra, then the CS Lagrangian leads to the Einstein equations in a certain limit of the coupling constant l, if and only if $m \geq p$.*

2.2.2 Even-dimensional General Relativity from Born-Infeld gravity

In this section, following ref. [11] we discuss how to recover General Relativity from a Born-Infeld gravity theory using the semigroup expansion method.

The four-dimensional Lovelock Lagrangian corresponds to the Born-Infeld gravity Lagrangian and can be seen as the bosonic MacDowell-Mansouri Lagrangian [17]. Then, the BI Lagrangian can be constructed from the 2-form curvature as

$$L_{BI} = \kappa \langle F \wedge F \rangle = \kappa F^A \wedge F^B \langle T_A T_B \rangle . \tag{2.44}$$

Let us note that if we choose $\langle T_A T_B \rangle$ as an invariant tensor for the $so\,(3,2)$ group, the the action (2.44) is a topological invariant and does not contribute to the dynamics. However, with the following choice of the invariant tensor

$$\langle T_A T_B \rangle = \langle J_{ab} J_{cd} \rangle = \epsilon_{abcd}, \tag{2.45}$$

the action (2.44) becomes

$$L_{BI} = \frac{\kappa}{4} \mathcal{R}^{ab} \mathcal{R}^{cd} \epsilon_{abcd}, \tag{2.46}$$

with

$$\mathcal{R}^{ab} = R^{ab} + \frac{1}{l^2} e^a e^b. \tag{2.47}$$

The choice of the invariant tensor, which is necessary in order to reproduce a dynamical action, breaks the $so\,(3,2)$ symmetry to its Lorentz subgroup.

One can note that, although the Einstein equations (with cosmological constant) can be obtained from a BI gravity theory, it is not possible to recover General Relativity in higher even dimensions. However, there is a particular choice of symmetry that allow to relate even-dimensional BI gravity theory and the Einstein dynamic.

Following ref. [11], let $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ be an abelian semigroup with the following multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 3, \\ \lambda_3, & \text{when } \alpha + \beta > 3, \end{cases} \tag{2.48}$$

where λ_3 plays the role of the zero element of the semigroup $S_E^{(2)}$. Let us consider the $S_E^{(2)}$-expansion of the $\mathfrak{so}\,(3,2)$ Lie algebra. The Maxwell algebra $\mathcal{M}$ ($\mathcal{M} = \mathcal{M}_4$)is obtained after extracting a resonant subalgebra and performing its 0_s-reduction [11]. The expanded algebra is generated by $\{J_{ab},\ P_a,\ Z_{ab}\}$ whose generators are related to the $\mathfrak{so}\,(4,2)$ generators through

$$
\begin{aligned}
J_{ab} &= \lambda_0 \tilde{J}_{ab}, &(2.49)\\
Z_{ab} &= \lambda_2 \tilde{J}_{ab}, &(2.50)\\
P_a &= \lambda_1 \tilde{P}_a, &(2.51)
\end{aligned}
$$

and satisfy the commutation relations given by eqs. $(2.16)-(2.19)$. In particular, as in the $so\,(3,2)$ symmetry, the Maxwell algebra has a Lorentz like subalgebra $\mathcal{L}_{\mathcal{M}}$ given by $\{J_{ab}, Z_{ab}\}$ which can be obtained directly as a reduced $S_0^{(2)}$-expansion of the Lorentz algebra $\mathfrak{so}\,(3,1)$. Using Theorem VII.2 from ref. [16], one can see that the only non-vanishing components of an invariant tensor for the $\mathcal{L}_{\mathcal{M}}$ subalgebra are given by

$$
\begin{aligned}
\langle J_{ab} J_{cd} \rangle_{\mathcal{L}_{\mathcal{M}}} &= \alpha_0 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle_{\mathcal{L}} \\
&= \alpha_0 \epsilon_{abcd}, &(2.52)\\
\langle J_{ab} Z_{cd} \rangle_{\mathcal{L}_{\mathcal{M}}} &= \alpha_2 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle_{\mathcal{L}} \\
&= \alpha_2 \epsilon_{abcd}. &(2.53)
\end{aligned}
$$

where α_0 and α_2 are arbitrary constants. Interestingly, The invariant tensor breaks the Maxwell symmetry to its Lorentz like subgroup.

The curvature 2-form for the Lorentz like algebra $\mathcal{L}_{\mathcal{M}}$ is given by

$$
F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{2} \left(D_\omega k^{ab} + \frac{1}{l^2} e^a e^b \right) Z_{ab}, \tag{2.54}
$$

where k^{ab} corresponds to the bosonic field associated to the abelian generator Z_{ab} and R^{ab} is the usual Lorentz curvature $R^{ab} = d\omega^{ab} + \omega^a_{\ c} \omega^{cb}$. Inserting the 2-form curvature (2.54) into the general expression of the BI expression (2.44) and using the invariant tensor $(2.52 - 2.53)$, we can write explicitly the Born-Infeld gravity Lagrangian for the Lorentz like $\mathcal{L}_{\mathcal{M}}$ algebra [11],

$$
L_{BI}^{\mathcal{L}_{\mathcal{M}}} = \frac{\alpha_0}{4} \epsilon_{abcd} R^{ab} R^{cd} + \frac{\alpha_2}{2} \epsilon_{abcd} \left(R^{ab} e^c e^d + D_\omega k^{ab} R^{cd} \right). \tag{2.55}
$$

Here we can note that the Lagrangian (2.55) is split into two independent terms. The piece proportional to α_0 corresponds to a topological boundary term known as the

Gauss Bonnet term. While the term proportional to α_2 contains the Einstein-Hilbert term and the coupling between the new gauge field k^{ab} and the Lorentz curvature R^{ab} which corresponds to a Gauss Bonnet like term. Although the topological Euler-Gauss-Bonnet term do not contribute to the field equations, it permits to regularize the action and the related conserved charges [18, 19, 20, 21, 22].

Interestingly, the variation of the Lagrangian, modulo boundary terms, leads to General Relativity equations when a solution without matter $\left(k^{ab}=0\right)$ is considered.

$$\delta L_{BI}^{\mathcal{L}_{\mathcal{M}}} = \alpha_2 \epsilon_{abcd} \left(R^{ab} e^c\right) \delta e^d + \alpha_2 \epsilon_{abcd} \delta\omega^{ab} \left(T^c e^d\right). \tag{2.56}$$

Nevertheless as was shown in ref. [11], in higher even dimensions ($D \geq 6$), an appropriate limit of the coupling constant l has to be considered in order to recover the field equations of General Relativity. In particular, it was pointed out in Theorem 5 of ref. [12] that only some Lorentz like algebras allow to recover General Relativity from a Born-Infeld type gravity theory.

Theorem 2 *Let $\mathcal{L}_{\mathcal{M}_{2m}}$ be the Lorentz like algebra obtained from the Lorentz algebra by a reduced $S_0^{(2m-2)}$-expansion, which corresponds to a subalgebra of the Maxwell type algebra $\mathcal{M}_{2m}$. If $L_{BI}^{(2p)}$ is a $(2p)$-dimensional Born-Infeld type Lagrangian constructed from the 2-form curvature of the $\mathcal{L}_{\mathcal{M}_{2m}}$ algebra, then the BI Lagrangian leads to the Einstein equations in a certain limit of the coupling constant l, if and only if $m \geq p$.*

2.3 Einstein-Lovelock-Cartan gravity theory

In the previous section we have seen that the S-expansion method permits to relate General Relativity with the Chern-Simons and Born-Infeld gravity theories using the Maxwell symmetries. This suggests a generalized formulation of the Lovelock-Cartan gravity action (2.1) which allows to recover the Einstein equations under a certain limit of a coupling constant l.

A generalized Lovelock action can be obtained considering the most general D-form invariant under a local lorentz type transformation constructed out of the vielbein e^a, the spin-connection ω^{ab} and the expanded fields [13]. The new action is given by

$$S_{GL} = \int \sum_{p=0}^{[D/2]} \mu_i \alpha_p L_{GL}^{(p,i)}, \tag{2.57}$$

where α_p and μ_i, with $i = 0, \ldots, D-2$, are arbitrary constants and $L_{GL}^{(p,i)}$ is given by

$$L_{GL}^{(p,i)} = l^{d-2} \delta^{i}_{i_1 + \cdots + i_{D-p}} \epsilon_{a_1 a_2 \cdots a_D} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2p-1}, a_{2p}, i_p)} e^{(a_{2p+1}, i_{p+1})} \cdots e^{\left(a_D, i_{D-p}\right)}, \quad (2.58)$$

with

$$R^{(ab,2i)} = d\omega^{(ab,2i)} + \eta_{cd} \omega^{(ac,2j)} \omega^{(db,2k)} \delta^{i}_{j+k}. \quad (2.59)$$

The expanded fields $\left\{\omega^{(ab,2i)}, e^{(a,2i+1)}\right\}$ are related to the $\mathfrak{so}\,(D-1,2)$ fields $\left\{\tilde{\omega}^{ab}, \tilde{e}^{a}\right\}$ through

$$\begin{aligned} \omega^{(ab,2i)} &= \lambda_{2i} \otimes \tilde{\omega}^{ab}, & (2.60) \\ e^{(a,2i+1)} &= \lambda_{2i+1} \otimes \tilde{e}^{a}, & (2.61) \end{aligned}$$

where $\lambda_\alpha \in S_E^{(D-2)}$obeys the following multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq D-1, \\ \lambda_{D-1}, & \text{when } \alpha + \beta > D-1. \end{cases} \quad (2.62)$$

Interestingly, there are different choices for the coefficients α_p leading to different theories with diverse numbers of degrees of freedom. In particular, as in ref. [5], it is possible to choose the α_p's coefficients according that the fields attain the maximum number of degrees of freedom. This fixes the α_p's parameters in terms of the gravitational and the cosmological constants [13].

In odd dimensions, the coefficients are given by

$$\alpha_p = \alpha_0 \frac{(2n-1)\,(2\gamma)^p}{(2n-2p-1)} \binom{n-1}{p}, \quad (2.63)$$

with

$$\begin{aligned} \alpha_0 &= \frac{\kappa}{D l^D}, & (2.64) \\ \gamma &= -sgn\,(\Lambda)\,\frac{l^2}{2}, & (2.65) \end{aligned}$$

where l is a length parameter related to the cosmological constant as in eq. (2.6). As in the original Lovelock gravity theory, the Lagrangian (2.57) may be written as the

Chern-Simons form[4]

$$L_{CS}^{(2n-1)} = \sum_{p=1}^{n-1} l^{2p-2} c_p \mu_i \delta^i_{i_1+\cdots+i_{2n-1-p}} \epsilon_{a_1 a_2 \cdots a_{2n-1}} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2p-1} a_{2p}, i_p)} \times e^{(a_{2p+1}, i_{p+1})} \cdots e^{(a_{2n-1}, i_{2n-1-p})}, \tag{2.66}$$

where

$$c_p = \frac{1}{2(n-p)-1} \binom{n-1}{p}. \tag{2.67}$$

The Lagrangian (2.66) corresponds to the Einstein-Chern-Simons Lagrangian and it is invariant not only local Lorentz type rotation but also under the Maxwell type algebra $\mathcal{M}_{2n-1}$. In particular, the $l \to 0$ limit permits to recover General Relativity dynamics as was shown in refs. [10, 12].

Meanwhile, in even dimensions the coefficients satisfying the requirement to have the maximum possible number of degrees of freedom are given by

$$\alpha_p = \alpha_0 (2\gamma)^p \binom{n}{p}. \tag{2.68}$$

With these coefficients the Lagrangian (2.66) take a Born-Infeld form,

$$L_{BI}^{(2n)} = \sum_{p=1}^{n} \frac{\kappa}{2n} l^{2p-2} \binom{n}{p} \mu_i \delta^i_{i_1+\cdots+i_{2n-p}} \epsilon_{a_1 a_2 \cdots a_{2n}} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2p-1} a_{2p}, i_p)} \times e^{(a_{2p+1}, i_{p+1})} \cdots e^{(a_{2n}, i_{2n-p})}. \tag{2.69}$$

The Lagrangian (2.69) is invariant under a local lorentz type algebra and corresponds to the Einstein-Born-Infeld Lagrangian found in ref. [11]. General Relativity is recovered when the $l \to 0$ limit is considered.

Unlike the Lanczos-Lovelock theory, the generalized Einstein-Lovelock gravity action allows to recover Einstein-Hilbert dynamics in a particular limit of the coupling constant l both in even and odd dimensions. Interestingly, as in ref. [5], torsional terms can be added to the Einstein-Lovelock Lagrangian leading to Pontryagin-Chern-Simons Lagrangians in $4k-1$ dimensions [13].

[4]The term with $p = 0$ does not contribute to the sum because $\delta^i_{i_1+\cdots+i_{2n-1}} = 0$ for any value of i and n.

Chapter 3

Supersymmetry and supergravity

3.1 Why Supersymmetry?

Three of the four forces of nature are successfully described by the Standard Model as a Yang-Mills theory. They are elegantly related to gauge symmetries allowing renormalizability and ensuring a viable quantum theory. On the other hand, gravity described by General Relativity, resists to the quantization. In spite of the huge success of the General Relativity theory, there is not a consistent quantum description of gravity which prevents a possible unification of gravity to the other interactions.

In order to unify gravity with the other interactions in a unique theory, it is necessary to put together the internal symmetries with the space-time symmetries. A good candidate for this purpose is the supersymmetry[1]. The presence of supersymmetry offers the possibility to solve the ultraviolet divergences cancelling the fermionic and bosonic contributions to divergent loop integrals. One of the phenomenological advantages of this theory is that it solves the hierarchy problem present in the Standard Model. In particular, supersymmetry requires the existence of super-partner for each particle, whose contributions allows to cancel quadratic divergences in quantum corrections to the Higgs mass.

Supersymmetry theories are remarkable theories since they unify space-time with internal symmetries relating bosonic and fermionic particles in an elegant way. The supersymmetry transformations generated by quantum operators Q have the interesting

[1]A general introduction to supersymmetry can be found in ref. [23].

property of mapping bosons into fermions and vice versa

$$Q\left|boson\right\rangle = \left|fermion\right\rangle ; \qquad Q\left|fermion\right\rangle = \left|boson\right\rangle . \tag{3.1}$$

Interestingly, a new algebraic structure known as the Lie superalgebra[2] is required in order to describe a supersymmetry theory. This permits to generalize the Poincaré algebra introducing besides the bosonic generators B, the fermionic generators Q. Thus the simplest supersymmetry extension of gravity corresponds to the Poincaré supergravity theory and can be viewed as the "gauge" theory of the Poincaré superalgebra. In the next section, we will briefly introduce the Lie superalgebra concept and review the simplest supersymmetric extension of General relativity.

3.2 Lie superalgebras

In the 1960s, there were various attempts to find a symmetry group which would relate different strongly interacting particles of different spins in a relativistic quantum theory. Nevertheless, Coleman and Mandula showed in their *no-go* theorem that the only possibility to unify the Poincaré symmetry and internal symmetries is given by the Lie algebra $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{s}$, where $\mathfrak{p}$ and $\mathfrak{s}$ correspond to the Poincaré and internal symmetry algebras, respectively [24]. A way to circumvent the *no-go* theorem is through supersymmetry using not only bosonic generators B, but also fermionic generators Q. Particularly, having both commutation and anticommutation relations forming a Lie superlgebra.

The Lie superalgebra $\mathfrak{L}$ can be decomposed in subspaces as

$$\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1, \tag{3.2}$$

where $\mathfrak{L}_0$ is the subspace generated by the bosonic generators and $\mathfrak{L}_1$ corresponds to the subspace generated by the fermionic ones . Then the product $\circ$ defined by

$$\circ \quad : \mathfrak{L} \times \mathfrak{L} \rightarrow \mathfrak{L} \tag{3.3}$$

satisfies the following properties [25]:

[2]Also known as a graded Lie algebra.

- Grading: $\forall x_i \in \mathfrak{L}_i;\ i = 0, 1;\ \ x_i \circ x_j \in \mathfrak{L}_{i+j \text{ mod}(2)}$ then $\mathfrak{L}$ is a graded Lie algebra.
- Supersymmetrization: $\forall x_i \in \mathfrak{L}_i$, $\forall x_j \in \mathfrak{L}_j$; $i, j = 0, 1$; $x_i \circ x_j = -\left(-1\right)^{ij} x_j \circ x_i = \left(-1\right)^{1+ij} x_j \circ x_i$.
- Generalized Jacobi identites: $\forall x_k \in \mathfrak{L}_k$, $\forall x_m \in \mathfrak{L}_m, \forall x_l \in \mathfrak{L}_l$; $k, l, m \in \{0, 1\}$;

$$x_k \circ (x_l \circ x_m) \left(-1\right)^{km} + x_l \circ (x_m \circ x_k) \left(-1\right)^{lk} + x_m \circ (x_k \circ x_l) \left(-1\right)^{ml} = 0. \quad (3.4)$$

Thus, the generators of a Lie superalgebra are closed under the (anti)commutation relations,

$$[B, B] = B, \quad (3.5)$$
$$[B, F] = F, \quad (3.6)$$
$$\{F, F\} = B. \quad (3.7)$$

One of the simplest supersymmetry algebras corresponds to the super Poincaré. In particular, the four-dimensional Poincaré superalgebra is given by the Lorentz transformations J_{ab}, the space-time translations P_a and the 4-component Majorana spinor charge Q_α. The super Poincaré (anti)commutation relations read

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \quad (3.8)$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = 0, \quad (3.9)$$
$$[J_{ab}, Q_\alpha] = -\frac{1}{2} \left(\gamma_{ab} Q\right)_\alpha, \qquad [P_a, Q_\alpha] = 0, \quad (3.10)$$
$$\{Q_\alpha, Q_\beta\} = \left(\gamma C\right)_{\alpha\beta} P_a. \quad (3.11)$$

Interestingly, this implies that the combination of two supersymmetry transformations corresponds to a space-time translation. On the other hand, the commutativity of the fermionic generators Q with the bosonic P implies that the supermultiplets contain one-particle states with the same mass but different spins. This particularity is crucial in order to unify the interactions with matter. In fact, gravity is described by the spin-2 particle known as the graviton while the matter is made of spin-1/2 particles.

It is important to clarify that given a Lie algebra, it is not always possible to extend into a closed superalgebra. As we have said, the generators have to satisfy the generalized Jacobi identity (3.4). In some cases, the introduction of a new set of bosonic generators and $\mathcal{N}$ fermionic generators are required in order to close the superalgebra. Along the thesis we will present diverse superalgebras and their geometrical consequences in the construction of a supergravity action. However, before to approach different superalgebras, it is useful to review one of the simplest models of a supergravity theory based on the Poincaré superalgebra.

3.3 Poincaré supergravity theory

In a supersymmetric extension of gravity, the invariance of the theory is generalized to an invariance under local supersymmetry transformations. Interestingly, there are several models[3] depending on the amount of supersymmetry charges $\mathcal{N}$ and on the choice of the space-time dimension D. The larger $\mathcal{N}$ and the larger D, more constraints are presents in the theory. It is known that increase $\mathcal{N}$ beyond 8 or the dimension D beyond 11 leads to spins higher than two which makes difficult a consistent coupling to gravity.

In the simplest version, a supergravity action consists of the coupling of the spin-3/2 field to gravity. This can be done considering the Einstein-Hilbert term plus a Rarita-Schwinger term [27, 28, 29]. The Rarita-Schwinger Lagrangian is given in term of forms by

$$\mathcal{L}_{RS} = \frac{1}{l^2}\bar{\psi}e^a\gamma_a\gamma_5 D\psi, \tag{3.12}$$

where ψ is a Majorana spinor (gravitino) which satisfies $\bar{\psi} = \psi^T C$, with C the charge conjugation matrix. This implies that ψ and $\bar{\psi}$ are not independent fields.

Then, the supergravity action describing the coupling of spin-2 and spin-3/2 fields is given by

$$S = \frac{1}{l^2}\int \epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi. \tag{3.13}$$

In a very similar way to the Einstein-Hilbert theory, the complete action (3.13) is not invariant under the Poincaré superalgebra. The non invariance of the supergravity

[3] An extended study of diverse supergravity theories in a geometrical formulation can be found in ref. [26].

action (3.13) under the Poincaré superalgebra can be viewed using the Poincaré gauge supersymmetry transformations

$$\delta_{gauge}A^{A}=\nabla\lambda^{A}, \tag{3.14}$$

where λ^{A} is the gauge parameter $\lambda^{a}\equiv\left(\rho^{a},\kappa^{ab},\epsilon^{\alpha}\right)$ and ∇ is the Poincaré covariant derivative. Then, using

$$\delta\left(A^{A}T_{A}\right)=d\lambda+\left[A^{B}T_{B},\lambda^{C}T_{C}\right], \tag{3.15}$$

the Poincaré gauge supersymmetry transformations are given by

$$\delta e^{a} = D\rho^{a}+e_{c}\kappa^{ca}+\bar{\epsilon}\gamma^{a}\psi, \tag{3.16}$$

$$\delta\omega^{ab} = D\kappa^{ab}, \tag{3.17}$$

$$\delta\psi = d\epsilon+\frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon=D\epsilon, \tag{3.18}$$

where D corresponds to the Lorentz covariant exterior derivative $D=d+\omega$. It is straightforward to see that the supergravity action (3.13) is not invariant under gauge supersymmetry. In fact, if we consider the variation of the action (3.13) under gauge supersymmetry, we find

$$\delta_{susy}S=-\frac{4}{l^{2}}\int R^{a}D\bar{\psi}\gamma_{a}\gamma_{5}\epsilon, \tag{3.19}$$

where $R^{a}=De^{a}-\frac{1}{2}\bar{\psi}\gamma^{a}\psi$ is the super torsion. Then, the invariance is obtained imposing the super torsion constraint

$$R^{a}=0. \tag{3.20}$$

This leads to write the spin connection ω^{ab} in terms of the veilbein and the gravitino fields yielding to the second order formalism. This may be solved considering the following decomposition,

$$\omega^{ab}=\mathring{\omega}^{ab}+\tilde{\omega}^{ab}, \tag{3.21}$$

where $\mathring{\omega}^{ab}$ is the solution of $De^{a}=0$ and it is given by

$$\mathring{\omega}^{ab}=\left(e_{\lambda}^{c}\partial_{[\mu}e_{\nu]}^{d}\eta_{cd}+e_{\nu}^{c}\partial_{[\lambda}e_{\mu]}^{d}\eta_{cd}-e_{\mu}^{c}\partial_{[\nu}e_{\lambda]}^{d}\eta_{cd}\right)e^{\lambda|a}e^{\nu|b}.$$

Then,

$$De^{a}=de^{a}+\mathring{\omega}^{ab}e_{b}+\tilde{\omega}^{ab}e_{b}=\frac{1}{2}\bar{\psi}\gamma^{a}\psi, \tag{3.22}$$

implies

$$\tilde{\omega}^{ab}_{[\mu}e_{\nu]b} = \frac{1}{2}\bar{\psi}_\mu\gamma^a\psi_\nu, \tag{3.23}$$

which permits to solve $\tilde{\omega}^{ab}$ in terms of the two other fields,

$$\tilde{\omega}^{ab}_\mu = \frac{1}{4}e^{a|\lambda}e^{b|\nu}\left(\bar{\psi}_\mu\gamma_\lambda\psi_\nu + \bar{\psi}_\lambda\gamma_\nu\psi_\mu - \bar{\psi}_\nu\gamma_\mu\psi_\lambda - \bar{\psi}_\mu\gamma_\nu\psi_\lambda - \bar{\psi}_\nu\gamma_\lambda\psi_\mu + \bar{\psi}_\lambda\gamma_\mu\psi_\nu\right). \tag{3.24}$$

Thus, the spin connection ω^{ab} is completely determined in terms of e^a_μ and ψ^α_μ and does not carry additional physical degrees of freedom.

Alternatively, the supersymmetry invariance can be recovered in the first order formalism modifying the supersymmetry transformation for the spin connection ω^{ab}. Indeed, considering the variation of the supergravity action (3.13) under an arbitrary ω^{ab} we have

$$\delta_\omega S = \frac{2}{l^2}\int \epsilon_{abcd}R^a e^b \delta\omega^{cd}, \tag{3.25}$$

Following ref. [30], it is possible to modify $\delta\omega^{cd}$ adding an extra piece such that the variation of the action have the following form

$$\delta S = -\frac{4}{l^2}\int R^a\left(D\bar{\psi}\gamma_a\gamma_5\epsilon - \frac{1}{2}\epsilon_{abcd}e^b\delta_{extra}\omega^{cd}\right). \tag{3.26}$$

The supersymmetry invariance of the action (3.13) is ensured when $\delta_{extra}\omega^{cd}$ has the following value

$$\delta_{extra}\omega^{cd} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon\right)e^e, \tag{3.27}$$

with $\bar{\Psi} = \bar{\Psi}_{ab}e^a e^b$.

Thus the supergravity action (3.13) is invariant under the following supersymmetry transformations:

$$\delta e^a = \bar{\epsilon}\gamma^a\psi, \tag{3.28}$$

$$\delta\omega^{ab} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon\right)e^e, \tag{3.29}$$

$$\delta\psi = D\epsilon. \tag{3.30}$$

It is important to emphasize that the action supersymmetry is not a gauge supersymmetry. In particular, one can see that the action (3.13) does not correspond to a

Yang-Mills action nor a topological invariant. Besides, the supersymmetry transformations leaving the action (3.13) invariant do not close off-shell. Meanwhile, the super Poincaré gauge variations close off-shell by construction.

The situation is completely different in three dimensions where the supergravity action is truly invariant under the Poincaré superalgebra. This occurs, of course, using the Chern-Simons formalism. A supersymmetry group of particular interest is the AdS supergroup since it allows to include the cosmological constant to the supergravity action. We will see that others superalgebras can be derived from the AdS superalgebra using the semigroup expansion procedure with interesting consequences in the construction of supergravity actions.

3.4 Geometric supergravity theory à la MacDowell-Mansouri

In this section, we briefly review the geometric formulation of the four-dimensional $\mathcal{N}=1$ supergravity theory presented in ref. [17]. In this unified geometric approach, the relevant gauge fields of the theory correspond to those of the $\mathfrak{osp}\left(4|1\right)$ superalgebra. The generators of this superalgebra satisfy the following (anti)commutation relations

$$\left[\tilde{J}_{ab},\tilde{J}_{cd}\right]=\eta_{bc}\tilde{J}_{ad}-\eta_{ac}\tilde{J}_{bd}-\eta_{bd}\tilde{J}_{ac}+\eta_{ad}\tilde{J}_{bc}, \tag{3.31}$$

$$\left[\tilde{J}_{ab},\tilde{P}_{c}\right]=\eta_{bc}\tilde{P}_{a}-\eta_{ac}\tilde{P}_{b}, \tag{3.32}$$

$$\left[\tilde{P}_{a},\tilde{P}_{b}\right]=\tilde{J}_{ab}, \tag{3.33}$$

$$\left[\tilde{J}_{ab},\tilde{Q}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{ab}\tilde{Q}\right)_{\alpha},\qquad\left[\tilde{P}_{a},\tilde{Q}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{a}\tilde{Q}\right)_{\alpha}, \tag{3.34}$$

$$\left\{\tilde{Q}_{\alpha},\tilde{Q}_{\beta}\right\}=-\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{J}_{ab}-2\left(\gamma^{a}C\right)_{\alpha\beta}\tilde{P}_{a}\right], \tag{3.35}$$

where $\tilde{J}_{ab}$ correspond to the Lorentz transformations, $\tilde{P}_a$ are the AdS boost generators and $\tilde{Q}_\alpha$ corresponds to the 4-component Majorana spinor generator.

The one-form gauge connection A is given by

$$A=A^{B}T_{B}=\frac{1}{2}\omega^{ab}\tilde{J}_{ab}+\frac{1}{l}e^{a}\tilde{P}_{a}+\frac{1}{\sqrt{l}}\psi^{\alpha}\tilde{Q}_{\alpha}, \tag{3.36}$$

and the associated curvature two-form $F = dA + A \wedge A$ is

$$F = F^A T_A = \frac{1}{2}\mathcal{R}^{ab}\tilde{J}_{ab} + \frac{1}{l}R^a\tilde{P}_a + \frac{1}{\sqrt{l}}\rho^\alpha\tilde{Q}_\alpha, \tag{3.37}$$

where

$$\mathcal{R}^{ab} = d\omega^{ab} + \omega^a_{\ c}\omega^{cb} + \frac{1}{l^2}e^a e^b + \frac{1}{2l}\bar{\psi}\gamma^{ab}\psi, \tag{3.38}$$

$$R^a = de^a + \omega^a_{\ b}e^b - \frac{1}{2}\bar{\psi}\gamma^a\psi, \tag{3.39}$$

$$\rho = d\psi + \frac{1}{4}\omega_{ab}\gamma^{ab}\psi + \frac{1}{2l}e^a\gamma_a\psi = D\psi + \frac{1}{2l}e^a\gamma_a\psi. \tag{3.40}$$

Here, the one-forms ω^{ab}, e^a and ψ are respectively the spin connection, the vierbein and the gravitino field (Majorana spinor). It is important to clarify that, since we have choosen the Lie algebra generators T_A and the one-form connection A dimensionless, the "true" gauge fields must be considered as e^a/l and $\psi/\sqrt{l}$.

The supergravity action can be constructed only with the 2-form curvatures (3.37) as

$$S = 2\int \langle F \wedge F \rangle = 2\int F^A \wedge F^B \langle T_A T_B \rangle . \tag{3.41}$$

In particular, if $\langle T_A T_B \rangle$ is an invariant tensor for the $Osp\,(4|1)$ supergroup then the action (3.41) corresponds to a topological invariant and does not contribute to the equations of motion. However, with a particular choice of the components of an invariant tensor

$$\langle T_A T_B \rangle = \begin{cases} \langle J_{ab} J_{cd} \rangle = \epsilon_{abcd} \\ \langle Q_\alpha Q_\beta \rangle = 2\left(\gamma_5\right)_{\alpha\beta} \end{cases} \tag{3.42}$$

the action (3.41) takes the following form

$$S = 2\int \frac{1}{4}\mathcal{R}^{ab}\mathcal{R}^{ab}\epsilon_{abcd} + \frac{2}{l}\bar{\rho}\gamma_5\rho. \tag{3.43}$$

The action (3.43) corresponds to the MacDowell-Mansouri supergravity action [17] whose bosonic part is equivalent to the four-dimensional Born-infeld gravity action (see eq. (2.46)). Let us note that this choice of the components of the invariant tensor reproduces not only a dynamical action but also breaks the $Osp\,(4|1)$ supergroup to its Lorentz subgroup. Considering the components of the curvature 2-form F the supergravity action can be written explicitly as

$$S = \int \frac{1}{2}\epsilon_{abcd}\left(R^{ab}R^{cd} + \frac{2}{l^2}R^{ab}e^c e^d + \frac{1}{l^4}e^a e^b e^c e^d + \frac{2}{l^3}\bar{\psi}\gamma^{ab}\psi e^c e^d\right) + \frac{4}{l^2}\bar{\psi}e^a\gamma_a\gamma_5 D\psi + \frac{4}{l}d\left(\bar{\psi}\gamma_5 D\psi\right). \tag{3.44}$$

Then, modulo boundary terms, we have

$$S = \int \frac{1}{l^2}\left(\epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi\right) + \frac{1}{2}\epsilon_{abcd}\left(\frac{1}{l^4}e^a e^b e^c e^d + \frac{2}{l^3}\bar{\psi}\gamma^{ab}\psi e^c e^d\right). \tag{3.45}$$

The supergravity action (3.45) is the MacDowell-Mansouri supergravity action for the $\mathfrak{osp}\,(4|1)$ superalgebra [17]. As in the Poincaré supersymmetries, the four-dimensional $\mathcal{N}=1$ supergravity action is not invariant under supersymmetry gauge transformations for the $Osp\,(4|1)$ supergroup. Nevertheless, the supersymmetry invariance of the action (3.45) can be obtained modifying the spin connection supersymmetry transformation [30].

Along this thesis, we will present diverse supergravity actions à la MacDowell-Mansouri using different superalgebras. In particular, following our results obtained in refs. [31, 32, 33], we will present in the next sections the geometric consequences of using different superalgebras in the construction of a $\mathcal{N}=1$ supergravity action. The generalization to $\mathcal{N}$-extended supergravity theory using the MacDowell-Mansouri formalism will not be approached in this thesis.

Chapter 4

Geometric theory of Supergravity and Maxwell superalgebras

4.1 Introduction

A well-known enlargement of the Poincaré algebra is the Maxwell algebra $\mathcal{M}$ where a constant electromagnetic field background is added to the Minkowski space [7, 8]. This algebra can be obtained by adding tensorial central charges Z_{ab} to the Poincaré generators (J_{ab}, P_a) modifying the commutation relation of the translation generators P_a,

$$[P_a, P_b] = Z_{ab}. \tag{4.1}$$

As shown in refs. [34, 35], the Maxwell algebra can be derived as an expansion of the AdS Lie algebra $\mathfrak{so}(3,2)$. Particularly in ref. [35], the Maxwell algebra can be obtained using the semigroup expansion method using $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ as the relevant abelian semigroup. Subsequently, the procedure was generalized to all Maxwell type algebra[1] $\mathcal{M}_m$ which can be derived as an $S_E^{(N)}$-expansion of the AdS Lie algebra [12]. As we have seen previously, the Maxwell type algebras are useful in order to recover General Relativity from a Chern-Simons and Born-Infeld gravity theory [10, 11, 12, 13].

As shown in ref. [36], a supersymmetric extension of the four-dimensional Maxwell

[1] Also known as generalized Poincar algebras $\mathfrak{B}_m$.

algebra can be derived as an enlargement of the Poincaré superalgebra. Interestingly, the $\mathcal{N}=1$, $D=4$ Maxwell superalgebra $s\mathcal{M}$ describes the geometry of a $\mathcal{N}=1$, $D=4$ superspace in the presence of a constant abelian supersymmetric field strength background. Recently, it was pointed out in ref. [34] that the minimal Maxwell superalgebra sM can be obtained from the AdS algebra using the Maurer-Cartan expansion method.

In the next section, following our results found in ref. [31], we show that the abelian semigroup expansion procedure can be used in order to derive the Maxwell superalgebras and its generalization using bigger semigroups. The construction of a supergravity action using a geometrical formulation is also considered.

4.2 Maxwell superalgebras and abelian semigroup expansion

In this section, we shall consider the AdS superalgebra $\mathfrak{osp}\left(4|1\right)$ as a starting point and present new interesting four-dimensional superalgebras using the semigroup expansion method. Before to apply the expansion procedure to the $\mathfrak{osp}\left(4|1\right)$ superalgebra, it is necessary to study the subspace decomposition of the original lie superalgebra $\mathfrak{g}$. In particular, the $\mathfrak{osp}\left(4|1\right)$ superalgebra $\mathfrak{g}$ can be decomposed as a direct sum of subspaces V_p as

$$\begin{aligned}\mathfrak{g}=\mathfrak{osp}\left(4|1\right)&=\mathfrak{so}\left(3,1\right)\oplus\frac{\mathfrak{osp}\left(4|1\right)}{\mathfrak{sp}\left(4\right)}\oplus\frac{\mathfrak{sp}\left(4\right)}{\mathfrak{so}\left(3,1\right)}\\&=V_0\oplus V_1\oplus V_2,\end{aligned}\tag{4.2}$$

where V_0 is the Lorentz subspace generated by the Lorentz transformations $\tilde{J}_{ab}$, V_1 corresponds to the supersymmetry translation generated by a 4-component Majorana spinor charge $\tilde{Q}_\alpha$ and V_2 is generated by $\tilde{P}_a$. The $\mathfrak{osp}\left(4|1\right)$ generators satisfy the following (anti)commutation relations

$$\left[\tilde{J}_{ab},\tilde{J}_{cd}\right]=\eta_{bc}\tilde{J}_{ad}-\eta_{ac}\tilde{J}_{bd}-\eta_{bd}\tilde{J}_{ac}+\eta_{ad}\tilde{J}_{bc}, \tag{4.3}$$

$$\left[\tilde{J}_{ab},\tilde{P}_{c}\right]=\eta_{bc}\tilde{P}_{a}-\eta_{ac}\tilde{P}_{b}, \tag{4.4}$$

$$\left[\tilde{P}_{a},\tilde{P}_{b}\right]=\tilde{J}_{ab}, \tag{4.5}$$

$$\left[\tilde{J}_{ab},\tilde{Q}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{ab}\tilde{Q}\right)_{\alpha},\qquad\left[\tilde{P}_{a},\tilde{Q}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{a}\tilde{Q}\right)_{\alpha}, \tag{4.6}$$

$$\left\{\tilde{Q}_{\alpha},\tilde{Q}_{\beta}\right\}=-\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{J}_{ab}-2\left(\gamma^{a}C\right)_{\alpha\beta}\tilde{P}_{a}\right], \tag{4.7}$$

where γ_a are the Dirac matrices and C stands for the charge conjugation matrix. Then, the subspace structure may be written as

$$[V_0,V_0]\subset V_0,\qquad [V_1,V_1]\subset V_0\oplus V_2, \tag{4.8}$$

$$[V_0,V_1]\subset V_1,\qquad [V_1,V_2]\subset V_1, \tag{4.9}$$

$$[V_0,V_2]\subset V_2,\qquad [V_2,V_2]\subset V_0. \tag{4.10}$$

Now, we have to find a subset decomposition of a semigroup S "resonant" with respect to (4.8) – (4.10). As shown in ref. [31], the choice of the semigroup leads to various superalgebras with interesting properties.

4.2.1 Minimal $D=4$ Maxwell superalgebra $s\mathcal{M}$

In this section we show, following ref. [31], that the four-dimensional minimal Maxwell superalgebra can be derived from the $\mathfrak{osp}\left(4|1\right)$ superalgebra using the abelian semigroup expansion method.

Let $S_E^{(4)}=\{\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\}$ be the relevant finite abelian semigroup with the following multiplication law

$$\lambda_\alpha\lambda_\beta=\begin{cases}\lambda_{\alpha+\beta}, & \text{when } \alpha+\beta\leq 5,\\ \lambda_5, & \text{when } \alpha+\beta>5.\end{cases} \tag{4.11}$$

Here, λ_5 plays the role of the zero element of the semigroup $S_E^{(4)}$ so that for each $\lambda_\alpha\in S_E^{(4)}$, $\lambda_5\lambda_\alpha=\lambda_5=0_S$. Let us consider the subset decomposition $S_E^{(4)}=S_0\cup S_1\cup S_2$, with

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\}\,, \tag{4.12}$$
$$S_1 = \{\lambda_1, \lambda_3, \lambda_5\}\,, \tag{4.13}$$
$$S_2 = \{\lambda_2, \lambda_4, \lambda_5\}\,. \tag{4.14}$$

One sees that this decomposition is said to be resonant since it satisfies the same structure as the subspaces V_p [compare with eqs (4.8) – (4.10)]

$$S_0 \cdot S_0 \subset S_0, \qquad S_1 \cdot S_1 \subset S_0 \cap S_2, \tag{4.15}$$
$$S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_2 \subset S_1, \tag{4.16}$$
$$S_0 \cdot S_2 \subset S_2, \qquad S_2 \cdot S_2 \subset S_0. \tag{4.17}$$

Following theorem IV.2 of ref. [16], we can say that the superalgebra

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{4.18}$$

is a resonant super-subalgebra of $S_E^{(4)} \times \mathfrak{g}$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\} \times \left\{\tilde{J}_{ab}\right\} = \left\{\lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab}, \lambda_4 \tilde{J}_{ab}, \lambda_5 \tilde{J}_{ab}\right\},$$
$$W_1 = (S_1 \times V_1) = \{\lambda_1, \lambda_3, \lambda_5\} \times \left\{\tilde{Q}_\alpha\right\} = \left\{\lambda_1 \tilde{Q}_\alpha, \lambda_3 \tilde{Q}_\alpha, \lambda_5 \tilde{Q}_\alpha\right\},$$
$$W_2 = (S_2 \times V_2) = \{\lambda_2, \lambda_4, \lambda_5\} \times \left\{\tilde{P}_a\right\} = \left\{\lambda_2 \tilde{P}_a, \lambda_4 \tilde{P}_a, \lambda_5 \tilde{P}_a\right\}.$$

As was pointed out in ref. [16], a smaller superalgebra can be extracted from the resonant super-subalgebra $\mathfrak{G}_R$. To this aim we have to apply the reduction procedure. Let us consider a decomposition of the semigroup $S_p = \hat{S}_p \cup \check{S}_p$ where $\hat{S}_p \cap \check{S}_p = \varnothing$,

$$\begin{aligned} \check{S}_0 &= \{\lambda_0, \lambda_2, \lambda_4\}\,, & \hat{S}_0 &= \{\lambda_5\}\,, \\ \check{S}_1 &= \{\lambda_1, \lambda_3\}\,, & \hat{S}_1 &= \{\lambda_5\}\,, \\ \check{S}_2 &= \{\lambda_2\}\,, & \hat{S}_2 &= \{\lambda_4, \lambda_5\}\,. \end{aligned} \tag{4.19}$$

In particular, the partition of the subsets $S_p \subset S$ satisfies [compare with eqs. (4.8) – (4.10)]

$$\check{S}_0 \cdot \hat{S}_0 \subset \hat{S}_0, \qquad \check{S}_1 \cdot \hat{S}_1 \subset \hat{S}_0 \cap \hat{S}_2, \tag{4.20}$$
$$\check{S}_0 \cdot \hat{S}_1 \subset \hat{S}_1, \qquad \check{S}_1 \cdot \hat{S}_2 \subset \hat{S}_1, \tag{4.21}$$
$$\check{S}_0 \cdot \hat{S}_2 \subset \hat{S}_2, \qquad \check{S}_2 \cdot \hat{S}_2 \subset \hat{S}_0. \tag{4.22}$$

Then, we have

$$\check{\mathfrak{G}}_R = \left(\check{S}_0 \times V_0\right) \oplus \left(\check{S}_1 \times V_1\right) \oplus \left(\check{S}_2 \times V_2\right), \tag{4.23}$$

$$\hat{\mathfrak{G}}_R = \left(\hat{S}_0 \times V_0\right) \oplus \left(\hat{S}_1 \times V_1\right) \oplus \left(\hat{S}_2 \times V_2\right), \tag{4.24}$$

where

$$\left[\check{\mathfrak{G}}_R, \hat{\mathfrak{G}}_R\right] \subset \hat{\mathfrak{G}}_R, \tag{4.25}$$

and therefore $\left|\check{\mathfrak{G}}_R\right|$ corresponds to a reduced algebra of $\mathfrak{G}_R$.

The new superalgebra obtained is generated by $\left\{J_{ab}, P_a, \tilde{Z}_{ab}, Z_{ab}, Q_\alpha, \Sigma_\alpha\right\}$ whose generators are related to the $\mathfrak{osp}\left(4|1\right)$ generators as

$$\begin{aligned} J_{ab} &= \lambda_0 \tilde{J}_{ab}, \quad P_a = \lambda_2 \tilde{P}_a, \\ \tilde{Z}_{ab} &= \lambda_2 \tilde{J}_{ab}, \quad Z_{ab} = \lambda_4 \tilde{J}_{ab}, \\ Q_\alpha &= \lambda_1 \tilde{Q}_\alpha, \quad \Sigma_\alpha = \lambda_3 \tilde{Q}_\alpha. \end{aligned}$$

The (anti)commutation relations read

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{4.26}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{4.27}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{4.28}$$

$$[P_a, Q_\alpha] = -\frac{1}{2} \left(\gamma_a \Sigma\right)_\alpha, \tag{4.29}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2} \left(\gamma_{ab} Q\right)_\alpha, \tag{4.30}$$

$$[J_{ab}, \Sigma_\alpha] = -\frac{1}{2} \left(\gamma_{ab} \Sigma\right)_\alpha, \tag{4.31}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left[\left(\gamma^{ab} C\right)_{\alpha\beta} \tilde{Z}_{ab} - 2 \left(\gamma^a C\right)_{\alpha\beta} P_a\right], \tag{4.32}$$

$$\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2} \left(\gamma^{ab} C\right)_{\alpha\beta} Z_{ab}, \tag{4.33}$$

$$\left[J_{ab}, \tilde{Z}_{ab}\right] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \tag{4.34}$$

$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{4.35}$$

$$\left[\tilde{Z}_{ab}, Q_\alpha\right] = -\frac{1}{2} \left(\gamma_{ab} \Sigma\right)_\alpha, \tag{4.36}$$

$$\text{others} = 0, \tag{4.37}$$

where we have used the (anti)commutation relations of the original superalgebra $\mathfrak{osp}(4|1)$ and the multiplication law of the semigroup (4.11). The superalgebra obtained after a reduced resonant $S_E^{(4)}$-expansion of the $\mathfrak{osp}(4|1)$ superalgebra corresponds to a generalized minimal four-dimensional Maxwell superalgebra $s\mathcal{M}_4$. In particular, the minimal Maxwell superalgebra $s\mathcal{M}$ introduced in ref. [36] can be recovered imposing $\tilde{Z}_{ab} = 0$. Set $\tilde{Z}_{ab}$ equals to zero does not violate the Jacobi identities (JI) for spinors generators. Indeed, the JI are satisfied due to the gamma matrix identity $(C\gamma^a)_{(\alpha\beta}(C\gamma_a)_{\gamma\delta)} = 0$ (cyclic permutations of α, β, γ).

It is interesting to note the presence of a new Majorna spinor charge Σ. The introduction of a second abelian spinor generator is not new in the literature and has been already proposed in ref. [37] in the context of $D = 11$ supergravity theory and subsequently in ref. [38] in the superstring theory context. On the other hand, the minimal Maxwell superalgebra contains the Maxwell algebra $\mathcal{M} = \{J_{ab}, P_a, Z_{ab}\}$ and the Lorentz type $\mathcal{L}^{\mathcal{M}} = \{J_{ab}, Z_{ab}\}$ as a subalgebras.

4.2.2 Minimal $D = 4$ Maxwell type superalgebras $s\mathcal{M}_{m+2}$

The procedure presented previously can be generalized to a family of Maxwell superalgebras. In this section, following ref. [31], we show that a minimal four-dimensional Maxwell type superalgebra $s\mathcal{M}_{m+2}$ can be defined from the $\mathfrak{osp}(4|1)$ superalgebra using the abelian semigroup expansion method.

Let $S_E^{(2m)} = \{\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{2m+1}\}$ be the relevant finite abelian semigroup with the following multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq \lambda_{2m+1}, \\ \lambda_{2m+1}, & \text{when } \alpha+\beta > \lambda_{2m+1}. \end{cases} \tag{4.38}$$

Here λ_{2m+1} plays the role of the zero element of the semigroup $S_E^{(2m)}$. As in the previous section, let us consider the decomposition $S_E^{(2m)} = S_0 \cup S_1 \cup S_2$ where the subsets S_p are given by

$$S_p = \left\{ \lambda_{2n+p}, \text{ with } n = 0, \cdots, \left[\frac{2m-p}{2}\right] \right\} \cup \{\lambda_{2m+1}\}, \qquad p = 0, 1, 2. \tag{4.39}$$

In particular, we said that this decomposition is said to be resonant since it satisfies [compare with eqs. (4.8) – (4.10)]

$$S_0 \cdot S_0 \subset S_0, \qquad S_1 \cdot S_1 \subset S_0 \cap S_2, \tag{4.40}$$

$$S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_2 \subset S_1, \tag{4.41}$$

$$S_0 \cdot S_2 \subset S_2, \qquad S_2 \cdot S_2 \subset S_0. \tag{4.42}$$

Then, according to theorem IV.2 of ref. [16], we have that

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{4.43}$$

with

$$W_p = S_p \times V_p, \tag{4.44}$$

is a resonant subalgebra of $\mathfrak{G} = S_E^{(2m)} \times \mathfrak{g}$.

In order to extract a smaller superalgebra from the resonant one $\mathfrak{G}_R$ we have to apply the reduction procedure. Let us consider $S_p = \hat{S}_p \cup \check{S}_p$ a partition of the subsets $S_p \subset S$ where $\hat{S}_p \cap \check{S}_p = \varnothing$,

$$\begin{array}{ll} \check{S}_0 = \left\{\lambda_{2n}, \text{ with } n = 0, \cdots, 2\left[m/2\right]\right\}, & \hat{S}_0 = \left\{\left(\lambda_{2m}\right), \lambda_{2m+1}\right\}, \\ \check{S}_1 = \left\{\lambda_{2n+1}, \text{ with } n = 0, \cdots, m-1\right\}, & \hat{S}_1 = \left\{\lambda_{2m+1}\right\}, \\ \check{S}_2 = \left\{\lambda_{2n+2}, \text{ with } n = 0, \cdots, 2\left[\left(m-1\right)/2\right]\right\}, & \hat{S}_2 = \left\{\left(\lambda_{2m}\right), \lambda_{2m+1}\right\}, \end{array} \tag{4.45}$$

and where

$$\lambda_{2m} \in \begin{cases} \hat{S}_0 & \text{if } m \text{ is odd} \\ \hat{S}_2 & \text{if } m \text{ is even.} \end{cases}$$

Then, one can see that the partition satisfies [compare with eqs. (4.8) – (4.10)]

$$\check{S}_0 \cdot \hat{S}_0 \subset \hat{S}_0, \qquad \check{S}_1 \cdot \hat{S}_1 \subset \hat{S}_0 \cap \hat{S}_2, \tag{4.46}$$

$$\check{S}_0 \cdot \hat{S}_1 \subset \hat{S}_1, \qquad \check{S}_1 \cdot \hat{S}_2 \subset \hat{S}_1, \tag{4.47}$$

$$\check{S}_0 \cdot \hat{S}_2 \subset \hat{S}_2, \qquad \check{S}_2 \cdot \hat{S}_2 \subset \hat{S}_0. \tag{4.48}$$

Then, following the definitions of ref. [16], we have that

$$\check{\mathfrak{G}}_R = \check{W}_0 \oplus \check{W}_1 \oplus \check{W}_2, \tag{4.49}$$

corresponds to a reduced superalgebra of $\mathfrak{G}_R$, where

$$\check{W}_0 = \left(\check{S}_0 \times V_0\right) = \{\lambda_{2n}, \text{ with } n = 0, \cdots, 2\,[m/2]\} \times \left\{\tilde{J}_{ab}\right\},$$
$$\check{W}_1 = \left(\check{S}_1 \times V_1\right) = \{\lambda_{2n+1}, \text{ with } n = 0, \cdots, m-1\} \times \left\{\tilde{Q}_{\alpha}\right\},$$
$$\check{W}_2 = \left(\check{S}_2 \times V_2\right) = \{\lambda_{2n+2}, \text{ with } n = 0, \cdots, 2\,[(m-1)/2]\} \times \left\{\tilde{P}_{a}\right\}.$$

Then, the new superalgebra obtained by the S-expansion procedure is generated by

$$\left\{J_{ab,(k)}, P_{a,(l)}, Q_{\alpha,(p)}\right\}, \tag{4.50}$$

where these new generators are related to the $\mathfrak{osp}\,(4|1)$ generators as

$$J_{ab,(k)} = \lambda_{2k}\tilde{J}_{ab},$$
$$P_{a,(l)} = \lambda_{2l}\tilde{P}_{a},$$
$$Q_{\alpha,(p)} = \lambda_{2p-1}\tilde{Q}_{\alpha},$$

with $k = 0, \ldots, m-1$; $l = 1, \ldots, m$; $p = 1, \ldots, m$ when m is odd and $k = 0, \ldots, m$; $l = 1, \ldots, m-1$; $p = 1, \ldots, m$ when m is even. The new generators satisfy the (anti)commutation relations

$$\left[J_{ab,(k)}, J_{cd,(j)}\right] = \eta_{bc}J_{ad,(k+j)} - \eta_{ac}J_{bd,(k+j)} - \eta_{bd}J_{ac,(k+j)} + \eta_{ad}J_{bc,(k+j)}, \tag{4.51}$$
$$\left[J_{ab,(k)}, P_{a,(l)}\right] = \eta_{bc}P_{a,(k+l)} - \eta_{ac}P_{b,(k+l)}, \tag{4.52}$$
$$\left[P_{a,(l)}, P_{b,(n)}\right] = J_{ab,(l+n)}, \tag{4.53}$$
$$\left[J_{ab,(k)}, Q_{\alpha,(p)}\right] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_{\alpha,(k+p)}, \tag{4.54}$$
$$\left[P_{a,(l)}, Q_{\alpha,(p)}\right] = -\frac{1}{2}\left(\gamma_{a}Q\right)_{\alpha,(l+p)}, \tag{4.55}$$
$$\left\{Q_{\alpha,(p)}, Q_{\beta,(q)}\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}J_{ab,(p+q)} - 2\left(\gamma^{a}C\right)_{\alpha\beta}P_{a,(p+q)}\right]. \tag{4.56}$$

The superalgebra obtained after a reduced resonant $S_E^{(2m)}$-expansion of the $\mathfrak{osp}\,(4|1)$ superalgebra corresponds to the four-dimensional minimal Maxwell type superalgebra $s\mathcal{M}_{m+2}$. Naturally, when $k+j > m$, the generators $T_A^{(k)}$ and $T_B^{(j)}$ become abelian. It is important to clarify that the indices p and q of the spinor charges correspond to the expansion labels and they do not define an $\mathcal{N}$-extended superalgebra. In particular, the (anti)commutation relations of ref. [31] can be written explicitly if we redefine the generators as

$$
\begin{aligned}
&J_{ab} = J_{ab,0} = \lambda_0 \tilde{J}_{ab}, && P_a = P_{a,2} = \lambda_2 \tilde{P}_a,\\
&Z_{ab}^{(i)} = J_{ab,4i} = \lambda_{4i} \tilde{J}_{ab}, && Z_a^{(j)} = P_{a,4j+2} = \lambda_{4j+2} \tilde{P}_a,\\
&\tilde{Z}_{ab}^{(i)} = J_{ab,4i-2} = \lambda_{4i-2} \tilde{J}_{ab}, && \tilde{Z}_a^{(j)} = P_{a,4j} = \lambda_{4j} \tilde{P}_a,\\
&Q_\alpha = Q_{\alpha,1} = \lambda_1 \tilde{Q}_\alpha, && \Sigma_\alpha^{(i)} = Q_{\alpha,4i-1} = \lambda_{4i-1} \tilde{Q}_\alpha,\\
&\Phi_\alpha^{(j)} = Q_{\alpha,4j+1} = \lambda_{4j+1} \tilde{Q}_\alpha, &&
\end{aligned}
$$

with $i = 1, \ldots, [m/2]$, $j = 1, \ldots, \left[\frac{m-1}{2}\right]$. A bosonic subalgebra of the $s\mathcal{M}_{m+2}$ superalgebra is the Maxwell type algebra $\mathcal{M}_{m+2} = \left\{ J_{ab}, P_a, Z_{ab}^{(i)}, Z_a^{(j)} \right\}$ whose generators satisfy [10, 12],

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{4.57}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}^{(1)}, \tag{4.58}$$

$$\left[J_{ab}, Z_{cd}^{(i)}\right] = \eta_{bc} Z_{ad}^{(i)} - \eta_{ac} Z_{bd}^{(i)} - \eta_{bd} Z_{ac}^{(i)} + \eta_{ad} Z_{bc}^{(i)}, \tag{4.59}$$

$$\left[Z_{ab}^{(i)}, P_c\right] = \eta_{bc} Z_a^{(i)} - \eta_{ac} Z_b^{(i)}, \qquad \left[J_{ab}, Z_c^{(j)}\right] = \eta_{bc} Z_a^{(j)} - \eta_{ac} Z_b^{(j)}, \tag{4.60}$$

$$\left[Z_{ab}^{(i)}, Z_c^{(j)}\right] = \eta_{bc} Z_a^{(i+j)} - \eta_{ac} Z_b^{(i+j)}, \tag{4.61}$$

$$\left[Z_{ab}^{(i)}, Z_{cd}^{(k)}\right] = \eta_{bc} Z_{ad}^{(i+k)} - \eta_{ac} Z_{bd}^{(i+k)} - \eta_{bd} Z_{ac}^{(i+k)} + \eta_{ad} Z_{bc}^{(i+k)}, \tag{4.62}$$

$$\left[P_a, Z_c^{(j)}\right] = Z_{ab}^{(j+1)}, \qquad \left[Z_a^{(j)}, Z_c^{(l)}\right] = Z_{ab}^{(j+l+1)}, \tag{4.63}$$

with $i, k = 1, \ldots, [m/2]$; $j, l = 1, \ldots, \left[\frac{m-1}{2}\right]$. As was pointed out in refs. [10, 11, 12, 13], the Maxwell type algebras are useful in order to recover the Einstein equations from Chern-Simons and Born-Infeld gravity theories in a certain limit of a coupling constant.

Interestingly, when we consider the $S_E^{(4)}$ as the relevant abelian semigroup ($m = 2$) and imposing $\tilde{Z}_{ab}^{(1)} = 0$, we recover the minimal Maxwell superalgebra $s\mathcal{M}$. The case $m = 1$ is the most trivial case corresponding to the four-dimensional Poincaré superalgebra $s\mathcal{P} = \{J_{ab}, P_a, Q_\alpha\}$ whose generators satisfy the following (anti)commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{4.64}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \qquad [P_a, P_b] = 0, \tag{4.65}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \qquad [P_a, Q_\alpha] = 0, \tag{4.66}$$

$$\{Q_\alpha, Q_\beta\} = \left(\gamma C\right)_{\alpha\beta} P_a. \tag{4.67}$$

This result is not a surprise since the Inönü-Wigner contraction of the four-dimensional AdS superalgebra can be seen as a reduced resonant $S_E^{(2)}$-expansion of the $\mathfrak{osp}(4|1)$ superalgebra.

One can see that the minimal Maxwell type superalgebra $s\mathcal{M}_{m+2}$ contains additional Majorana spinors generators $Q_{\alpha,(p)}$ which transform as spinors under Lorentz transformations. In particular, all the anticommutators of fermionic generators satisfy the Jacobi identities by virtue of the gamma matrix identity $\left(C\gamma^a\right)_{(\alpha\beta}\left(C\gamma_a\right)_{\gamma\delta)} = \left(C\gamma^{ab}\right)_{(\alpha\beta}\left(C\gamma_{ab}\right)_{\gamma\delta)} = 0$ (cyclic permutations of α, β, γ). In fact, the JI are satisfied for all the generators since they correspond to an S-expansion of the original JI of the $\mathfrak{osp}(4|1)$ superalgebra.

The introduction of additional Majorana spinors charges in the minimal Maxwell type superalgebra $s\mathcal{M}_{m+2}$ can be seen as a generalization of the D'auria-Fré superalgebra and the Green algebras introduced in refs. [37, 38], respectively. Naturally, in presence of only one 4-component Majorana spinor generator ($m = 1$) the Maxwell superalgebra $s\mathcal{M}_3$ corresponds trivially to the superPoincaré one $s\mathcal{P}$.

The construction of a four-dimensional supergravity action using the minimal Maxwell type superalgebra will be considered later. In the next section, following ref. [31], we will approach the $\mathcal{N}$-extended Maxwell superalgebra using the semigroup expansion procedure.

4.2.3 $\mathcal{N}$-extended Maxwell superalgebras

In the previous section, we have shown that the S-expansion of the AdS superalgebra $\mathfrak{osp}(4|1)$ allows to derive diverse minimal Maxwell superalgebras. Then, it seems natural to consider the $\mathfrak{osp}(4|\mathcal{N})$ superalgebra as a starting point in order to obtain the four-dimensional $\mathcal{N}$-extended Maxwell superalgebra [31].

Before to apply the semigroup expansion procedure it is necessary to consider a decomposition of the original superalgebra $\mathfrak{osp}\left(4|\mathcal{N}\right)=\left\{\tilde{J}_{ab},\tilde{P}_{a},T^{ij},\tilde{Q}_{\alpha}^{i}\right\}$ as a direct sum of subspaces V_p,

$$\begin{aligned}\mathfrak{g}=\mathfrak{osp}\left(4|\mathcal{N}\right) &= \left(\mathfrak{so}\left(3,1\right)\oplus\mathfrak{so}\left(\mathcal{N}\right)\right)\oplus\frac{\mathfrak{osp}\left(4|\mathcal{N}\right)}{\mathfrak{sp}\left(4\right)\oplus\mathfrak{so}\left(\mathcal{N}\right)}\oplus\frac{\mathfrak{sp}\left(4\right)}{\mathfrak{so}\left(3,1\right)} \\ &= V_0\oplus V_1\oplus V_2, \end{aligned} \tag{4.68}$$

where V_0 is the subspace generated by Lorentz transformations $\tilde{J}_{ab}$ and by $\frac{\mathcal{N}(\mathcal{N}-1)}{2}$ internal symmetry generators T^{ij}, V_1 corresponds to the supersymmetry translation generated by $\mathcal{N}$ Majorana spinor generators $\tilde{Q}_{\alpha}^{i}$ $(i=1,\cdots,\mathcal{N}\ ;\ \alpha=1,\cdots,4)$ and V_2 is associated to the $\tilde{P}_a$ generators. The $\mathfrak{osp}\left(4|\mathcal{N}\right)$ generators satisfy the following (anti)commutation relations

$$\left[\tilde{J}_{ab},\tilde{J}_{cd}\right]=\eta_{bc}\tilde{J}_{ad}-\eta_{ac}\tilde{J}_{bd}-\eta_{bd}\tilde{J}_{ac}+\eta_{ad}\tilde{J}_{bc}, \tag{4.69}$$

$$\left[T^{ij},T^{kl}\right]=\delta^{jk}T^{il}-\delta^{ik}T^{jl}-\delta^{jl}T^{ik}+\delta^{il}T^{jk}, \tag{4.70}$$

$$\left[\tilde{J}_{ab},\tilde{P}_{c}\right]=\eta_{bc}\tilde{P}_{a}-\eta_{ac}\tilde{P}_{b}, \tag{4.71}$$

$$\left[\tilde{P}_{a},\tilde{P}_{b}\right]=\tilde{J}_{ab}, \tag{4.72}$$

$$\left[\tilde{J}_{ab},\tilde{Q}_{\alpha}^{i}\right]=-\frac{1}{2}\left(\gamma_{ab}\tilde{Q}^{i}\right)_{\alpha},\qquad \left[\tilde{P}_{a},\tilde{Q}_{\alpha}^{i}\right]=-\frac{1}{2}\left(\gamma_{a}\tilde{Q}^{i}\right)_{\alpha}, \tag{4.73}$$

$$\left[T^{ij},\tilde{Q}_{\alpha}^{k}\right]=\left(\delta^{jk}\tilde{Q}_{\alpha}^{i}-\delta^{ik}\tilde{Q}_{\alpha}^{i}\right), \tag{4.74}$$

$$\left\{\tilde{Q}_{\alpha}^{i},\tilde{Q}_{\beta}^{j}\right\}=-\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{J}_{ab}-2\left(\gamma^{a}C\right)_{\alpha\beta}\tilde{P}_{a}\right]+C_{\alpha\beta}T^{ij}, \tag{4.75}$$

where $i,j,k,l=1,\ldots,\mathcal{N}$; γ_a are the Dirac matrices and C stands for the charge conjugation matrix. Then, the subspace structure may be written as

$$\left[V_0,V_0\right]\subset V_0,\qquad \left[V_1,V_1\right]\subset V_0\oplus V_2, \tag{4.76}$$

$$\left[V_0,V_1\right]\subset V_1,\qquad \left[V_1,V_2\right]\subset V_1, \tag{4.77}$$

$$\left[V_0,V_2\right]\subset V_2,\qquad \left[V_2,V_2\right]\subset V_0. \tag{4.78}$$

Following ref. [31], let $S_E^{(4)}=\{\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\}$ be the relevant finite abelian semigroup with the following multiplication law

$$\lambda_\alpha\lambda_\beta=\begin{cases}\lambda_{\alpha+\beta}, & \text{cuando } \alpha+\beta\leq 5,\\ \lambda_5, & \text{cuando } \alpha+\beta>5.\end{cases} \tag{4.79}$$

Here λ_5 corresponds to the zero element of the semigroup $S_E^{(4)}$. Let us consider the decomposition $S_E^{(4)} = S_0 \cup S_1 \cup S_2$ where

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\}\,, \tag{4.80}$$
$$S_1 = \{\lambda_1, \lambda_3, \lambda_5\}\,, \tag{4.81}$$
$$S_2 = \{\lambda_2, \lambda_4, \lambda_5\}\,. \tag{4.82}$$

Such decomposition is said to be resonant since it satisfies [compare with eqs. (4.76) – (4.78)]

$$S_0 \cdot S_0 \subset S_0, \qquad S_1 \cdot S_1 \subset S_0 \cap S_2, \tag{4.83}$$
$$S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_2 \subset S_1, \tag{4.84}$$
$$S_0 \cdot S_2 \subset S_2, \qquad S_2 \cdot S_2 \subset S_0. \tag{4.85}$$

Then, according to the definitions of ref. [16], we have that

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{4.86}$$

is a resonant super-subalgebra of $S_E^{(4)} \times \mathfrak{g}$, where

$$\begin{aligned}
W_0 &= (S_0 \times V_0) = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\} \times \left\{\tilde{J}_{ab}, T^{ij}\right\} \\
&= \left\{\lambda_0\tilde{J}_{ab}, \lambda_2\tilde{J}_{ab}, \lambda_4\tilde{J}_{ab}, \lambda_5\tilde{J}_{ab}, \lambda_0 T^{ij}, \lambda_2 T^{ij}, \lambda_4 T^{ij}, \lambda_5 T^{ij}\right\}, \\
W_1 &= (S_1 \times V_1) = \{\lambda_1, \lambda_3, \lambda_5\} \times \left\{\tilde{Q}_\alpha\right\} = \left\{\lambda_1\tilde{Q}_\alpha, \lambda_3\tilde{Q}_\alpha, \lambda_5\tilde{Q}_\alpha\right\}, \\
W_2 &= (S_2 \times V_2) = \{\lambda_2, \lambda_4, \lambda_5\} \times \left\{\tilde{P}_a\right\} = \left\{\lambda_2\tilde{P}_a, \lambda_4\tilde{P}_a, \lambda_5\tilde{P}_a\right\}.
\end{aligned}$$

The 0_S-reduced resonant superalgebra is obtained imposing the reduction condition $\lambda_5 T_A = 0$. The resulting superalgebra is then generated by $\left\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, \tilde{Z}_a, Q_\alpha^i, \Sigma_\alpha^i, T^{ij}, Y^{ij}, \tilde{Y}^{ij}\right\}$ whose generators are related to the $\mathfrak{osp}\,(4|\mathcal{N})$ ones through

$$\begin{aligned}
J_{ab} &= \lambda_0\tilde{J}_{ab}, & Q_\alpha^i &= \lambda_1\tilde{Q}_\alpha^i, \\
P_a &= \lambda_2\tilde{P}_a, & \Sigma_\alpha^i &= \lambda_3\tilde{Q}_\alpha^i, \\
Z_{ab} &= \lambda_4\tilde{J}_{ab}, & T^{ij} &= \lambda_0 T^{ij}, \\
\tilde{Z}_{ab} &= \lambda_2\tilde{J}_{ab}, & Y^{ij} &= \lambda_4 T^{ij}, \\
\tilde{Z}_a &= \lambda_4\tilde{P}_a, & \tilde{Y}^{ij} &= \lambda_2 T^{ij}.
\end{aligned}$$

In particular, the bosonic generators $\left\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, \tilde{Z}_a, T^{ij}, Y^{ij}, \tilde{Y}^{ij}\right\}$ satisfy the following commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{4.87}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{4.88}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{4.89}$$

$$\left[J_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \tag{4.90}$$

$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{4.91}$$

$$\left[J_{ab}, \tilde{Z}_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \qquad \left[\tilde{Z}_{ab}, P_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \tag{4.92}$$

$$\left[T^{ij}, T^{kl}\right] = \delta^{jk} T^{il} - \delta^{ik} T^{jl} - \delta^{jl} T^{ik} + \delta^{il} T^{jk}, \tag{4.93}$$

$$\left[T^{ij}, Y^{kl}\right] = \delta^{jk} Y^{il} - \delta^{ik} Y^{jl} - \delta^{jl} Y^{ik} + \delta^{il} Y^{jk}, \tag{4.94}$$

$$\left[T^{ij}, \tilde{Y}^{kl}\right] = \delta^{jk} \tilde{Y}^{il} - \delta^{ik} \tilde{Y}^{jl} - \delta^{jl} \tilde{Y}^{ik} + \delta^{il} \tilde{Y}^{jk}, \tag{4.95}$$

$$\left[\tilde{Y}^{ij}, \tilde{Y}^{kl}\right] = \delta^{jk} Y^{il} - \delta^{ik} Y^{jl} - \delta^{jl} Y^{ik} + \delta^{il} Y^{jk}, \tag{4.96}$$

$$\text{others } = 0. \tag{4.97}$$

Meanwhile the fermionic generators $\{Q^i_\alpha, \Sigma^i_\alpha\}$ satisfy the following (anti)commutation relations

$$\left[J_{ab}, Q^i_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab} Q^i\right)_\alpha, \qquad \left[\tilde{Z}_{ab}, Q^i_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab} \Sigma^i\right)_\alpha, \tag{4.98}$$

$$\left[J_{ab}, \Sigma^i_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab} \Sigma^i\right)_\alpha, \qquad \left[T^{ij}, Q^i_\alpha\right] = \left(\delta^{jk} Q^i_\alpha - \delta^{ik} Q^i_\alpha\right), \tag{4.99}$$

$$\left[T^{ij}, \Sigma^k_\alpha\right] = \left(\delta^{jk} \Sigma^i_\alpha - \delta^{ik} \Sigma^i_\alpha\right), \qquad \left[\tilde{Y}^{ij}, Q^k_\alpha\right] = \left(\delta^{jk} \Sigma^i_\alpha - \delta^{ik} \Sigma^i_\alpha\right), \tag{4.100}$$

$$\left[P_a, Q^i_\alpha\right] = -\frac{1}{2}\left(\gamma_a \Sigma^i\right)_\alpha, \tag{4.101}$$

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab} C\right)_{\alpha\beta} \tilde{Z}_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta} P_a\right] + C_{\alpha\beta} \tilde{Y}^{ij}, \tag{4.102}$$

$$\left\{Q^i_\alpha, \Sigma^j_\beta\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab} C\right)_{\alpha\beta} Z_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta} \tilde{Z}_a\right] + C_{\alpha\beta} Y^{ij}, \tag{4.103}$$

$$\text{others } = 0. \tag{4.104}$$

These (anti)commutation relations can be obtained using the commutation relations of the $\mathfrak{osp}(4|\mathcal{N})$ superalgebra and the multiplication law of the semigroup $S^{(4)}_E$. In particular, the 0_S-reduced resonant $S^{(4)}_E$-expansion of the $\mathfrak{osp}(4|\mathcal{N})$ superalgebra leads to the four-dimensional $\mathcal{N}$-extended Maxwell superalgebra $s\mathcal{M}_4^{(\mathcal{N})}$ [31]. This superalgebra contains the generalized Maxwell algebra $g\mathcal{M} = \left\{ J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, \tilde{Z}_a \right\}$ as a bosonic subalgebra (see eqs. (4.87) – (4.92)). One can see that the presence of additional bosonic generators modifies the anticommutator of the minimal Maxwell superalgebra. Interestingly, it is possible to recover the simplest four-dimensional $\mathcal{N}$-extended Maxwell superalgebra $s\mathcal{M}^{(\mathcal{N})} = \{J_{ab}, P_a, Z_{ab}, Q_\alpha^i, \Sigma_\alpha^i, T^{ij}\}$ imposing $\tilde{Z}_a = \tilde{Z}_{ab} = \tilde{Y}^{ij} = Y^{ij} = 0$. Naturally, the minimal Maxwell superlgebra $s\mathcal{M}$ is recovered when $T^{ij} = 0$. It is important to clarify that, due to properties of the gamma matrices in four dimensions, impose some generators equals to zero does not break the Jacobi identity.

As in the minimal case, this procedure can be generalized in order to derive the $\mathcal{N}$-extended Maxwell type superalgebra $s\mathcal{M}_{m+2}^{(\mathcal{N})}$ from the $\mathfrak{osp}\left(4|\mathcal{N}\right)$ superalgebra. Following ref. [31], let us consider the $S_E^{(2m)} = \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{2m+1}\}$ as the relevant abelian semigroup. Let $S_E^{(2m)} = S_0 \cup S_1 \cup S_2$ be a resonant subset decomposition where

$$S_p = \left\{ \lambda_{2n+p}, \text{ with } n = 0, \cdots, \left[\frac{2m-p}{2} \right] \right\} \cup \{\lambda_{2m+1}\}, \qquad p = 0, 1, 2, \tag{4.105}$$

and let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ with

$$\check{S}_0 = \{\lambda_{2n}, \text{ with } n = 0, \cdots, 2\left[m/2\right]\}, \quad \hat{S}_0 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \tag{4.106}$$

$$\check{S}_1 = \{\lambda_{2n+1}, \text{ with } n = 0, \cdots, m-1\}, \quad \hat{S}_1 = \{\lambda_{2m+1}\}, \tag{4.107}$$

$$\check{S}_2 = \{\lambda_{2n+2}, \text{ with } n = 0, \cdots, 2\left[\left(m-1\right)/2\right]\}, \quad \hat{S}_2 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \tag{4.108}$$

where

$$\lambda_{2m} \in \begin{cases} \hat{S}_0 & \text{if } m \text{ is odd} \\ \hat{S}_2 & \text{if } m \text{ is even.} \end{cases}$$

This partition satisfies the resonant conditions for any value of m and $\hat{S}_p \cap \check{S}_p = \varnothing$. Then, according to the definitions of ref. [16],

$$\check{\mathfrak{G}}_R = \left(\check{S}_0 \times V_0\right) \oplus \left(\check{S}_1 \times V_1\right) \oplus \left(\check{S}_2 \times V_2\right), \tag{4.109}$$

corresponds to a reduced resonant superalgebra. The new superalgebra obtained is generated by

$$\left\{ J_{ab,(k)}, P_{a,(l)}, Q_{\alpha,(p)}^i, Y_{(k)}^{ij} \right\}, \tag{4.110}$$

whose generators are related to the $\mathfrak{osp}\left(4|\mathcal{N}\right)$ generators as

$$\begin{aligned}
J_{ab,(k)} &= \lambda_{2k}\tilde{J}_{ab}, \\
P_{a,(l)} &= \lambda_{2l}\tilde{P}_{a}, \\
Q_{\alpha,(p)} &= \lambda_{2p-1}\tilde{Q}_{\alpha}, \\
Y_{(k)}^{ij} &= \lambda_{2k}T^{ij},
\end{aligned}$$

with $k=0,\ldots,m-1$; $l=1,\ldots,m$; $p=1,\ldots,m$ when m is odd and $k=0,\ldots,m$; $l=1,\ldots,m-1$; $p=1,\ldots,m$ when m is even. The new generators satisfy the (anti)commutation relations

$$\left[J_{ab,(k)},J_{cd,(j)}\right] = \eta_{bc}J_{ad,(k+j)}-\eta_{ac}J_{bd,(k+j)}-\eta_{bd}J_{ac,(k+j)}+\eta_{ad}J_{bc,(k+j)}, \tag{4.111}$$

$$\left[Y_{(k)}^{ij},Y_{(j)}^{gh}\right] = \delta^{jg}Y_{(k+j)}^{ih}-\delta^{ig}Y_{(k+j)}^{jh}-\delta^{jh}Y_{(k+j)}^{ig}+\delta^{ih}Y_{(k+j)}^{jg}, \tag{4.112}$$

$$\left[J_{ab,(k)},P_{c,(l)}\right] = \eta_{bc}P_{a,(k+l)}-\eta_{ac}P_{b,(k+l)}, \tag{4.113}$$

$$\left[P_{a,(l)},P_{b,(n)}\right] = J_{ab,(l+n)}, \tag{4.114}$$

$$\left[J_{ab,(k)},Q_{\alpha,(p)}\right] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_{\alpha,(k+p)}, \tag{4.115}$$

$$\left[P_{a,(l)},Q_{\alpha,(p)}\right] = -\frac{1}{2}\left(\gamma_{a}Q\right)_{\alpha,(l+p)}, \tag{4.116}$$

$$\left[T_{(k)}^{ij},Q_{\alpha,(p)}^{g}\right] = \left(\delta^{jg}Q_{\alpha,(k+p)}^{i}-\delta^{ig}Q_{\alpha,(k+p)}^{i}\right), \tag{4.117}$$

$$\left\{Q_{\alpha,(p)},Q_{\beta,(q)}\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}J_{ab,(p+q)}-2\left(\gamma^{a}C\right)_{\alpha\beta}P_{a,(p+q)}\right]+C_{\alpha\beta}Y_{(p+q)}^{ij}. \tag{4.118}$$

The superalgebra obtained after a reduced resonant $S_E^{(2m)}$-expansion of the $\mathfrak{osp}\left(4|\mathcal{N}\right)$ superalgebra corresponds to the four-dimensional $\mathcal{N}$-extended Maxwell type superalgebra $s\mathcal{M}_{m+2}^{(\mathcal{N})}$ [31]. Naturally, when $k+j>m$, the generators $T_A^{(k)}$ and $T_B^{(j)}$ become abelian. As in the minimal case, this $\mathcal{N}$-extended superalgebra contains additional Majorana spinors generators $Q_{\alpha,(p)}^{i}$ which transform as spinors under Lorentz transformations. Interestingly, the $s\mathcal{M}_3^{(\mathcal{N})}$ superalgebra obtained after a reduced resonant $S_E^{(2)}$-expansion of the $\mathfrak{osp}\left(4|\mathcal{N}\right)$ superalgebra corresponds to the four-dimensional N-

extended Poincaré superalgebra $s\mathcal{P}^{(\mathcal{N})} = \{J_{ab}, P_a, Q^i_\alpha, T^{ij}\}$ whose generators satisfy

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{4.119}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \qquad [P_a, P_b] = 0, \tag{4.120}$$

$$\left[J_{ab}, Q^i_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}Q^i\right)_\alpha, \qquad \left[P_a, Q^i_\alpha\right] = 0, \tag{4.121}$$

$$\left[T^{ij}, T^{kl}\right] = \delta^{jk}T^{il} - \delta^{ik}T^{jl} - \delta^{jl}T^{ik} + \delta^{il}T^{jk}, \tag{4.122}$$

$$\left[T^{ij}, Q^i_\alpha\right] = \left(\delta^{jk}Q^i_\alpha - \delta^{ik}Q^i_\alpha\right), \tag{4.123}$$

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = \delta^{ij}\left(\gamma^a C\right)_{\alpha\beta} P_a. \tag{4.124}$$

This result is not a surprise since the Inönü-Wigner contraction of the four-dimensional $\mathcal{N}$-extended AdS superalgebra can be seen as a reduced resonant $S^{(2)}_E$-expansion of the $\mathfrak{osp}\left(4|\mathcal{N}\right)$ superalgebra.

The construction of an four-dimensional $\mathcal{N}$-extended supergravity action based on the $\mathcal{N}$-extended Maxwell type superalgebras remains an open problem and will not be considered in the present thesis.

4.3 $D = 4$ supergravity from minimal Maxwell superalgebra $s\mathcal{M}_4$

In this section, following ref. [32], we present a geometric construction of a supergravity action using the minimal Maxwell superalgebra $s\mathcal{M}_4$.

In the previous section, we have shown that after extracting a reduced resonant $S^{(4)}_E$-expansion of the $\mathfrak{osp}\left(4|1\right)$ superalgebra we find the minimal Maxwell superalgebra $s\mathcal{M}_4 = \left\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, Q_\alpha, \Sigma_\alpha\right\}$ whose generators satisfy the (anti)commutation relations (4.26) − (4.37).

The one-form gauge connection for the $s\mathcal{M}_4$ superalgebra is given by

$$A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{2}\tilde{k}^{ab}\tilde{Z}_{ab} + \frac{1}{2}k^{ab}Z_{ab} + \frac{1}{l}e^a P_a + \frac{1}{\sqrt{l}}\psi^\alpha Q_\alpha + \frac{1}{\sqrt{l}}\xi^\alpha\Sigma_\alpha, \tag{4.125}$$

where the one-form gauge fields can be written in terms of the components of the

$\mathfrak{osp}\,(4|1)$ connection as

$$\begin{aligned} \omega^{ab} &= \lambda_0 \tilde{\omega}^{ab}, & e^a &= \lambda_2 \tilde{e}^a, \\ \tilde{k}^{ab} &= \lambda_2 \tilde{\omega}^{ab}, & \psi^\alpha &= \lambda_1 \tilde{\psi}^\alpha, \\ k^{ab} &= \lambda_4 \tilde{\omega}^{ab}, & \xi^\alpha &= \lambda_3 \tilde{\psi}^\alpha. \end{aligned}$$

The associated curvature two-form $F = dA + A \wedge A$ is given by

$$F = F^A T_A = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} R^a P_a + \frac{1}{2} \tilde{F}^{ab} \tilde{Z}_{ab} + \frac{1}{2} F^{ab} Z_{ab} + \frac{1}{\sqrt{l}} \Psi^\alpha Q_\alpha + \frac{1}{\sqrt{l}} \Xi^\alpha \Sigma_\alpha, \quad (4.126)$$

where

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \omega^a_{\ c} \omega^{cb}, \\ R^a &= de^a + \omega^a_{\ b} e^b - \frac{1}{2} \bar{\psi} \gamma^a \psi, \\ \tilde{F}^{ab} &= d\tilde{k}^{ab} + \omega^a_{\ c} \tilde{k}^{cb} - \omega^b_{\ c} \tilde{k}^{ca} + \frac{1}{2l} \bar{\psi} \gamma^{ab} \psi, \\ F^{ab} &= dk^{ab} + \omega^a_{\ c} k^{cb} - \omega^b_{\ c} k^{ca} + \tilde{k}^a_{\ c} \tilde{k}^{cb} + \frac{1}{l^2} e^a e^b + \frac{1}{l} \bar{\xi} \gamma^{ab} \psi, \\ \Psi &= d\psi + \frac{1}{4} \omega_{ab} \gamma^{ab} \psi = D\psi, \\ \Xi &= d\xi + \frac{1}{4} \omega_{ab} \gamma^{ab} \xi + \frac{1}{4} \tilde{k}_{ab} \gamma^{ab} \psi + \frac{1}{2l} e^a \gamma_a \psi \\ &= D\xi + \frac{1}{4} \tilde{k}_{ab} \gamma^{ab} \psi + \frac{1}{2l} e^a \gamma_a \psi. \end{aligned}$$

The one-forms ω^{ab}, e^a, ψ and ξ are the spin connection, the vielbein, the gravitino field and an additional Majorana fermionic field[2], respectively. While the k^{ab} and $\tilde{k}^{ab}$ fields describe bosonic "matter" fields.

On the other hand, the Lorentz covariant exterior derivatives $D = d + \omega$ of the curvatures can be derived from the Bianchi identity $\nabla F = 0$ (where ∇ is the gauge

[2] A Majorana spinor ψ satisfies the Majorana condition $\bar{\psi} = \psi C$, where C is the charge conjugation matrix.

covariant derivative given by $\nabla = d + [A, \cdot]$) leading to

$$DR^{ab} = 0, \tag{4.127}$$

$$DR^{a} = R^{a}_{\ b}e^{b} + \bar{\psi}\gamma^{a}\Psi, \tag{4.128}$$

$$D\tilde{F}^{ab} = R^{a}_{\ c}\tilde{k}^{cb} - R^{b}_{\ c}\tilde{k}^{ca} - \frac{1}{l}\bar{\psi}\gamma^{ab}\Psi, \tag{4.129}$$

$$DF^{ab} = R^{a}_{\ c}k^{cb} - R^{b}_{\ c}k^{ca} + \tilde{F}^{a}_{\ c}\tilde{k}^{cb} - \tilde{F}^{b}_{\ c}\tilde{k}^{ca} + \frac{1}{l^2}R^{a}e^{b} - \frac{1}{l^2}e^{a}R^{b} \tag{4.130}$$

$$+ \frac{1}{l}\bar{\Xi}\gamma^{ab}\psi - \frac{1}{l}\bar{\xi}\gamma^{ab}\Psi, \tag{4.131}$$

$$D\Psi = \frac{1}{4}R_{ab}\gamma^{ab}\psi, \tag{4.132}$$

$$D\Xi = \frac{1}{4}R_{ab}\gamma^{ab}\xi - \frac{1}{4}\tilde{k}_{ab}\gamma^{ab}\Psi + \frac{1}{4}\tilde{F}_{ab}\gamma^{ab}\psi + \frac{1}{2l}R^{a}\gamma_{a}\psi - \frac{1}{2l}e^{a}\gamma_{a}\Psi. \tag{4.133}$$

Then, using the MacDowell-Mansouri geometrical formalism [17] and following ref. [32], a supergravity action can be constructed out of the 2-form curvatures of the minimal Maxwell superalgebra $s\mathcal{M}_4$ as

$$S = 2\int \langle F \wedge F \rangle = 2\int F^{A} \wedge F^{B} \langle T_A T_B \rangle_{s\mathcal{M}_4}. \tag{4.134}$$

Here, $\langle T_A T_B \rangle_{s\mathcal{M}_4}$ can be obtained using the useful properties of the semigroup expansion procedure. Indeed, using theorem VII.1 of ref. [16], one can see that the components of an invariant tensor for the $s\mathcal{M}_4$ superalgebra can be written in terms of a particular choice of the original invariant tensor,

$$\langle J_{ab}J_{cd} \rangle_{s\mathcal{M}_4} = \alpha_0 \left\langle \tilde{J}_{ab}\tilde{J}_{cd} \right\rangle, \tag{4.135}$$

$$\left\langle J_{ab}\tilde{Z}_{cd} \right\rangle_{s\mathcal{M}_4} = \alpha_2 \left\langle \tilde{J}_{ab}\tilde{J}_{cd} \right\rangle, \tag{4.136}$$

$$\left\langle \tilde{Z}_{ab}\tilde{Z}_{cd} \right\rangle_{s\mathcal{M}_4} = \alpha_4 \left\langle \tilde{J}_{ab}\tilde{J}_{cd} \right\rangle, \tag{4.137}$$

$$\langle J_{ab}Z_{cd} \rangle_{s\mathcal{M}_4} = \alpha_4 \left\langle \tilde{J}_{ab}\tilde{J}_{cd} \right\rangle, \tag{4.138}$$

$$\langle Q_{\alpha}Q_{\beta} \rangle_{s\mathcal{M}_4} = \alpha_2 \left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta} \right\rangle, \tag{4.139}$$

$$\langle Q_{\alpha}\Sigma_{\beta} \rangle_{s\mathcal{M}_4} = \alpha_4 \left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta} \right\rangle, \tag{4.140}$$

where

$$\left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle = \epsilon_{abcd},$$
$$\left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta}\right\rangle = 2\left(\gamma_5\right)_{\alpha\beta},$$

and the α's are dimensionless arbitrary independent constants. It is important to clarify that this choice of the invariant tensor breaks the Maxwell supergroup to its Lorentz like subgroup. This is not a surprise since we have considered the S-expansion of a particular choice of an invariant tensor which breaks the $Osp\,(4|1)$ supergroups to its Lorentz subgroup. This construction of a supergravity action can be seen as a supersymmetric generalization of the four-dimensional Born-Infeld gravity action in which the action is constructed from the AdS two-form curvatures using $\langle T_A T_B\rangle$ as an invariant tensor for the Lorentz group.

Then, considering the two-form curvature of the minimal Maxwell superalgebra $s\mathcal{M}_4$ (4.126) and the non-vanishing components of the invariant tensor (4.135)−(4.140), the supergravity action (4.134) becomes

$$\begin{aligned} S = 2\int \Bigg(& \frac{1}{4}\alpha_0\epsilon_{abcd}R^{ab}R^{cd} + \frac{1}{2}\alpha_2\epsilon_{abcd}R^{ab}\tilde{F}^{cd} + \frac{1}{2}\alpha_4\epsilon_{abcd}R^{ab}F^{cd} \\ & + \frac{1}{4}\alpha_4\epsilon_{abcd}\tilde{F}^{ab}\tilde{F}^{cd} + \frac{2}{l}\alpha_2\bar{\Psi}\gamma_5\Psi + \frac{4}{l}\alpha_4\bar{\Psi}\gamma_5\Xi\Bigg). \end{aligned} \tag{4.141}$$

The action (4.141) can be written explicitly in terms of the different components of the curvature two-form as

$$\begin{aligned} S = \int & \frac{\alpha_0}{2}\epsilon_{abcd}R^{ab}R^{cd} + \alpha_2\epsilon_{abcd}\left(R^{ab}D\tilde{k}^{cd} + \frac{1}{2l}R^{ab}\bar{\psi}\gamma^{cd}\psi\right) \\ & + \frac{4}{l}\alpha_2 D\bar{\psi}\gamma_5 D\psi + \alpha_4\epsilon_{abcd}\Bigg(R^{ab}Dk^{cd} + \frac{1}{2}D\tilde{k}^{ab}D\tilde{k}^{cd} + \frac{1}{l^2}R^{ab}e^c e^d \\ & + \frac{1}{2l}D\tilde{k}^{ab}\bar{\psi}\gamma^{cd}\psi + R^{ab}\tilde{k}^{c}{}_{f}\tilde{k}^{fd} + \frac{1}{l}R^{ab}\bar{\xi}\gamma^{cd}\psi\Bigg) \\ & + \frac{8}{l}\alpha_4 D\bar{\psi}\gamma_5 D\xi + \frac{2}{l}\alpha_4 D\bar{\psi}\gamma_5\tilde{k}_{ab}\gamma^{ab}\psi + \frac{4}{l^2}\alpha_4\bar{\psi}e^a\gamma_a\gamma_5 D\psi. \end{aligned} \tag{4.142}$$

Interestingly, using the gravitino Bianchi identity and the gamma matrix identity

$$2\gamma_{ab}\gamma_5 = -\epsilon_{abcd}\gamma^{cd}, \tag{4.143}$$

it is possible to combine some expressions as boundary terms. In fact, following ref. [32], we have

$$\frac{1}{2}\epsilon_{abcd}R^{ab}\bar{\psi}\gamma^{ab}\psi + 4D\bar{\psi}\gamma_5 D\psi = d\left(4D\bar{\psi}\gamma_5\psi\right),$$
$$\epsilon_{abcd}R^{ab}\bar{\xi}\gamma^{cd}\psi + 8D\bar{\xi}\gamma_5 D\psi = d\left(8D\bar{\xi}\gamma_5\psi\right),$$
$$\frac{1}{2}\epsilon_{abcd}D\tilde{k}^{ab}\bar{\psi}\gamma^{cd}\psi + 2\bar{\psi}\tilde{k}^{ab}\gamma_{ab}\gamma_5 D\psi = d\left(\bar{\psi}\tilde{k}^{ab}\gamma_{ab}\gamma_5\psi\right).$$

Thus the MacDowell-Mansouri geometrical formulation of a supergravity action for the $s\mathcal{M}_4$ superalgebra is given by

$$\begin{aligned} S = \int \frac{\alpha_0}{2}\epsilon_{abcd}R^{ab}R^{cd} + \alpha_2 d\left(\epsilon_{abcd}R^{ab}\tilde{k}^{cd} + \frac{4}{l}D\bar{\psi}\gamma_5\psi\right) \\ + \alpha_4\left[\frac{1}{l^2}\epsilon_{abcd}R^{ab}e^c e^d + \frac{4}{l^2}\bar{\psi}e^a\gamma_a\gamma_5 D\psi \right. \\ \left. + d\left(\epsilon_{abcd}\left(R^{ab}k^{cd} + \frac{1}{2}D\tilde{k}^{ab}\tilde{k}^{cd}\right) + \frac{8}{l}\bar{\xi}\gamma_5 D\psi + \frac{1}{l}\bar{\psi}\tilde{k}^{ab}\gamma_{ab}\gamma_5\psi\right)\right]. \end{aligned} \tag{4.144}$$

The supergravity action is split into three independent terms proportional to α_0, α_2 and α_4, respectively. The first term corresponds to the topological Euler Lagrangian and does not contribute to the dynamics. The piece proportional to α_2 is also a boundary term and contains explicitly the coupling between the new bosonic gauge fields $\tilde{k}^{ab}$ and the Lorentz curvatures R^{ab}. The last term is proportional to α_4 and contains the Einstein-Hilbert Lagrangian $\epsilon_{abcd}R^{ab}e^c e^d$, the Rarita-Schwinger Lagrangian $4\bar{\psi}e^a\gamma_a\gamma_5 D\psi$ and boundary terms.

Interestingly, the supergravity action obtained using the MacDowell-Mansouri geometrical approach and the minimal Maxwell superalgebra $s\mathcal{M}_4$ describes pure supergravity in four dimensions. Indeed, the new Maxwell gauge fields k^{ab} and $\tilde{k}^{ab}$ appear only in the boundary terms and do not contribute to the dynamics. Moreover, as a consequence of the semigroup expansion method, the cosmological constant term disappears completely from the supergravity action similarly to the bosonic case using the Maxwell algebra[3]. Then, this result can be seen as the supersymmetric extension of the results found in refs. [11, 12] where General Relativity is recovered from Maxwell algebra as Born-Infeld gravity action.

[3]Also known as $\mathfrak{B}_4$ algebra.

A particular case can be derived when we consider $\tilde{k}^{ab}=0$. In fact, the action found in ref. [39] corresponds to the term proportional to α_4, namely

$$S|_{\tilde{k}^{ab}=0}=\alpha_4\int\frac{1}{l^2}\left(\epsilon_{abcd}R^{ab}e^ce^d+4\bar{\psi}e^a\gamma_a\gamma_5D_\omega\psi\right)+d\left(\epsilon_{abcd}R^{ab}k^{cd}+\frac{8}{l}\bar{\xi}\gamma_5D_\omega\psi\right). \tag{4.145}$$

This results is not a surprise since we have previously seen that setting $\tilde{Z}_{ab}=0$ in $s\mathcal{M}_4$ leads to the simplest minimal Maxwell algebra [31], which allows to construct the action (4.145) as shown in ref. [39].

It is tempting to argue that the presence of the new bosonic gauge fields k^{ab} and $\tilde{k}^{ab}$ in the boundary would allow to recover the supersymmetry invariance in the rheonomic approach. It seems that the supergravity action obtained here could be obtained using the geometric approach considered in ref. [40] where $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravities are constructed on a manifold with boundary.

4.3.1 $s\mathcal{M}_4$ gauge transformations and supersymmetry

In this section, following ref. [32], we analyze the supersymmetry invariance of the action (4.144). Although the supergravity action à la MacDowell-Mansouri (4.144) is constructed out of the 2-form curvatures of the minimal Maxwell superalgebra $s\mathcal{M}_4$, it is not invariant under the gauge transformations. Indeed, the supergravity action does not correspond to a topologial invariant, nor a Yang-Mills action.

The $s\mathcal{M}_4$ gauge transformation of the one-form gauge connection A is given by

$$\delta_\rho A=D\rho=d\rho+[A,\rho]$$

where ρ is the $s\mathcal{M}_4$ gauge parameter,

$$\rho=\frac{1}{2}\rho^{ab}J_{ab}+\frac{1}{2}\tilde{\kappa}^{ab}\tilde{Z}_{ab}+\frac{1}{2}\kappa^{ab}Z_{ab}+\frac{1}{l}\rho^aP_a+\frac{1}{\sqrt{l}}\epsilon^\alpha Q_\alpha+\frac{1}{\sqrt{l}}\varrho^\alpha\Sigma_\alpha. \tag{4.146}$$

Then, we have explicitly for each component the following gauge transformations,

$$\delta\omega^{ab} = D\rho^{ab}, \tag{4.147}$$

$$\delta\tilde{k}^{ab} = D\tilde{\kappa}^{ab} - \left(\tilde{k}^{a}_{\ c}\,\rho^{b}_{\ c} - \tilde{k}^{bc}\rho^{a}_{\ c}\right) - \frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi, \tag{4.148}$$

$$\begin{aligned} \delta k^{ab} = {} & D\kappa^{ab} - \left(k^{ac}\rho^{b}_{\ c} - k^{bc}\rho^{a}_{\ c}\right) - \left(\tilde{k}^{ac}\tilde{\kappa}^{b}_{\ c} - \tilde{k}^{bc}\tilde{\kappa}^{a}_{\ c}\right) \\ & + \frac{2}{l^2}e^{a}\rho^{b} - \frac{1}{l}\bar{\varrho}\gamma^{ab}\psi - \frac{1}{l}\bar{\epsilon}\gamma^{ab}\xi, \end{aligned} \tag{4.149}$$

$$\delta e^{a} = D\rho^{a} + e^{b}\rho_{b}^{\ a} + \bar{\epsilon}\gamma^{a}\psi, \tag{4.150}$$

$$\delta\psi = d\epsilon + \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon - \frac{1}{4}\rho^{ab}\gamma_{ab}\psi, \tag{4.151}$$

$$\begin{aligned} \delta\xi = {} & d\varrho + \frac{1}{4}\omega^{ab}\gamma_{ab}\varrho + \frac{1}{2l}e^{a}\gamma_{a}\epsilon - \frac{1}{2l}\rho^{a}\gamma_{a}\psi - \frac{1}{4}\rho^{ab}\gamma_{ab}\xi \\ & + \frac{1}{4}\tilde{k}^{ab}\gamma_{ab}\epsilon - \frac{1}{4}\tilde{\kappa}^{ab}\gamma_{ab}\psi. \end{aligned} \tag{4.152}$$

Similarly, the gauge transformations of the curvature F can be obtained from $\delta_{\rho}F = [F,\rho]$ leading to

$$\delta R^{ab} = R^{ac}\rho_{c}^{\ b} - R^{cb}\rho^{a}_{\ c}, \tag{4.153}$$

$$\delta\tilde{F}^{ab} = \left(R^{ac}\tilde{\kappa}_{c}^{\ b} - R^{bc}\tilde{\kappa}^{a}_{\ c}\right) - \left(\tilde{F}^{ac}\rho^{b}_{\ c} - \tilde{F}^{bc}\rho^{a}_{\ c}\right) - \frac{1}{l}\bar{\epsilon}\gamma^{ab}\Psi, \tag{4.154}$$

$$\begin{aligned} \delta F^{ab} = {} & \left(R^{ac}\kappa_{c}^{\ b} - R^{bc}\kappa^{a}_{\ c}\right) - \left(F^{ac}\rho^{b}_{\ c} - F^{bc}\rho^{a}_{\ c}\right) - \left(\tilde{F}^{ac}\tilde{\kappa}^{b}_{\ c} - \tilde{F}^{ac}\tilde{\kappa}^{a}_{\ c}\right) \\ & + \frac{2}{l^2}R^{a}\rho^{b} - \frac{1}{l}\bar{\varrho}\gamma^{ab}\Psi - \frac{1}{l}\bar{\epsilon}\gamma^{ab}\Xi, \end{aligned} \tag{4.155}$$

$$\delta R^{a} = R^{a}_{\ b}\rho^{b} + R^{b}\rho_{b}^{\ a} + \bar{\epsilon}\gamma^{a}\Psi, \tag{4.156}$$

$$\delta\Psi = \frac{1}{4}R^{ab}\gamma_{ab}\epsilon - \frac{1}{4}\rho^{ab}\gamma_{ab}\Psi, \tag{4.157}$$

$$\delta\Xi = \frac{1}{4}R^{ab}\gamma_{ab}\varrho + \frac{1}{2l}R^{a}\gamma_{a}\epsilon - \frac{1}{2l}\rho^{a}\gamma_{a}\Psi - \frac{1}{4}\rho^{ab}\gamma_{ab}\Xi + \frac{1}{4}\tilde{F}^{ab}\gamma_{ab}\epsilon - \frac{1}{4}\tilde{\kappa}^{ab}\gamma_{ab}\Psi. \tag{4.158}$$

Let us note that the variation of the action (4.144) under gauge supersymmetry is

$$\delta_{susy}S = -\frac{4}{l^2}\alpha_4\int R^{a}\bar{\Psi}\gamma_{a}\gamma_{5}\epsilon. \tag{4.159}$$

As in the Poincaré and $\mathfrak{osp}\,(4|1)$ superalgebra, the gauge supersymmetry invariance of the action is obtained imposing the supertorsion constraint

$$R^{a} = 0.$$

This leads us to express the spin connection ω^{ab} in terms of the other fields (second order formalism). Nevertheless, the supersymmetry invariance of the action in the first formalism can be recovered adding an extra piece to the gauge transformation of the spin connection $\delta\omega^{ab}$. Then, the variation of the action can be written as

$$\delta S = -\frac{4}{l^2}\alpha_4 \int R^a \left(\bar{\Psi}\gamma_a\gamma_5\epsilon - \frac{1}{2}\epsilon_{abcd}e^b\delta_{extra}\omega^{cd} \right). \tag{4.160}$$

The supersymmetry invariance of the supergravity action is obtained imposing

$$\delta_{extra}\omega^{ab} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon \right) e^e, \tag{4.161}$$

with $\bar{\Psi} = \bar{\Psi}_{ab}e^a e^b$.

Thus, the supergravity action (4.144) is invariant under the following supersymmetry transformations

$$\delta\omega^{ab} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon \right) e^e, \tag{4.162}$$

$$\delta\tilde{k}^{ab} = -\frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi, \tag{4.163}$$

$$\delta k^{ab} = -\frac{1}{l}\bar{\epsilon}\gamma^{ab}\xi, \tag{4.164}$$

$$\delta e^a = \bar{\epsilon}\gamma^a\psi, \tag{4.165}$$

$$\delta\psi = d\epsilon + \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon = D\epsilon, \tag{4.166}$$

$$\delta\xi = \frac{1}{2l}e^a\gamma_a\epsilon + \frac{1}{4}\tilde{k}^{ab}\gamma_{ab}\epsilon. \tag{4.167}$$

It is important to clarify that the susy transformations are not gauge symmetries of the action. Additionally, the supersymmetry transformations leaving the action (4.144) invariant do not close off-shell, meanwhile, the $s\mathcal{M}_4$ gauge variation close off-shell by construction.

The situation is quite different when we consider the gauge supersymmetry transformations related to the spinor generator Σ_α. From (4.147) – (4.152), we have that the new supersymmetry transformations are given by

$$\delta\omega^{ab} = 0, \tag{4.168}$$

$$\delta\tilde{k}^{ab} = 0, \tag{4.169}$$

$$\delta k^{ab} = -\frac{1}{l}\bar{\varrho}\gamma^{ab}\psi, \tag{4.170}$$

$$\delta e^{a} = 0, \tag{4.171}$$

$$\delta\psi = 0, \tag{4.172}$$

$$\delta\xi = d\varrho + \frac{1}{4}\omega^{ab}\gamma_{ab}\varrho = D\varrho. \tag{4.173}$$

Interestingly, the action (4.144) is invariant under these transformations,

$$\delta S = 0. \tag{4.174}$$

In particular, the supergravity action à la MacDowell-Mansouri (4.144) is off-shell invariant under a particular subalgebra of $s\mathcal{M}_4$ which are generated by $\left\{ J_{ab}, \tilde{Z}_{ab}, Z_{ab}, \Sigma_{\alpha} \right\}$ and corresponds to a Lorentz type superalgebra.

Our results show that the Poincaré supersymmetries are not the only supersymmetries of the pure supergravity action. The invariance of the pure supergravity action under additional supersymmetry transformations could not be guessed trivially. The procedure used here could be useful in order to derive new supersymmetry structures related to standard supergravity. It seems that it should be possible to recover higher-dimensional standard supergravity from the Maxwell superalgebras.

4.4 $D = 4$ supergravity from minimal Maxwell type superalgebra $s\mathcal{M}_{m+2}$

In this section, following ref. [32], we present a geometric construction of a supergravity action using the minimal Maxwell type superalgebra $s\mathcal{M}_{m+2}$.

In the previous section, we have shown that after extracting a reduced resonant $S_E^{(2m)}$-expansion of the $\mathfrak{osp}\,(4|1)$ superalgebra we find the minimal Maxwell type superalgebra $s\mathcal{M}_{m+2} = \left\{ J_{ab,(k)}, P_{a,(l)}, Q_{\alpha,(p)} \right\}$, whose generators satisfy the (anti)commutation relations (4.51) – (4.56).

The one-form gauge connection for the $s\mathcal{M}_{m+2}$ superalgebra is given by

$$A=\frac{1}{2}\sum_{k}\omega^{ab,(k)}J_{ab,(k)}+\frac{1}{l}\sum_{l}e^{a,(l)}P_{a,(l)}+\frac{1}{\sqrt{l}}\sum_{p}\psi^{\alpha,(p)}Q_{\alpha,(p)}, \tag{4.175}$$

where the one-form gauge fields can be written in terms of the components of the $\mathfrak{osp}\,(4|1)$ connection as

$$\omega^{ab,(k)}=\lambda_{2k}\tilde{\omega}^{ab}, \tag{4.176}$$

$$e^{a,(l)}=\lambda_{2l}\tilde{e}^{a}, \tag{4.177}$$

$$\psi^{\alpha,(p)}=\lambda_{2p-1}\tilde{\psi}^{\alpha}. \tag{4.178}$$

The associated curvature two-form $F=dA+A\wedge A$ is given by

$$F=F^{A}T_{A}=\frac{1}{2}\sum_{k}\mathcal{R}^{ab,(k)}J_{ab,(k)}+\frac{1}{l}\sum_{l}R^{a,(l)}P_{a,(l)}+\frac{1}{\sqrt{l}}\sum_{p}\Psi^{\alpha,(p)}Q_{\alpha,(p)}, \tag{4.179}$$

where

$$\begin{aligned}
\mathcal{R}^{ab,(k)}&=d\omega^{ab,(k)}+\omega_{\ c}^{a\ (i)}\wedge\omega^{cb,(j)}\delta_{i+j}^{k}+\frac{1}{l^{2}}e^{a,(l)}e^{b,(n)}\delta_{l+n}^{k}\\
&\quad+\frac{1}{2l}\bar{\psi}^{(p)}\gamma^{ab}\wedge\psi^{(q)}\delta_{p+q}^{2k},\\
R^{a,(l)}&=de^{a,(l)}+\omega_{\ b}^{a\ (k)}\wedge e^{b,(n)}\delta_{k+n}^{l}-\frac{1}{2}\bar{\psi}^{(p)}\gamma^{a}\wedge\psi^{(q)}\delta_{p+q}^{2l},\\
\Psi^{(p)}&=d\psi^{(p)}+\frac{1}{4}\omega_{ab}^{\ \ (k)}\gamma^{ab}\wedge\psi^{(q)}\delta_{k+q}^{p}+\frac{1}{2l}e^{a,(l)}\gamma_{a}\wedge\psi^{(q)}\delta_{l+q}^{p},
\end{aligned}$$

with $k=0,\dots,m;\,l,p=1,\dots,m$. The one-forms $\omega^{ab}=\omega^{ab,(0)}$, $e^{a}=e^{a,(2)}$ and $\psi=\psi^{(1)}$ are the spin connection, the vielbein and the gravitino field, respectively.

On the other hand, the Lorentz covariant exterior derivatives $D=d+\omega$ of the curvatures can be derived from the Bianchi identity $\nabla F=0$ leading to

$$\begin{aligned}
D\mathcal{R}^{ab,(k)}&=\left(\mathcal{R}^{ac,(i)}\omega_{c}^{\ b,(j+1)}-\mathcal{R}^{bc,(i)}\omega_{c}^{\ a,(j+1)}\right)\delta_{i+j+1}^{k}\\
&\quad+\frac{1}{l}\left(R^{a,(l)}e^{b,(n)}-e^{a,(n)}R^{b,(l)}\right)\delta_{l+n}^{k}-\frac{1}{l}\bar{\psi}^{(p)}\gamma^{ab}\Psi^{(q)}\delta_{p+q}^{2k},
\end{aligned} \tag{4.180}$$

$$DR^{a,(l)}=\mathcal{R}^{ab,(i)}e_{b}^{\ ,(j)}\delta_{i+j}^{l}+R^{c,(n)}\omega_{c}^{\ a,(j+1)}\delta_{n+j+1}^{l}+\bar{\psi}^{(p)}\gamma^{a}\Psi^{(q)}\delta_{p+q}^{2l}, \tag{4.181}$$

$$\begin{aligned}
D\Psi^{(p)}&=\frac{1}{4}\left(\mathcal{R}^{ab,(i)}\gamma_{ab}\psi^{(q)}\right)\delta_{i+q}^{p}-\frac{1}{4}\left(\omega^{ab,(i+1)}\gamma_{ab}\Psi^{(q)}\right)\delta_{i+1+q}^{p}\\
&\quad+\frac{1}{2l}\left(T^{a,(l)}\gamma_{a}\psi^{(q)}\right)\delta_{l+q}^{p}-\frac{1}{2l}\left(e^{a,(l)}\gamma_{a}\Psi^{(q)}\right)\delta_{l+q}^{p},
\end{aligned} \tag{4.182}$$

Then, using the MacDowell-Mansouri geometrical formalism [17] and following ref. [32], a supergravity action can be constructed out of the curvature 2-forms of the minimal Maxwell superalgebra $s\mathcal{M}_{m+2}$ as

$$S = 2\int \langle F\wedge F\rangle = 2\int F^{A}\wedge F^{B}\left\langle T_{A}T_{B}\right\rangle_{s\mathcal{M}_{m+2}}. \tag{4.183}$$

Here, $\left\langle T_{A}T_{B}\right\rangle_{s\mathcal{M}_{m+2}}$ can be obtained using theorem VII.1 of ref. [16]. Indeed, it is possible to show that the components of an invariant tensor for the $s\mathcal{M}_{m+2}$ superalgebra can be written in terms of a particular choice of the original invariant tensor,

$$\left\langle J_{ab,(k)}J_{cd,(j)}\right\rangle_{s\mathcal{M}_{m+2}} = \alpha_{2(k+j)}\left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{4.184}$$

$$\left\langle Q_{\alpha,(p)}Q_{\beta,(q)}\right\rangle_{s\mathcal{M}_{m+2}} = \alpha_{2(p+q-1)}\left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta}\right\rangle, \tag{4.185}$$

which can be written as

$$\left\langle J_{ab,(k)}J_{cd,(j)}\right\rangle_{s\mathcal{M}_{m+2}} = \alpha_{2(k+j)}\epsilon_{abcd}, \tag{4.186}$$

$$\left\langle Q_{\alpha,(p)}Q_{\beta,(q)}\right\rangle_{s\mathcal{M}_{m+2}} = 2\alpha_{2(p+q-1)}\left(\gamma_{5}\right)_{\alpha\beta}. \tag{4.187}$$

Here the α's are dimensionless arbitrary independent constants. Similarly to the previous case, this choice of the invariant tensor breaks the Maxwell type supergroup to its Lorentz like subgroup. This is not a surprise since we have considered the S-expansion of a particular choice of an invariant tensor which breaks the $Osp\left(4|1\right)$ supergroups to its Lorentz subgroup.

Then, considering the two-form curvature of the minimal Maxwell type superalgebra $s\mathcal{M}_{m+2}$ (4.179) and the non-vanishing components of the invariant tensor (4.186) – (4.187), the supergravity action (4.183) becomes

$$S = 2\int \sum_{k,j}\frac{\alpha_{2(k+j)}}{2}\epsilon_{abcd}\mathcal{R}^{ab,(k)}\mathcal{R}^{cd,(j)} + \sum_{p,q}\alpha_{2(p+q-1)}\frac{4}{l}\bar{\Psi}^{(p)}\wedge\gamma_{5}\Psi^{(q)}, \tag{4.188}$$

with $k,j = 0,\ldots,m$; $p,q = 1,\ldots,m$.

Interestingly, the term proportional to α_4 describes pure supergravity,

$$S = 2\alpha_{4}\int\left(\frac{1}{2}\epsilon_{abcd}\mathcal{R}^{ab,(0)}\mathcal{R}^{cd,(2)} + \frac{1}{4}\epsilon_{abcd}\mathcal{R}^{ab,(1)}\mathcal{R}^{cd,(1)} + \frac{4}{l}\bar{\Psi}^{(2)}\wedge\gamma_{5}\Psi^{(1)}\right), \tag{4.189}$$

which can be written explicitly as

$$
\begin{aligned}
S = \alpha_4 \int \epsilon_{abcd} \frac{1}{l^2} \left(\mathcal{R}^{ab,(0)} e^{c,(1)} e^{d,(1)} + 4 \bar{\psi}^{(1)} e^{a,(1)} \gamma_a \gamma_5 D_\omega \psi^{(1)} \right) \\
+ d \left(\epsilon_{abcd} \left(\mathcal{R}^{ab,(0)} \omega^{ab,(2)} + \frac{1}{2} D_\omega \omega^{ab,(1)} \omega^{cd,(1)} \right) \right. \\
\left. + \frac{8}{l} D_\omega \bar{\psi}^{(1)} \gamma_5 \psi^{(2)} + \frac{1}{l} \bar{\psi}^{(1)} \omega^{ab,(1)} \gamma_{ab} \gamma_5 \psi^{(1)} \right)
\end{aligned}
\tag{4.190}
$$

Then using the gravitino Bianchi identity $D\Psi^{(1)} = \frac{1}{4} R^{ab} \gamma_{ab} \Psi^{(1)}$, the gamma matrix identity (4.143), and using the following identification,

$$
\begin{aligned}
\omega^{ab,(0)} &= \omega^{ab}, & \omega^{ab,(1)} &= \tilde{k}^{ab}, \\
\omega^{ab,(2)} &= k^{ab}, & e^{a,(1)} &= e^{a}, \\
\mathcal{R}^{ab,(0)} &= R^{ab}, & \psi^{(1)} &= \psi, \\
\psi^{(2)} &= \xi,
\end{aligned}
$$

it is possible to write the pure supergravity action plus boundary terms,

$$
\begin{aligned}
S = \alpha_4 \int \epsilon_{abcd} \frac{1}{l^2} \left(R^{ab} e^c e^d + 4 \bar{\psi} e^a \gamma_a \gamma_5 D_\omega \psi \right) \\
+ d \left(\epsilon_{abcd} \left(R^{ab} k^{cd} + \frac{1}{2} D_\omega \tilde{k}^{ab} \tilde{k}^{cd} \right) + \frac{8}{l} \bar{\xi} \gamma_5 D_\omega \psi + \frac{1}{l} \bar{\psi} \tilde{k}^{ab} \gamma_{ab} \gamma_5 \psi \right).
\end{aligned}
\tag{4.191}
$$

Let us note that, as in the previous result, the cosmological constant does not appear explicitly in the α_4 term. In particular, the presence of the cosmological term requires the components $\left\langle J_{ab,(2)} J_{cd,(2)} \right\rangle$ which is proportional to α_8.

On the other hand, the case $m = 1$ reproduces the four-dimensional Poincaré supergravity à la MacDowell-Mansouri. Nevertheless, it is not possible to recover the pure supergravity action from the MacDowell-Mansouri formalism using the $s\mathcal{P}$ superalgebra since the Einstein-Hilbert term cannot be construct from the component $\langle J_{ab} J_{cd} \rangle_{s\mathcal{P}}$.

4.4.1 $s\mathcal{M}_{m+2}$ gauge transformations and supersymmetry

In this section, following ref. [32], we analyze the supersymmetry invariance of the action (4.188). Although the supergravity action à la MacDowell-Mansouri (4.188) is constructed out of the 2-form curvatures of the minimal Maxwell type superalgebra $s\mathcal{M}_{m+2}$, it is not invariant under the gauge transformations. Indeed, the supergravity action does not correspond to a topological invariant, nor a Yang-Mills action.

The $s\mathcal{M}_{m+2}$ gauge transformation of the one-form gauge connection A is given by

$$\delta_{\rho}A = D\rho = d\rho + [A, \rho]$$

where ρ is the $s\mathcal{M}_{m+2}$ gauge parameter,

$$\rho = \frac{1}{2}\sum_{k}\rho^{ab,(k)}J_{ab,(k)} + \frac{1}{l}\sum_{l}\rho^{a,(l)}P_{a,(l)} + \frac{1}{\sqrt{l}}\sum_{p}\epsilon^{\alpha,(p)}Q_{\alpha,(p)}. \tag{4.192}$$

Here, the components of the gauge parameter are related to the components of the $\mathfrak{osp}\,(4|1)$ gauge parameter as

$$\begin{aligned} \rho^{ab,(k)} &= \lambda_{2k}\tilde{\rho}^{ab}, \\ \rho^{a,(l)} &= \lambda_{2l}\tilde{\rho}^{a}, \\ \epsilon^{\alpha,(p)} &= \lambda_{2p-1}\tilde{\epsilon}^{\alpha}, \end{aligned}$$

with $k = 0, \ldots, m$; $l, p = 1, \ldots, m$. Then, we have the following gauge transformations

$$\begin{aligned} \delta\omega^{ab,(k)} &= D\rho^{ab,(k)} - \left(\omega^{ac,(i+1)}\rho_{\ c}^{b\ ,(j)} - \omega^{bc,(i+1)}\rho_{\ c}^{a\ ,(j)}\right)\delta_{i+j+1}^{k} \\ &\quad + \frac{2}{l^{2}}e^{a,(l)}\rho^{b,(n)}\delta_{l+n}^{k} - \frac{1}{l}\bar{\epsilon}^{(p)}\gamma^{ab}\psi^{(q)}\delta_{p+q}^{2k}, \end{aligned} \tag{4.193}$$

$$\delta e^{a,(l)} = D\rho^{a,(l)} + \omega_{\ b}^{a\ ,(k+1)}\rho^{b,(n)}\delta_{k+n+1}^{l} + e^{b,(n)}\rho_{b}^{\ a,(k)}\delta_{n+k}^{l} + \bar{\epsilon}^{(p)}\gamma^{a}\psi^{(q)}\delta_{p+q}^{2l}, \tag{4.194}$$

$$\begin{aligned} \delta\psi^{(p)} &= d\epsilon^{(p)} + \frac{1}{4}\omega^{ab,(k)}\gamma_{ab}\epsilon^{(q)}\delta_{k+q}^{p} + \frac{1}{2l}e^{a,(l)}\gamma_{a}\epsilon^{(q)}\delta_{l+q}^{p} \\ &\quad - \frac{1}{4}\rho^{ab,(k)}\gamma_{ab}\psi^{(q)}\delta_{k+q}^{p} - \frac{1}{2l}\rho^{a,(l)}\gamma_{a}\psi^{(q)}\delta_{l+q}^{p}. \end{aligned} \tag{4.195}$$

Similarly, the gauge transformations of the curvature F can be obtained from $\delta_{\rho}F =$

$[F,\rho]$ leading to

$$\delta\mathcal{R}^{ab,(k)} = \left(\mathcal{R}^{ac,(i)}\rho_c^{\ b,(j)} - \mathcal{R}^{cb,(i)}\rho^a_{\ c}{}^{,(j)}\right)\delta^k_{i+j} + \frac{2}{l^2}R^{a,(l)}\rho^{b,(n)}\delta^k_{l+n}$$
$$- \frac{1}{l}\bar{\epsilon}^{(p)}\gamma^{ab}\Psi^{(q)}\delta^{2k}_{p+q}, \tag{4.196}$$
$$\delta R^{a,(l)} = \mathcal{R}^{a\ ,(k)}_{\ b}\rho^{b,(n)}\delta^l_{k+n} + R^{b,(n)}\rho_b^{\ a,(k)}\delta^l_{k+n} + \bar{\epsilon}^{(p)}\gamma^a\Psi^{(q)}\delta^{2l}_{p+q}, \tag{4.197}$$
$$\delta\Psi^{(p)} = \frac{1}{4}\mathcal{R}^{ab,(k)}\gamma_{ab}\epsilon^{(q)}\delta^p_{k+q} + \frac{1}{2l}R^{a,(l)}\gamma_a\epsilon^{(q)}\delta^p_{l+q} - \frac{1}{4}\rho^{ab,(k)}\gamma_{ab}\Psi^{(q)}\delta^p_{k+q}$$
$$- \frac{1}{2l}\rho^{a,(l)}\gamma_a\Psi^{(q)}\delta^p_{l+q}. \tag{4.198}$$

Let us note that the variation of the action (4.188) under gauge supersymmetry is

$$\delta_{susy}S = -\frac{4}{l^2}\int\sum_k \alpha_{2k}R^{a,(l)}\bar{\Psi}^{(p)}\gamma_a\gamma_5\epsilon\delta^k_{l+p}, \tag{4.199}$$

As in the previous case, the gauge supersymmetry invariance of the action is obtained imposing the expanded supertorsion constraint

$$R^{a,(l)} = 0.$$

This leads us to express the bosonic fields $\omega^{ab,(k)}$ in terms of the other fields (second order formalism).

Interestingly, since the α constants are independent and arbitrary, the study of the supersymmetry invariance can be approached in each term separately. Let us consider the variation of the term proportional to α_{2k} under gauge supersymmetry transformations related to the $Q_{(k-1)}$ generator,

$$\delta_{susy}S = -\frac{4}{l^2}\alpha_{2k}\int R^a\bar{\Psi}\gamma_a\gamma_5\epsilon^{(k-1)}, \tag{4.200}$$

with $k = 0,\ldots,m$. Here $\epsilon^{(k-1)}$ corresponds to the gauge parameter associated to the spinor generator $Q_{(k-1)}$ and R^a and Ψ correspond to $R^{a,(1)}$ and $\Psi^{(1)}$, respectively. The supersymmetry invariance of the α_{2k} term in the first formalism can be recovered adding an extra piece to the gauge transformation of $\omega^{ab,(k-2)}$. Then, the variation of the action proportional to α_{2k} can be written as

$$\delta S = -\frac{4}{l^2}\alpha_{2k}\int R^a\left(\bar{\Psi}\gamma_a\gamma_5\epsilon^{(k-1)} - \frac{1}{2}\epsilon_{abcd}e^b\delta_{extra}\omega^{cd,(k-2)}\right). \tag{4.201}$$

The supersymmetry invariance of the supergravity action proportional to α_{2k} is obtained imposing

$$\delta_{extra}\omega^{ab,(k-2)} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon^{(k-1)} + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon^{(k-1)} - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon^{(k-1)}\right)e^e, \qquad (4.202)$$

with $\bar{\Psi} = \bar{\Psi}_{ab}e^a e^b$.

It is important to clarify that the supersymmetry transformation leaving the action proportional to α_{2k} invariant is not a gauge symmetry. Additionally, these supersymmetry transformations do not close off-shell, meanwhile, the $s\mathcal{M}_{m+2}$ gauge variation close off-shell by construction.

However, the term proportional to α_{2k} is truly invariant under gauge supersymmetry transformations related to the $Q_{(q)}$ generators if $q \geq k$. Naturally, when $m = 2$, we recover the previous results.

Chapter 5

Generalized supersymmetric cosmological term in $\mathcal{N}=1$ supergravity

5.1 Introduction

In the literature, it was pointed out that a good candidate to describe the dark energy is the cosmological constant [41, 42]. In the geometric approach, the cosmological term can be introduced in a four-dimensional gravity theory using the AdS algebra. The introduction of a cosmological term in the supersymmetric extension of gravity can be performed in the MacDowell-Mansouri geometric formalism. In this framework, as we have seen previously, the construction of the supergravity action is based only on the $\mathfrak{osp}\left(4|1\right)$ curvatures [17].

An alternative method to introduce a generalized cosmological constant term using the Maxwell algebra has been presented in ref. [9]. Nevertheless, as we have shown in the previous section, the geometric construction of a supergravity action using the Maxwell superalgebras does not reproduce the generalized cosmological term. An alternative superalgebra have to be considered in order to introduce a generalized supersymmetric cosmological constant to a supergravity action.

An interesting deformations of the Maxwell algebras consist in the $\mathfrak{so}\left(D-1,2\right)\oplus\mathfrak{so}\left(D-1,1\right)$ or $\mathfrak{so}\left(D,1\right)\oplus\mathfrak{so}\left(D-1,1\right)$ algebra introduced in refs. [43, 44]. This algebra, also known as AdS-Lorentz ($AdS - \mathcal{L}_4$) algebra, has been used in order to reproduce the generalized cosmological constant term from a Born-Infeld gravity action [35]. In particular, as shown in refs. [45, 17], the AdS-Lorentz algebra can be derived applying the semigroup expansion procedure to the AdS algebra.

Then, it seems that the supersymmetric extension of the AdS-Lorentz algebra is the appropriate superalgebra in order to reproduce the generalized supersymmetric cosmological term in a supergravity theory. In this chapter, we present different AdS-Lorentz superalgebras using the abelian semigroup expansion procedure. The construction of supergravity actions à la MacDowell-Mansouri is also proposed.

5.2 AdS-Lorentz superalgebras and abelian semigroup expansion

5.2.1 The AdS-Lorentz superalgebra

In the present section, following the method used in ref. [46], we present the construction of the four-dimensional AdS-Lorent superalgebra as an S-expansion of the $\mathfrak{osp}(4|1)$ superalgebra.

As we have said previously, the original superalgebra has to be decomposed in subspaces before to apply the semigroup expansion procedure. Let us consider a decomposition of the $\mathfrak{osp}(4|1)$ superalgebra as

$$\begin{aligned}\mathfrak{g} = \mathfrak{osp}(4|1) &= \mathfrak{so}(3,1) \oplus \frac{\mathfrak{osp}(4|1)}{\mathfrak{sp}(4)} \oplus \frac{\mathfrak{sp}(4)}{\mathfrak{so}(3,1)} \\ &= V_0 \oplus V_1 \oplus V_2, \qquad (5.1)\end{aligned}$$

where V_0, V_1 and V_2 satisfy $(4.8) - (4.10)$ and correspond to the Lorentz, subspace, the fermionic subspace and the AdS-boost, respectively.

Following the properties and definitions of ref. [16], let us consider $S_M^{(2)} = \{\lambda_0, \lambda_1, \lambda_2\}$ as the relevant finite abelian semigroup which satisfy the following multiplication law,

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{if } \alpha + \beta \leq 2 \\ \lambda_{\alpha+\beta-2}, & \text{if } \alpha + \beta > 2 \end{cases} \qquad (5.2)$$

Let us consider the subset decomposition $S_M^{(2)} = S_0 \cup S_1 \cup S_2$ where

$$S_0 = \{\lambda_0, \lambda_2\}\,, \tag{5.3}$$

$$S_1 = \{\lambda_1\}\,, \tag{5.4}$$

$$S_2 = \{\lambda_2\}\,. \tag{5.5}$$

In particular, this subset decomposition is "resonant" since it satisfies the same structure as the subspaces V_p of the $\mathfrak{osp}\,(4|1)$ superalgebra [compare with eqs. $(4.8) - (4.10)$]

$$S_0 \cdot S_0 \subset S_0, \qquad S_1 \cdot S_1 \subset S_0 \cap S_2, \tag{5.6}$$

$$S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_2 \subset S_1, \tag{5.7}$$

$$S_0 \cdot S_2 \subset S_2, \qquad S_2 \cdot S_2 \subset S_0. \tag{5.8}$$

Then, according to theorem IV.2 of ref. [16], we can say that the superalgebra

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{5.9}$$

is a resonant subalgebra of $S_{\mathcal{M}}^{(2)} \times \mathfrak{g}$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2\} \times \left\{\tilde{J}_{ab}\right\} = \left\{\lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab}\right\}, \tag{5.10}$$

$$W_1 = (S_1 \times V_1) = \{\lambda_1\} \times \left\{\tilde{Q}_\alpha\right\} = \left\{\lambda_1 \tilde{Q}_\alpha\right\}, \tag{5.11}$$

$$W_2 = (S_2 \times V_2) = \{\lambda_2\} \times \left\{\tilde{P}_a\right\} = \left\{\lambda_2 \tilde{P}_a\right\}. \tag{5.12}$$

Then, the new superalgebra obtained by the S-expansion procedure is generated by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha\}$ whose generators are related to the $\mathfrak{osp}\,(4|1)$ generators as

$$J_{ab} = \lambda_0 \tilde{J}_{ab},$$

$$Z_{ab} = \lambda_2 \tilde{J}_{ab},$$

$$P_a = \lambda_2 \tilde{P}_a,$$

$$Q_\alpha = \lambda_1 \tilde{Q}_\alpha.$$

The (anti)commutation relations read

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{5.13}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{5.14}$$

$$[Z_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{5.15}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{5.16}$$

$$[Z_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{5.17}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2} \left(\gamma_{ab} Q\right)_\alpha, \qquad [P_a, Q_\alpha] = -\frac{1}{2} \left(\gamma_a Q\right)_\alpha, \tag{5.18}$$

$$[Z_{ab}, Q_\alpha] = -\frac{1}{2} \left(\gamma_{ab} Q\right)_\alpha, \tag{5.19}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left[\left(\gamma^{ab} C\right)_{\alpha\beta} Z_{ab} - 2 \left(\gamma^a C\right)_{\alpha\beta} P_a \right], \tag{5.20}$$

where we have used the (anti)commutation relations of the $\mathfrak{osp}(4|1)$ superalgebra and the multiplication law of the semigroup (5.2). The superalgebra obtained after a resonant $S_M^{(2)}$-expansion of the $\mathfrak{osp}(4|1)$ superalgebra corresponds to the four-dimensional AdS-Lorentz superalgebra $sAdS - \mathcal{L}_4$. This superalgebra has the usual AdS-Lorentz algebra[1] $AdS - L_4 = \{J_{ab}, P_a, Z_{ab}\}$ as a bosonic subalgebra which allows to introduce a generalized cosmological term to a Born-Infeld gravity action. Unlike the Maxwell symmetries, the Z_{ab} generators are not abelian and behave as Lorentz generators.

It is interesting to note that the Inönü-Wigner (IW) contraction [48, 49, 50] of the $sAdS - \mathcal{L}_4$ superalgebra leads us to the Maxwell superalgebra. Indeed, considering the rescaling

$$Z_{ab} \to \mu^2 Z_{ab}, \quad P_a \to \mu P_a \;\text{ and }\; Q_\alpha \to \mu Q_\alpha \tag{5.21}$$

[1]Also known as Poincaré semi-simple extended algebra.

the Maxwell superalgebra is recovered in the limit $\mu \to \infty$ [47].

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{5.22}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{5.23}$$

$$[Z_{ab}, Z_{cd}] = 0, \qquad [Z_{ab}, P_c] = 0, \tag{5.24}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \tag{5.25}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \qquad [P_a, Q_\alpha] = 0, \tag{5.26}$$

$$[Z_{ab}, Q_\alpha] = 0, \qquad [P_a, P_b] = Z_{ab}, \tag{5.27}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(\gamma^{ab}C\right)_{\alpha\beta} Z_{ab}, \tag{5.28}$$

It is important to clarify that this Maxwell superalgebra is quite different from the minimal Maxwell supperalgebra $s\mathcal{M}$. In particular, it does not have additional Majorana spinor generators and cannot be obtained directly as an S-expansion of the $\mathfrak{osp}(4|1)$ superalgebra.

5.2.2 The generalized minimal AdS-Lorentz superalgebra

In this section, following ref. [33], we present the construction of a four-dimensional generalized minimal AdS-Lorentz superalgebra using the abelian semigroup expansion method.

Following the definitions of ref. [16], let us consider $S_M^{(4)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ as the relevant finite abelian semigroup with the following multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{if } \alpha+\beta \leq 4 \\ \lambda_{\alpha+\beta-4}, & \text{if } \alpha+\beta > 4 \end{cases} \tag{5.29}$$

Let $S = S_0 \cup S_1 \cup S_2$ be the subset decomposition where

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4\}, \tag{5.30}$$

$$S_1 = \{\lambda_1, \lambda_3\}, \tag{5.31}$$

$$S_2 = \{\lambda_2, \lambda_4\}. \tag{5.32}$$

One can see that this subset decomposition is said to be "resonant" since it satisfies the same structure as the original subspaces V_p [compare with eqs. (4.8) – (4.10)]

$$S_0 \cdot S_0 \subset S_0, \qquad S_1 \cdot S_1 \subset S_0 \cap S_2, \tag{5.33}$$

$$S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_2 \subset S_1, \tag{5.34}$$

$$S_0 \cdot S_2 \subset S_2, \qquad S_2 \cdot S_2 \subset S_0. \tag{5.35}$$

Then, according to theorem IV.2 of ref. [16], we can say that the superalgebra

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{5.36}$$

is a resonant subalgebra of $S_{\mathcal{M}}^{(4)} \times \mathfrak{osp}\left(4|1\right)$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2, \lambda_4\} \times \left\{\tilde{J}_{ab}\right\} = \left\{\lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab}, \lambda_4 \tilde{J}_{ab}\right\}, \tag{5.37}$$

$$W_1 = (S_1 \times V_1) = \{\lambda_1, \lambda_3\} \times \left\{\tilde{Q}_\alpha\right\} = \left\{\lambda_1 \tilde{Q}_\alpha, \lambda_3 \tilde{Q}_\alpha\right\}, \tag{5.38}$$

$$W_2 = (S_2 \times V_2) = \{\lambda_2, \lambda_4\} \times \left\{\tilde{P}_a\right\} = \left\{\lambda_2 \tilde{P}_a, \lambda_4 \tilde{P}_a\right\}. \tag{5.39}$$

The resulting superalgebra is then generated by $\left\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, \tilde{Z}_a, Q_\alpha, \Sigma_\alpha\right\}$ whose generators are related to the $\mathfrak{osp}\left(4|1\right)$ ones through

$$\begin{aligned}
J_{ab} &= \lambda_0 \tilde{J}_{ab}, \quad & P_a &= \lambda_2 \tilde{P}_a, \\
\tilde{Z}_{ab} &= \lambda_2 \tilde{J}_{ab}, \quad & \tilde{Z}_a &= \lambda_4 \tilde{P}_a, \\
Z_{ab} &= \lambda_4 \tilde{J}_{ab}, \quad & Q_\alpha &= \lambda_1 \tilde{Q}_\alpha, \\
\Sigma_\alpha &= \lambda_3 \tilde{Q}_\alpha.
\end{aligned}$$

In particular, these new generators satisfy the (anti)commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{5.40}$$

$$[Z_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{5.41}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{5.42}$$

$$\left[J_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc}\tilde{Z}_{ad} - \eta_{ac}\tilde{Z}_{bd} - \eta_{bd}\tilde{Z}_{ac} + \eta_{ad}\tilde{Z}_{bc}, \tag{5.43}$$

$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{5.44}$$

$$\left[\tilde{Z}_{ab}, Z_{cd}\right] = \eta_{bc}\tilde{Z}_{ad} - \eta_{ac}\tilde{Z}_{bd} - \eta_{bd}\tilde{Z}_{ac} + \eta_{ad}\tilde{Z}_{bc}, \tag{5.45}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \qquad [Z_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \tag{5.46}$$

$$\left[\tilde{Z}_{ab}, P_c\right] = \eta_{bc}\tilde{Z}_a - \eta_{ac}\tilde{Z}_b, \qquad \left[J_{ab}, \tilde{Z}_c\right] = \eta_{bc}\tilde{Z}_a - \eta_{ac}\tilde{Z}_b, \tag{5.47}$$

$$\left[\tilde{Z}_{ab}, \tilde{Z}_c\right] = \eta_{bc}P_a - \eta_{ac}P_b, \qquad \left[Z_{ab}, \tilde{Z}_c\right] = \eta_{bc}\tilde{Z}_a - \eta_{ac}\tilde{Z}_b, \tag{5.48}$$

$$[P_a, P_b] = Z_{ab}, \qquad \left[\tilde{Z}_a, P_b\right] = \tilde{Z}_{ab}, \qquad \left[\tilde{Z}_a, \tilde{Z}_b\right] = Z_{ab}, \tag{5.49}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \qquad [P_a, Q_\alpha] = -\frac{1}{2}\left(\gamma_a\Sigma\right)_\alpha, \tag{5.50}$$

$$\left[\tilde{Z}_{ab}, Q_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}\Sigma\right)_\alpha, \qquad \left[\tilde{Z}_a, Q_\alpha\right] = -\frac{1}{2}\left(\gamma_a Q\right)_\alpha, \tag{5.51}$$

$$[Z_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \qquad [P_a, \Sigma_\alpha] = -\frac{1}{2}\left(\gamma_a Q\right)_\alpha, \tag{5.52}$$

$$[J_{ab}, \Sigma_\alpha] = -\frac{1}{2}\left(\gamma_{ab}\Sigma\right)_\alpha, \qquad \left[\tilde{Z}_a, \Sigma_\alpha\right] = -\frac{1}{2}\left(\gamma_a\Sigma\right)_\alpha, \tag{5.53}$$

$$\left[\tilde{Z}_{ab}, \Sigma_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \qquad [Z_{ab}, \Sigma_\alpha] = -\frac{1}{2}\left(\gamma_{ab}\Sigma\right)_\alpha, \tag{5.54}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta}P_a\right], \tag{5.55}$$

$$\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}Z_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta}\tilde{Z}_a\right], \tag{5.56}$$

$$\{\Sigma_\alpha, \Sigma_\beta\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta}P_a\right], \tag{5.57}$$

The superalgebra obtained after a resonant $S_E^{(4)}$-expansion of the $\mathfrak{osp}\left(4|1\right)$ superalgebra corresponds to the four-dimensional generalized minimal *AdS*-Lorentz superalgebra. Unlike the usual *AdS*-Lorentz superalgebra, this superalgebra contains an additional

4-component Majorana spinor charge Σ. The introduction of a second spinor generator has been already proposed in ref. [37] in the context of $D = 11$ supergravity theory and subsequently in ref. [38] in the superstring theory context.

A particular bosonic subalgebra of this superalgebra is the generalized AdS-Lorentz algebra generated by $\left\{ J_{ab}, P_a, \tilde{Z}_{ab}, Z_{ab}, \tilde{Z}_a \right\}$ and can be confused with the $AdS - \mathcal{L}_6$ algebra introduced in ref. [35]. Indeed, one could identify the $\tilde{Z}_{ab}$, Z_{ab} and $\tilde{Z}_a$ generators with the $Z_{ab}^{(1)}$, $Z_{ab}^{(2)}$ and Z_a generators of the $AdS - \mathcal{L}_6$ algebra, respectively. However, the commutators (5.49) are subtly different of those of the $AdS - \mathcal{L}_6$ algebra. On the other hand, the usual $AdS - \mathcal{L}_4$ algebra generated by $\{J_{ab}, P_a, Z_{ab}\}$ is a subalgebra of the generalized minimal AdS-Lorentz superalgebra.

Interestingly, a generalized minimal Maxwell superalgebra can be recovered as an Inönü-Wigner contraction [48, 49, 50] of the generalized minimal AdS-Lorentz superalgebra. Indeed, considering the rescaling

$$\begin{gathered} \tilde{Z}_{ab} \rightarrow \mu^2 \tilde{Z}_{ab}, \quad Z_{ab} \rightarrow \mu^4 Z_{ab}, \quad P_a \rightarrow \mu^2 P_a, \\ \tilde{Z}_a \rightarrow \mu^4 \tilde{Z}_a, \quad Q_\alpha \rightarrow \mu Q_\alpha \quad \text{and} \quad \Sigma \rightarrow \mu^3 \Sigma, \end{gathered}$$

and the limit $\mu \rightarrow \infty$, we found a generalized minimal Maxwell superalgebra [33]. Naturally, when we consider $\tilde{Z}_a = 0$ we recover the usual minimal Maxwell superalgebra $s\mathcal{M}_4$ defined in the previous section.

The construction of a four-dimensional supergravity action using the generalized minimal AdS-Lorentz superalgebra will be considered later. In the next section, following the method presented in ref. [33], we will approach the $\mathcal{N}$-extended AdS-Lorentz superalgebra using the semigroup expansion procedure.

5.2.3 $\mathcal{N}$-extended AdS-Lorentz superalgebras

In the previous sections, we have shown that the S-expansion of the AdS superalgebra $\mathfrak{osp}(4|1)$ allows to derive diverse AdS-Lorentz superalgebras. Then, it seems natural to consider the $\mathfrak{osp}(4|\mathcal{N})$ superalgebra as a starting point in order to obtain the four-dimensional $\mathcal{N}$-extended AdS-Lorentz superalgebra.

Before to apply the semigroup expansion method, the original superalgebra has to be decomposed in subspaces. Let us consider a decomposition of the $\mathfrak{osp}(4|\mathcal{N})$

superalgebra as

$$\mathfrak{g} = \mathfrak{osp}\left(4|\mathcal{N}\right) = \left(\mathfrak{so}\left(3,1\right) \oplus \mathfrak{so}\left(\mathcal{N}\right)\right) \oplus \frac{\mathfrak{osp}\left(4|\mathcal{N}\right)}{\mathfrak{sp}\left(4\right) \oplus \mathfrak{so}\left(\mathcal{N}\right)} \oplus \frac{\mathfrak{sp}\left(4\right)}{\mathfrak{so}\left(3,1\right)}$$
$$= V_0 \oplus V_1 \oplus V_2, \tag{5.58}$$

where V_0, V_1 and V_2 satisfy (4.76) − (4.78). Here, V_0 corresponds to the subspace generated by Lorentz transformations $\tilde{J}_{ab}$ and by $\frac{\mathcal{N}(\mathcal{N}-1)}{2}$ internal symmetry generators T^{ij}, V_1 corresponds to the supersymmetry translation generated by $\mathcal{N}$ Majorana spinor generators $\tilde{Q}^i_\alpha$ ($i = 1, \cdots, \mathcal{N}$; $\alpha = 1, \cdots, 4$) and V_2 is associated to the $\tilde{P}_a$ generators.

Following the definitions of ref. [16], let us consider $S_M^{(2)} = \{\lambda_0, \lambda_1, \lambda_2\}$ as the relevant abelian semigroup whose elements satisfy the following multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{if } \alpha+\beta \leq 2 \\ \lambda_{\alpha+\beta-2}, & \text{if } \alpha+\beta > 2 \end{cases} \tag{5.59}$$

Let us consider the subset decomposition $S_M^{(2)} = S_0 \cup S_1 \cup S_2$ where

$$S_0 = \{\lambda_0, \lambda_2\}, \tag{5.60}$$
$$S_1 = \{\lambda_1\}, \tag{5.61}$$
$$S_2 = \{\lambda_2\}. \tag{5.62}$$

In particular, this subset decomposition is said to "resonant" since it satisfies the same structure as the subspaces V_p of the $\mathfrak{osp}\left(4|\mathcal{N}\right)$ superalgebra [compare with eqs. (4.76) − (4.78)]

$$S_0 \cdot S_0 \subset S_0, \qquad S_1 \cdot S_1 \subset S_0 \cap S_2, \tag{5.63}$$
$$S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_2 \subset S_1, \tag{5.64}$$
$$S_0 \cdot S_2 \subset S_2, \qquad S_2 \cdot S_2 \subset S_0. \tag{5.65}$$

Then, according to theorem IV.2 of ref. [16], the superalgebra

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{5.66}$$

is a resonant subalgebra of $S_{\mathcal{M}}^{(2)} \times \mathfrak{osp}\left(4|\mathcal{N}\right)$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2\} \times \left\{\tilde{J}_{ab}, T^{ij}\right\} = \left\{\lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab}, \lambda_0 T^{ij}, \lambda_2 T^{ij}\right\}, \tag{5.67}$$

$$W_1 = (S_1 \times V_1) = \{\lambda_1\} \times \left\{\tilde{Q}_{\alpha}^{i}\right\} = \left\{\lambda_1 \tilde{Q}_{\alpha}^{i}\right\}, \tag{5.68}$$

$$W_2 = (S_2 \times V_2) = \{\lambda_2\} \times \left\{\tilde{P}_a\right\} = \left\{\lambda_2 \tilde{P}_a\right\}. \tag{5.69}$$

The resulting superalgebra is then generated by $\{J_{ab}, P_a, Z_{ab}, Q_{\alpha}^{i}, T^{ij}, Y^{ij}\}$ whose generators are related to the $\mathfrak{osp}\left(4|\mathcal{N}\right)$ ones through

$$\begin{aligned} J_{ab} &= \lambda_0 \tilde{J}_{ab}, & Q_{\alpha}^{i} &= \lambda_1 \tilde{Q}_{\alpha}^{i}, \\ P_a &= \lambda_2 \tilde{P}_a, & T^{ij} &= \lambda_0 T^{ij}, \\ Z_{ab} &= \lambda_2 \tilde{J}_{ab}, & Y^{ij} &= \lambda_2 T^{ij}. \end{aligned}$$

These generators satisfy the (anti)commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{5.70}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{5.71}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{5.72}$$

$$[Z_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{5.73}$$

$$[Z_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{5.74}$$

$$\left[T^{ij}, T^{kl}\right] = \delta^{jk} T^{il} - \delta^{ik} T^{jl} - \delta^{jl} T^{ik} + \delta^{il} T^{jk}, \tag{5.75}$$

$$\left[T^{ij}, Y^{kl}\right] = \delta^{jk} Y^{il} - \delta^{ik} Y^{jl} - \delta^{jl} Y^{ik} + \delta^{il} Y^{jk}, \tag{5.76}$$

$$\left[Y^{ij}, Y^{kl}\right] = \delta^{jk} Y^{il} - \delta^{ik} Y^{jl} - \delta^{jl} Y^{ik} + \delta^{il} Y^{jk}, \tag{5.77}$$

$$\left[J_{ab}, Q_{\alpha}^{i}\right] = -\frac{1}{2}\left(\gamma_{ab} Q^{i}\right)_{\alpha}, \qquad \left[Z_{ab}, Q_{\alpha}^{i}\right] = -\frac{1}{2}\left(\gamma_{ab} Q^{i}\right)_{\alpha}, \tag{5.78}$$

$$\left[T^{ij}, Q_{\alpha}^{i}\right] = \left(\delta^{jk} Q_{\alpha}^{i} - \delta^{ik} Q_{\alpha}^{i}\right), \tag{5.79}$$

$$\left[Y^{ij}, Q_{\alpha}^{k}\right] = \left(\delta^{jk} Q_{\alpha}^{i} - \delta^{ik} Q_{\alpha}^{i}\right), \tag{5.80}$$

$$\left[P_a, Q_{\alpha}^{i}\right] = -\frac{1}{2}\left(\gamma_a Q^{i}\right)_{\alpha}, \tag{5.81}$$

$$\left\{Q_{\alpha}^{i}, Q_{\beta}^{j}\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab} C\right)_{\alpha\beta} Z_{ab} - 2\left(\gamma^{a} C\right)_{\alpha\beta} P_a\right] + C_{\alpha\beta} Y^{ij}, \tag{5.82}$$

$$\text{others } = 0. \tag{5.83}$$

The superalgebra obtained after a resonant $S_M^{(2)}$-expansion of the $\mathfrak{osp}(4|\mathcal{N})$ superalgebra corresponds to the four-dimensional $\mathcal{N}$-extended AdS-Lorentz superalgebra. In particular, the J_{ab} generators form the Lorentz algebra $\mathfrak{so}(3,1)$ while the Z_{ab}, P_a, Y^{ij} and Q_α^i generators form the $\mathfrak{osp}(4|\mathcal{N})$ superalgebra. Then, the $\mathcal{N}$-extended AdS-Lorentz superalgebra corresponds to a direct sum of the Lorentz algebra $\mathfrak{so}(3,1)$ and the AdS superalgebra $\mathfrak{osp}(4|\mathcal{N})$.

On the other hand, one can see that the AdS-Lorentz algebra generated by $\{J_{ab}, Z_{ab}, P_a\}$ is contained as a bosonic subalgebra of the N-extended AdS-Lorentz superalgebra.

The generalization of this procedure to $(\mathcal{N})$-extended AdS-Lorentz type superalgebras and the construction of a $\mathcal{N}$-extended supergravity action à la MacDowell-Mansouri remains an interesting problem to approach and will not be considered in the present thesis.

5.3 Geometric theory of supergravity with a generalized cosmological constant

It is the purpose of this section, following ref. [33], to construct a supergravity action using the MacDowell-Mansouri geometric formalism which contains a generalized supersymmetric cosmological constant. To this aim, we consider diverse AdS-Lorentz superalgebras and propose a supergravity action based only on the two-form curvature. In particular, as we have seen previously, the AdS-Lorentz superalgebra contains non abelian Z_{ab} generators which implies the presence of additional bosonic fields k^{ab}.

Our main motivation of considering the AdS-Lorentz symmetries is that we are interested in investigate the geometric consequences of the presence of the generators $Z_{ab} = [P_a, P_b]$ in the construction of a supergravity action. Although a similar non-commutativity appears in the Maxwell superalgebra, as shown in ref. [32], the supergravity action à la MacDowell-Mansouri based on the Maxwell supersymmetries does not reproduce the cosmological term in the supergravity action.

5.3.1 $D=4$ supergravity from the AdS-Lorentz superalgebra

In this section, following ref. [33], we present an alternative way of introducing the supersymmetric cosmological term to the four-dimensional supergravity action using the MacDowell-Mansouri geometrical approach. In particular, analogously to the previous chapter, we propose a supergravity action constructed out the curvature two-form of the AdS-Lorent superalgebra using the semigroup expansion method. The study of the supersymmetry invariance is also considered in the present section.

Let us consider the connection one-form

$$A = A^A T_A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{l}e^a P_a + \frac{1}{2}k^{ab}Z_{ab} + \frac{1}{\sqrt{l}}\psi^\alpha Q_\alpha, \tag{5.84}$$

whose components are related to the $\mathfrak{osp}\,(4|1)$ ones as follows

$$\begin{aligned}
\omega^{ab} &= \lambda_0 \tilde{\omega}^{ab}, \\
e^a &= \lambda_2 \tilde{e}^a, \\
k^{ab} &= \lambda_2 \tilde{\omega}^{ab}, \\
\psi^\alpha &= \lambda_1 \tilde{\psi}^\alpha.
\end{aligned}$$

Let $F = dA + A \wedge A$ be the associated curvature two-form given by

$$F = F^A T_A = \frac{1}{2}R^{ab}J_{ab} + \frac{1}{l}R^a P_a + \frac{1}{2}F^{ab}Z_{ab} + \frac{1}{\sqrt{l}}\Psi^\alpha Q_\alpha, \tag{5.85}$$

with

$$\begin{aligned}
R^{ab} &= d\omega^{ab} + \omega^a_{\ c}\omega^{cb}, \\
R^a &= de^a + \omega^a_{\ b}e^b + k^a_{\ b}e^b - \frac{1}{2}\bar{\psi}\gamma^a\psi, \\
F^{ab} &= dk^{ab} + \omega^a_{\ c}k^{cb} - \omega^b_{\ c}k^{ca} + k^a_{\ c}k^{cb} + \frac{1}{l^2}e^a e^b + \frac{1}{2l}\bar{\psi}\gamma^{ab}\psi, \\
\Psi &= d\psi + \frac{1}{4}\omega_{ab}\gamma^{ab}\psi + \frac{1}{2l}e^a\gamma_a\psi + \frac{1}{4}k_{ab}\gamma^{ab}\psi.
\end{aligned}$$

Here, the one-forms ω^{ab}, e^a, and ψ are the spin connection, the vielbein and the gravitino field, respectively. While the k^{ab} fields describe additional bosonic "matter" fields. Let us note that the Maurer-Cartan equations for the AdS-Lorentz superalgebra are satisfied when $F = 0$.

On the other hand, the Lorentz covariant exterior derivatives $D = d + \omega$ of the curvatures can be obtained from the Bianchi identity $\nabla F = 0$ (where ∇ is the gauge covariant derivative given by $\nabla = d + [A, \cdot]$) leading to

$$\begin{aligned} DR^{ab} &= 0, && (5.86)\\ DR^{a} &= R^{a}_{\ b}e^{b} + F^{a}_{\ b}e^{b} + R^{c}k_{c}^{\ a} + \bar{\psi}\gamma^{a}\Psi, && (5.87)\\ DF^{ab} &= R^{a}_{\ c}k^{cb} - R^{b}_{\ c}k^{ca} + F^{a}_{\ c}k^{cb} - F^{b}_{\ c}k^{ca} + \frac{1}{l^{2}}\left(R^{a}e^{b} - e^{a}R^{b}\right) \\ &\quad + \frac{1}{l}\bar{\Psi}\gamma^{ab}\psi, && (5.88)\\ D\Psi &= \frac{1}{4}R_{ab}\gamma^{ab}\psi + \frac{1}{4}F_{ab}\gamma^{ab}\psi - \frac{1}{4}k_{ab}\gamma^{ab}\Psi + \frac{1}{2l}R^{a}\gamma_{a}\psi \\ &\quad - \frac{1}{2l}e^{a}\gamma_{a}\Psi. && (5.89) \end{aligned}$$

In order to construct a supergravity action à la MacDowell-Mansouri for the AdS-Lorentz superalgebra we shall consider the semigroup expansion of a particular choice of the invariant tensor $\langle T_A T_B \rangle$ and the curvature two-form (5.85). Then the MacDowell-Mansouri type action for the $sAdS - \mathcal{L}_4$ superalgebra can be written as

$$S = 2 \int F^{A} \wedge F^{B} \left\langle T_A T_B \right\rangle_{sAdS-\mathcal{L}_4} . \tag{5.90}$$

Here, $\left\langle T_A T_B \right\rangle_{sAdS-\mathcal{L}_4}$ can be obtained using the useful properties of the semigroup expansion procedure. Indeed, according to the theorem VII.1 of ref. [16], the components of an invariant tensor for the AdS-Lorentz superalgebra can be written in terms of a particular choice of the original invariant tensor,

$$\left\langle J_{ab} J_{cd} \right\rangle_{sAdS-\mathcal{L}_4} = \alpha_0 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle , \tag{5.91}$$

$$\left\langle J_{ab} Z_{cd} \right\rangle_{sAdS-\mathcal{L}_4} = \alpha_2 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle , \tag{5.92}$$

$$\left\langle Z_{ab} Z_{cd} \right\rangle_{sAdS-\mathcal{L}_4} = \alpha_2 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle , \tag{5.93}$$

$$\left\langle Q_{\alpha} Q_{\beta} \right\rangle_{sAdS-\mathcal{L}_4} = \alpha_2 \left\langle \tilde{Q}_{\alpha} \tilde{Q}_{\beta} \right\rangle , \tag{5.94}$$

with

$$\left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle = \epsilon_{abcd},$$
$$\left\langle \tilde{Q}_{\alpha} \tilde{Q}_{\beta} \right\rangle = 2 \left(\gamma_5\right)_{\alpha\beta} ,$$

and where the α's are arbitrary dimensionless independent constants. It is important to clarify that this choice of the invariant tensor breaks the AdS-Lorentz superalgebra to its Lorentz like subalgebra generated by $\{J_{ab}, Z_{ab}\}$. This is not a surprise since we have considered the S-expansion of a particular choice of an invariant tensor which breaks the $Osp\,(4|1)$ supergroups to its Lorentz subgroup.

Then, considering the curvature two-form of the AdS-Lorentz superalgebra $sAdS-\mathcal{L}_4$ (5.85) and the non-vanishing components of the invariant tensor (5.91) – (5.94), the MacDowell-Mansouri type supergravity action (5.90) becomes

$$S = 2\int\left(\frac{1}{4}\alpha_0\epsilon_{abcd}R^{ab}R^{cd} + \frac{1}{2}\alpha_2\epsilon_{abcd}R^{ab}F^{cd} + \frac{1}{4}\alpha_2\epsilon_{abcd}F^{ab}F^{cd} + \frac{2}{l}\alpha_2\bar{\Psi}\gamma_5\Psi\right). \quad (5.95)$$

The action (5.95) can be written explicitly in terms of the different components of the two-form curvature as

$$\begin{aligned} S = \int & \frac{\alpha_0}{2}\epsilon_{abcd}R^{ab}R^{cd} + \alpha_2\epsilon_{abcd}\left(R^{ab}Dk^{cd} + R^{ab}k^c_{\ e}k^{ed} + \frac{1}{l^2}R^{ab}e^ce^d\right. \\ & + \frac{1}{2l}R^{ab}\bar{\psi}\gamma^{cd}\psi + \frac{1}{2}Dk^{ab}Dk^{cd} + Dk^{ab}k^c_{\ e}k^{ed} + \frac{1}{l^2}Dk^{ab}e^ce^d \\ & + \frac{1}{2l}Dk^{ab}\bar{\psi}\gamma^{cd}\psi + \frac{1}{2}k^a_{\ f}k^{fb}k^c_{\ g}k^{gd} + \frac{1}{l^2}k^a_{\ f}k^{fb}e^ce^d + \frac{1}{2l}k^a_{\ f}k^{fb}\bar{\psi}\gamma^{cd}\psi\Big) \\ & \left. + \frac{1}{2l^3}e^ae^b\bar{\psi}\gamma^{cd}\psi + \frac{1}{2l^4}e^ae^be^ce^d\right) + \alpha_2\left(\frac{4}{l}D\bar{\psi}\gamma_5D\psi + \frac{4}{l^2}\bar{\psi}e^a\gamma_a\gamma_5D\psi\right. \\ & + \frac{2}{l}D\bar{\psi}\gamma_5k_{ab}\gamma^{ab}\psi + \frac{1}{l^3}\bar{\psi}e^a\gamma_a\gamma_5e^b\gamma_b\psi + \frac{1}{l^2}\bar{\psi}e^a\gamma_a\gamma_5k^{bc}\gamma_{bc}\psi \\ & \left. + \frac{1}{4l}\bar{\psi}k_{ab}\gamma^{ab}\gamma_5k_{cd}\gamma^{cd}\psi\right). \end{aligned} \quad (5.96)$$

Interestingly, using the gravitino Bianchi identity and the gamma matrix identity

$$2\gamma_{ab}\gamma_5 = -\epsilon_{abcd}\gamma^{cd}, \quad (5.97)$$

it is possible to combine some expressions as boundary terms. Indeed, following ref. [33], we have

$$\frac{1}{2}\epsilon_{abcd}R^{ab}\bar{\psi}\gamma^{cd}\psi + 4D\bar{\psi}\gamma_5D\psi = d\left(4D\bar{\psi}\gamma_5\psi\right),$$
$$\frac{1}{2}\epsilon_{abcd}Dk^{ab}\bar{\psi}\gamma^{cd}\psi + 2D\bar{\psi}\gamma_5k^{ab}\gamma_{ab}\psi = d\left(\bar{\psi}k^{ab}\gamma_{ab}\gamma_5\psi\right).$$

Moreover, using the useful gamma matrix identities [see Appendix B], it is possible to show

$$\begin{aligned}
\bar{\psi} e^a \gamma_a \gamma_5 e^b \gamma_b \psi &= \frac{1}{2} e^a e^b \bar{\psi} \gamma^{cd} \psi \epsilon_{abcd}, \\
\frac{1}{4} \bar{\psi} k_{ab} \gamma^{ab} \gamma_5 k_{cd} \gamma^{cd} \psi &= -\frac{1}{2} k^a{}_f k^{fb} \bar{\psi} \gamma^{cd} \psi \epsilon_{abcd}, \\
\bar{\psi} e^a \gamma_a \gamma_5 k^{bc} \gamma_{bc} \psi &= \epsilon_{abcd} k^{ab} e^c \bar{\psi} \gamma^d \psi,
\end{aligned}$$

Thus, the supergravity action for the AdS-Lorentz superalgebra can be finally written as

$$\begin{aligned}
S = \int & \frac{\alpha_0}{2} \epsilon_{abcd} R^{ab} R^{cd} + \frac{\alpha_2}{l^2} \left(\epsilon_{abcd} R^{ab} e^c e^d + 4 \bar{\psi} e^a \gamma_a \gamma_5 D\psi \right) \\
& + \alpha_2 \epsilon_{abcd} \left(R^{ab} D k^{cd} + R^{ab} k^c{}_e k^{ed} + \frac{1}{2} D k^{ab} D k^{cd} + D k^{ab} k^c{}_e k^{ed} + \frac{1}{2} k^a{}_f k^{fb} k^c{}_g k^{gd} \right) \\
& + \alpha_2 \epsilon_{abcd} \left(\frac{1}{l^2} D k^{ab} e^c e^d + \frac{1}{l^2} k^a{}_f k^{fb} e^c e^d + \frac{1}{l^3} e^a e^b \bar{\psi} \gamma^{cd} \psi \right. \\
& \left. + \frac{1}{l^2} k^{ab} e^c \bar{\psi} \gamma^d \psi + \frac{1}{2l^4} e^a e^b e^c e^d \right) + \alpha_2 d \left(4 D \bar{\psi} \gamma_5 \psi + \bar{\psi} k^{ab} \gamma_{ab} \gamma_5 \psi \right). \qquad (5.98)
\end{aligned}$$

The supergravity action (5.98) is split intentionally into five terms. The first term corresponds to the topological Gauss-Bonnet term and does not contribute to the dynamics. The second piece is proportional to α_2 and contains the Einstein-Hilbert and the Rarita-Schwinger terms . The third and last term is also a boundary term and contains explicitly the coupling between the new bosonic gauge fields k^{ab} and the usual fields. Interestingly, the fourth term contains a generalized supersymmetric cosmological term which contains not only the usual supersymmetric cosmological constant, but also additional terms containing the new fields k^{ab}.

The procedure used here corresponds to an alternative method to include a cosmological term to a supergravity action à la MacDowell-Mansouri. Interestingly, the bosonic part of the action (5.98) corresponds to the Born-Infeld gravity action for the AdS-Lorentz algebra presented in ref. [35]. On the other hand, the bosonic cosmological term introduced here coincides with the one appearing in ref. [9].

It is important to clarify that although there are many four-dimensional supergravity theories with cosmological constant, the formalism used here could be useful in the

AdS/CFT correspondence. In particular, the presence of the new bosonic fields k^{ab} in the boundary could play an important role in the well celebrated duality between superstring theory realized on an AdS space-time and the conformal field theory on its boundary [51, 52, 53, 54]. As was pointed out in ref. [55], the introduction of an appropriate topological boundary term in a four-dimensional bosonic action is equivalent to the holographic renormalization in the AdS/CFT context. Then, it seems that the presence of the k^{ab} fields in the boundary would allow to regularize the supergravity action in the holographic renormalization language.

Additionally, as shown in ref. [55, 56], the bosonic MacDowell-Mansouri action is on-shell equivalent to the square of the Weyl tensor describing conformal gravity. This would suggest a superconformal structure in the MacDowell-Mansouri geometrical formalism of supergravity theory.

The supergravity action (5.98) can be rewritten omitting the boundary contributions as

$$\begin{aligned} S = \int & \frac{\alpha_2}{l^2} \left(\epsilon_{abcd} R^{ab} e^c e^d + 4 \bar{\psi} e^a \gamma_a \gamma_5 D\psi \right) \\ & + \alpha_2 \epsilon_{abcd} \left(\frac{2}{l^2} k^{ab} \hat{T}^c e^d + \frac{1}{l^2} k^a{}_f k^{fb} e^c e^d + \frac{1}{l^3} e^a e^b \bar{\psi} \gamma^{cd} \psi + \frac{1}{2l^4} e^a e^b e^c e^d \right), \end{aligned} \tag{5.99}$$

where

$$\epsilon_{abcd} D k^{ab} e^c e^d = 2 \epsilon_{abcd} k^{ab} T^c e^d + d \left(\frac{1}{l^2} \epsilon_{abcd} k^{ab} e^c e^d \right),$$
$$\hat{T}^a = De^a - \frac{1}{2} \bar{\psi} \gamma^a \psi = T^a - \frac{1}{2} \bar{\psi} \gamma^a \psi.$$

In particular, the usual MacDowell-Mansouri supergravity action for the $\mathfrak{osp}(4|1)$ superalgebra can be recover in the limit $k^{ab} = 0$.

Although the supergravity action (5.98) is constructed out of the curvature 2-forms of the AdS-Lorentz superalgebra, it is not invariant under the gauge transformations. Indeed, the supergravity action does not correspond to a topological invariant, nor a Yang-Mills action.

The AdS-Lorentz gauge transformation of the one-form gauge connection A is given by

$$\delta_\rho A = D\rho = d\rho + [A, \rho]$$

where ρ is the gauge parameter given by

$$\rho=\frac{1}{2}\rho^{ab}J_{ab}+\frac{1}{2}\kappa^{ab}Z_{ab}+\frac{1}{l}\rho^{a}P_{a}+\frac{1}{\sqrt{l}}\epsilon^{\alpha}Q_{\alpha}. \tag{5.100}$$

Then, we have explicitly for each component the following gauge transformations,

$$\delta\omega^{ab}=D\rho^{ab}, \tag{5.101}$$

$$\begin{aligned}\delta k^{ab}=&\,D\kappa^{ab}-\left(k^{ac}\rho^{b}_{\ c}-k^{bc}\rho^{a}_{\ c}\right)-\left(k^{ac}\kappa^{b}_{\ c}-k^{bc}\kappa^{a}_{\ c}\right)\\&+\frac{2}{l^{2}}e^{a}\rho^{b}-\frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi,\end{aligned} \tag{5.102}$$

$$\delta e^{a}=D\rho^{a}\,+e^{b}\rho_{b}^{\ a}+k^{a}_{\ b}\rho^{b}+e^{b}\kappa_{b}^{\ a}+\bar{\epsilon}\gamma^{a}\psi, \tag{5.103}$$

$$\begin{aligned}\delta\psi=&\,d\epsilon+\frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon-\frac{1}{4}\rho^{ab}\gamma_{ab}\psi+\frac{1}{2l}e^{a}\gamma_{a}\epsilon-\frac{1}{2l}\rho^{a}\gamma_{a}\psi\\&+\frac{1}{4}\tilde{k}^{ab}\gamma_{ab}\epsilon-\frac{1}{4}\tilde{\kappa}^{ab}\gamma_{ab}\psi.\end{aligned} \tag{5.104}$$

Similarly, the gauge transformations of the curvature F can be obtained from $\delta_{\rho}F=[F,\rho]$ leading to

$$\delta R^{ab}=R^{ac}\rho_{c}^{\ b}-R^{cb}\rho^{a}_{\ c}, \tag{5.105}$$

$$\begin{aligned}\delta F^{ab}=&\left(R^{ac}\kappa_{c}^{\ b}-R^{bc}\kappa^{a}_{\ c}\right)-\left(F^{ac}\rho^{b}_{\ c}-F^{bc}\rho^{a}_{\ c}\right)-\left(F^{ac}\kappa^{b}_{\ c}-F^{ac}\kappa^{a}_{\ c}\right)\\&+\frac{2}{l^{2}}R^{a}\rho^{b}-\frac{1}{l}\bar{\epsilon}\gamma^{ab}\Psi,\end{aligned} \tag{5.106}$$

$$\delta R^{a}=R^{a}_{\ b}\rho^{b}+R^{b}\rho_{b}^{\ a}+F^{a}_{\ b}\rho^{b}+R^{b}\kappa_{b}^{\ a}+\bar{\epsilon}\gamma^{a}\Psi, \tag{5.107}$$

$$\begin{aligned}\delta\Psi=&\,\frac{1}{4}R^{ab}\gamma_{ab}\epsilon-\frac{1}{4}\rho^{ab}\gamma_{ab}\Psi+\frac{1}{2l}R^{a}\gamma_{a}\epsilon-\frac{1}{2l}\rho^{a}\gamma_{a}\Psi\\&+\frac{1}{4}F^{ab}\gamma_{ab}\epsilon-\frac{1}{4}\kappa^{ab}\gamma_{ab}\Psi.\end{aligned} \tag{5.108}$$

Then, one can see that the variation of the action (5.98) under gauge supersymmetry is

$$\delta_{susy}S=-\frac{4}{l^{2}}\alpha_{2}\int R^{a}\bar{\Psi}\gamma_{a}\gamma_{5}\epsilon. \tag{5.109}$$

This implies that the supertorsion constraint

$$R^{a}=De^{b}+k^{a}_{\ b}e^{b}-\frac{1}{2}\bar{\psi}\gamma^{a}\psi=0,$$

has to be imposed in order to obtain the gauge AdS-Lorentz supersymmetry invariance. This leads us to express the spin connection ω^{ab} and the bosonic field k^{ab} in terms of the other fields (second order formalism). Interestingly, following ref. [33], it is possible to define a new bosonic field as the combination of the spin connection and the k^{ab} fields as

$$\varpi^{ab} = \omega^{ab} + k^{ab}, \tag{5.110}$$

and its respective covariant derivative,

$$\mathcal{D} = d + \varpi. \tag{5.111}$$

Then, the supertorsion constraint can be written as

$$\left(\mathcal{D}e^{a} - \frac{1}{2}\bar{\psi}\gamma^{a}\psi \right) = 0, \tag{5.112}$$

allowing to express the bosonic field ϖ^{ab} in terms of the vielbein e^{a} and gravitino field ψ^{α}. Let us consider the following decomposition,

$$\varpi^{ab} = \mathring{\varpi}^{ab} + \tilde{\varpi}^{ab}, \tag{5.113}$$

where $\mathring{\varpi}^{ab}$ corresponds to the solution of $\mathcal{D}e^{c} = 0$ and it is given by

$$\mathring{\varpi}^{ab}_{\mu} = \left(e^{c}_{\lambda}\partial_{[\mu}e^{d}_{\nu]}\eta_{cd} + e^{c}_{\nu}\partial_{[\lambda}e^{d}_{\mu]}\eta_{cd} - e^{c}_{\mu}\partial_{[\nu}e^{d}_{\lambda]}\eta_{cd} \right) e^{\lambda|a}e^{\nu|b}. \tag{5.114}$$

Thus,

$$\mathcal{D}e^{a} = de^{a} + \mathring{\varpi}^{ab}e_{b} + \tilde{\varpi}^{ab}e_{b} = \frac{1}{2}\bar{\psi}\gamma^{a}\psi, \tag{5.115}$$

implies

$$\tilde{\varpi}^{ab}_{[\mu}e_{\nu]b} = \frac{1}{2}\bar{\psi}_{\mu}\gamma^{a}\psi_{\nu}. \tag{5.116}$$

This may be solved in terms of the two other fields,

$$\tilde{\varpi}^{ab}_{\mu} = \frac{1}{4}e^{a|\lambda}e^{b|\nu} \left(\bar{\psi}_{\mu}\gamma_{\lambda}\psi_{\nu} + \bar{\psi}_{\lambda}\gamma_{\nu}\psi_{\mu} - \bar{\psi}_{\nu}\gamma_{\mu}\psi_{\lambda} - \bar{\psi}_{\mu}\gamma_{\nu}\psi_{\lambda} - \bar{\psi}_{\nu}\gamma_{\lambda}\psi_{\mu} + \bar{\psi}_{\lambda}\gamma_{\mu}\psi_{\nu} \right). \tag{5.117}$$

Here, the bosonic field ϖ^{ab} does not carry additional physical degrees of freedom. In particular, the number of bosonic degrees of freedom is two when the supertorsion is set equal to zero.

On the other, the supersymmetry invariance of the action (5.98) can be obtained in the first formalism adding an extra piece to the gauge transformation of the spin connection ω^{ab}. Then, the variation of the action is given by

$$\delta S = -\frac{4}{l^{2}}\alpha_{4} \int R^{a} \left(\bar{\Psi}\gamma_{a}\gamma_{5}\epsilon - \frac{1}{2}\epsilon_{abcd}e^{b}\delta_{extra}\omega^{cd} \right). \tag{5.118}$$

The supersymmetry invariance of the supergravity action is fulfilled imposing

$$\delta_{extra}\omega^{ab} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_{d}\gamma_{5}\epsilon + \bar{\Psi}_{de}\gamma_{c}\gamma_{5}\epsilon - \bar{\Psi}_{cd}\gamma_{e}\gamma_{5}\epsilon\right)e^{e}, \tag{5.119}$$

with $\bar{\Psi} = \bar{\Psi}_{ab}e^{a}e^{b}$.

Thus, the supergravity action à la MacDowell-Mansouri (5.98) is invariant under the following supersymmetry transformations

$$\delta\omega^{ab} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_{d}\gamma_{5}\epsilon + \bar{\Psi}_{de}\gamma_{c}\gamma_{5}\epsilon - \bar{\Psi}_{cd}\gamma_{e}\gamma_{5}\epsilon\right)e^{e}, \tag{5.120}$$

$$\delta k^{ab} = -\frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi, \tag{5.121}$$

$$\delta e^{a} = \bar{\epsilon}\gamma^{a}\psi, \tag{5.122}$$

$$\delta\psi = D\epsilon + \frac{1}{4}k^{ab}\gamma_{ab}\epsilon + \frac{1}{2l}e^{a}\gamma_{a}\epsilon. \tag{5.123}$$

It is important to clarify that these supersymmetry transformations do not correspond to gauge symmetries of the action, since it is broken to a Lorentz like symmetry.

5.3.2 $D = 4$ supergravity from the generalized minimal AdS-Lorentz superalgebra

In this section, we present the construction of a supergravity action with a generalized supersymmetric cosmological term using the generalized minimal AdS-Lorentz superalgebra introduced in ref. [33]. In particular, the MacDowell-Mansouri geometric formalism is considered.

First, let us consider the one-form gauge connection,

$$A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{l}e^{a}P_{a} + \frac{1}{2}\tilde{k}^{ab}\tilde{Z}_{ab} + \frac{1}{2}k^{ab}Z_{ab} + \frac{1}{l}\tilde{h}^{a}\tilde{Z}_{a} + \frac{1}{\sqrt{l}}\psi^{\alpha}Q_{\alpha} + \frac{1}{\sqrt{l}}\xi^{\alpha}\Sigma_{\alpha}, \tag{5.124}$$

whose components are related to the $\mathfrak{osp}(4|1)$ ones through the elements of the semigroup $S_{M}^{(4)}$

$$\begin{aligned} \omega^{ab} &= \lambda_0\tilde{\omega}^{ab}, & e^{a} &= \lambda_2\tilde{e}^{a}, \\ \tilde{k}^{ab} &= \lambda_2\tilde{\omega}^{ab}, & \psi^{\alpha} &= \lambda_1\tilde{\psi}^{\alpha}, \\ k^{ab} &= \lambda_4\tilde{\omega}^{ab}, & \xi^{\alpha} &= \lambda_3\tilde{\psi}^{\alpha}, \\ \tilde{h}^{a} &= \lambda_4 e^{a}. \end{aligned}$$

Let $F = dA + A \wedge A$ be the associated curvature two-form given by

$$F = \frac{1}{2}R^{ab}J_{ab} + \frac{1}{l}R^{a}P_{a} + \frac{1}{2}\tilde{F}^{ab}\tilde{Z}_{ab} + \frac{1}{2}F^{ab}Z_{ab} + \frac{1}{l}\tilde{H}^{a}\tilde{Z}_{a} + \frac{1}{\sqrt{l}}\Psi^{\alpha}Q_{\alpha} + \frac{1}{\sqrt{l}}\Xi^{\alpha}\Sigma_{\alpha}, \quad (5.125)$$

with

$$\begin{aligned}
R^{ab} &= d\omega^{ab} + \omega^{a}{}_{c}\omega^{cb}, \\
R^{a} &= de^{a} + \omega^{a}{}_{b}e^{b} + k^{a}{}_{b}e^{b} + \tilde{k}^{a}{}_{b}\tilde{h}^{b} - \frac{1}{2}\bar{\psi}\gamma^{a}\psi - \frac{1}{2}\bar{\xi}\gamma^{a}\xi, \\
\tilde{H}^{a} &= d\tilde{h}^{a} + \omega^{a}{}_{b}\tilde{h}^{b} + \tilde{k}^{a}{}_{b}e^{b} + k^{a}{}_{b}\tilde{h}^{b} - \bar{\psi}\gamma^{a}\xi, \\
\tilde{F}^{ab} &= d\tilde{k}^{ab} + \omega^{a}{}_{c}\tilde{k}^{cb} - \omega^{b}{}_{c}\tilde{k}^{ca} + k^{a}{}_{c}\tilde{k}^{cb} - k^{b}{}_{c}\tilde{k}^{ca} + \frac{2}{l^{2}}e^{a}\tilde{h}^{b} + \frac{1}{2l}\bar{\psi}\gamma^{ab}\psi + \frac{1}{2l}\bar{\xi}\gamma^{ab}\xi, \\
F^{ab} &= dk^{ab} + \omega^{a}{}_{c}k^{cb} - \omega^{b}{}_{c}k^{ca} + \tilde{k}^{a}{}_{c}\tilde{k}^{cb} + k^{a}{}_{c}k^{cb} + \frac{1}{l^{2}}e^{a}e^{b} + \frac{1}{l^{2}}\tilde{h}^{a}\tilde{h}^{b} + \frac{1}{l}\bar{\xi}\gamma^{ab}\psi, \\
\Psi &= d\psi + \frac{1}{4}\omega_{ab}\gamma^{ab}\psi + \frac{1}{4}k_{ab}\gamma^{ab}\psi + \frac{1}{4}\tilde{k}_{ab}\gamma^{ab}\xi + \frac{1}{2l}e_{a}\gamma^{a}\xi + \frac{1}{2}\tilde{h}_{a}\gamma^{a}\psi, \\
\Xi &= d\xi + \frac{1}{4}\omega_{ab}\gamma^{ab}\xi + \frac{1}{4}k_{ab}\gamma^{ab}\xi + \frac{1}{4}\tilde{k}_{ab}\gamma^{ab}\psi + \frac{1}{2l}e_{a}\gamma^{a}\psi + \frac{1}{2l}\tilde{h}_{a}\gamma^{a}\xi.
\end{aligned}$$

Here, the one-forms ω^{ab}, e^{a}, ψ and ξ are the spin connection, the vielbein, the gravitino field and an additional Majorana fermionic field[2], respectively. While the k^{ab}, $\tilde{k}^{ab}$ and $\tilde{h}^{a}$ fields describe bosonic fields.

In order to construct a supergravity action à la MacDowell-Mansouri for the generalized minima AdS-Lorentz superalgebra we shall consider the semigroup expansion of a particular choice of the invariant tensor $\langle T_A T_B \rangle$ and the two-form curvature (5.125). Then the supergravity action for this generalized AdS-Lorentz superalgebra can be written as

$$S = 2\int F^{A} \wedge F^{B} \left\langle T_A T_B \right\rangle_{\mathcal{S}} . \quad (5.126)$$

Here, $\langle T_A T_B \rangle_{\mathcal{S}}$ can be obtained using the useful properties of the semigroup expansion method. In fact, according to the theorem VII.1 of ref. [16], the components of an invariant tensor for the generalized minimal AdS-Lorentz superalgebra can be written

[2] A Majorana spinor ψ satisfies the Majorana condition $\bar{\psi} = \psi C$, where C is the charge conjugation matrix.

in terms of a particular choice of the original invariant tensor,

$$\langle J_{ab}J_{cd}\rangle_{\mathcal{S}} = \alpha_0 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \qquad \left\langle \tilde{Z}_{ab}\tilde{Z}_{cd}\right\rangle_{\mathcal{S}} = \alpha_4 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{5.127}$$

$$\left\langle J_{ab}\tilde{Z}_{cd}\right\rangle_{\mathcal{S}} = \alpha_2 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \qquad \langle Z_{ab}Z_{cd}\rangle_{\mathcal{S}} = \alpha_4 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{5.128}$$

$$\left\langle \tilde{Z}_{ab}Z_{cd}\right\rangle_{\mathcal{S}} = \alpha_2 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \qquad \langle J_{ab}Z_{cd}\rangle_{\mathcal{S}} = \alpha_4 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{5.129}$$

$$\langle Q_{\alpha}Q_{\beta}\rangle_{\mathcal{S}} = \alpha_2 \left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta}\right\rangle, \qquad \langle \Sigma_{\alpha}\Sigma_{\beta}\rangle_{\mathcal{S}} = \alpha_2 \left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta}\right\rangle, \tag{5.130}$$

$$\langle Q_{\alpha}\Sigma_{\beta}\rangle_{\mathcal{S}} = \alpha_4 \left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta}\right\rangle, \tag{5.131}$$

with

$$\left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle = \epsilon_{abcd},$$
$$\left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta}\right\rangle = 2\left(\gamma_5\right)_{\alpha\beta},$$

and where the α's are arbitrary dimensionless independent constants. Let us note that this choice of the invariant tensor breaks the generalized AdS-Lorentz superalgebra to its Lorentz like subalgebra generated by $\left\{J_{ab}, Z_{ab}, \tilde{Z}_{ab}\right\}$. This is not a surprise since we have considered the S-expansion of a particular choice of an invariant tensor which breaks the $Osp\left(4|1\right)$ supergroups to its Lorentz subgroup.

Then, considering the curvature two-form of the generalized minimal AdS-Lorentz superalgebra(5.125) and the non-vanishing components of the invariant tensor (5.127)−(5.131), the MacDowell-Mansouri type supergravity action (5.126) becomes

$$\begin{aligned} S = 2\int \Big(& \frac{\alpha_0}{4}\epsilon_{abcd}R^{ab}R^{cd} + \frac{\alpha_2}{2}\epsilon_{abcd}R^{ab}\tilde{F}^{cd} + \frac{\alpha_2}{2}\epsilon_{abcd}\tilde{F}^{ab}F^{cd} + \frac{\alpha_4}{2}\epsilon_{abcd}R^{ab}F^{cd} \\ & \frac{\alpha_4}{4}\epsilon_{abcd}\tilde{F}^{ab}\tilde{F}^{cd} + \frac{\alpha_4}{2}\epsilon_{abcd}F^{ab}F^{cd} + \frac{2}{l}\alpha_2\bar{\Psi}\gamma_5\Psi + \frac{2}{l}\alpha_2\bar{\Xi}\gamma_5\Xi + \frac{4}{l}\alpha_4\bar{\Psi}\gamma_5\Xi\Big). \end{aligned} \tag{5.132}$$

Interestingly, the term proportional to α_4 contains the Einstein-Hilbert and the Rarita-Schwinger type Lagrangian in presence of a generalized supersymmetric cosmological term. Indeed, using the gamma matrix identities and the Bianchi identities

$(dF + [A, F] = 0)$, the term proportional to α_4 can be written explicitly as

$$
\begin{aligned}
\mathcal{S} = \alpha_4 \int \epsilon_{abcd} & \left(R^{ab}\mathcal{K}^{cd} + \frac{1}{2}\tilde{\mathcal{K}}^{ab}\tilde{\mathcal{K}}^{cd} + \frac{1}{2}\mathcal{K}^{ab}\mathcal{K}^{cd} \right) \\
& + \frac{1}{l^2}\left(\epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi + 4\bar{\xi}e^a\gamma_a\gamma_5 D\xi \right) \\
& + \frac{1}{l^2}\left(\epsilon_{abcd}R^{ab}\tilde{h}^c\tilde{h}^d + 4\bar{\psi}\tilde{h}^a\gamma_a\gamma_5 D\xi + 4\bar{\xi}\tilde{h}^a\gamma_a\gamma_5 D\psi \right) \\
& + \frac{1}{l^2}\epsilon_{abcd}\left(2\tilde{\mathcal{K}}^{ab}e^c\tilde{h}^d + \mathcal{K}^{ab}e^c e^d + \mathcal{K}^{ab}\tilde{h}^c\tilde{h}^d + \frac{1}{l^2}e^a e^b e^c e^d \right. \\
& + \frac{6}{l^2}e^a e^b\tilde{h}^c\tilde{h}^d + \frac{1}{l^2}\tilde{h}^a\tilde{h}^b\tilde{h}^c\tilde{h}^d + \frac{2}{l}\bar{\psi}\gamma^{ab}\psi e^c\tilde{h}^d + \frac{2}{l}\bar{\psi}\gamma^{ab}\xi e^c e^d + \frac{2}{l}\bar{\psi}\gamma^{ab}\xi\tilde{h}^c\tilde{h}^d \\
& + \frac{2}{l}\bar{\xi}\gamma^{ab}\xi e^c\tilde{h}^d + k^{ab}e^c\left\{ \bar{\psi}\gamma^d\psi + \bar{\xi}\gamma^d\xi \right\} + 2\tilde{k}^{ab}e^c\bar{\psi}\gamma^d\xi + 2k^{ab}\tilde{h}^a\bar{\psi}\gamma^d\xi \\
& \left. + \tilde{k}^{ab}\tilde{h}^c\left\{ \bar{\psi}\gamma^d\psi + \bar{\xi}\gamma^d\xi \right\} \right) + d\left(\frac{8}{l}\bar{\xi}\gamma_5\nabla\psi \right),
\end{aligned}
\tag{5.133}
$$

where we have defined

$$
\begin{aligned}
\tilde{\mathcal{K}}^{ab} &= D\tilde{k}^{cb} + k^a_{\ c}\tilde{k}^{cb} + k^b_{\ c}\tilde{k}^{ac}, \\
\mathcal{K}^{ab} &= Dk^{ca} + \tilde{k}^a_{\ c}\tilde{k}^{cb} + k^a_{\ c}k^{cb}.
\end{aligned}
$$

A notorious difference with the previous supergravity action (see eq. (5.98)) is the presence of the bosonic field $\tilde{h}^a$ related to the generator $\tilde{Z}_a$. Interestingly, set $\tilde{Z}_a$ equals to zero does not violate the Jacobi identities (JI). Indeed, the JI are satisfied due to the gamma matrix identity $(C\gamma^a)_{(\alpha\beta}(C\gamma_a)_{\gamma\delta)} = 0$ (cyclic permutations of α, β, γ). Then, setting h^a equals to zero and omitting boundary contributions, the supergravity action proportional to α_4 (5.133) can be written as

$$
\begin{aligned}
\mathcal{S} = \alpha_4 \int & \frac{1}{l^2}\left(\epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5\nabla\psi + 4\bar{\xi}e^a\gamma_a\gamma_5\nabla\xi \right) \\
& + \frac{1}{l^2}\epsilon_{abcd}\left(\mathcal{K}^{ab}e^c e^d + \frac{1}{l^2}e^a e^b e^c e^d + \frac{2}{l}\bar{\psi}\gamma^{ab}\xi e^c e^d \right),
\end{aligned}
\tag{5.134}
$$

with

$$
\begin{aligned}
\nabla\psi &= D\psi + \frac{1}{4}k_{ab}\gamma^{ab}\psi + \frac{1}{4}\tilde{k}_{ab}\gamma^{ab}\xi, \\
\nabla\xi &= D\xi + \frac{1}{4}k_{ab}\gamma^{ab}\xi + \frac{1}{4}\tilde{k}_{ab}\gamma^{ab}\psi.
\end{aligned}
$$

The supergravity action (5.134) obtained here corresponds to a four-dimensional geometric supergravity action in presence of a generalized cosmological term. Naturally, the procedure used here can be generalized using bigger semigroups leading to more complicated actions.

Interestingly, the four-dimensional pure supergravity action presented in the previous chapter can be recovered as an Inönü-Wigner contraction. Indeed, considering the rescaling

$$\begin{gathered}\omega_{ab} \rightarrow \omega_{ab}, \quad \tilde{k}_{ab} \rightarrow \mu^2 \tilde{k}_{ab}, \quad k_{ab} \rightarrow \mu^4 k_{ab}, \\ e_a \rightarrow \mu^2 e_a, \quad \psi \rightarrow \mu\psi \quad \text{and} \quad \xi \rightarrow \mu^3 \xi,\end{gathered}$$

the pure supergravity action is obtained dividing by μ^4 and taking the limit $\mu \rightarrow 0$,

$$\mathcal{S} = \alpha_4 \int \frac{1}{l^2} \left(\epsilon_{abcd} R^{ab} e^c e^d + 4\bar{\psi} e^a \gamma_a \gamma_5 D\psi \right). \tag{5.135}$$

This result is not a surprise since the minimal Maxwell superalgebra $s\mathcal{M}_4$ can be obtained as a Inönü-Wigner contraction of the generalized AdS-Lorentz superalgebra. In particular, pure supergravity can be viewed as the geometric formulation of a supergravity theory invariant under the minimal Maxwell superalgebra.

Chapter 6

Chern-Simons formulation of supergravity and Maxwell superalgebras

6.1 Introduction

The four-dimensional supergravity theories in the MacDowell-Mansouri geometric framework are not gauge theories for a given superalgebra. In this framework, as we have seen, the supersymmetry algebra closes only on shell. A way to close off shell the superalgebra is through the introduction of auxiliary fields. However, this procedure cannot be reproduce for all dimensions and $\mathcal{N}$ and cannot be related to a fiber bundle structure. An interesting formalism which allows to construct a gauge theory of supergravity in odd dimensions is the Chern-Simons approach.

In particular, the Chern-Simons action in three dimensions [14, 15] is given by

$$S_{CS}^{(2+1)} = k \int \left\langle A \left(dA + \frac{2}{3} A^2 \right) \right\rangle , \tag{6.1}$$

where A is the one-form gauge connection and the bracket $\langle \cdots \rangle$ stands for the non-vanishing components of an invariant tensor.

A good candidate to describe a three-dimensional CS supergravity theory in presence of a cosmological constant is the AdS supergroup. The most generalized susy extension of the three-dimensional AdS algebra corresponds to the direct product

$$\mathfrak{osp}\left(2|p\right)\otimes\mathfrak{osp}\left(2|q\right), \tag{6.2}$$

and allows to construct a (p,q)-type AdS-CS supergravity action [57]. The minimal three-dimensional AdS-CS supergravity action occurs for $p=1$ and $q=0$ ($\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)$) whose supergravity action with cosmological term is given by

$$\begin{aligned} S_{CS}^{(2+1)} = k\int_M \frac{\mu_0}{2}\left(\omega^a{}_b d\omega^b{}_a + \frac{2}{3}\omega^a{}_c\omega^c{}_b\omega^b{}_a + \frac{2}{l^2}e^aT_a + \frac{2}{l}\bar{\psi}\Psi\right) \\ + \frac{\mu_1}{l}\left(\epsilon_{abc}\left(R^{ab}e^c + \frac{1}{3l^2}e^ae^be^c\right) - \bar{\psi}\Psi\right) - d\left(\frac{\mu_1}{2l}\epsilon_{abc}\omega^{ab}e^c\right) \end{aligned} \tag{6.3}$$

where $T^a = de^a + \omega^a_b e^b$ is the torsion 2-form and ψ is a Majorana spinor[58].

There is a particular interest in supergravity theories to explore new superalgebras. In particular, the minimal Maxwell superalgebra $s\mathcal{M}$ describes the supersymmetries of generalized four-dimensional $\mathcal{N}=1$ superspace in the presence of a constant abelian spersymmetric field strength background [36]. Interestingly, the minimal Maxwell superalgebra has the particularity to have more than one spinor charge. The generalization to diverse minimal Maxwell superalgebras through the semigroup expansion procedure has been subsequently studied in ref. [31] and has been approached in detail in the chapter 4 of this thesis.

In this chapter, following ref. [59], we present the construction of the minimal CS supergravity action (without cosmological constant) using the minimal Maxwell superalgebra $s\mathcal{M}_3$. In the following section, according to ref. [60], we first consider an useful algebraic construction of a three-dimensional supersymmetric action from the non-standard Maxwell superalgebra.

6.2 $D=3$ CS exotic supersymmetric theory from non-standard Maxwell superlagebra

In this section we present the construction of a three-dimensional Chern-Simons supersymmetric action based on the non-standard Maxwell superalgebra. An essential ingredient in order to construct a Chern-Simons action is the invariant tensor. Besides, as was pointed out in ref. [16], the invariant tensor of an S-expanded (super)algebra can be obtained from the invariant tensor of the original algebra. Nevertheless, it seems that the non-standard Maxwell superalgebra cannot be obtained as an semigroup expansion of an known algebra and the components of an invariant tensor remain unknown. This difficulty has been elegantly solved in ref. [60], combining the semigroup expansion method with the Inönü-Wigner contraction.

As shown in ref. [46], the three-dimensional AdS-Lorentz superalgebra can be obtained as an S-expansion of the $\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)$ superalgebra. On the other hand, an Inönü-Wigner contraction of the AdS-Lorentz superalgebra leads to the non-standard Maxwell superalgebra [61, 35]. Then it seems natural to combine the semigroup expansion method with the Inönü-Wigner contraction in order to obtain the non-standard Maxwell superalgebra and its respective invariant tensor.

6.2.1 The non-standard Maxwell superalgebra

Let us consider first the AdS-Lorentz superalgebra as an S-expansion of the $\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)$ superalgebra using S_{Λ} as the relevant abelian semigroup. Before to apply the S-expansion method it is necessary to consider a decomposition of the $\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)$ superalgebra in subspaces V_p,

$$\mathfrak{g}=\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)=V_0\oplus V_1\oplus V_2 \tag{6.4}$$

where V_0 is generated by the Lorentz generator $\tilde{J}_{ab}$, V_1 corresponds to the fermionic subspace generated by a 3-component Majorana spinor charge $\tilde{Q}_{\alpha}$ and V_2 is generated by $\tilde{P}_a$. The $\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)$ generators satisfy the following (anti)commutation relations

$$\left[\tilde{J}_{ab},\tilde{J}_{cd}\right]=\eta_{bc}\tilde{J}_{ad}-\eta_{ac}\tilde{J}_{bd}-\eta_{bd}\tilde{J}_{ac}+\eta_{ad}\tilde{J}_{bc}, \tag{6.5}$$

$$\left[\tilde{J}_{ab},\tilde{P}_{c}\right]=\eta_{bc}\tilde{P}_{a}-\eta_{ac}\tilde{P}_{b}, \tag{6.6}$$

$$\left[\tilde{P}_{a},\tilde{P}_{b}\right]=\tilde{J}_{ab}, \tag{6.7}$$

$$\left[\tilde{J}_{ab},\tilde{Q}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{ab}\tilde{Q}\right)_{\alpha},\qquad\left[\tilde{P}_{a},\tilde{Q}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{a}\tilde{Q}\right)_{\alpha}, \tag{6.8}$$

$$\left\{\tilde{Q}_{\alpha},\tilde{Q}_{\beta}\right\}=-\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{J}_{ab}-2\left(\gamma^{a}C\right)_{\alpha\beta}\tilde{P}_{a}\right], \tag{6.9}$$

where γ_a are the Dirac matrices, C stands for the charge conjugation matrix and $a, b, c, d = 0, 1, 2$. Then, the subspace structure may be written as

$$[V_0, V_0] \subset V_0, \qquad [V_1, V_1] \subset V_0 \oplus V_2, \tag{6.10}$$

$$[V_0, V_1] \subset V_1, \qquad [V_1, V_2] \subset V_1, \tag{6.11}$$

$$[V_0, V_2] \subset V_2, \qquad [V_2, V_2] \subset V_0. \tag{6.12}$$

Let $S_\Lambda = \{\lambda_0, \lambda_1\}$ be the abelian semigroup with the following multiplication law,

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_1, & \text{if } \alpha = \beta = 1 \\ \lambda_0, & \text{all others} \end{cases} \tag{6.13}$$

Let $S_\wedge = S_0 \cup S_1 \cup S_2$ be a subset decomposition with

$$S_0 = \{\lambda_0, \lambda_1\}, \tag{6.14}$$

$$S_1 = \{\lambda_0\}, \tag{6.15}$$

$$S_2 = \{\lambda_0\}. \tag{6.16}$$

This decomposition is said to be "resonant " since it satisfies the same structure as the subspaces V_p [compare with eqs. (6.10) – (6.12)]. According to the theorem IV. 2 of ref. [16], the superalgebra

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{6.17}$$

is a resonant super subalgebra of $S_\Lambda \times \mathfrak{g}$, with

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_1\} \times \left\{\tilde{J}_{ab}\right\} = \left\{\lambda_0 \tilde{J}_{ab}, \lambda_1 \tilde{J}_{ab}\right\}, \tag{6.18}$$

$$W_1 = (S_1 \times V_1) = \{\lambda_0\} \times \left\{\tilde{Q}_\alpha\right\} = \left\{\lambda_0 \tilde{Q}_\alpha\right\}, \tag{6.19}$$

$$W_2 = (S_2 \times V_2) = \{\lambda_0\} \times \left\{\tilde{P}_a\right\} = \left\{\lambda_0 \tilde{P}_a\right\}. \tag{6.20}$$

Thus, the new superalgebra obtained is generated by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha\}$ whose generators are related to the $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ through

$$J_{ab} = \lambda_1 \tilde{J}_{ab}, \qquad P_a = \lambda_0 \tilde{P}_a, \tag{6.21}$$

$$Z_{ab} = \lambda_0 \tilde{J}_{ab}, \qquad Q_\alpha = \lambda_0 \tilde{Q}_\alpha. \tag{6.22}$$

The (anti)commutation relations read

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{6.23}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{6.24}$$

$$[Z_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{6.25}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{6.26}$$

$$[Z_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \tag{6.27}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}\left(\Gamma_{ab}Q\right)_\alpha, \qquad [P_a, Q_\alpha] = -\frac{1}{2}\left(\Gamma_a Q\right)_\alpha, \tag{6.28}$$

$$[Z_{ab}, Q_\alpha] = -\frac{1}{2}\left(\Gamma_{ab}Q\right)_\alpha, \tag{6.29}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left[\left(\Gamma^{ab}C\right)_{\alpha\beta} Z_{ab} - 2\left(\Gamma^a C\right)_{\alpha\beta} P_a\right], \tag{6.30}$$

The new superalgebra obtained after a resonant $S_\wedge$-expansion of the AdS superalgebra corresponds to the three-dimensional $AdS-$Lorentz superalgebra. As we have seen in the previous chapter, this superalgebra has an interesting application in four-dimensional supergravity allowing an alternative method to include the cosmological term [33].

Now, let us consider an Inönü-Wigner contraction to this superalgebra applying the following rescaling [60],

$$Z_{ab} \to \sigma^2 Z_{ab}, \quad P_a \to \sigma P_a \;\text{ and }\; Q_\alpha \to \sigma Q_\alpha. \tag{6.31}$$

Then the limit $\sigma \to \infty$ provides the three-dimensional non-standard Maxwell superalgebra,

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{6.32}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{6.33}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{6.34}$$

$$[Z_{ab}, Z_{cd}] = 0, \qquad [Z_{ab}, P_c] = 0, \tag{6.35}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}\left(\Gamma_{ab}Q\right)_\alpha, \tag{6.36}$$

$$[Z_{ab}, Q_\alpha] = 0, \qquad [P_a, Q_\alpha] = 0, \tag{6.37}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(\Gamma^{ab}C\right)_{\alpha\beta} Z_{ab}. \tag{6.38}$$

In particular, this superalgebra has the Maxwell algebra $\mathcal{M} = \{J_{ab}, P_a, Z_{ab}\}$ and the Lorentz type algebra $\mathcal{L}^{\mathcal{M}} = \{J_{ab}, Z_{ab}\}$ as subalgebras. Let us note that this superalgebra does not have a necessary relation to supergravity. Indeed, from eq. (6.38), one can note that the combination of two subsequent supersymmetry transformations does not amount to a space-time translation. The situation is completely different in the case of the minimal Maxwell superalgebra. However, before to approach the construction of CS supergravity action for the minimal case, we shall consider the explicit construction of a supersymmetric CS action for the usual Maxwell superalgebra.

6.2.2 $D = 3$ supersymmetric action

It seems that a Chern-Simons supersymmetric action for the non-standard Maxwell superalgebra can be constructed combining the S_Λ-expansion of the AdS superalgebra with the appropiate rescaling of the generators. However, as it was pointed out in ref. [60], the arbitrary constants of an invariant tensor have also to be rescaled in order to avoid a trivial Chern-Simons action.

Following Theorem VII.2 of ref. [16], the non-vanishing components of an invariant tensor for the AdS-Lorentz superalgebra are related to the $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ through,

$$\begin{aligned}
&\langle J_{ab}J_{cd}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_1 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \quad \langle Z_{ab}P_c\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{J}_{ab}\tilde{P}_c\right\rangle, \\
&\langle J_{ab}Z_{cd}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \quad \langle P_aP_b\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{P}_a\tilde{P}_b\right\rangle, \\
&\langle Z_{ab}Z_{cd}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \quad \langle Q_\alpha Q_\beta\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{Q}_\alpha\tilde{Q}_\beta\right\rangle, \\
&\langle J_{ab}P_c\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{J}_{ab}\tilde{P}_c\right\rangle,
\end{aligned} \tag{6.39}$$

where

$$\begin{aligned}
\left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle &= \mu_0 \left(\eta_{bc}\eta_{ad} - \eta_{ac}\eta_{bd}\right), \\
\left\langle \tilde{J}_{ab}\tilde{P}_c\right\rangle &= \mu_1 \epsilon_{abc}, \\
\left\langle \tilde{P}_a\tilde{P}_b\right\rangle &= \mu_0 \eta_{ab}, \\
\left\langle \tilde{Q}_\alpha\tilde{Q}_\beta\right\rangle &= \left(\mu_0 - \mu_1\right) C_{\alpha\beta}.
\end{aligned}$$

Then, defining

$$\beta_0 \equiv \tilde{\alpha}_0\mu_0, \qquad \alpha_0 \equiv \tilde{\alpha}_0\mu_1, \qquad \beta_1 \equiv \tilde{\alpha}_1\mu_0, \tag{6.40}$$

the components of an invariant tensor for the AdS-Lorentz superalgebra can be written as

$$\begin{aligned}
&\langle J_{ab}J_{cd}\rangle_{AdS-\mathcal{L}} = \beta_1\left(\eta_{bc}\eta_{ad}-\eta_{ac}\eta_{bd}\right), \quad \langle Z_{ab}P_c\rangle_{AdS-\mathcal{L}} = \alpha_0\epsilon_{abc}, \\
&\langle J_{ab}Z_{cd}\rangle_{AdS-\mathcal{L}} = \beta_0\left(\eta_{bc}\eta_{ad}-\eta_{ac}\eta_{bd}\right), \quad \langle P_aP_b\rangle_{AdS-\mathcal{L}} = \beta_0\eta_{ab}, \\
&\langle Z_{ab}Z_{cd}\rangle_{AdS-\mathcal{L}} = \beta_0\left(\eta_{bc}\eta_{ad}-\eta_{ac}\eta_{bd}\right), \quad \langle Q_\alpha Q_\beta\rangle_{AdS-\mathcal{L}} = \left(\beta_0-\alpha_0\right)C_{\alpha\beta}, \\
&\langle J_{ab}P_c\rangle_{AdS-\mathcal{L}} = \alpha_0\epsilon_{abc}.
\end{aligned} \tag{6.41}$$

The next step consists in considering a rescaling which preserves the structure of the curvatures in the supergravity action. There is one rescaling with this particularity and it is given by

$$\beta_0 \to \sigma^2\beta_0, \qquad \alpha_0 \to \sigma\alpha_0, \qquad \beta_1 \to \beta_1. \tag{6.42}$$

Thus, the components of an invariant tensor for the non-standard Maxwell superalgebra is obtained considering the rescaling of not only the generators (7.109) but also of the constants (6.42) [60]. Indeed, the limit $\sigma \to \infty$ leads

$$\langle J_{ab}J_{cd}\rangle_{s\mathcal{M}} = \beta_1\left(\eta_{bc}\eta_{ad}-\eta_{ac}\eta_{bd}\right), \tag{6.43}$$

$$\langle J_{ab}Z_{cd}\rangle_{s\mathcal{M}} = \beta_0\left(\eta_{bc}\eta_{ad}-\eta_{ac}\eta_{bd}\right), \tag{6.44}$$

$$\langle J_{ab}P_c\rangle_{s\mathcal{M}} = \alpha_0\epsilon_{abc}, \tag{6.45}$$

$$\langle P_aP_b\rangle_{s\mathcal{M}} = \beta_0\eta_{ab}, \tag{6.46}$$

$$\langle Q_\alpha Q_\beta\rangle_{s\mathcal{M}} = \beta_0 C_{\alpha\beta}. \tag{6.47}$$

An additional ingredient in order to construct a Chern-Simons action is the one-form gauge connection which is given by

$$A = A^AT_A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{l}e^aP_a + \frac{1}{2}k^{ab}Z_{ab} + \frac{1}{\sqrt{l}}\psi^\alpha Q_\alpha. \tag{6.48}$$

The associated curvature two-form $F = dA + A\wedge A$ is given by

$$F = F^AT_A = \frac{1}{2}R^{ab}J_{ab} + \frac{1}{l}R^aP_a + \frac{1}{2}F^{ab}Z_{ab} + \frac{1}{\sqrt{l}}\Psi^\alpha Q_\alpha, \tag{6.49}$$

with

$$
\begin{aligned}
R^{ab} &= d\omega^{ab} + \omega^{a}_{\ c}\omega^{cb}, \\
R^{a} &= de^{a} + \omega^{a}_{\ b}e^{b} = T^{a}, \\
F^{ab} &= dk^{ab} + \omega^{a}_{\ c}k^{cb} - \omega^{b}_{\ c}k^{ca} + \frac{1}{l^{2}}e^{a}e^{b} + \frac{1}{l}\bar{\psi}\Gamma^{ab}\psi, \\
\Psi &= d\psi + \frac{1}{4}\omega_{ab}\Gamma^{ab}\psi = D\psi.
\end{aligned}
$$

The one-forms ω^{ab}, e^{a}, ψ are the spin connection, the vielbein and the gravitino field, respectively. While the k^{ab} fields describe bosonic "matter" fields. On the other hand, the Lorentz covariant exterior derivatives $D = d + \omega$ of the curvatures can be derived from the Bianchi identity $\nabla F = 0$ (where ∇ is the gauge covariant derivative given by $\nabla = d + [A, \cdot]$) leading to

$$DR^{ab} = 0, \tag{6.50}$$

$$DR^{a} = R^{a}_{\ b}e^{b}, \tag{6.51}$$

$$DF^{ab} = R^{a}_{\ c}k^{cb} - R^{b}_{\ c}k^{ca} + \frac{1}{l^{2}}R^{a}e^{b} - \frac{1}{l^{2}}e^{a}R^{b} - \frac{1}{l}\bar{\psi}\gamma^{ab}\Psi, \tag{6.52}$$

$$D\Psi = \frac{1}{4}R_{ab}\gamma^{ab}\psi. \tag{6.53}$$

Then, considering the one-form connection of the Maxwell superalgebra (6.48) and the non-vanishing components of the invariant tensor (6.43) – (6.47), the supergravity action (6.1) becomes

$$
\begin{aligned}
S_{CS}^{(2+1)} = k\int_{M}\Bigg[&\frac{1}{2}\beta_{1}\left(\omega^{a}_{\ b}d\omega^{b}_{\ a} + \frac{2}{3}\omega^{a}_{\ b}\omega^{b}_{\ c}\omega^{c}_{\ a}\right) + \frac{\alpha_{0}}{l}\left(\epsilon_{abc}R^{ab}e^{c}\right) \\
&+\beta_{0}\left(R^{a}_{\ b}k^{b}_{\ a} + \frac{1}{l^{2}}e^{a}T_{a} + \frac{1}{l}\bar{\psi}\Psi\right) - \frac{1}{2}d\left(\beta_{0}\omega^{a}_{\ b}k^{b}_{\ a} + \frac{\alpha_{0}}{l}\epsilon_{abc}\omega^{ab}e^{c}\right)\Bigg],
\end{aligned} \tag{6.54}
$$

where $\Psi = d\psi + \frac{1}{4}\omega_{ab}\Gamma^{ab}\psi$ is the covariant derivative of the spinor ψ. Here, the term proportional to β_{1} corresponds to the exotic Lagrangian [62, 63] and it is constructed exclusively out of the spin connection. The α_{0} piece corresponds to the Einstein-Hilbert term. Unlike the $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ supergravity action, the cosmological term is not present in the superMaxwell case. It is important to note that the action (6.54) is related not only to the Euler invariant through the term proportional to α_{0} but also to the Pontryagin invariant through the β_0 and β_1 terms. The full supersymmetric action is invariant (modulo boundary terms) under the local gauge transformations of the non-standard Maxwell superalgebra,

$$\delta\omega^{ab} = D\rho^{ab}, \tag{6.55}$$

$$\delta k^{ab} = D\kappa^{ab} + \frac{2}{l^2}e^a\rho^b - \frac{1}{l}\bar{\epsilon}\Gamma^{ab}\psi, \tag{6.56}$$

$$\delta e^a = D\rho^a + e^b\rho_b{}^a, \tag{6.57}$$

$$\delta\psi = d\epsilon + \frac{1}{4}\omega^{ab}\Gamma_{ab}\epsilon = D\epsilon. \tag{6.58}$$

Here, the gauge parameter ρ is given by

$$\rho = \frac{1}{2}\rho^{ab}J_{ab} + \frac{1}{2}\kappa^{ab}Z_{ab} + \frac{1}{l}\rho^a P_a + \frac{1}{\sqrt{l}}\epsilon^\alpha Q_\alpha.$$

Interestingly, the bosonic part of the action (6.54) corresponds to the Maxwell-Chern-Simons gravity action found in ref. [64, 65]. Nevertheless, the supersymmetric action (6.54) does not describe a supergravity action due principally to eq. (6.38). It is tempting to argue that the IW contraction used here can be seen as a low-energy limit $\sigma \to \infty$ where the EH term is decoupled from the rigid supersymmetric Lagrangian. Naturally, in the $\sigma = 1$ case, we obtain the Chern-Simons supergravity action for the $AdS-$Lorentz superalgebra presented in ref. [46].

6.3 $D = 3$ CS supergravity from the minimal Maxwell superalgebra

In the present section, following ref. [59], we present the construction of a three-dimensional Chern-Simons supergravity action using the minimal Maxwell superalgebra $s\mathcal{M}_3$.

Following the definitions of the semigroup expansion procedure [16] and the method used in ref. [59], it is possible to derive a minimal Maxwell superalgebra after extracting a 0_s-reduced resonant $S_E^{(4)}$-expansion of the $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ superalgebra. The new superalgebra is generated by $\left\{J_{ab}, P_a, \tilde{Z}_{ab}, Z_{ab}, \tilde{Z}_a, Q_\alpha, \Sigma_\alpha\right\}$ whose generators obey the following (anti)commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{6.59}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{6.60}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{6.61}$$

$$[P_a, Q_\alpha] = -\frac{1}{2}\left(\Gamma_a\Sigma\right)_\alpha, \tag{6.62}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}\left(\Gamma_{ab}Q\right)_\alpha, \tag{6.63}$$

$$[J_{ab}, \Sigma_\alpha] = -\frac{1}{2}\left(\Gamma_{ab}\Sigma\right)_\alpha, \tag{6.64}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left[\left(\Gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab} - 2\left(\Gamma^a C\right)_{\alpha\beta}P_a\right], \tag{6.65}$$

$$\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2}\left[\left(\Gamma^{ab}C\right)_{\alpha\beta}Z_{ab} - 2\left(\Gamma^a C\right)_{\alpha\beta}\tilde{Z}_a\right], \tag{6.66}$$

$$\left[J_{ab}, \tilde{Z}_{ab}\right] = \eta_{bc}\tilde{Z}_{ad} - \eta_{ac}\tilde{Z}_{bd} - \eta_{bd}\tilde{Z}_{ac} + \eta_{ad}\tilde{Z}_{bc}, \tag{6.67}$$

$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{6.68}$$

$$\left[J_{ab}, \tilde{Z}_c\right] = \eta_{bc}\tilde{Z}_a - \eta_{ac}\tilde{Z}_b, \qquad \left[\tilde{Z}_{ab}, P_c\right] = \eta_{bc}\tilde{Z}_a - \eta_{ac}\tilde{Z}_b, \tag{6.69}$$

$$\left[\tilde{Z}_{ab}, Q_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}\Sigma\right)_\alpha, \tag{6.70}$$

$$\text{others} = 0. \tag{6.71}$$

The Maxwell superalgebra $s\mathcal{M}_3$ has the particularity to have an additional spinor generator as we have seen in chapter 4. This superalgebra can be seen as the supersymmetric extension of a generalized Maxwell algebra introduced in ref. [31]. Interestingly, we recover the usual three-dimensional minimal Maxwell superalgebra setting $\tilde{Z}_{a,}\ \tilde{Z}_{ab} = 0$. Set $\tilde{Z}_{ab}$ and $\tilde{Z}_a$ equals to zero does not violate the Jacobi identities. Indeed, the JI are satisfied due to the gamma matrix identity $\left(C\gamma^a\right)_{(\alpha\beta}\left(C\gamma_a\right)_{\gamma\delta)} = 0$ (cyclic permutations of α, β, γ).

In order to write down an Chern-Simons supergravity action for a minimal Maxwell superalgebra we consider the one-gauge connection

$$A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{2}\tilde{k}^{ab}\tilde{Z}_{ab} + \frac{1}{2}k^{ab}Z_{ab} + \frac{1}{l}e^aP_a + \frac{1}{l}\tilde{h}^a\tilde{Z}_a + \frac{1}{\sqrt{l}}\psi^\alpha Q_\alpha + \frac{1}{\sqrt{l}}\xi^\alpha\Sigma_\alpha, \tag{6.72}$$

whose components are related to the $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ ones $\left(\tilde{\omega}^{ab}, \tilde{e}^{a}, \tilde{\psi}\right)$ through

$$\begin{aligned}
&\omega^{ab} = \lambda_0 \tilde{\omega}^{ab}, \quad && \tilde{k}^{ab} = \lambda_2 \tilde{\omega}^{ab} \quad && k^{ab} = \lambda_4 \tilde{\omega}^{ab}, \\
&e^{a} = \lambda_2 \tilde{e}^{a}, && \tilde{h}^{a} = \lambda_4 \tilde{e}^{a}, && \psi^{\alpha} = \lambda_1 \tilde{\psi}^{\alpha}, \\
&\xi^{\alpha} = \lambda_3 \tilde{\psi}^{\alpha}. && &&
\end{aligned}$$

The corresponding curvature two-form $F = dA + A \wedge A$ is given by

$$\begin{aligned}
F = F^{A} T_{A} = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} R^{a} P_{a} + \frac{1}{2} \tilde{F}^{ab} \tilde{Z}_{ab} + \frac{1}{2} F^{ab} Z_{ab} + \frac{1}{l} \tilde{H}^{a} \tilde{Z}_{a} \\
+ \frac{1}{\sqrt{l}} \Psi^{\alpha} Q_{\alpha} + \frac{1}{\sqrt{l}} \Xi^{\alpha} \Sigma_{\alpha},
\end{aligned} \tag{6.73}$$

with

$$\begin{aligned}
R^{ab} &= d\omega^{ab} + \omega^{a}_{\ c} \omega^{cb}, \\
R^{a} &= de^{a} + \omega^{a}_{\ b} e^{b} - \frac{1}{2} \bar{\psi} \Gamma^{a} \psi, \\
\tilde{H}^{a} &= d\tilde{h}^{a} + \omega^{a}_{\ b} \tilde{h}^{b} + \tilde{k}^{a}_{\ c} e^{c} - \bar{\xi} \Gamma^{a} \psi, \\
\tilde{F}^{ab} &= d\tilde{k}^{ab} + \omega^{a}_{\ c} \tilde{k}^{cb} - \omega^{b}_{\ c} \tilde{k}^{ca} + \frac{1}{2l} \bar{\psi} \Gamma^{ab} \psi, \\
F^{ab} &= dk^{ab} + \omega^{a}_{\ c} k^{cb} - \omega^{b}_{\ c} k^{ca} + \tilde{k}^{a}_{\ c} \tilde{k}^{cb} + \frac{1}{l^{2}} e^{a} e^{b} + \frac{1}{l} \bar{\xi} \Gamma^{ab} \psi, \\
\Psi &= d\psi + \frac{1}{4} \omega_{ab} \Gamma^{ab} \psi, \\
\Xi &= d\xi + \frac{1}{4} \omega_{ab} \Gamma^{ab} \xi + \frac{1}{4} \tilde{k}_{ab} \Gamma^{ab} \psi + \frac{1}{2l} e^{a} \Gamma_{a} \psi.
\end{aligned}$$

The one-forms ω^{ab}, e^{a}, ψ and ξ are the spin connection, the vielbein, the gravitino field and an additional Majorana fermionic field[1], respectively. While the k^{ab}, $\tilde{k}^{ab}$ and $\tilde{h}^{a}$ fields describe bosonic "matter" fields.

According to the Theorem VII.2 of ref. [16], it is possible to show that the components of an invariant tensor for the Maxwell superalgebra can be written in terms of an

[1] A Majorana spinor ψ satisfies the Majorana condition $\bar{\psi} = \psi C$, where C is the charge conjugation matrix.

$\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ invariant tensor, leading

$$\langle J_{ab}J_{cd}\rangle_{s\mathcal{M}^g} = \tilde{\alpha}_0 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle \tag{6.74}$$

$$\left\langle J_{ab}\tilde{Z}_{cd}\right\rangle_{s\mathcal{M}^g} = \tilde{\alpha}_2 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle \tag{6.75}$$

$$\left\langle \tilde{Z}_{ab}\tilde{Z}_{cd}\right\rangle_{s\mathcal{M}^g} = \langle J_{ab}Z_{cd}\rangle = \tilde{\alpha}_4 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle \tag{6.76}$$

$$\langle J_{ab}P_c\rangle_{s\mathcal{M}^g} = \tilde{\alpha}_1 \left\langle \tilde{J}_{ab}\tilde{P}_c\right\rangle \tag{6.77}$$

$$\left\langle \tilde{Z}_{ab}P_c\right\rangle_{s\mathcal{M}^g} = \left\langle J_{ab}\tilde{Z}_c\right\rangle = \tilde{\alpha}_3 \left\langle J_{ab}\tilde{P}_c\right\rangle \tag{6.78}$$

$$\langle P_aP_b\rangle_{s\mathcal{M}^g} = \tilde{\alpha}_4 \left\langle \tilde{P}_a\tilde{P}_b\right\rangle \tag{6.79}$$

$$\langle Q_\alpha Q_\beta\rangle_{s\mathcal{M}^g} = (\tilde{\alpha}_2 - \tilde{\alpha}_1) \left\langle \tilde{Q}_\alpha\tilde{Q}_\beta\right\rangle \tag{6.80}$$

$$\langle Q_\alpha \Sigma_\beta\rangle_{s\mathcal{M}^g} = (\tilde{\alpha}_4 - \tilde{\alpha}_3) \left\langle \tilde{Q}_\alpha\tilde{Q}_\beta\right\rangle \tag{6.81}$$

with

$$\begin{aligned}
\left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle &= \mu_0 \left(\eta_{bc}\eta_{ad} - \eta_{ac}\eta_{bd}\right), \\
\left\langle \tilde{J}_{ab}\tilde{P}_c\right\rangle &= \mu_1 \epsilon_{abc}, \\
\left\langle \tilde{P}_a\tilde{P}_b\right\rangle &= \mu_0 \eta_{ab}, \\
\left\langle \tilde{Q}_\alpha\tilde{Q}_\beta\right\rangle &= (\mu_0 - \mu_1)\, C_{\alpha\beta}.
\end{aligned}$$

Then, considering the following definitions

$$\begin{aligned}
&\alpha_0 \equiv \tilde{\alpha}_0\mu_0, \quad \alpha_1 \equiv \tilde{\alpha}_2\mu_1, \quad \alpha_2 \equiv \tilde{\alpha}_2\mu_0 \\
&\alpha_3 \equiv \tilde{\alpha}_4\mu_1, \quad \alpha_4 \equiv \tilde{\alpha}_4\mu_0,
\end{aligned}$$

the components of an invariant tensor for the minimal Maxwell superalgebra $s\mathcal{M}_3$ can be written as

$$\langle J_{ab}J_{cd}\rangle_{s\mathcal{M}_3} = \alpha_0 \left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right) \tag{6.82}$$

$$\left\langle J_{ab}\tilde{Z}_{cd}\right\rangle_{s\mathcal{M}_3} = \alpha_2 \left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right) \tag{6.83}$$

$$\left\langle \tilde{Z}_{ab}\tilde{Z}_{cd}\right\rangle_{s\mathcal{M}_3} = \langle J_{ab}Z_{cd}\rangle = \alpha_4 \left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right) \tag{6.84}$$

$$\langle J_{ab}P_c\rangle_{s\mathcal{M}_3} = \alpha_1\epsilon_{abc} \tag{6.85}$$

$$\left\langle \tilde{Z}_{ab}P_c\right\rangle_{s\mathcal{M}_3} = \left\langle J_{ab}\tilde{Z}_c\right\rangle = \alpha_3\epsilon_{abc} \tag{6.86}$$

$$\langle P_aP_b\rangle_{s\mathcal{M}_3} = \alpha_4\eta_{ab} \tag{6.87}$$

$$\langle Q_\alpha Q_\beta\rangle_{s\mathcal{M}_3} = (\alpha_2-\alpha_1)\,C_{\alpha\beta} \tag{6.88}$$

$$\langle Q_\alpha \Sigma_\beta\rangle_{s\mathcal{M}_3} = (\alpha_4-\alpha_3)\,C_{\alpha\beta} \tag{6.89}$$

Then, considering the connection one-form (6.72) and the non-vanishing components of the invariant tensor (6.82)−(6.89) in the general expression of a Chern-Simons action, we find

$$\begin{aligned} S_{CS}^{(2+1)} = k\int_M &\left[\frac{\alpha_0}{2}\left(\omega^a{}_b d\omega^b{}_a + \frac{2}{3}\omega^a{}_c\omega^c{}_b\omega^b{}_a\right) + \frac{\alpha_1}{l}\left(\epsilon_{abc}R^{ab}e^c - \bar{\psi}\Psi\right)\right. \\ &+ \alpha_2\left(R^a{}_b\tilde{k}^b{}_a + \frac{1}{l}\bar{\psi}\Psi\right) + \frac{\alpha_3}{l}\left(\epsilon_{abc}\left(R^{ab}\tilde{h}^c + D_\omega\tilde{k}^{ab}e^c\right) - \bar{\xi}\Psi - \bar{\psi}\Xi\right) \\ &+ \alpha_4\left(R^a{}_b k^b{}_a + \frac{1}{l^2}e^aT_a + \frac{1}{l}\bar{\xi}\Psi + \frac{1}{l}\bar{\psi}\Xi\right) \\ &\left. -d\left(\frac{\alpha_1}{2l}\epsilon_{abc}\omega^{ab}e^c + \frac{\alpha_3}{2l}\epsilon_{abc}\left(\tilde{k}^{ab}e^c + \omega^{ab}\tilde{h}^c\right) + \frac{\alpha_2}{2}\omega^a{}_b\tilde{k}^b{}_a + \frac{\alpha_4}{2}\omega^a{}_b k^b{}_a\right)\right]. \end{aligned} \tag{6.90}$$

The three-dimensional action (6.90) describes a supergravity theory without cosmological constant and can be seen as a supersymmetric extension of the results in refs [64, 65]. where new extra fields have been added in order to have well defined invariant tensors.

The supergravity action (6.90) is split into five independent terms proportional to α_0, α_1, α_2, α_3 and α_4, respectively. The term proportional to α_0 describes the so called exotic Lagrangian [62, 63]. The second term consists of the Einstein-Hilbert term plus the fermionic contribution describing a pure supergravity action without cosmological constant invariant under the Poincaré symmetries. The others terms contain the coupling of the new gauge fields to the original ones. Unlike the action for the usual Maxwell superalgebra, this action contains an additional spinor field ξ which appears in the α_3 and α_4 terms.

Let us note that the new bosonic fields $\left(k^{ab},\tilde{k}^{ab},\tilde{h}^a\right)$ appear also in the boundary term. Although the boundary terms do not contribute to the dynamics of the theory, they play an essential role in the study of the AdS/CFT correspondence [51, 52, 53, 54].

The presence of boundary terms in (super)gravity theories has been extensively studied in refs. [18, 20, 40, 66].

One can see that the minimal Maxwell superalgebra $s\mathcal{M}_3$ enlarges the previous action adding new terms to the action allowing to construct a Maxwell-Chern-Simons Supergravity action which is off-shell invariant under the local gauge transformations of the minimal Maxwell superalgebra,

$$\delta\omega^{ab} = D\rho^{ab}, \tag{6.91}$$

$$\delta\tilde{k}^{ab} = D\tilde{\kappa}^{ab} - \left(\tilde{k}^{a}_{\ c}\,\rho^{b}_{\ c} - \tilde{k}^{bc}\rho^{a}_{\ c}\right) - \frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi, \tag{6.92}$$

$$\begin{aligned}\delta k^{ab} &= D\kappa^{ab} - \left(k^{ac}\rho^{b}_{\ c} - k^{bc}\rho^{a}_{\ c}\right) - \left(\tilde{k}^{ac}\tilde{\kappa}^{b}_{\ c} - \tilde{k}^{bc}\tilde{\kappa}^{a}_{\ c}\right) \\ &\quad + \frac{2}{l^2}e^a\rho^b - \frac{1}{l}\bar{\varrho}\gamma^{ab}\psi - \frac{1}{l}\bar{\epsilon}\gamma^{ab}\xi,\end{aligned} \tag{6.93}$$

$$\delta e^a = D\rho^a + e^b\rho_b^{\ a} + \bar{\epsilon}\gamma^a\psi, \tag{6.94}$$

$$\delta\tilde{h}^a = D\tilde{\rho}^a + \tilde{h}^b\rho_b^{\ a} + \tilde{\kappa}^a_{\ c}e^c + \tilde{k}^a_{\ c}\,\rho^c + \bar{\varrho}\gamma^a\psi + \bar{\epsilon}\gamma^a\xi \tag{6.95}$$

$$\delta\psi = d\epsilon + \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon - \frac{1}{4}\rho^{ab}\gamma_{ab}\psi, \tag{6.96}$$

$$\begin{aligned}\delta\xi &= d\varrho + \frac{1}{4}\omega^{ab}\gamma_{ab}\varrho + \frac{1}{2l}e^a\gamma_a\epsilon - \frac{1}{2l}\rho^a\gamma_a\psi - \frac{1}{4}\rho^{ab}\gamma_{ab}\xi \\ &\quad + \frac{1}{4}\tilde{k}^{ab}\gamma_{ab}\epsilon - \frac{1}{4}\tilde{\kappa}^{ab}\gamma_{ab}\psi.\end{aligned} \tag{6.97}$$

where the gauge parameter is given by

$$\rho = \frac{1}{2}\rho^{ab}J_{ab} + \frac{1}{2}\tilde{\kappa}^{ab}\tilde{Z}_{ab} + \frac{1}{2}\kappa^{ab}Z_{ab} + \frac{1}{l}\rho^a P_a + \frac{1}{l}\tilde{\rho}^a\tilde{Z}_a + \frac{1}{\sqrt{l}}\epsilon^\alpha Q_\alpha + \frac{1}{\sqrt{l}}\varrho^\alpha\Sigma_\alpha.$$

This result provides one more example of the advantage of the semigroup expansion in the construction of new (super)algebras and new (super)gravity theories. In particular, the procedure used here can be useful in order to construct supergravity action in higher odd dimensions. It should be possible to recover standard odd-dimensional supergravity from the Maxwell supersymmetries. On the other hand, the same procedure could be applied to the construction of (p, q)-type Chern-Simons models and to the construction of matter-supergravity theories.

Chapter 7

Supersymmetric Born-infeld theory from $\mathcal{N} = 2$ Supergravity theory

7.1 Introduction

Recently, there has been growing interest in the study of the supersymmetric Born-Infeld theory and its multi-vector generalization. The Born-Infeld theory [67] describes a non-linear electrodynamics in four dimensional space-time. The supersymmetric extension of the BI theory was constructed in [68, 69]. In particular, the BI theory and its multi-vector generalization emerge from a low energy limit of partially broken $U(1)^n$ rigid $\mathcal{N} = 2$ supersymmetric theory [70]. As shown by I. Antoniadis, H. Partouche, T.R. Taylor (APT model) [71], the partially supersymmetry breaking requires the introduction of magnetic Fayet-Iliopoulos (FI) terms besides the electric ones. Interestingly, as shown in ref. [72], the partially broken $\mathcal{N} = 2$ rigid theory to $\mathcal{N} = 1$ in presence of one vector multiplet, corresponding to ref. [71], can be obtained as a rigid limit of a $\mathcal{N} = 2$ supergravity theory.

The purpose of this chapter, following ref. [73], is to generalize the procedure of ref. [72] to n vector multiplets. In particular, we are interested in relate the partially broken $\mathcal{N} = 2$ rigid theory of n abelian vector multiplets to supergravity. This would clarify the supergravity origin of the multi-vector generalization of the BI theory of ref. [74].

In the rigid limit of ref. [72], the partial breaking of supersymmetry required the use of a specific choice of symplectic frame. In particular, in this frame the prepotential of the special geometry does not exist. This restriction is forced within the framework of santard electric gaugins due to some no-go theorems [75, 76]. Nevertheless, as shown in ref. [77], the partial supersymmetry breaking can be achieved in any symplectic frame using an embedding tensor [78, 79, 80] with both electric and magnetic components. In particular, a frame in which the prepotential exists can be chosen.

A generalization to n vector multiplets of ref. [72], leads us to relate the FI terms of the rigid theory not only to the components of the embedding tensor, but also to constants entering the geometry of the scalar manifold. Interestingly, we shall show that we can reformulate the theory in a symplectic frame leading to a more clear interpretation of the FI terms. In particular, in this new frame, the manifest symplectic invariance is preserved after the rigid limit. Besides, the electric and magnetic FI terms are related only to the components of the embedding tensor. Indeed, denoting by $A^\Lambda_\mu = (A^0_\mu,\, A^I_\mu)$, the $n+1$ supergravity vector fields in the new frame, A^0_μ is identified with the graviphoton while A^I_μ corresponds to the vector fields of the resulting rigid theory.

In our approach, we shall consider the construction of a suitable dyonic gauging of an $\mathcal{N} = 2$ supergravity model coupled to n vector multiplets and to a single hypermultiplet which, in the rigid limit, leads us to a multi-vector generalization of the APT model and ref. [72].

Before to present the $\mathcal{N} = 1$ rigid supersymmetric theory as a rigid limit of a $\mathcal{N} = 2$ supergravity partially broken, we give the relevant identities related to the most general gauging of special Kähler and quaternionic Kähler isometries in a general $\mathcal{N} = 2$ supergravity model. Some of these identities are already known and have been proven only for electric gaugings [81, 82] or within superconformal calculus [83]. Here, following ref. [73], we present some compact proof for the generic dyonic gaugings, based on the symplectic-covariant description of the local special-geometry and on the general constraints on the embedding tensor. In particular, a detailed study of the potential Ward identity [84, 85] for generic dyonic gaugings, which is required by the supersymmetry invariance of the action, is presented.

7.2 Geometry of $\mathcal{N} = 2$ matter-coupled supergravity theory

In the present section we study the underlying geometry of the general $\mathcal{N} = 2$ supergravity theory. To this aim, we briefly review some useful formulae for special and quaternionic geometry following refs. [81, 82, 86]. Our purpose is to clarify the general structure of the four-dimensional $\mathcal{N} = 2$ supergravity coupled to n vector multiplets gauging some Lie group G and n_H hypermultiplets. In particular, the scalar sector of the vector multiplets is described by a special Kähler manifold $\mathcal{M}_{SK}$. On the other hand, the scalar sector of the hypermultiplets is described by a quaternionic Kähler manifold $\mathcal{M}_{QK}$. Then, the more general $\mathcal{N} = 2$ supergravity theory coupled to matter contains $2n + 4n_H$ scalar fields interacting through a σ-model based on the following scalar manifold:

$$\mathcal{M}_{scalar} = \mathcal{M}_{SK} \times \mathcal{M}_{QK}. \tag{7.1}$$

7.2.1 Special Kähler geometry

A special Kähler manifold $\mathcal{M}_{SK}$ is a Hodge-Kähler manifold endowed with a flat, holomorphic, symplectic bundle satisfying certain properties. Interestingly, there are two kinds of special Kähler geometry. The local one describes a $\mathcal{N} = 2$ Supergravity coupled to vector multiplets. While the rigid one describes the scalar field sector of vector multiplets in rigid $\mathcal{N} = 2$ Yang-Mills theories. In particular, a special Kähler manifold has a complex structure and a hermitian metric

$$ds^2 = g_{i\bar{\jmath}}\left(z, \bar{z}\right) dz^i \otimes d\bar{z}^{\bar{\jmath}}, \tag{7.2}$$

such that the 2-form

$$K = i g_{i\bar{\jmath}}\left(z, \bar{z}\right) dz^i \wedge d\bar{z}^{\bar{\jmath}},$$

is closed $dK = 0$. The Kähler potential $\mathcal{K}\left(z, \bar{z}\right)$ can be defined such that

$$g_{i\bar{\jmath}} = \partial_i \partial_j \mathcal{K}, \tag{7.3}$$

$$K = d\mathcal{Q}, \qquad \mathcal{Q} = -\frac{i}{2}\left(\partial_i \mathcal{K} dz^i - \partial_i \mathcal{K} d\bar{z}^{\bar{\imath}}\right). \tag{7.4}$$

In particular, under a Kähler transformation $\mathcal{K} \to \mathcal{K} + f(z) + f(\bar{z})$, the one-form $\mathcal{Q}$ transforms as an $U(1)$ connection $\mathcal{Q} \to \mathcal{Q} + d(\text{Im } f)$.

Let now $\Phi(z, \bar{z})$ be a field of weight p. Then, its $U(1)$ covariant derivative is given by

$$D\Phi = (d + ipQ)\,\Phi$$

or, in components

$$D_i\Phi = \left(\partial_i + \frac{1}{2}p\partial_i\mathcal{K}\right)\Phi, \qquad D_{\bar{\imath}}\Phi = \left(\partial_{\bar{\imath}} - \frac{1}{2}p\partial_{\bar{\imath}}\mathcal{K}\right)\Phi. \tag{7.5}$$

A covariantly holomorphic field of weight p is defined by the equation

$$D_{\bar{\imath}}\Phi = 0. \tag{7.6}$$

On the other hand, setting

$$\tilde{\Phi} = e^{-pG/2}\Phi, \tag{7.7}$$

we have

$$D_i\tilde{\Phi} = \left(\partial_i + \frac{1}{2}p\partial_i\mathcal{K}\right)\Phi, \qquad D_{\bar{\imath}}\tilde{\Phi} = \partial_{\bar{\imath}}\Phi. \tag{7.8}$$

Then, $\tilde{\Phi}$ is a holomorphic section with respect to the holomorphic connection $\partial_i\mathcal{K}$.

A more intrinsic and useful definition of a special Kähler manifold can be given introducing a (n)-dimensional holomorphic tensor whose holomorphic section are denoted by

$$\Omega(z) = \Omega^M(z) = \begin{pmatrix} X^\Lambda(z) \\ F_\Sigma(z) \end{pmatrix} \tag{7.9}$$

with $\Lambda, \Sigma = 0, \ldots, n$

We say that a Hodge-Kähler manifold $\mathcal{M}$ is special Kähler of the local type if, for some section Ω, the Kähler two-form is given by

$$K = \frac{i}{2\pi}\partial\bar{\partial}\log\left(i\left\langle\Omega|\bar{\Omega}\right\rangle\right), \tag{7.10}$$

where $\langle\ |\ \rangle$ denotes a symplectic inner product given by

$$i\left\langle\Omega|\bar{\Omega}\right\rangle \equiv -i\Omega^T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\bar{\Omega}. \tag{7.11}$$

This definition implies that it is possible to relate the Kähler potential $\mathcal{K}$ with the holomorphic section Ω,

$$\mathcal{K} = -\log\left(i\left\langle\Omega|\bar{\Omega}\right\rangle\right) = -\log\left[i\left(\bar{X}^{\Lambda}F_{\Lambda} - \bar{F}_{\Sigma}X^{\Sigma}\right)\right]. \tag{7.12}$$

Introducing now a covariantly holomorphic section V such that

$$V(z,\bar{z}) = (V^{M}(z,\bar{z})) = \begin{pmatrix} L^{\Lambda} \\ M_{\Sigma} \end{pmatrix} = e^{\mathcal{K}/2}\Omega = e^{\mathcal{K}/2}\begin{pmatrix} X^{\Lambda} \\ F_{\Sigma} \end{pmatrix}, \tag{7.13}$$

which satisfies the condition

$$1 = i\left\langle V|\bar{V}\right\rangle = i\left(\bar{L}^{\Lambda}M_{\Lambda} - \bar{M}_{\Sigma}L^{\Sigma}\right). \tag{7.14}$$

Since V is related to a holomorphic section it follows

$$D_{\bar{\imath}}V = \left(\partial_{\bar{\imath}} - \frac{1}{2}\partial_{\bar{\imath}}\mathcal{K}\right)V = 0. \tag{7.15}$$

On the other hand, defining the $U(1)$-covariant derivatives on V,

$$U_{i} = D_{i}V = \left(\partial_{i} + \frac{1}{2}\partial_{i}\mathcal{K}\right)V \equiv \begin{pmatrix} f_{i}^{\Lambda} \\ h_{\Sigma|i} \end{pmatrix}, \tag{7.16}$$

and introducing a symmetric three-tensor C_{ijk}, it follows that

$$D_{i}U_{j} = \partial_{i}U_{j} + \frac{\partial_{i}\mathcal{K}}{2}U_{j} - \Gamma_{ij}^{k}U_{k} = iC_{ijk}g^{k\bar{l}}\bar{U}_{\bar{l}}. \tag{7.17}$$

Then, the special geometry is defined by the following set of differential equations:

$$D_{i}V = U_{i}, \tag{7.18}$$
$$D_{i}U_{j} = iC_{ijk}g^{k\bar{l}}\bar{U}_{\bar{l}}, \tag{7.19}$$
$$D_{\bar{\imath}}U_{j} = g_{\bar{\imath}j}V, \tag{7.20}$$
$$D_{\bar{\imath}}V = 0. \tag{7.21}$$

Let us construct, using V and its covariant derivatives, the following matrix

$$\mathbb{L}(z,\bar{z})^{M}{}_{\underline{N}} \equiv (V, \mathsf{e}_{\bar{I}}{}^{\bar{\imath}}\overline{U}_{\bar{\imath}}^{M}, \overline{V}^{M}, \mathsf{e}_{I}{}^{i}U_{i}^{M}),$$

where $\mathsf{e}_I{}^i$ are the inverse vielbein matrices $g_{i\bar{\jmath}} = \sum_{I=\bar{I}=1}^{n} \mathsf{e}_i{}^I \bar{\mathsf{e}}_{\bar{\jmath}}{}^{\bar{I}}$, and $\underline{N}$ is a holonomy group index. Then, eqs. (7.18) – (7.21) imply

$$\mathbb{L}^\dagger \mathbb{C} \mathbb{L} = \varpi \,, \tag{7.22}$$

with

$$\varpi \equiv -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{7.23}$$

Interestingly, a symplectic matrix can be define in terms of $\mathbb{L}$,

$$\begin{aligned} \mathcal{M}(z,\bar{z}) &= (\mathcal{M}_{MN}) \equiv \mathbb{C}\mathbb{L}\mathbb{L}^\dagger\mathbb{C} = \mathcal{M}(z,\bar{z})^T \,, \\ \mathcal{M}\mathbb{C}\mathcal{M} &= \mathbb{C} \,. \end{aligned} \tag{7.24}$$

In particular, $\mathcal{M}$ encodes all the information about the coupling of the vector fields to the scalars. From the above properties of V and U_i we find the following general symplectic covariant relation

$$U^{MN} \equiv g^{i\bar{\jmath}}\, U_i^M U_{\bar{\jmath}}^N = -\frac{1}{2}\mathcal{M}^{MN} - \frac{i}{2}\,\mathbb{C}^{MN} - \overline{V}^M V^N \,, \tag{7.25}$$

where$\mathcal{M}^{MN}$ are the components of

$$M^{-1} = -\mathbb{L}\mathbb{L}^\dagger . \tag{7.26}$$

Let us now consider the Killing vectors $k_a = k_a^i\,(z)\,\partial_i + k_a^{\bar{\imath}}\,(\bar{z})\,\partial_{\bar{\imath}}$ defining an infinitesimal isometry and satisfying

$$[k_a, k_b] = -f_{ab}{}^c k_c . \tag{7.27}$$

The invariance of the Kähler 2-form K implies

$$\ell_a K = d(\iota_a K) = 0 \;\Rightarrow\; \iota_a K = -d\mathcal{P}_a \,, \tag{7.28}$$

where $\ell_a = \ell_{k_a}$ and $\iota_a \equiv \iota_{k_a}$ denote, respectively, the Lie derivative and the contraction along k_a. Then, we can define the momentum map $\mathcal{P}_a$ (the details can be found in Appendix C) such that

$$\iota_a K = -d\mathcal{P}_a \,. \tag{7.29}$$

On the other hand, we say that a Hodge-Kähler manifold $\mathcal{M}$ is special Kähler of the rigid type if for some section Ω, the Kähler two form is given by

$$K = -\frac{i}{2\pi}\partial\bar{\partial}\left(i\left\langle\Omega|\bar{\Omega}\right\rangle\right), \tag{7.30}$$

where the holomorphic section Ω have the following structure

$$\Omega = \begin{pmatrix} Y^I \\ F_J \end{pmatrix} \tag{7.31}$$

with $I, J = 1, \ldots, n$. As in the local case, the Kähler potential $\mathcal{K}$ can be related to the holomorphic section Ω,

$$\mathcal{K} = \left(i\left\langle\Omega|\bar{\Omega}\right\rangle\right) = \left[i\left(\bar{Y}^I F_I - \bar{F}_J Y^J\right)\right] \tag{7.32}$$

The rigid special geometry is then defined by the following set of differential equations:

$$\begin{aligned} \partial_{\bar{i}}\Omega &= 0, \\ U_i &= \partial_i\Omega, \\ \nabla_i U_j &= iC_{ijk}g^{k\bar{l}}\bar{U}_{\bar{l}}, \end{aligned}$$

where ∇_i is the covariant derivative with respect to the Levi-Civita connection.

7.2.2 Hypergeometry

In a $\mathcal{N} = 2$ four-dimensional supergravity theory coupled to hypermultiplets, there are 4 real scalar fields for each hypermultiplet which can be seen locally as the four components of a quaternion. As in the special geometry, there are two kinds of hypergeometry. The local one is described by a Quaternionic geometry, meanwhile the rigid one is described by a HyperKähler geometry. Both manifolds correspond to a $4n_H$-dimensional real manifold endowed with a metric h

$$ds^2 = h_{uv}(q)\, dq^u \otimes dq^v \tag{7.33}$$

with $u, v = 1, \ldots, 4n_H$; and three complex structures satisfying the quaternionic algebra

$$J^x J^y = -\delta^{xy} + \epsilon^{xyz} J^z. \tag{7.34}$$

Let us consider a triplet of K^x 2-form named the HyperKähler form,

$$K^x = K^x_{uv} dq^u \wedge dq^v \quad , \quad K^x_{uv} = h_{uw} \left(J^x\right)^w_v . \tag{7.35}$$

The HyperKähler 2-form is covariantly closed

$$\nabla K^x \equiv dK^x + \epsilon^{xyz} \omega^y \wedge K^z = 0 \tag{7.36}$$

with respect to an $SU(2)$ connection ω^x.

We say that a quaternionic manifold is a $4n_H$-dimensional manifold such that the curvature associated to the $SU(2)$ connection is proportional to the HyperKähler 2-form

$$\Omega^x \equiv d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z = \lambda K^x. \tag{7.37}$$

While a HyperKähler manifold is a $4n_H$-dimensional manifold such that

$$\Omega^x = 0. \tag{7.38}$$

In particular, the quaternionic Kähler manifold $\mathcal{M}_{QK}$ has a holonomy group $Hol\left(\mathcal{M}_{QK}\left(n_H\right)\right) = SU(2) \otimes \mathcal{H}$ where $\mathcal{H} \subset Sp\left(2n_H, \mathbb{R}\right)$ is some subgroup of the symplectic group in $D = 2n_H$. Then, introducing flat indices $\{A, B, C = 1, 2\}$, $\{\alpha, \beta, \gamma = 1, \ldots, 2n_H\}$ that run in the fundamental representation of $SU(2)$ and $Sp\left(2n_H, \mathbb{R}\right)$, respectively, we can find a vielbein 1-form

$$\mathcal{U}^{A\alpha} = \mathcal{U}^{A\alpha}_u (q)\, dq^u, \tag{7.39}$$

such that

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta} \epsilon_{AB}, \tag{7.40}$$

where $\epsilon_{AB} = -\epsilon_{BA}$ and $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ are the $Sp(2) \sim SU(2)$ and the flat $Sp\left(2n_H, \mathbb{R}\right)$ invariant matrix, respectively.

The vielbein 1-form $\mathcal{U}^{A\alpha}$ is covariantly closed with respect to the $SU(2)$-connection ω^z and to some $Sp\left(2n_H\right)$ Lie algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$

$$\nabla \mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} + \frac{1}{2} i \omega^x \left(\epsilon \sigma_x \epsilon^{-1}\right)^A{}_B \wedge \mathcal{U}^{B\alpha} + \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma} \mathbb{C}_{\beta\gamma} = 0. \tag{7.41}$$

Additionally, it satisfies the following relations

$$\begin{aligned} \mathcal{U}_{A\alpha} &\equiv \left(\mathcal{U}^{A\alpha}\right)^* = \epsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}^{B\beta}, & (7.42) \\ \mathcal{U}_{A\alpha\, u} \mathcal{U}^{B\alpha}_v &= \frac{1}{2} h_{uv}\, \delta^B_A + \frac{i}{2} K^x_{uv} \left(\sigma^x\right)_A{}^B . & (7.43) \end{aligned}$$

7.3 General $\mathcal{N} = 2$ gauging

In the present section, following ref. [73], we briefly review the most general gauging of $\mathcal{N} = 2$ supergravity involving both electric and magnetic charges. In particular, the discussion presented here generalize the identities given in ref. [81] to electric-magnetic gaugings. The results discussed here will be relevant to the very specific electric-magnetic abelian gaugings in which the rigid limit of a spontaneously broken $\mathcal{N} = 2$ supergravity is approached.

Let us first consider an $\mathcal{N} = 2$ supergravity model coupled to n vector multiplets and n_H hypermultiplets. The theory consist of n complex scalars z^i and $4n_H$ hyperscalars q^u parameterizing a special Kähler manifold $\mathcal{M}_{SK}$ and a quaternionic Kähler manifold $\mathcal{M}_{QK}$, respectively, such that

$$\mathcal{M}_{scalar} = \mathcal{M}_{SK} \times \mathcal{M}_{QK} \tag{7.44}$$

Let us now consider the general gauging of a gauge group G in the isometry of the scalar manifold $\mathcal{M}_{scalar}$. According to ref. [83], it is possible to write the gauge generators as components of an electric-magnetic vector $X_M = \left(X_\Lambda, X^\Lambda\right)$. Let t_a, t_n be the generators of the isometry groups of $\mathcal{M}_{SK}$ and $\mathcal{M}_{QK}$, respectively, and let $\theta_M{}^a$ be the embedding tensor such that

$$X_M = \theta_M{}^a\, t_a + \theta_M{}^n\, t_n\,. \tag{7.45}$$

The symplectic matrices $X_{MN}{}^P = \theta_M{}^a t_{aN}{}^P$ describe the symplectic electric-magnetic duality action of X_M. In particular, the following set of linear and quadratic constraints on the embedding tensor

$$\begin{aligned} X_{(MNP)} \equiv X_{(MN}{}^Q \mathbb{C}_{Q|P)} &= 0\,, & (7.46)\\ \Theta_M{}^a \Theta_N{}^b f_{ab}{}^c + X_{MN}{}^P\, \Theta_P{}^c &= 0\,, & (7.47)\\ \Theta_M{}^m \Theta_N{}^n f_{mn}{}^p + X_{MN}{}^P\, \Theta_P{}^p &= 0\,, & (7.48)\\ \Theta_M{}^a \mathbb{C}^{MN} \Theta_N{}^b = \Theta_M{}^a \mathbb{C}^{MN} \Theta_N{}^n &= \Theta_M{}^m \mathbb{C}^{MN} \Theta_N{}^n = 0\,. & (7.49) \end{aligned}$$

assures the consistency of the gauging. One can see that the conditions (7.47) and (7.48) are equivalent to

$$[X_M,\, X_N] = -X_{MN}{}^P\, X_P\,. \tag{7.50}$$

It is possible to define gauge Killing vectors and momentum maps as

$$k_M \equiv \Theta_M{}^a\, k_a \ , \ \ \mathcal{P}_M \equiv \Theta_M{}^a\, \mathcal{P}_a \ , \ \ \mathcal{P}^x_M \equiv \Theta_M{}^m\, \mathcal{P}^x_m \, . \tag{7.51}$$

Additionally, the equivariance conditions,

$$i g_{i\bar{\jmath}}\, k^i_{[M}\, k^{\bar{\jmath}}_{N]} \ = \ \frac{1}{2}\, X_{MN}{}^P\, \mathcal{P}_P \, , \tag{7.52}$$

$$2\, K^x_{uv}\, k^u_M\, k^v_N + \epsilon^{xyz}\, \mathcal{P}^y_M\, \mathcal{P}^z_N \ = \ X_{MN}{}^P\, \mathcal{P}^x_P \, , \tag{7.53}$$

can be found from the quadratic constraints and the following eqs. (see Appendix C),

$$i g_{i\bar{\jmath}}\, k^i_{[a}\, k^{\bar{\jmath}}_{b]} \ = \ -\frac{1}{2}\, f_{ab}{}^c\, (\mathcal{P}_c - C_c) \, , \tag{7.54}$$

$$2\, K_{uv}\, k^u_n\, k^v_m - \lambda\, \epsilon^{xyz}\, \mathcal{P}^y_n\, \mathcal{P}^z_m \ = \ -f_{mn}{}^p\, \mathcal{P}^x_p \, . \tag{7.55}$$

Interestingly, using the linear constraint (7.46) on the embedding tensor it is possible to prove the following identities

$$\mathcal{P}_M \Omega^M = 0 \ , \ \ k^i_M\, \Omega^M = 0 \, . \tag{7.56}$$

Indeed, using some relevant relations of the Appendix C,

$$\mathcal{P}_a = -V^N\, t_{aNM} \overline{V}^M = -\overline{V}^N\, t_{aNM}\, V^P \, , \tag{7.57}$$

we find for the gauge-momentum maps the following relation,

$$\mathcal{P}_M = -e^{\mathcal{K}}\, X_{MNP} \overline{\Omega}^N \Omega^P \, .$$

Then, contracting both sides with Ω^M and using the linear constraint (7.46), we have

$$\Omega^M \mathcal{P}_M = -e^{\mathcal{K}}\, \Omega^M X_{MNP} \overline{\Omega}^N \Omega^P = \frac{e^{\mathcal{K}}}{2}\, \overline{\Omega}^N X_{NMP} \Omega^M \Omega^P \, , \tag{7.58}$$

where we have used the symplectic property of the matrices $X_{MN}{}^P$ given by

$$2 X_{(MP)N} = -X_{NMP} \, , \tag{7.59}$$

with $X_{MNP} \equiv X_{MN}{}^Q \mathbb{C}_{QP}$. Then, using the general property,

$$t_{a\,MN} \Omega^M \Omega^N = 0 \ , \ \ \forall t_a \, ,$$

we finally find the first identity,

$$\Omega^M \mathcal{P}_M = 0.$$

On the other hand, the second identity of (7.56) can be proven using the first identity as

$$\Omega^M\, k^i_M = i\, g^{i\bar{\jmath}}\, \Omega^M\, \partial_{\bar{\jmath}} \mathcal{P}_M = i\, g^{i\bar{\jmath}}\, \partial_{\bar{\jmath}} (\Omega^M\, \mathcal{P}_M) = 0\,. \tag{7.60}$$

Interestingly, from (7.56), the following relations can be deduced

$$D_i(V^M \mathcal{P}_M) = 0 \;\Rightarrow\; U^M_i \mathcal{P}_M + V^M \partial_i \mathcal{P} = 0 \;\Rightarrow\; U^M_i \mathcal{P}_M + i\, g_{i\bar{\jmath}}\, k^{\bar{\jmath}}_M V^M = 0\,. \tag{7.61}$$

Then, contracting the following equation

$$k^i_a\, U^M_i = -t_{aN}{}^M\, V^N + i\, \mathcal{P}_a\, V^M\,,$$

with the embedding tensor, we find

$$k^i_M\, U^P_i = -X_{MN}{}^P\, V^N + i\, \mathcal{P}_M\, V^P\,. \tag{7.62}$$

Besides, using the first identity of (7.56) and contracting both sides with $\bar{V}^M$, we find the relation

$$\overline{V}^M k^i_M\, U^P_i = -X_{MN}{}^P\, \overline{V}^M V^N\,. \tag{7.63}$$

Eventually, using the quadratic constraints (7.50) and contracting both sides with Θ_P we have

$$\overline{V}^M k^i_M\, U^P_i \Theta_P = -X_{MN}{}^P\, \overline{V}^M V^N \Theta_P = X_{NM}{}^P\, \overline{V}^M V^N \Theta_P = -V^M k^{\bar{\imath}}_M\, \overline{U}^P_{\bar{\imath}}\, \Theta_P\,. \tag{7.64}$$

7.3.1 The general Ward identity

The supersymmetry variation terms of the gauged Lagrangian, which are quadratic in the embedding tensor, are canceled by the supersymmetry Ward identity [84, 85]. The Ward identity expresses a relation between the scalar potential $\mathcal{V}\,(z, z, q)$ and the fermionic shift matrices in the following way

$$g_{i\bar{\jmath}}\, W^{i\,AC} \overline{W}^{\bar{\jmath}}_{BC} + 2\, N_\alpha{}^A\, N^\alpha{}_B - 12\, S^{AC} S_{BC} = \delta^B_A\, \mathcal{V}(z, \bar{z}, q)\,, \tag{7.65}$$

where $W^{i\,AC}$, N^{α}_{B}, S_{AB} are the supersymmetry shift-matrices of the chiral gaugini λ^i, hyperini ζ^α and gravitini ψ_A, respectively. In particular, we use the following convention

$$v_A = \epsilon_{AB}\, v^B\,,\;\; v^A = \epsilon^{BA}\, v_B\,,\;\; v_\alpha = \mathbb{C}_{\alpha\beta}\, v^\beta\,,\;\; v^\alpha = \mathbb{C}^{\beta\alpha}\, v_\beta\,. \tag{7.66}$$

Let us consider the generic dyonic gauging of $\mathcal{N}=2$ supergravity and let us prove the Ward identity for this particular case. Here, the fermionic shifts is generalized to the following symplectically-invariant expressions

$$\begin{aligned} W^{i\,AB} &= \epsilon^{AB}\, k^i_M\, \overline{V}^M - i\,(\sigma^x)_C{}^B \epsilon_{CA} \mathcal{P}^x_M\, g^{i\bar{\jmath}} \overline{U}^M_{\bar{\jmath}}\,, & (7.67)\\ S_{AB} &= \frac{i}{2}\,(\sigma^x)_A{}^C \epsilon_{BC}\, \mathcal{P}^x_M\, V^M\,, & (7.68)\\ N_\alpha{}^A &= 2\mathcal{U}^A_{u\,\alpha}\, k^u_M\, \overline{V}^M\,,\quad N^\alpha{}_A \equiv (N_\alpha{}^A)^* = -2\,\mathcal{U}_{u\,A}{}^\alpha\, k^u_M\, V^M\,. & (7.69) \end{aligned}$$

Then, the right hand side of eq. (7.65) can be decomposed explicitly in a singlet and a triplet of $SU(2)$

$$g_{i\bar{\jmath}}\, W^{i\,AC} \overline{W}^{\bar{\jmath}}_{BC} + 2\, N_\alpha{}^A\, N^\alpha{}_B - 12\, S^{AC} S_{BC} = \delta^A_B\, \mathcal{V}(z,\bar{z},q) + i\, Z^x\, (\sigma^x)_B{}^A\,, \tag{7.70}$$

where the general symplectic invariant expression of the scalar potential is given by

$$\mathcal{V}(z,\bar{z},q) = (k^i_M k^{\bar{\jmath}}_N g_{i\bar{\jmath}} + 4\, h_{uv} k^u_M k^v_N) \overline{V}^M\, V^N + (U^{MN} - 3\, V^M \overline{V}^N) \mathcal{P}^x_N \mathcal{P}^x_M\,,$$

generalizing to dyonic gaugings. On the other hand,

$$Z^x = (-2\, X_{MN}{}^P\, \mathcal{P}^x_P + 2\,\epsilon^{xyz}\, \mathcal{P}^y_M \mathcal{P}^z_N + 4\, K^x_{uv} k^u_M\, k^v_N) \overline{V}^M V^N\,,$$

which, from the equivariance condition (7.53), it equals to zero so that the Ward identity is proven. The explicit expression of the left hand side of the eq. (7.65) can be found in Appendix D.

7.3.2 Abelian gauging of quaternionic isometries

Let us now consider the gauging considered in ref. [73], which involves an abelian group of quaternionic isometries. Since we consider only gauging of quaternionic isometries, such that the generalized structure constants vanish

$$X_{MN}{}^P = 0, \tag{7.71}$$

we have that eq. (7.53) implies

$$K^x_{uv} k^u_M \, k^v_N = -\frac{1}{2}\, \epsilon^{xyz}\, \mathcal{P}^y_M \mathcal{P}^z_N \,. \tag{7.72}$$

Using eq. (7.72), it is possible to show that the three fermionic-shifts cancel against one another

$$g_{i\bar{\jmath}}\, W^{i\,AC} \overline{W}^{\bar{\jmath}}_{BC} \quad \to \quad -\epsilon^{xyz}\, \mathcal{P}^y_M \mathcal{P}^z_N \overline{V}^M V^N \,, \tag{7.73}$$

$$2\, N_\alpha{}^A\, N^\alpha{}_B \quad \to \quad -2\, \epsilon^{xyz}\, \mathcal{P}^y_M \mathcal{P}^z_N \overline{V}^M V^N \,, \tag{7.74}$$

$$-12\, S^{AC} S_{BC} \quad \to \quad 3\, \epsilon^{xyz}\, \mathcal{P}^y_M \mathcal{P}^z_N \overline{V}^M V^N \,. \tag{7.75}$$

The objective of the present chapter it to consider the rigid limit of the Ward identity (7.65) to a rigid supersymmetric theory of vector multiplets [70, 71, 72, 87, 88]. In particular, according to refs. [72, 88], the Ward identity of the rigid theory in presence of n vector multiplets is given by

$$\mathring{g}_{i\bar{\jmath}}\, \mathring{W}^{i\,AC} \overline{\mathring{W}}^{\bar{\jmath}}_{BC} = \delta^A_B\, \mathcal{V}^{(APT)}(z,\bar{z}) + C_B{}^A \,. \tag{7.76}$$

Here, $V^{(APT)}(z,\bar{z})$ corresponds to the $\mathcal{N}=2$ scalar potential in the spontaneously broken rigid theory which reproduce the APT one in the case $n=1$ and $\mathring{g}_{i\bar{\jmath}}$ is the metric of the rigid special Kähler manifold parameterized by the scalar fields z^i. On the other hand, $C_B{}^A$ is a $SU(2)$-traceless matrix which allows partial breaking of supersymmetry if $C_B{}^A \neq 0$. Interestingly, this occurs for gauging involving non-commuting electric and magnetic charges [71].

The relations (7.73), (7.74) and (7.75) allow us to understand the meaning of the matrix $C_B{}^A$ by relating the supergravity Ward identity (7.65) to the rigid one (7.76). In fact, rewriting the Ward identity (7.65) as

$$g_{i\bar{\jmath}}\, W^{i\,AC} \overline{W}^{\bar{\jmath}}_{BC} = \delta^B_A\, \mathcal{V}(z,\bar{z},q) - 2\, N_\alpha{}^A\, N^\alpha{}_B + 12\, S^{AC} S_{BC} \,, \tag{7.77}$$

it is possible to show that all squared fermionic shift matrices survive in the rigid limit in which the Planck mass M_{Pl} is sent to infinity. In particular, the left-hand side of (7.77) corresponds to the left-hand side of (7.76). While the constant matrix $C_B{}^A$ have contributions from the hyperini and gravitini shift-matrices proportional to σ^x. Then, using eqs. (7.74) and (7.75), we find that

$$C_B{}^A = \lim_{M_{Pl}\to\infty} \frac{M^4_{Pl}}{\Lambda^4} \left(-i\, \epsilon^{xyz}\, \mathcal{P}^y_M \mathcal{P}^z_N \overline{V}^M V^N (\sigma^z)_B{}^A \right) , \tag{7.78}$$

where Λ corresponds to the supersymmetry breaking scale. The same fermionic shift-matrices also contribute to the terms proportional to δ^A_B affecting the scalar potential form in the rigid theory. Indeed, according to the explicit value of $N_\alpha{}^A\, N^\alpha{}_B$, $S^{AC} S_{BC}$ (see Appendix D), we have

$$\mathcal{V}^{(APT)}_{\mathcal{N}=2} = \lim_{M_{Pl}\to\infty} \frac{M^4_{Pl}}{\Lambda^4} \left[\mathcal{V}(z,\bar{z},q) - (4\, h_{uv}\, k^u_M k^v_N - 3\,\mathcal{P}^x_M \mathcal{P}^x_N)\overline{V}^M V^N\right] . \tag{7.79}$$

We shall see in the next section that, in the rigid limit, the leading order terms in $\Theta_N{}^n V^N$ depend only on the hyperscalars q^u, such that

$$\mathcal{V}^{(APT)}_{\mathcal{N}=2} = \lim_{M_{Pl}\to\infty} \frac{M^4_{Pl}}{\Lambda^4} \left[\mathcal{V}(z,\bar{z},q)\right] + A(q)\,. \tag{7.80}$$

Thus, the $\mathcal{N}=2$ scalar potential of the rigid theory $\mathcal{V}^{(APT)}_{\mathcal{N}=2}$ is given by the rigid limit of the supergravity potential $\mathcal{V}$ modulo an unphysical additive constant. Indeed, the fluctuations of the hyperscalars q^u are suppressed in the rigid theory by a factor M^{-1}_{Pl} so that they are non-dynamical.

7.4 Partial breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ supersymmetry in presence of n vector multiplets

In this section, following ref. [73] we present a partial breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ supersymmetry. In particular, we consider a supergravity model which, in the low energy limit, gives rise to a rigid supersymmetric theory. The rigid supersymmetric theory obtained here corresponds to a generalization of the APT model [71] to a generic number n of vector multiplets. Interestingly, the procedure approached here admits a well defined limit to the supersymmetric Born-Infeld theory generalized to n vector multiplets.

The $\mathcal{N}=2$ supergravity model considered here, consists of n vector multiplets and a single hypermultiplet, whose scalars parameterize the quaternionic manifold

$$\mathcal{M}_{QK} = \frac{SO\,(4,1)}{SO\,(4)}. \tag{7.81}$$

Following the procedure of ref. [72], let us consider a symplectic section

$$\Omega^M\left(z^i\right) = \begin{pmatrix} X^\Lambda\left(z^i\right) \\ F_\Lambda\left(z^i\right)\end{pmatrix} \qquad \Lambda = 0, I, \qquad I, i = 1,\ldots,n, \tag{7.82}$$

in a symplectic frame in the presence of a holomorphic prepotential. Then using special coordinates $z^i = X^i/X^0\delta^i_I$, we have

$$F\left(X^\Lambda\right) = -i\left(X^0\right)^2 f\left(X^i/X^0\right)\ , \tag{7.83}$$

so that, choosing

$$X^\Lambda = \begin{cases} X^0 = 1 \\ X^i = z^i \end{cases}\ , \tag{7.84}$$

we found

$$F_\Lambda = \begin{cases} F_0 = \partial F/\partial X^0 = -i\left(2f - z^i\partial_i f\right) \\ F_i = \partial F/\partial X^i = -i\partial_i f \end{cases}\ , \tag{7.85}$$

$$\Omega = \begin{pmatrix} 1 \\ z^i \\ -i\left(2f - z^i\partial_i f\right) \\ -i\partial_i f \end{pmatrix}. \tag{7.86}$$

In particular the Kähler potential is given by

$$\begin{aligned} \mathcal{K} &= -\ln\left[i\left(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda\right)\right], \\ &= -\ln\left[2\left(f+\bar{f}\right) - (z-\bar{z})^i\left(\partial_i f - \overline{\partial_i f}\right)\right]. \end{aligned}$$

In order to generalize the procedure of ref. [72] to the n vector multiplets case, it is necessary to consider a rigid limit $\mu = M_{Pl}/\Lambda \to \infty$, leading to partial breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in a rigid supersymmetric theory. In particular, M_{Pl} denotes the Planck scale and Λ the scale of partial supersymmetry breaking. As shown in ref. [72], the presence of a linear term in the expansion of the prepotential $f(z)$ in powers of $\frac{1}{\mu}$ was crucial in the derivation of partial breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$. In particular, in ref. [72], the prepotenial $f(z)$ was given by

$$f\left(z\right) = \frac{1}{4} + \frac{z}{2\mu} + \frac{\phi(z)}{2\mu^2} + O\left(\frac{1}{\mu^3}\right). \tag{7.87}$$

In our case, the generalization is obtained by introducing a set of n constant parameters η_i, so that the holomorphic prepotential $f\left(z^i\right)$ takes the form

$$f\left(z^i\right) = \frac{1}{4} + \frac{\eta_i z^i}{2\mu} + \frac{\phi(z^i)}{2\mu^2} + O\left(\frac{1}{\mu^3}\right). \tag{7.88}$$

Using the definition introduced in the previous section, we found for the Kähler potential, up to order μ^{-3}

$$\begin{aligned} \mathcal{K} &= \frac{\kappa^{(1)}}{\mu} + \frac{\mathring{\kappa}}{\mu^2} \\ &= -\frac{\eta_i \left(z+\bar{z}\right)^i}{\mu} - \frac{1}{\mu^2}\left[\phi + \bar{\phi} - (z-\bar{z})^i \left(\frac{\partial_i\phi - \overline{\partial_i\phi}}{2}\right) - \frac{\left(\eta_i \left(z+\bar{z}\right)^i\right)^2}{2}\right] \end{aligned} \quad (7.89)$$

Then, from eq. 7.89, one finds

$$\begin{aligned} g_{i\bar{\jmath}} &= \partial_i \partial_{\bar{\jmath}} \mathcal{K} \\ &= \frac{1}{\mu^2}\mathring{g}_{i\bar{\jmath}} = \frac{1}{\mu^2}\left\{\eta_i\eta_j - \frac{1}{2}\left(\overline{\partial_{ij}\phi} + \partial_{ij}\phi\right)\right\}, \end{aligned} \quad (7.90)$$

where $\mathring{g}_{i\bar{\jmath}}$ corresponds to the rigid special Kähler metric, which can be derived, in terms of the (rigid) $Sp(2n)$-symplectic section

$$\hat{\Omega}^{\mathcal{M}} = \begin{pmatrix} z^i \\ \partial_i \mathcal{F} \end{pmatrix} = \begin{pmatrix} z^i \\ \frac{i}{2}(\eta_i\eta_j z^j - \partial_i\phi) \end{pmatrix}, \quad \mathcal{M} = 1, \cdots, 2n\,, \quad (7.91)$$

from the (rigid) prepotential

$$\mathcal{F} = \frac{i}{4}\left[\left(\eta_i z^i\right)^2 - 2\phi\right]. \quad (7.92)$$

In fact,

$$\begin{aligned} \mathcal{F}_{i\bar{\jmath}} &= \partial_i\partial_{\bar{\jmath}}\mathcal{F} = \frac{i}{2}\left(\eta_i\eta_{\bar{\jmath}} - \partial_i\partial_{\bar{\jmath}}\phi\right) \\ &= \frac{i}{4}\left(\overline{\partial_i\partial_{\bar{\jmath}}\phi} - \partial_i\partial_{\bar{\jmath}}\phi\right) + \frac{i}{2}\left(\eta_i\eta_j - \frac{1}{2}\left(\overline{\partial_i\partial_{\bar{\jmath}}\phi} + \partial_i\partial_{\bar{\jmath}}\phi\right)\right) \\ &= \frac{i}{4}\left(\overline{\partial_i\partial_{\bar{\jmath}}\phi} - \partial_i\partial_{\bar{\jmath}}\phi\right) + \frac{i}{2}\mathring{g}_{i\bar{\jmath}}, \end{aligned}$$

which can be written as

$$\mathcal{F}_{i\bar{\jmath}} = \tau_{1i\bar{\jmath}} + i\tau_{2i\bar{\jmath}},$$

and where we have defined

$$\begin{aligned} \tau_{1i\bar{\jmath}} &\equiv \frac{i}{4}\left(\overline{\partial_i\partial_{\bar{\jmath}}\phi} - \partial_i\partial_{\bar{\jmath}}\phi\right), \\ \tau_{2i\bar{\jmath}} &\equiv \frac{\mathring{g}_{i\bar{\jmath}}}{2}. \end{aligned}$$

The covariantly holomorphic symplectic section $V^M \equiv e^{\mathcal{K}/2}\Omega^M$ has the following expansion

$$V^M = \begin{pmatrix} 1 - \frac{1}{2\mu}\eta_i\left(z+\bar{z}\right)^i + O\left(1/\mu^2\right) \\ z^j - \frac{1}{2\mu}\eta_i\left(z+\bar{z}\right)^i z^j + O\left(1/\mu^2\right) \\ -i\left[\frac{1}{2} + \frac{1}{2\mu}\left\{\eta_i z^i - \frac{1}{2}\eta_i\left(z+\bar{z}\right)^i\right\}\right] + O\left(1/\mu^2\right) \\ -\frac{i}{2\mu}\eta_j + O\left(1/\mu^2\right) \end{pmatrix}. \tag{7.93}$$

On the other hand, the symplectic section U_i^M denotes the Kähler-covariant derivative of the symplectic section given by

$$U_i^M = D_i V^M = \partial_i V^M + \frac{\partial_i \mathcal{K}}{2} V^M.$$

Then using special coordinates (eq. (7.82)) we can write

$$\begin{aligned} U_i^M = &\left(1 + \frac{\mathring{\mathcal{K}}^{(1)}}{2\mu} + \frac{1}{2\mu^2}\left[\mathring{\mathcal{K}} + \mathring{\mathcal{K}}^{(1)^2}/4\right]\right) \\ &\times \left\{ \begin{pmatrix} 0 \\ \delta_i^j \\ -i\left(\frac{\eta_i}{2\mu} + \frac{2}{2\mu^2}\left[\partial_i\phi - \partial_{ij}\phi\ z^j\right] + O\left(1/\mu^3\right)\right) \\ -i\frac{\partial_{ij}\phi}{2\mu^2} \end{pmatrix} \right. \\ &+ \left[-\frac{\eta_i}{\mu} - \frac{1}{\mu^2}\left(\frac{1}{2}\left[\partial_i\phi + \overline{\partial_i\phi}\right] - \left[z-\bar{z}\right]^k \frac{\partial_{ik}\phi}{2} - \eta_i\eta_k\left[z+\bar{z}\right]^k\right)\right] \\ &\left. \times \begin{pmatrix} 1 \\ z^j \\ -i\left(\frac{1}{2} + \frac{\eta_i z^i}{2\mu} + O\left(1/\mu^2\right)\right) \\ -i\left(\frac{\eta_j}{2\mu} + O\left(1/\mu^2\right)\right) \end{pmatrix} \right\}, \end{aligned}$$

$$U_i^M = \begin{pmatrix} -\frac{\eta_i}{\mu} + \frac{1}{2\mu^2}\left(-\left[\partial_i\phi + \overline{\partial_i\phi}\right] + \partial_{ij}\phi\left[z-\bar{z}\right]^j + 3\eta_i\eta_j\left[z+\bar{z}\right]^j\right) + O\left(1/\mu^3\right) \\ \delta_i^j - \frac{1}{\mu}\left(\frac{1}{2}\eta_k\left(z+\bar{z}\right)^k\delta_i^j + \eta_i z^j\right) + O\left(1/\mu^3\right) \\ -\frac{i}{4\mu^2}\left(\left[\partial_i\phi - \overline{\partial_i\phi}\right] - \partial_{ij}\phi\left[z+\bar{z}\right]^j + 2\eta_i\eta_j z^j\right) + O\left(1/\mu^3\right) \\ -\frac{i}{2\mu^2}\left(\partial_{ij}\phi - \eta_i\eta_j\right) + O\left(1/\mu^3\right) \end{pmatrix}. \tag{7.94}$$

The physical meaning of the constant parameters η_i appearing in the symplectic section $\hat{\Omega}^{\mathcal{M}}$ and in the metric $\mathring{g}_{i\bar{\jmath}}$ of the rigid theory will be clarified in the following subsection.

Let us now consider a gauging of two translational isometries in the hypermultiplet sector involving both electric and magnetic charges [89, 90]. To this purpose, we express the gauge generators $X_M \equiv (X_\Lambda,\, X^\Lambda)$ in terms of the isometry generators t_m, $m = 1, \ldots, \dim G$, of the quaternionic Kähler manifold $\mathcal{M}_{QK}$ through an embedding tensor [80, 83],

$$X_M = \theta_M{}^m\, t_m\,. \tag{7.95}$$

Then, we choose an abelian gauging involving only two translational isometries t_m $(m = 1, 2)$ and the embedding tensor as

$$\Theta_M^{\;m} = \left(\Theta_M^{\;1}, \Theta_M^{\;2}\right) = \begin{pmatrix} \Theta_0{}^1 & \Theta_0{}^2 \\ \Theta_i{}^1 & \Theta_i{}^2 \\ \Theta^{0\,1} & \Theta^{0\,2} \\ \Theta^{i\,1} & \Theta^{i\,2} \end{pmatrix} = \begin{pmatrix} e/\mu^2 & \sigma/\mu^2 \\ 0 & 0 \\ 0 & 0 \\ m^i/\mu & 0 \end{pmatrix}. \tag{7.96}$$

The embedding tensor $\Theta_M^{\;m}$ depends on constant charges e, σ, m^i and satisfies the locality condition

$$\mathbb{C}^{MN}\Theta_M^{\;m}\Theta_N^{\;n} = 0\,, \text{ where } \mathbb{C}^{MN} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7.97}$$

The embedding tensor $\Theta_M^{\;m}$ relates the embedded Killing vectors $k_M^{\;u} = \left(k_\Lambda{}^u, k^{\Lambda\,u}\right)$ to the geometrical Killing vectors $k_m^{\;u}$ $(m = 1, \ldots, \dim G)$ generating the isometry group G of $\mathcal{M}_{QK}$ through

$$k_M^{\;u} = \Theta_M^{\;m} k_m^{\;u}. \tag{7.98}$$

In particular, the fermionic shifts $\delta_\epsilon^{(\Theta)}$ of the supersymmetry transformation laws can be written in a symplectic covariant way using the embedding tensor $\Theta_M^{\;m}$. Indeed, in our $\mathcal{N} = 2$ matter-coupled supergravity, we have

$$\delta_\epsilon^{(\Theta)}\lambda^{i\,A} = W^{i\,AB}\epsilon_B, \tag{7.99}$$

$$\delta_\epsilon^{(\Theta)}\psi_{A\,\mu} = iS_{AB}\gamma_\mu\epsilon^B, \tag{7.100}$$

$$\delta_\epsilon^{(\Theta)}\zeta^\alpha = N_A^\alpha\epsilon^A, \tag{7.101}$$

where

$$W^{i\ AB} = ig^{i\bar{\jmath}}\left(\sigma^x\right)_C{}^B \epsilon^{CA} U_{\bar{\jmath}}{}^M \Theta_M{}^\alpha \mathcal{P}^x_\alpha, \tag{7.102}$$

$$S_{AB} = \frac{i}{2}\left(\sigma^x\right)_A{}^C \epsilon_{BC} V^M \Theta_M{}^\alpha \mathcal{P}^x_\alpha, \tag{7.103}$$

$$N^\alpha_A = -2\mathcal{U}^\alpha_{A|u} k^u_\alpha V^M \Theta_M{}^\alpha. \tag{7.104}$$

Let us note that, since our gauging does not involve special Kähler isometries, we have set $k^i_M = 0$. Here, $\mathcal{U}^\alpha_{A|u}$ is the vielbein of the quaternionic manifold, which can be parametrized as [72]:

$$\mathcal{U}^\alpha_A = \mathcal{U}^\alpha_{A|u} dq^u = \frac{1}{2}\epsilon^{\alpha\beta}\left[-d\varphi\delta_{\beta A} - ie^\varphi d\vec{q}\cdot\vec{\sigma}_\beta{}^A\right]. \tag{7.105}$$

On the other hand, $\left(\sigma^x\right)_A{}^C$ are the standard Pauli matrices and $\mathcal{P}^x_m$ correspond to the quaternionic momentum maps associated with the quaternionic isometries through (see Appendix C)

$$\mathcal{P}^x_m = -k_m{}^u \omega^x_u. \tag{7.106}$$

Here ω^x_u denotes the $SU(2)$-connection on the quaternionic Kähler manifold $\mathcal{M}_{QK}$. Let us note that the eigenvalues of the mass matrix S_{AB} correspond to the gravitino masses [1].

In particular, the momentum maps can be chosen as

$$\mathcal{P}^x_m = \left(\mathcal{P}^x_1, \mathcal{P}^x_2\right) = \delta^x_m e^\varphi,$$

with

$$\mathcal{P}^x_1 = (0,1,0)\, e^\varphi, \tag{7.107}$$

$$\mathcal{P}^x_2 = (0,0,1)\, e^\varphi. \tag{7.108}$$

7.4.1 Partial supersymmetry breaking

The partial supersymmetry breaking is recovered considering the limit $\mu = \frac{M_{Pl}}{\Lambda} \to \infty$. To explicitly perform the limit on the fermionic shifts (which are written in natural

[1] In the supersymmetry partially broken case, only one of them, $m_{3/2}$is different from zero.

units $c=\hbar=M_{Pl}=1$) it is convenient to reintroduce the appropriate dependence on the Planck Mass M_{Pl} and on the supersymmetry breaking scale Λ. On the other hand, Taking into account that the gravitino mass is related to the scale Λ through $\Lambda^2 = M_{Pl}\, m_{\frac{3}{2}}$, and that the Special-Kähler metric rescales according to (7.90), the canonically normalized kinetic terms are recovered by the following rescaling [72],

$$\begin{aligned} & x^\mu \to M_{Pl} x^\mu, \qquad \epsilon \to M_{Pl}^{1/2}\epsilon, \\ & \psi_\mu \to M_{Pl}^{-3/2}\psi_\mu, \qquad \lambda \to \left(M_{Pl}\Lambda^2\right)^{-1/2}\lambda, \qquad \zeta^\alpha \to M_{Pl}^{-3/2}\zeta^\alpha. \end{aligned} \tag{7.109}$$

Them, using the rescaling of eq. (7.109) we find in the rigid limit that the fermionic shifts read

$$\begin{aligned} \delta\lambda^{i\,A} &= -i\Lambda^2\epsilon^{CA}\left[\mathring{g}^{i\bar{\jmath}}\left(e^x_{\bar{\jmath}} - \tau_{1\bar{\jmath}k}m^{k\,x}\right) + \frac{i}{2}m^{i\,x}\right]\left(\sigma^x\right)_C^{\;B} e^\varphi \epsilon_B, \\ \delta\psi_{A\,\mu} &= -\frac{\Lambda^2}{2}\epsilon_{BC}\left[e^x - i\frac{\eta_j}{2}m^{j\,x}\right]\left(\sigma^x\right)_A^{\;C} e^\varphi\epsilon^B, \\ \delta\zeta^\alpha &= -i\Lambda^2\epsilon^{\alpha\beta}\left[e^x - i\frac{\eta_j}{2}m^{j\,x}\right]\left(\sigma^x\right)_A^{\alpha} e^\varphi\epsilon^A, \end{aligned} \tag{7.110}$$

where we have defined

$$\begin{aligned} e^x &= (0,e,\sigma) = (0,e^m)\,, \\ m^{i\,x} &= \left(0,m^i,0\right) = (0,m^{im})\,, \\ e^x_i &= \eta_i e^x. \end{aligned} \tag{7.111}$$

Let us note that the hypermultiplet decouples in the rigid theory meanwhile the momentum maps $\mathcal{P}^{xM}$ reduce to constant Fayet-Iliopoulos terms

$$\mathbb{P}^x_{\mathcal{M}} = \left(m^{ix}, e^x_i\right).$$

The relation between them can be read explicitly from the gaugino shift:

$$\mathring{g}^{i\bar{\jmath}}\bar{U}^M_{\bar{\jmath}}\mathcal{P}^x_M = \frac{e^\varphi}{\mu}\left[\mathring{g}^{i\bar{\jmath}}\left(e^x_{\bar{\jmath}} - \tau_{1\bar{\jmath}k}m^{k\,x}\right) + \frac{i}{2}m^{i\,x}\right] = \frac{e^\varphi}{\mu}\mathring{g}^{i\bar{\jmath}}\bar{U}^{\mathcal{M}}_{\bar{\jmath}}\mathbb{P}^x_{\mathcal{M}}\,. \tag{7.112}$$

Here $U^{\mathcal{M}}_j$ are related to the rigid symplectic sections by $U^{\mathcal{M}}_j = \partial_j\hat{\Omega}^{\mathcal{M}}$.

Interestingly, the case of one vector multiplet ($n = 1$), is recovering setting $\eta_i = 1$ in the eq. (7.110) reproducing the results of ref. [72] leading to the APT model.

It is important to clarify that the FI terms are expressed not only in terms of the components of the embedding tensor (e, σ, m^i) but also in terms of the parameters η_i characterizing the special geometry. We shall show in the next subsection that we can reformulate the theory in a symplectic frame leading to a more clear interpretation of the FI terms.

7.4.2 Interpretation of the constant parameters η_i

It is well known that partial supersymmetry breaking in rigid supersymmetry can occur, provided one evades previously stated no-go theorems [75, 76]. Indeed, the partial breaking of supersymmetry crucially requires that the quantity ξ^x, defined by

$$\xi^x \equiv \frac{1}{2}\epsilon^{xyz}\mathbb{P}^{y\mathcal{M}}\mathbb{P}^{z\mathcal{N}}\mathbb{C}_{\mathcal{M}\mathcal{N}} = \epsilon^{xyz}e^y_i m^{zi} \neq 0, \tag{7.113}$$

be different from zero. As shown in [88], this condition is also necessary to achieve a multi-field generalization of the Born-Infeld theory in the low energy limit. This relation seems to be an non-locality condition. Nevertheless, the locality condition is satisfied in the rigid theory due to the choice of the embedding tensor (7.96)

$$\Theta^m_{\mathcal{M}}\Theta^n_{\mathcal{N}}\mathbb{C}_{\mathcal{M}\mathcal{N}} = 2\Theta^{i[m}\Theta^{n]}_i = 0. \tag{7.114}$$

Thus, recalling the definition of the momentum maps $\mathcal{P}^x_{\mathcal{M}} = \mathcal{P}^x_m\Theta^m_{\mathcal{M}}$, the condition $\epsilon^{xyz}\mathcal{P}^{y\mathcal{M}}\mathcal{P}^{z\mathcal{N}}\mathbb{C}_{\mathcal{M}\mathcal{N}} = 0$ is satisfied in the chosen frame. Indeed, the momentum maps in supergravity $\mathcal{P}^x_{\mathcal{M}}$ and the Fayet-Iliopoulos terms $\mathbb{P}^x_{\mathcal{M}}$ of the rigid theory are related through (7.112) which involves the contribution from the index 0 of the symplectic section, keeping a memory of the graviphoton. On the other hand, the geometry of the rigid theory in the chosen coordinate frame depends in a non trivial way on the constant parameters η_i.

Interestingly, the η_i required in order to implement partial supersymmetry breaking (with its BI low energy limit) can be traded with charges through a symplectic rotation. This involves a redefinition of the special coordinates in the underlying supergravity theory.

In fact, let us consider the symplectic transformation in supergravity

$$S(\eta,\mu) = \begin{pmatrix} 1 & \eta_i/\mu & 0 & 0 \\ 0 & \frac{1}{\mu}\mathbf{1_n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\eta_i & \mu\mathbf{1_n} \end{pmatrix} \tag{7.115}$$

which induces the following rotation in the symplectic section Ω (7.82):

$$\tilde{\Omega} = S\cdot\Omega = \begin{pmatrix} X^0 + \frac{1}{\mu}\eta_i X^i \\ \frac{1}{\mu}X^i \\ F_0 \\ \mu F_i - \eta_i F_0 \end{pmatrix} = \begin{pmatrix} \tilde{X}^0 \\ \tilde{X}^i \\ \tilde{F}_0 \\ \tilde{F}_i \end{pmatrix} . \tag{7.116}$$

The new holomorphic prepotential is then $\tilde{F}(\tilde{X}) = F(X)$. Since the new special coordinates $\tilde{z}^i$ are related to the old ones by $\tilde{z}^i = \frac{z^i}{\mu+\eta_j z^j} = \frac{1}{\mu}\omega^i$, then the reduced prepotential $\tilde{f}(\tilde{z})$ is related to $f(z)$ by (see (7.83)):

$$\tilde{f}(\tilde{z}) = (1 + \frac{1}{\mu}\eta_j z^j)^{-2} f(z)$$

that gives

$$\tilde{f}(\tilde{z}) = \left(\frac{1}{4} + \frac{1}{2\mu^2}\tilde{\phi}(\tilde{z}) + O(\frac{1}{\mu^3})\right) \tag{7.117}$$

where $\tilde{\phi}(\tilde{z})$ is related to $\phi(z)$ by

$$\tilde{\phi}(\tilde{z}) = \phi(z) - \frac{1}{2}(\eta_i \tilde{z}^i)^2 \equiv \Phi(\omega).$$

Interestingly, after the symplectic rotation, we note that the covariantly holomorphic symplectic sections $\tilde{V}^M$ and $\tilde{U}_i^M$ can be written in the rigid limit $\mu \to \infty$ as

$$\tilde{V}^M = \begin{pmatrix} X^0 \\ 0 \\ F_0 \\ 0 \end{pmatrix} + \frac{1}{\mu}\begin{pmatrix} 0 \\ \mathring{X}^I(\omega) \\ 0 \\ \mathring{F}_I(\omega) \end{pmatrix} + O\left(1/\mu^2\right) ; \tag{7.118}$$

$$\tilde{U}_i^M = \frac{1}{\mu}\begin{pmatrix} 0 \\ \partial_i \mathring{X}^I \\ 0 \\ \partial_i \mathring{F}_I \end{pmatrix} + O\left(1/\mu^2\right) . \tag{7.119}$$

Here, the special coordinates $\mathring{X}^I(\omega) = \omega^i$, $\mathring{F}_I(\omega) = \frac{\partial\Phi}{\partial\omega^i}$ describe the symplectic section of the rigid theory $\mathring{\Omega}^{\mathcal{M}} \equiv (\mathring{X}^I, \mathring{F}_I)$ $(I = 1, \cdots n)$. Let us note that in the new frame the symplectic structure $Sp(2n+2)$ of the supergravity theory flows in the rigid limit to a manifest $Sp(2n)$ structure. Interestingly, the 0-directions have a different μ-rescaling with respect to the $\mathcal{M}$ directions.

On the other hand, the embedding tensor (7.96) is also modified by the symplectic transformation (7.115)

$$\tilde{\Theta}^m_M = \Theta^m_N \cdot (S^{-1})^N{}_M = \frac{1}{\mu^2}\left(e^m, -\eta_i e^m, \eta_i m^{im}, m^{im}\right) = \frac{1}{\mu^2}\mathring{\Theta}^m_M\,, \tag{7.120}$$

where $\mathring{\Theta}^m_M$ corresponds to the embedding tensor of the rigid theory.

In this new frame the parameters η_i play the role of charges, since $\tilde{\Theta}^m_i = \eta_i e^m$ are the electric charges associated with the vector multiplets. While $\tilde{\Theta}^{0m} = \eta_i m^{im}$ is the magnetic charge associated with the graviphoton. Note that in the old frame both of them were zero.

Consequently, the new embedding tensor (7.120) obeys the same locality condition (7.97) as the old one. The difference is given by

$$\tilde{\Theta}^{\Lambda[m}\tilde{\Theta}^{n]}_\Lambda = 0 \quad \Rightarrow \quad \tilde{\Theta}^{0[m}\tilde{\Theta}^{n]}_0 = -\tilde{\Theta}^{i[m}\tilde{\Theta}^{n]}_i = \frac{1}{\mu^4}e^m\eta_i m^{in} \neq 0\,. \tag{7.121}$$

Additionally, unlike the old frame, in the new frame the graviphoton is identified with the 0 direction of the vector field strengths. Due to the decoupling of the graviphoton from the spectrum, we find that the rigid supersymmetric theory found as low energy limit of supergravity in the new frame is actually non local. Thus, with the new embedding tensor, eq. (7.113) express indeed the non locality of the rigid theory.

The effects of the non-locality (7.121) is intimately related to the supersymmetric structure. Indeed, the supergravity modes associated with the underlying $\mathcal{N} = 2$ supergravity theory still freely propagate in the rigid theory. Consequently, the $SU(2)$-Lie algebra valued term $C_A{}^B$ which appears in the rigid Ward identity (7.76) can be understood as the contribution to the Ward identity from gravitini and hyperini, still propagating in the rigid theory.

This non-locality of the rigid theory hints toward a high-energy interpretation in terms of a non-triviality of the fiber bundle associated with the graviphoton. Interestingly, this non-locality poses no obstruction to a correct definition of the vector fields

A^I_μ in the rigid theory, by virtue of an interesting mechanism. According to refs. [83, 89, 91, 92], the natural symplectic frame to deal with magnetic charges $m^{\Lambda n}$ is rotated with respect to the purely electric frame. In particular, this allows the presence of antisymmetric tensors $B_{n\mu\nu}$, coupled to the gauge fields A^Λ in the combinations

$$\hat{F}^\Lambda_{\mu\nu} = F^\Lambda_{\mu\nu} + 2m^{\Lambda n} B_{n\mu\nu}. \tag{7.122}$$

A generic feature of magnetic gaugings in supergravity is the fact that the vector fields A^Λ_μ corresponding to non-vanishing magnetic components of the embedding tensor $\Theta^{\Lambda m}$, are not well defined since the corresponding field strengths $F^\Lambda_{\mu\nu}$ are not covariantly closed. This poses no problem because such vector fields, in a vacuum, are "eaten" by the tensor ones B_n and become their longitudinal components by virtue of the "anti-Higgs" mechanism [93]. In the rigid limit, as we shall show, the antisymmetric tensor fields decouple, thus preventing the anti-Higgs mechanism from taking place, so that the vectors A^I_μ survive and become well defined.

The $\mathcal{N} = 2$ supersymmetric Free Differential Algebra in four dimensions contains, in the case where the antisymmetric tensors dualize scalars in the quaternionic sector

$$\begin{aligned} \hat{F}^{(2)\Lambda} &\equiv dA^\Lambda + 2m^{\Lambda n} B_n + (L^\Lambda(z)\bar{\psi}_A \wedge \psi_B \epsilon^{AB} + h.c.) && (7.123) \\ H^{(3)}_n &\equiv dB_n + \frac{i}{2}\mathcal{P}^x_n \, (\sigma^x)_A^{\ B} \, \bar{\psi}_B \wedge \gamma_a \psi^A \wedge V^a && (7.124) \end{aligned}$$

where

$$V^M = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix}$$

and $\mathcal{P}^x_n$ are functions of the hyperscalars [90]. Then, one can see that the closure of the free differential algebra requires

$$d\hat{F}^\Lambda = \Theta^{\Lambda n} \left(2H_n - i\mathcal{P}^x_n \, (\sigma^x)_A^{\ B} \, \bar{\psi}_B \wedge \gamma_a \psi^A \wedge V^a \right). \tag{7.125}$$

In the low energy limit the hyperscalars are not suppresed but tend to constants. Indeed, $\Theta_M^{\ n} \mathcal{P}^x_n$ becomes constants $\Theta_M^{\ n} \mathbb{P}^x_n \neq 0$ whose non-zero indices $\Theta_{\mathcal{M}}^{\ n} \mathbb{P}^x_n$ yield the FI parameters. Then, from the expression (7.125), taking account the decoupling of the tensor fields, the clousure of the supersymmetric free differential algebra gives

$$d\hat{F}^I \propto i\Theta^{I\ n} \mathbb{P}^x_n \, (\sigma^x)_A^{\ B} \, \bar{\psi}_B \wedge \gamma_a \psi^A \wedge V^a + \cdots \neq 0 \, . \tag{7.126}$$

Thus, the magnetic FI terms parametrize a non-locality only along the fermionic directions of superspace, thus not affecting the well-definiteness of A^I_μ. The eq. (7.126) is the superspace counterpart of the fact that, on space-time, the commutator of two supersymmetries acts on the gauge field A^I_μ as a gauge transformation, as observed in [72] and, in the multi-vector field case, in [88].

7.5 The rigid limit: $\mathcal{N} = 1$ Supersymmetric Lagrangian

In this section we present the rigid limit of the $\mathcal{N} = 2$ Supergravity action corresponding to partial breaking of supersymmetry.

Following ref. [73], we shall consider the symplectic frame defined in the previous section whose gauging structure involve the presence of magnetic charges. Then, the natural framework to perform the limit is the formalism in which the scalars of the hypermultiplets are Hodge-dualized to antisymmetric tensor $B_{I\mu\nu}$ [83, 89, 90, 91, 92].

Before to perform the rigid limit, it is convenient to introduce the appropriate scale dimensions in the Lagrangian. In particular, we shall first explicitly write the correct Planck-mass M_{Pl} dependence of the physical fields in the $\mathcal{N} = 2$ supergravity Lagrangian. This leads us, after perform the low energy limit $\mu = \frac{M_{Pl}}{\Lambda} \to \infty$, to the appropriate redefinitions of the physical fields appearing in the $\mathcal{N} = 1$ rigid supersymmetric theory.

The canonical scale dimensions of the fields of the theory in natural units $c = \hbar = 1$ are

$$
\begin{aligned}
&[x^\mu] = M^{-1}\,, \quad [\partial_\mu] = M\,, \quad [A^\Lambda_\mu] = [B^x_{\mu\nu}] = M\,, \quad [z^i_{(can.)}] = [q^u_{(can.)}] = M, \\
&[\psi^A_\mu] = [\lambda^A] = [\zeta^\alpha] = M^{3/2}\,, \quad [\epsilon^A] = M^{-1/2}\,.
\end{aligned}
$$

On the other hand, the symplectic-covariant embedding tensor Θ^m_M given by eq. (7.120) is adimensional. Since the scalars z^i, q^u appear in the theory through non-linear sigma-models, we will keep them adimensional so that $z^i \equiv z^i_{(can.)}/M_{Pl}$, $q^u \equiv q^u_{(can.)}/M_{Pl}$.

Then, the Lagrangian of ref. [90] can be reorganized in terms of Planck-scale powers, up to four fermions terms,

$$
\mathcal{L} = \mathcal{L}_{(4)} + \mathcal{L}_{(2)} + \mathcal{L}_{(1)} + \mathcal{L}_{(0)} + \mathcal{L}_{(-1)}. \qquad (7.127)
$$

In particular, we have

$$\mathcal{L}_{(4)} = M_{Pl}^4 \mathcal{V}(z,q) \tag{7.128}$$

$$\mathcal{L}_{(2)} = M_{Pl}^2 \left(-\frac{R}{2} + g_{i\bar{\jmath}} \partial^\mu z^i \partial_\mu \bar{z}^{\bar{\jmath}} + h_{uv} \partial_\mu q^u \partial^\mu q^v \right) \tag{7.129}$$

$$\begin{aligned} \mathcal{L}_{(1)} = & M_{Pl} \Big\{ (-\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}) \Big[2\mathcal{H}_{m|\nu\rho\sigma} A_u^m \partial_\mu q^u + \frac{1}{2} B_{m|\mu\nu} \Theta_\Lambda^{\ m} \Big(\hat{\mathcal{F}}_{\rho\sigma}^\Lambda - M_{Pl} \frac{1}{2} \Theta^{\ \Lambda n} B_{n|\rho\sigma} \Big) \Big] + \\ & + \big(2S_{AB} \bar{\psi}_\mu^A \gamma^{\mu\nu} \psi_\nu^B + i g_{i\bar{\jmath}} W^{iAB} \bar{\lambda}_A^{\bar{\jmath}} \gamma_\mu \psi_B^\mu + 2i N_\alpha^A \bar{\zeta}^\alpha \gamma_\mu \psi_A^\mu \\ & + \mathcal{M}^{\alpha\beta} \bar{\zeta}_\alpha \zeta_\beta + \mathcal{M}_{iB}^\alpha \bar{\zeta}_\alpha \lambda^{iB} + \mathcal{M}_{iAjB} \bar{\lambda}^{iA} \lambda^{jB} + h.c. \big) \Big\} \end{aligned} \tag{7.130}$$

$$\begin{aligned} \mathcal{L}_{(0)} = & i \left(\bar{\mathcal{N}}_{\Lambda\Sigma} \hat{\mathcal{F}}_{\mu\nu}^{-\Lambda} \hat{\mathcal{F}}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma} \hat{\mathcal{F}}_{\mu\nu}^{+\Lambda} \hat{\mathcal{F}}^{+\Sigma\mu\nu} \right) + 6\mathcal{M}^{mn} \mathcal{H}_{m\mu\nu\rho} \mathcal{H}_n^{\ \mu\nu\rho} + \\ & + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}} \left(\bar{\psi}_\mu^A \gamma_\nu \rho_{A|\lambda\sigma} - \bar{\psi}_{A|\mu} \gamma_\nu \rho_{\lambda\sigma}^A \right) - \frac{i}{2} g_{i\bar{\jmath}} \left(\bar{\lambda}^{iA} \gamma^\mu \nabla_\mu \lambda_A^{\bar{\jmath}} + \bar{\lambda}_A^{\bar{\jmath}} \gamma^\mu \nabla_\mu \lambda^{iA} \right) + \\ & - i \left(\bar{\zeta}^\alpha \gamma^\mu \nabla_\mu \zeta_\alpha + \bar{\zeta}_\alpha \gamma^\mu \nabla_\mu \zeta^\alpha \right) + \\ & - g_{i\bar{\jmath}} \partial_\mu \bar{z}^{\bar{\jmath}} \left(\bar{\psi}_A^\mu \lambda^{iA} - \bar{\lambda}^{iA} \gamma^{\mu\nu} \psi_{A\nu} + h.c. \right) - 2\mathcal{U}_u^{\alpha A} \partial_\mu q^u \left(\bar{\psi}_A^\mu \zeta_\alpha - \bar{\zeta}_\alpha \gamma^{\mu\nu} \psi_{A\nu} + h.c. \right) \end{aligned} \tag{7.131}$$

$$\begin{aligned} \mathcal{L}_{(-1)} = & M_{Pl}^{-1} \Big\{ \hat{\mathcal{F}}_{\mu\nu}^{-\Lambda} I_{\Lambda\Sigma} \Big[L^\Sigma \bar{\psi}^{A\mu} \psi^{B\nu} \epsilon_{AB} - 4i f_{\bar{\imath}}^\Sigma \bar{\lambda}_A^{\bar{\imath}} \gamma^\nu \psi_B^\mu \epsilon^{AB} + \frac{1}{2} \nabla_i f_j^\Sigma \bar{\lambda}^{iA} \gamma^{\mu\nu} \lambda^{jB} \epsilon_{AB} + \\ & - L^\Sigma \bar{\zeta}_\alpha \gamma^{\mu\nu} \zeta_\beta \mathbb{C}^{\alpha\beta} \Big] + h.c. + \\ & + 2\mathcal{M}^{mn} \mathcal{H}_m^{\ \mu\nu\rho} \left[\mathcal{U}_n^{\ A\alpha} \left(3i \bar{\psi}_{A\mu} \gamma_{\nu\rho} \zeta_\alpha + \bar{\psi}_{A\mu} \zeta_\alpha \right) + i \Delta_{n\alpha}^{\ \ \beta} \zeta_\beta \gamma_{\mu\nu\rho} \zeta^\alpha \right] \Big\} \ . \end{aligned} \tag{7.132}$$

Here, h_{uv}, A_u^m and $\mathcal{M}^{mn}$ correspond to the components of the quaternionic metric after dualization of the scalars q^m to antisymmetric tensors $B_{m|\mu\nu}$. On the other hand $\mathcal{F}_{\mu\nu}^{\pm\Lambda} = \frac{1}{2} \left(\mathcal{F}_{\mu\nu}^\Lambda \pm \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\Lambda\rho\sigma} \right)$ and $\hat{\mathcal{F}}_{\mu\nu}^\Lambda := \mathcal{F}_{\mu\nu}^\Lambda + \frac{1}{2} \Theta^{\Lambda m} B_{\mu\nu m}$ are the gauge-field-strengths undergoing the anti-Higgs mechanism introduced in (7.123) [2]. The mass-matrices are

[2] In a symplectic frame, where the gauge fields undergo the standard Higgs-mechanism by coupling to the scalars in the quaternionic sector (not dualized to antisymmetric tensors), the gauge-covariant derivative in the quaternionic sector is defined as

$$\nabla_\mu q^u = \partial_\mu q^u + M_{Pl}^{-1} A_\mu^\Lambda \Theta_\Lambda^{\ \alpha} k_\alpha^{\ u} \, .$$

given by

$$W^{i\,AB} = ig^{i\bar{\jmath}}\,(\sigma^x)_C{}^B\,\epsilon^{CA}U_{\bar{\jmath}}{}^M\Theta_M{}^m\mathcal{P}^x_m, \tag{7.133}$$

$$S_{AB} = \frac{i}{2}\,(\sigma^x)_A{}^C\,\epsilon_{BC}V^M\Theta_M{}^m\mathcal{P}^x_m, \tag{7.134}$$

$$N^\alpha_A = -2\mathcal{U}^\alpha_{A|u}k^u_m V^M\Theta_M{}^m, \tag{7.135}$$

$$\mathcal{M}^{\alpha\beta} = -\mathcal{U}^{\alpha A}_u\mathcal{U}^{\beta B}_v\epsilon_{AB}\Theta_M{}^m\nabla^{[u}k^{v]}_m V^M, \tag{7.136}$$

$$\mathcal{M}^\alpha_{iB} = -4\mathcal{U}^\alpha_{Bu}\Theta_M{}^m k^u_m U_i{}^M, \tag{7.137}$$

$$\mathcal{M}_{iAjB} = \frac{i}{3}\left(\sigma_x\epsilon^{-1}\right)_{AB}\Theta_M{}^m\mathcal{P}^x_m\nabla_j U_i{}^M. \tag{7.138}$$

To perform the rigid limit $\mu \to \infty$ of the Lagrangian, we have to consider the limit of the various couplings in the Lagrangian. We shall identify the fields of the rigid supersymmetric theory with a ring, in order to distinguish them from the supergravity fields. In particular, the special-geometry sigma-model metric in supergravity is related to its counterpart $\mathring{g}_{i\bar{\jmath}}$ in the rigid limit by:

$$g_{i\bar{\jmath}} = \frac{1}{\mu^2}\mathring{g}_{i\bar{\jmath}}\,, \tag{7.139}$$

so that the kinetic terms of scalars and spinors in the vector multiplets read

$$\frac{1}{\mu^2}\mathring{g}_{i\bar{\jmath}}\left[M^2_{Pl}\partial^\mu z^i\partial_\mu\bar{z}^{\bar{\jmath}} - \frac{i}{2}\left(\bar{\lambda}^{iA}\gamma^\mu\nabla_\mu\lambda^{\bar{\jmath}}_A + \bar{\lambda}^{\bar{\jmath}}_A\gamma^\mu\nabla_\mu\lambda^{iA}\right)\right].$$

This implies that the gaugini of the rigid supersymmetric theory should be related to the supergravity one as

$$\mathring{\lambda}^{iA} = \frac{1}{\mu}\lambda^{iA}. \tag{7.140}$$

On the other hand, the holomorphic scalars should not be rescaled

$$\mathring{z}^i = z^i.$$

Furthermore, the relations of special geometry imply a low-energy rescaling of the vector-kinetic-matrix $\mathcal{N}_{\Lambda\Sigma}$ corresponding to the following identification of the matrix $\mathring{\mathcal{N}}_{\Lambda\Sigma}$ of the rigid theory:

$$\mathcal{N}_{00} = \mathring{\mathcal{N}}_{00}\,,\quad \mathcal{N}_{IJ} = \mathring{\mathcal{N}}_{IJ}\,,\quad \mathcal{N}_{0I} = \frac{1}{\mu}\mathring{\mathcal{N}}_{0I}\,. \tag{7.141}$$

In this way, the gauge vector should not be redefined

$$\mathring{A}^{\Lambda}_{\mu} = A^{\Lambda}_{\mu}\,, \tag{7.142}$$

and the gauge kinetic term is given by, at low energies

$$I_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma|\mu\nu} = \mathring{I}_{00}F^{0}_{\mu\nu}F^{0|\mu\nu} + \mathring{I}_{IJ}F^{I}_{\mu\nu}F^{J|\mu\nu} + \frac{2}{\mu}\mathring{I}_{0I}F^{0}_{\mu\nu}F^{I|\mu\nu} + \mathcal{O}(1/\mu^2),$$

where $I_{\Lambda\Sigma} \equiv Im(\mathcal{N}_{\Lambda\Sigma})$.

The rescaling of the special geometry sector, in a generic coordinate frame, is given by

$$V^M = \begin{pmatrix} X^0 \\ 0 \\ F_0 \\ 0 \end{pmatrix} + \frac{1}{\mu}\begin{pmatrix} 0 \\ \mathring{X}^I(z,\bar{z}) \\ 0 \\ \mathring{F}_I(z,\bar{z}) \end{pmatrix} + \mathcal{O}\left(1/\mu^2\right)\,; \tag{7.143}$$

$$\tilde{U}^M_i = \frac{1}{\mu}\begin{pmatrix} 0 \\ \partial_i\mathring{X}^I \equiv \mathring{f}^I_i \\ 0 \\ \partial_i\mathring{F}_I \equiv \mathring{h}_{Ii} \end{pmatrix} + \mathcal{O}\left(1/\mu^2\right)\,, \tag{7.144}$$

while the embedding tensor,

$$\Theta^m_M = \frac{1}{\mu^2}\mathring{\Theta}^m_M. \tag{7.145}$$

Then, following the low energy limit of the symplectic sections and embedding tensor discussed in the previous section, we have that the rescalings of the fermion shifts and spinor mass matrices are given by

$$W^{i\;AB} = \frac{1}{\mu}\mathring{W}^{i\;AB}\,, \qquad \mathcal{M}^{\alpha\beta} = \frac{1}{\mu^2}\mathring{\mathcal{M}}^{\alpha\beta}\,, \tag{7.146}$$

$$S_{AB} = \frac{1}{\mu^2}\mathring{S}_{AB}\,, \qquad \mathcal{M}^{\alpha}_{iB} = \frac{1}{\mu^3}\mathring{\mathcal{M}}^{\alpha}_{iB}\,, \tag{7.147}$$

$$N^{\alpha}_{\;A} = \frac{1}{\mu^2}\mathring{N}^{\alpha}_{\;A}\,, \qquad \mathcal{M}_{iAjB} = \frac{1}{\mu^3}\mathring{\mathcal{M}}_{iAjB}\,. \tag{7.148}$$

Thus, the scalar potential rescales as

$$\mathcal{V} = \frac{1}{\mu^4}\mathring{\mathcal{V}}. \tag{7.149}$$

Consequently, using the rescaled fields, the various contributions to the Lagrangian (7.127) can be written as

$$\mathcal{L}_{(4)} = \Lambda^4 \mathring{\mathcal{V}}(z,q) \tag{7.150}$$

$$\mathcal{L}_{(2)} = M_{Pl}^2 \left(-\frac{R}{2} + h_{uv}\partial_\mu q^u \partial^\mu q^v\right) + \Lambda^2 \mathring{g}_{i\bar{\jmath}} \partial^\mu \mathring{z}^i \partial_\mu \mathring{\bar{z}}^{\bar{\jmath}} \tag{7.151}$$

$$\begin{aligned}
\mathcal{L}_{(1)} = {} & M_{Pl}\Big\{(-\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}})\left[2\mathcal{H}_{m|\nu\rho\sigma}A^m_u\partial_\mu q^u + \frac{1}{2\mu^2}B_{m|\mu\nu}\mathring{\Theta}_\Lambda^{\ m}\left(\hat{\mathcal{F}}^\Lambda_{\rho\sigma} - \frac{M_{Pl}}{\mu^2}\frac{1}{2}\mathring{\Theta}^{\ \Lambda n}B_{n|\rho\sigma}\right)\right] + \\
& + \frac{1}{\mu^2}\left(2\mathring{S}_{AB}\bar{\psi}^A_\mu\gamma^{\mu\nu}\psi^B_\nu + i\mathring{g}_{i\bar{\jmath}}\mathring{W}^{iAB}\mathring{\bar{\lambda}}^{\bar{\jmath}}_A\gamma_\mu\psi^\mu_B + 2i\mathring{N}^A_\alpha\bar{\zeta}^\alpha\gamma_\mu\psi^\mu_A + \text{h.c.}\right) + \\
& + \frac{1}{\mu^2}\left(\mathring{\mathcal{M}}^{\alpha\beta}\bar{\zeta}_\alpha\zeta_\beta + \mathring{\mathcal{M}}^\alpha_{iB}\bar{\zeta}_\alpha\mathring{\lambda}^{iB} + \text{h.c.}\right)\Big\} + \Lambda\left(\mathring{\mathcal{M}}_{iAjB}\mathring{\bar{\lambda}}^{iA}\mathring{\lambda}^{jB} + \text{h.c.}\right).
\end{aligned} \tag{7.152}$$

$$\begin{aligned}
\mathcal{L}_{(0)} = {} & i\left(\bar{\mathcal{N}}_{\Lambda\Sigma}\hat{\mathcal{F}}^{-\Lambda}_{\mu\nu}\hat{\mathcal{F}}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\hat{\mathcal{F}}^{+\Lambda}_{\mu\nu}\hat{\mathcal{F}}^{+\Sigma\mu\nu}\right) + 6\mathcal{M}^{mn}\mathcal{H}_{m|\mu\nu\rho}\mathcal{H}_n^{\ \mu\nu\rho} + \\
& + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\bar{\psi}^A_\mu\gamma_\nu\rho_{A|\lambda\sigma} - \bar{\psi}_{A|\mu}\gamma_\nu\rho^A_{\lambda\sigma}\right) - \frac{i}{2}\mathring{g}_{i\bar{\jmath}}\left(\mathring{\bar{\lambda}}^{iA}\gamma^\mu\nabla_\mu\mathring{\lambda}^{\bar{\jmath}}_A + \mathring{\bar{\lambda}}^{\bar{\jmath}}_A\gamma^\mu\nabla_\mu\mathring{\lambda}^{iA}\right) + \\
& - i\left(\bar{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha + \bar{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\right) + \\
& - \frac{1}{\mu}\mathring{g}_{i\bar{\jmath}}[\partial_\mu\mathring{\bar{z}}^{\bar{\jmath}}\left(\bar{\psi}^\mu_A\mathring{\lambda}^{iA} - \mathring{\bar{\lambda}}^{iA}\gamma^{\mu\nu}\psi_{A\nu}\right) + h.c.] - 2\mathcal{U}^{\alpha A}_u\partial_\mu q^u\left(\bar{\psi}^\mu_A\zeta_\alpha - \bar{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\nu} + h.c.\right)
\end{aligned} \tag{7.153}$$

$$\begin{aligned}
\mathcal{L}_{(-1)} = {} & \Lambda^{-1}\mathcal{F}^{-I}_{\mu\nu}\mathring{I}_{IJ}\left[\frac{1}{2}\nabla_i\mathring{f}^J_j\mathring{\bar{\lambda}}^{iA}\gamma^{\mu\nu}\mathring{\lambda}^{jB}\epsilon_{AB} + h.c.\right] + \\
& + M_{Pl}^{-1}\Big\{\mathcal{F}^{-0}_{\mu\nu}\mathring{I}_{00}\mathring{L}^0\left[\bar{\psi}^{A\mu}\psi^{B\nu}\epsilon_{AB} - \bar{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta\mathbb{C}^{\alpha\beta} + h.c.\right] + \\
& \quad - \mathcal{F}^{-I}_{\mu\nu}\mathring{I}_{IJ}\left[4i\mathring{\bar{f}}^J_{\bar{\imath}}\mathring{\bar{\lambda}}^{\bar{\imath}}_A\gamma^\nu\psi^\mu_B\epsilon^{AB} + h.c.\right] \\
& \quad + 2\mathcal{M}^{mn}\mathcal{H}_m^{\ \mu\nu\rho}\left[\mathcal{U}_n^{\ A\alpha}\left(3i\bar{\psi}_{A\mu}\gamma_{\nu\rho}\zeta_\alpha + \bar{\psi}_{A\mu}\zeta_\alpha\right) + i\Delta_{n\alpha}^{\ \ \beta}\,\zeta_\beta\gamma_{\mu\nu\rho}\zeta^\alpha\right]\Big\}\ .
\end{aligned} \tag{7.154}$$

Then, performing the rigid limit $\mu = M_{Pl}/\Lambda \to \infty$, we find

$$\mathcal{L}_{(4)} = \Lambda^4 \mathring{\mathcal{V}}(z,q) \tag{7.155}$$

$$\mathcal{L}_{(2)} = M_{Pl}^2 \left(-\frac{R}{2} + h_{uv}\partial_\mu q^u \partial^\mu q^v\right) + \Lambda^2 \mathring{g}_{i\bar{\jmath}}\partial^\mu z^i \partial_\mu \bar{z}^{\bar{\jmath}} \tag{7.156}$$

$$\mathcal{L}_{(1)} = -2\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} M_{Pl}\mathcal{H}_{m|\nu\rho\sigma}A_u^m \partial_\mu q^u + \Lambda\left(\mathring{\mathcal{M}}_{iAjB}\mathring{\bar{\lambda}}^{iA}\mathring{\lambda}^{jB} + \text{h.c.}\right). \tag{7.157}$$

$$\begin{aligned}\mathcal{L}_{(0)} = {} & i\left(\mathring{\bar{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma\mu\nu} - \mathring{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma\mu\nu}\right) + 6\mathcal{M}^{mn}\mathcal{H}_{m\mu\nu\rho}\mathcal{H}_n^{\ \mu\nu\rho} + \\ & +\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\bar{\psi}^A_\mu\gamma_\nu\rho_{A|\lambda\sigma} - \bar{\psi}_{A|\mu}\gamma_\nu\rho^A_{\lambda\sigma}\right) - \frac{i}{2}\mathring{g}_{i\bar{\jmath}}\left(\mathring{\bar{\lambda}}^{iA}\gamma^\mu\nabla_\mu\mathring{\lambda}^{\bar{\jmath}}_A + \mathring{\bar{\lambda}}^{\bar{\jmath}}_A\gamma^\mu\nabla_\mu\mathring{\lambda}^{iA}\right) + \\ & -i\left(\bar{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha + \bar{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\right) - 2\mathcal{U}_u^{\alpha A}\partial_\mu q^u\left(\bar{\psi}^\mu_A\zeta_\alpha - \bar{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\nu} + h.c.\right)\end{aligned} \tag{7.158}$$

$$\mathcal{L}_{(-1)} = \Lambda^{-1}\mathring{\mathcal{F}}^{-I}_{\mu\nu}\mathring{I}_{IJ}\left[\frac{1}{2}\nabla_i\mathring{f}^J_j\mathring{\bar{\lambda}}^{iA}\gamma^{\mu\nu}\mathring{\lambda}^{jB}\epsilon_{AB} + h.c.\right]. \tag{7.159}$$

The observable sector corresponds to the $\mathcal{N}=1$ rigid supersymmetric Lagrangian obtained as a rigid limit of a $\mathcal{N}=2$ supergravity Lagragian partially broken. Let us note that the $\mathcal{N}=2$ supergravity Lagrangian reduce to the multi-vector generalization of the rigid Lagrangian of the APT model [71]. On the other hand, the hidden sector is still propagating but fully decoupled from the observable one,

$$\mathcal{L}_{sugra} \to \mathcal{L}_{APT} + \mathcal{L}_{hidden},$$

where

$$\begin{aligned}\mathcal{L}_{APT} = {} & \Lambda^2\mathring{g}_{i\bar{\jmath}}\partial^\mu z^i\partial_\mu\bar{z}^{\bar{\jmath}} - \frac{i}{2}\mathring{g}_{i\bar{\jmath}}\left(\mathring{\bar{\lambda}}^{iA}\gamma^\mu\nabla_\mu\mathring{\lambda}^{\bar{\jmath}}_A + \mathring{\bar{\lambda}}^{\bar{\jmath}}_A\gamma^\mu\nabla_\mu\mathring{\lambda}^{iA}\right) + \\ & +i\left(\mathring{\bar{N}}_{IJ}\mathcal{F}^{-I}_{\mu\nu}\mathcal{F}^{-J\mu\nu} - \mathring{\mathcal{N}}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J\mu\nu}\right) + \\ & +\Lambda^4\mathring{\mathcal{V}} + \Lambda\left(\mathring{\mathcal{M}}_{iAjB}\mathring{\bar{\lambda}}^{iA}\mathring{\lambda}^{jB} + \text{h.c.}\right) + \\ & +\Lambda^{-1}\mathring{\mathcal{F}}^{-I}_{\mu\nu}\mathring{I}_{IJ}\left[\frac{1}{2}\nabla_i\mathring{f}^J_j\mathring{\bar{\lambda}}^{iA}\gamma^{\mu\nu}\mathring{\lambda}^{jB}\epsilon_{AB} + h.c.\right],\end{aligned} \tag{7.160}$$

$$
\begin{aligned}
\mathcal{L}_{hidden} \;=\;& M_{Pl}^2\left(-\frac{R}{2}+h_{uv}\partial_\mu q^u\partial^\mu q^v\right)+i\left(\bar{\mathring{N}}_{00}\mathcal{F}^{-0}_{\mu\nu}\mathcal{F}^{-0\mu\nu}-\mathring{\mathcal{N}}_{00}\mathcal{F}^{+0}_{\mu\nu}\mathcal{F}^{+0\mu\nu}\right)+\\
&+6\mathcal{M}^{mn}\mathcal{H}_{m|\mu\nu\rho}\mathcal{H}_n^{\mu\nu\rho}-2\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}M_{Pl}\mathcal{H}_{m|\nu\rho\sigma}A^m_u\partial_\mu q^u+\\
&+\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\bar{\psi}^A_\mu\gamma_\nu\rho_{A|\lambda\sigma}-\bar{\psi}_{A|\mu}\gamma_\nu\rho^A_{\lambda\sigma}\right)-i\left(\bar{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha+\bar{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\right)+\\
&-2\mathcal{U}^{\alpha A}_u\partial_\mu q^u\left(\bar{\psi}^\mu_A\zeta_\alpha-\bar{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\nu}+h.c.\right).
\end{aligned}
\tag{7.161}
$$

Thus, the high-energy supergravity Lagrangian is characterized by a visible sector surviving the rigid limit and by a hidden one consisting of the gravitational multiplet and by a hypermultiplet, which decouple when the Planck mass is sent to infinity.

Chapter 8

Conclusion

In the present thesis we have presented, using a geometrical formalism, diverse supergravity theories in different frameworks. In particular, we have incorporated diverse interesting features to supergravity models like enlarged symmetries, three-dimensional space-time, matter couplings and the presence of cosmological constant.

In chapter 4, we have shown that the Maxwell superalgebras can be derived by the semigroup expansion procedure. In particular, the minimal Maxwell superalgebra and its generalizations can be obtained as an S-expansion of the $\mathfrak{osp}\left(4|1\right)$ superalgebra with a suitable semigroup S [31]. Interestingly, using the MacDowell-Mansouri approach, we showed that the supergravity action, constructed out of the curvatures of a minimal Maxwell superalgebra $s\mathcal{M}_4$, describes pure supergravity in four dimensions [32]. This result can be seen as a supersymmetric generalization of ref. [11] (see chapter 2) in which four-dimensional General Relativity can be derived from Maxwell algebra as a Born-Infeld gravity action. Additionally, we presented an analyze of the invariance of the supergravity action under the Maxwell supersymmetry transformations. The Maxwell symmetries could play an important role in higher dimensions supergravity theories. Indeed, it seems that it should be possible to recover standard odd-dimensional supergravity from the Maxwell superalgebras.

In chapter 5, we have presented an alternative method of introducing the supersymmetric cosmological term to a supergravity action à la MacDowell-Mansouri [33]. In particular, we showed the the AdS-Lorentz superalgebra allows to add new terms to the supergravity action, describing a generalized supersymmetric cosmological constant. This superalgebra and its $\mathcal{N}$-extended generalization has been derived through the S-expansion procedure. Interestingly, this expansion method gives us the components of an invariant tensor in terms of particular choice of the invariant tensor of the $\mathfrak{osp}\,(4|1)$ superalgebra, allowing to construct a supergravity action in the geometric formalism. Although there already exists supergravity theories with cosmological constant, the supergravity action à la MacDowell-Mansouri suggests a superconformal structure which represents an additional motivation in our construction.

In chapter 6, we analyzed the construction of a three-dimensional Chern-Simons supergravity action using a minimal Maxwell superalgebra [59]. To this purpose, we used the S-expansion method in order to obtain the Maxwell superalgebra $s\mathcal{M}_3$ from the $\mathfrak{osp}\,(2|1) \otimes \mathfrak{sp}\,(2)$ superalgebra. Additionally, using the usual Maxwell superalgebra, we briefly studied an exotic supersymmetric action combining the expansion and contraction procedures. Interestingly, the model considered here represents a toy model in order to approach richer theories in higher dimensions or in higher $\mathcal{N}$-extended supersymmetric theories.

Eventually, in chapter 7, we presented a multi-vector generalization of a rigid partially broken $\mathcal{N}=2$ supersymmetric theory as a rigid limit of a suitable gauged $\mathcal{N}=2$ supergravity theory in presence of electric and magnetic charges [73]. Interestingly, the $\mathcal{N}=1$ rigid supersymmetric theory corresponds to a generalization of the APT model [71] to a generic number n of vector multiplets. The purpose of this chapter was to elucidate the supergravity origin of the multifield supersymmetric Born-Infeld theory and to understand the origin of the dyonic Fayet-Iliopoulos terms. Furthermore, the $\mathcal{N}=2$ supergravity Ward identity for generic dyonic gaugings and its rigid limit was also approached.

Appendix

Appendix A

Abelian semigroup expansion procedure

The derivation of new Lie (super)algebras from a given one is an interesting problem in Physics since it allows to derive new physical theories from an already known. Nowadays, there are four different ways to relate and obtain diverse Lie (super)algebras. Interestingly, the expansion procedure leads to higher-dimensional Lie algebra from a known one. The expansion method was initially proposed in the context of AdS superstring by M. Hadsuda and M. Sakaguchi in ref. [94]. Subsequently, a method based on the Maurer-Cartan (MC) forms power-series expansion has been presented in ref. [95] and subsequently developed in refs. [96, 97] with interesting physical implications.

Recently, F. Izaurieta, E. Rodríguez and P. Salgado have proposed an alternative expansion method in ref. [16]. Unlike the Maurer-Cartan expansion, the expansion procedure introduced in ref. [16] is based on operations performed on the (super)algebra generators. Basically, it consists in combining the structure constants of a Lie (super)algebra $\mathfrak{g}$ with the inner multiplication law of a semigroup S leading to the Lie brackets of a new Lie (super)algebra $\mathfrak{G} = S \times \mathfrak{g}$.

Let $\mathfrak{g}$ a Lie (super)algebra with basis T_A and structure constants $C_{AB}^{\ \ C}$ and let $S = \{\lambda_\alpha\}$ be a finite abelian semigroup with 2-selector $K_{\alpha\beta}^{\ \ \gamma}$. Then, the direct product $S \times \mathfrak{g}$ is also a Lie algebra given by

$$\left[T_{(A,\alpha)}, T_{(B,\beta)}\right] = K_{\alpha\beta}^{\ \ \gamma} C_{AB}^{\ \ C} T_{(C,\gamma)}. \tag{A.1}$$

The Lie (super)algebra $\mathfrak{G}$ defined by $\mathfrak{G} = S \times \mathfrak{g}$ is called the S-expanded (super)algebra of $\mathfrak{g}$ and is generated by $T_{(A,\alpha)} = \lambda_\alpha T_A$.

Interestingly, smaller (super)algebras can be extracted from an S-expanded (super)algebra $\mathfrak{G} = S \times \mathfrak{g}$. However, it is first necessary to consider a decomposition of the original (super)algebra $\mathfrak{g}$ in subspaces V_p such that $\mathfrak{g} = \bigoplus_{p \in I} V_p$, where I is a set of indices. For each $p, q \in I$, one can define $i_{(p,q)} \subset I$ such that

$$[V_p, V_q] \subset \bigoplus_{r \in i_{(p,q)}} V_r. \tag{A.2}$$

According to the definitions of ref. [16], it is possible to consider a particular subset decomposition of the semigroup $S = \bigcup_{p \in I} S_p$ such that

$$S_p \cdot S_q \subset \bigcap_{r \in i_{(p,q)}} S_r. \tag{A.3}$$

When such decomposition exists, we say that this subset decomposition is in resonant with the subspace decomposition of the Lie (super)algebra $\mathfrak{g}$ and

$$\mathfrak{G}_R = \bigoplus_{p \in I} S_p \times V_p,$$

is a resonant (super) subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$. Therefore, in order to derive a resonant S-expanded (super)algebra, we just need to solve the resonance condition for a finite abelian semigroup S.

A smaller (super)algebra can be obtained when the semigroup has a zero element $0_S \in S$ such that for all $\lambda_\alpha \in S$, we have $0_S \lambda_\alpha = 0_S$. In particular, the (super)algebra derived by imposing the 0_S-reduction condition $0_S T_A = 0$ on $\mathfrak{G}$ is called the 0_S-reduced algebra of $\mathfrak{G} = S \times \mathfrak{g}$.

On the other hand, it is possible to extract a reduced (super)algebra from a resonant (super) subalgebra. Indeed, let us consider $\mathfrak{G}_R = \bigoplus_p S_p \times V_p$ as the resonant (super) subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$. Let $S_p = \hat{S}_p \cup \check{S}_p$ be a subset decomposition with $S_p \subset S$ such that

$$\hat{S}_p \cap \check{S}_p = \varnothing, \tag{A.4}$$

$$\check{S}_p \cdot \hat{S}_q \subset \bigcap_{r \in i_{(p,q)}} \hat{S}_r. \tag{A.5}$$

Then, the conditions $(A.4)-(A.5)$ induce the partition

$$\check{\mathfrak{G}}_R = \bigoplus_{p\in I} \check{S}_p \times V_p, \tag{A.6}$$

$$\hat{\mathfrak{G}}_R = \bigoplus_{p\in I} \hat{S}_p \times V_p, \tag{A.7}$$

with

$$\left[\check{\mathfrak{G}}_R, \hat{\mathfrak{G}}_R\right] \subset \hat{\mathfrak{G}}_R. \tag{A.8}$$

Thus, $\left|\check{\mathfrak{G}}_R\right|$ corresponds to a reduced (super)algebra of $\mathfrak{G}_R$ [16].

A useful property of this expansion mechanism is that it provides us with the components of an invariant tensor for the S-expanded (super)algebra in terms of the components of an invariant tensor for the original (super)algebra $\mathfrak{g}$. Following theorem VII.1 of ref. [16], let S be an abelian semigroup with the n-selector $K_{\alpha_1\cdots\alpha_n}{}^{\gamma}$, $\mathfrak{g}$ a Lie (super)algebra of basis $\{T_A\}$ and let $\langle T_{A_1}\cdots T_{A_n}\rangle$ be an invariant tensor for the original algebra g. Then,

$$\left\langle T_{(A_1,\alpha_1)}\cdots T_{(A_n,\alpha_n)}\right\rangle = \alpha_\gamma K_{\alpha_1\cdots\alpha_n}{}^{\gamma} \left\langle T_{A_1}\cdots T_{A_n}\right\rangle \tag{A.9}$$

corresponds to the invariant tensor for the S-expanded (super)algebra $\mathfrak{G} = S\times\mathfrak{g}$.

Appendix B

Gamma matrices identities and conventions

In this Appendix we briefly review some gamma matrices identities and the conventions used in the present thesis. The Dirac gamma matrices in a four-dimensional space-time are defined through the relation

$$\{\ \gamma_a, \gamma_b\} = -2\eta_{ab}, \tag{B.1}$$

where $\eta_{ab} = (-1, 1, 1, 1)$ is the Minkowski metric. This gamma matrices satisfy the Clifford Algebra:

$$\begin{aligned}
[\gamma_a, \gamma_b] &= 2\gamma_{ab}, &\text{(B.2)}\\
\gamma_5 &\equiv -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4, &\text{(B.3)}\\
\gamma_5^2 &= -1, &\text{(B.4)}\\
\{\gamma_5, \gamma_a\} &= \quad [\gamma_5, \gamma_{ab}] = 0, &\text{(B.5)}\\
\gamma_{ab}\gamma_5 &= -\frac{1}{2}\epsilon_{abcd}\gamma^{cd}, &\text{(B.6)}\\
\gamma_a\gamma_b &= \gamma_{ab} - \eta_{ab}, &\text{(B.7)}\\
\gamma^{ab}\gamma_{cd} &= \epsilon^{ab}{}_{cd}\gamma_5 - 4\delta^{[a}_{[c}\gamma^{b]}_{d]} - 2\delta^{ab}_{cd}, &\text{(B.8)}\\
\gamma^{ab}\gamma^c &= 2\gamma^{[a}\delta^{b]}_c - \epsilon^{abcd}\gamma_5\gamma_d, &\text{(B.9)}\\
\gamma^c\gamma^{ab} &= -2\gamma^{[a}\delta^{b]}_c - \epsilon^{abcd}\gamma_5\gamma_d. &\text{(B.10)}
\end{aligned}$$

In particular, in the present thesis we are working with the Majorana spinors which satisfy the Majorana condition $\bar{\psi} = \psi^T C$, where C is the charge conjugation matrix.

Furthermore, the gamma matrices satisfy

$$(C\gamma_a)^T = C\gamma_a \ , \tag{B.11}$$

$$(C\gamma_{ab})^T = C\gamma_{ab}, \tag{B.12}$$

while

$$C^T = -C, \tag{B.13}$$

$$(C\gamma_5)^T = -C\gamma_5, \tag{B.14}$$

$$(C\gamma_5\gamma_a)^T = -C\gamma_5\gamma_a, \tag{B.15}$$

which means that $C\gamma_a$ and $C\gamma_{ab}$ are symmetric, while C, $C\gamma_5$ and $C\gamma_5\gamma_a$ are antisymmetric gamma matrices. This leads to the following identities for the p-form ψ and q-form ξ:

$$\bar{\psi}\xi = (-1)^{pq}\,\bar{\xi}\psi, \tag{B.16}$$

$$\psi S\xi = -(-1)^{pq}\,\xi\psi, \tag{B.17}$$

$$\psi A\xi = (-1)^{pq}\,\xi\psi, \tag{B.18}$$

where S and A are symmetric and antisymmetric matrices, respectively. This properties allows to write some useful Fierz identities:

$$\psi\bar{\psi} = \frac{1}{2}\gamma_a\bar{\psi}\gamma^a\psi - \frac{1}{8}\gamma_{ab}\bar{\psi}\gamma^{ab}\psi, \tag{B.19}$$

$$\gamma_a\psi\bar{\psi}\gamma^a\psi = 0, \tag{B.20}$$

$$\gamma_{ab}\psi\bar{\psi}\gamma^{ab}\psi = 0. \tag{B.21}$$

Appendix C

Relevant relations on the sigma-model geometry

In the present appendix, following ref. [73], we present some important relations required to a good understanding of the $\mathcal{N}=2$ supergravity model considered in the present thesis.

A special Kähler manifold $\mathcal{M}_{SK}$ is locally described by a holomorphic section Ω and a choice of complex coordinates z^i,

$$\Omega(z) = \begin{pmatrix} X^{\Lambda}(z) \\ F_{\Lambda}(z) \end{pmatrix}, \quad \Lambda = 0, \ldots, n, \tag{C.1}$$

such that the Kähler potential is given by

$$\mathcal{K}(z,\bar{z}) = -\log[i\,\overline{\Omega}(\bar{z})^T \mathbb{C}\Omega(z)]\,. \tag{C.2}$$

We define the covariantly holomorphic section V^M in terms of Ω and κ as

$$V^M \equiv e^{\frac{\mathcal{K}}{2}}\,\Omega^M. \tag{C.3}$$

One can associate a holomorphic function $f_g(z)$ and a symplectic matrix $\mathbb{M}[g] = (\mathbb{M}[g]_M{}^N)$ to each element g of the identity component G_{SK} of the isometry group of the special Kähler manifold $\mathcal{M}_{SK}$. Indeed, if $g: z^i \to z'^i = z'^i(z)$, we have

$$\Omega\,(z') = e^{f_g(z)}\,\mathbb{M}[g]^{-T}\,\Omega(z) \;\Leftrightarrow\; \mathcal{K}(z',\bar{z}') = \mathcal{K}(z,\bar{z}) - f_g(z) - \bar{f}_g(\bar{z})\,. \tag{C.4}$$

Let $\{t_a\}$ be the infinitesimal generators of G_{SK} and let $k_a = k_a^i\,(z)\,\partial_i + k_a^{\bar\imath}\,(\bar z)\,\partial_{\bar\imath}$ be the Killing vectors satisfying

$$[t_a,\, t_b] = f_{ab}{}^c\, t_c\ ,\ \ [k_a,\, k_b] = -f_{ab}{}^c\, k_c\,,$$

then eqs. $(C.4)$ imply

$$\begin{aligned}\ell_a\Omega^M &= k_a^i\partial_i\Omega^M = -t_{aN}{}^M\,\Omega^N + f_a(z)\Omega^M\ , && \text{(C.5)}\\ \ell_a\mathcal{K} &= k_a^i\partial_i\mathcal{K} + k_a^{\bar\imath}\partial_{\bar\imath}\mathcal{K} = -(f_a + \bar f_a)\,\mathcal{K}\,, && \text{(C.6)}\\ \ell_a V^M &= (k_a^i\partial_i + k_a^{\bar\imath}\partial_{\bar\imath})V^M = -t_{aN}{}^M\,V^N + \frac{f_a - \bar f_a}{2}\,V^M\,. && \text{(C.7)}\end{aligned}$$

Here $t_{aN}{}^M$ corresponds to the symplectic matrix representation of the generator t_a on covariant vectors,

$$t_{a[N}{}^P\mathbb{C}_{M]P} = 0\,,\ (t_a\Omega)^M = -t_{aN}{}^M\,\Omega^N. \qquad \text{(C.8)}$$

Let $\mathcal{P}_a(z,\bar z)$ be the momentum map corresponding to k_a, defined as [81],

$$k_a^i = i\,g^{i\bar\jmath}\,\partial_{\bar\jmath}\mathcal{P}_a\,,\ \ k_a^{\bar\imath} = -i\,g^{\bar\imath i}\,\partial_i\mathcal{P}_a\,, \qquad \text{(C.9)}$$

and satisfying

$$ig_{i\bar\jmath}\,k^i_{[a}\,k^{\bar\jmath}_{b]} = -\frac{1}{2}\,f_{ab}{}^c\,(\mathcal{P}_c - C_c)\,. \qquad \text{(C.10)}$$

Here, C_c is a constant vector which can be reabsorbed by a redefinition of $\mathcal{P}_c$. In what follows we shall consider the following redefinition,

$$\mathcal{P}_c - C_c \to \mathcal{P}_c. \qquad \text{(C.11)}$$

In particular, using eq. $(C.6)$, eqs. $(C.9)$ are solved by

$$\begin{aligned}\mathcal{P}_a &= -\frac{i}{2}\,\left(k_a^i\partial_i\mathcal{K} - k_a^{\bar\imath}\partial_{\bar\imath}\mathcal{K}\right) + \mathrm{Im}(f_a)\\ &= i\,k_a^{\bar\imath}\partial_{\bar\imath}\mathcal{K} + i\,\bar f_a = -i\,k_a^i\partial_i\mathcal{K} - i\,f_a\,. && \text{(C.12)}\end{aligned}$$

Then, using eqs. $(C.12)$ and $(C.7)$, we find

$$k_a^i\,U_i^M = -t_{aN}{}^M\,V^N + i\,\mathcal{P}_a\,V^M\,. \qquad \text{(C.13)}$$

Interestingly, contracting eq. $(C.13)$ with $\mathbb{C}\bar V$ and using the relations $V^T\mathbb{C}\overline{V} = i$, $V^T\mathbb{C}U_i = 0$, we obtain

$$\mathcal{P}_a = -V^N\,t_{aNM}\overline{V}^M = -\overline{V}^N\,t_{aNM}\,V^P\,, \qquad \text{(C.14)}$$

where

$$t_{aNM} \equiv t_{aN}{}^{P} C_{PM} = t_{aMN}. \tag{C.15}$$

The general property

$$t_{a\,MN} \Omega^M \Omega^N = 0\ , \quad \forall t_a\,, \tag{C.16}$$

follows by contracting eq. $(C.5)$ by $\mathbb{C}\Omega$ and using $V^T \mathbb{C} U_i = 0$, which implies

$$\Omega^T \mathbb{C} \partial_i \Omega = 0\,. \tag{C.17}$$

Let us now consider infinitesimal isometries of the quaternionic Kähler manifold $\mathcal{M}_{QK}$. These isometries are generated by t_m whose action on the scalar fields is described by Killing vectors $k_m = k_m^u \partial_u$. In particular, they satisfy the isometry algebra

$$[t_m,\, t_n] = f_{mn}{}^{p}\, t_p \ \ , \quad [k_m,\, k_n] = -f_{mn}{}^{p}\, k_p\,,$$

and leave the 4-form $\sum_{x=1}^{3} K^x \wedge K^x$ invariant [81], which amounts to requiring

$$\ell_n K^x = \epsilon^{xyz}\, K^y\, W_n^z\,. \tag{C.18}$$

Here, W_n^z corresponds to an $SU\,(2)$-compensator. Writing the Killing vectors k_n in term of tri-holomorphic momentum maps $\mathcal{P}_n^x$ it is possible to solve eq. $(C.18)$ as

$$\iota_n K^x = -\nabla \mathcal{P}_n^x = -(d\mathcal{P}_n^x + \epsilon^{xyz} \omega^y\, \mathcal{P}_n^z)\,, \tag{C.19}$$

provided

$$\mathcal{P}_n^x = \lambda^{-1}(\iota_n \omega^x - W_n^x) = W_n^x - \iota_n \omega^x\,, \tag{C.20}$$

where we have defined $\lambda = -1$. In particular, in the case of vanishing compensator, $W_n^x = 0$, the momentum maps have the simple expression

$$\mathcal{P}_n^x = -k_n^u\, \omega_u^x. \tag{C.21}$$

As for the special Kähler manifolds, the momentum maps satisfy Poisson brackets described by the following condition

$$2\, K_{uv}\, k_n^u\, k_m^v - \lambda\, \epsilon^{xyz}\, \mathcal{P}_n^y\, \mathcal{P}_m^z = -f_{mn}{}^{p}\, \mathcal{P}_p^x\,. \tag{C.22}$$

Appendix D

The General Ward Identity for a generic $\mathcal{N}=2$ supergravity gauging

In the present appendix, we prove the general Ward identity for the generic dyonic gauging of $\mathcal{N}=2$ supergravity. To this aim, we will evaluate each term in the left hand side of eq. (7.65),

$$g_{i\bar{\jmath}}\,W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC}+2\,N_{\alpha}{}^{A}\,N^{\alpha}{}_{B}-12\,S^{AC}S_{BC}=\delta^{B}_{A}\,V(z,\bar{z},q)\,. \tag{D.1}$$

Let us consider the symplectically-invariant generalization of the fermionic shifts

$$\begin{aligned} W^{i\,AB} &= \epsilon^{AB}\,k^{i}_{M}\,\overline{V}^{M}-i\,(\sigma^{x})_{C}{}^{B}\epsilon_{CA}\mathcal{P}^{x}_{M}\,g^{i\bar{\jmath}}\overline{U}^{M}_{\bar{\jmath}}\,, && \text{(D.2)}\\ S_{AB} &= \frac{i}{2}\,(\sigma^{x})_{A}{}^{C}\epsilon_{BC}\,\mathcal{P}^{x}_{M}\,V^{M}\,, && \text{(D.3)}\\ N_{\alpha}{}^{A} &= 2\,\mathcal{U}^{A}_{u\,\alpha}\,k^{u}_{M}\,\overline{V}^{M}\ ,\quad N^{\alpha}{}_{A}\equiv(N_{\alpha}{}^{A})^{*}=-2\,\mathcal{U}_{u\,A}{}^{\alpha}\,k^{u}_{M}\,V^{M}\,. && \text{(D.4)} \end{aligned}$$

Then we have

$$\begin{aligned} W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC}g_{i\bar{\jmath}} &= \delta^{A}_{B}\,k^{i}_{M}k^{\bar{\jmath}}_{N}g_{i\bar{\jmath}}\overline{V}^{M}\,V^{N}\\ &\quad -i\,(\sigma^{x})_{B}{}^{A}\left(k^{\bar{\jmath}}_{M}\,V^{M}\,\overline{U}^{N}_{\bar{\jmath}}-k^{i}_{M}\,\overline{V}^{M}\,U^{N}_{i}\right)\mathcal{P}^{x}_{N}\\ &\quad +(\sigma^{x}\sigma^{y})_{B}{}^{A}\,\mathcal{P}^{x}_{M}\mathcal{P}^{y}_{N}U^{MN}\,, \end{aligned} \tag{D.5}$$

with

$$U^{MN}\equiv U^{N}_{i}\,g^{i\bar{\jmath}}\,\overline{U}^{N}_{\bar{\jmath}}\,. \tag{D.6}$$

Splitting the terms proportional to δ_B^A from those proportional to $(\sigma^x)_B{}^A$ and using eq. (7.64),

$$\overline{V}^M k^i_M\, U^P_i \Theta_P = -X_{MN}{}^P\, \overline{V}^M V^N \Theta_P = X_{NM}{}^P\, \overline{V}^M V^N \Theta_P = -V^M k^{\bar{\imath}}_M\, \overline{U}^P_{\bar{\imath}}\, \Theta_P\,, \quad \text{(D.7)}$$

we find

$$\begin{aligned} W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC} g_{i\bar{\jmath}} &= \delta^A_B \left(k^i_M k^{\bar{\jmath}}_N g_{i\bar{\jmath}} \overline{V}^M\, V^N + \mathcal{P}^x_N \mathcal{P}^x_M U^{MN} \right) \\ &\quad + i\,(\sigma^x)_B{}^A \left(-2\, X_{MN}{}^P \overline{V}^M\, V^N\, \mathcal{P}^x_P + \epsilon^{xyz}\, \mathcal{P}^y_M \mathcal{P}^z_N U^{[MN]} \right)\,. \quad \text{(D.8)} \end{aligned}$$

Let us now consider the general symplectic covariant relation given by eq. (7.25),

$$U^{MN} \equiv g^{i\bar{\jmath}}\, U^M_i U^N_{\bar{\jmath}} = -\frac{1}{2}\mathcal{M}^{MN} - \frac{i}{2}\,\mathbb{C}^{MN} - \overline{V}^M V^N\,, \quad \text{(D.9)}$$

and let be the locality constraint given by eq. (7.49),

$$\Theta_M{}^a \mathbb{C}^{MN} \Theta_N{}^b = \Theta_M{}^a \mathbb{C}^{MN} \Theta_N{}^n = \Theta_M{}^m \mathbb{C}^{MN} \Theta_N{}^n = 0\,. \quad \text{(D.10)}$$

Then we can write

$$\mathcal{P}^y_M \mathcal{P}^z_N U^{[MN]} = -\frac{i}{2}\,\mathcal{P}^y_M \mathcal{P}^z_N \mathbb{C}^{MN} - \mathcal{P}^y_M \mathcal{P}^z_N \overline{V}^{[M} V^{N]} = -\mathcal{P}^y_M \mathcal{P}^z_N \overline{V}^{[M} V^{N]}\,, \quad \text{(D.11)}$$

leading to

$$\begin{aligned} W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC} g_{i\bar{\jmath}} &= \delta^A_B \left(k^i_M k^{\bar{\jmath}}_N g_{i\bar{\jmath}} \overline{V}^M\, V^N + \mathcal{P}^x_N \mathcal{P}^x_M U^{MN} \right) \\ &\quad + i\,(\sigma^x)_B{}^A \left(-2\, X_{MN}{}^P \overline{V}^M\, V^N\, \mathcal{P}^x_P - \epsilon^{xyz}\, \mathcal{P}^y_M \mathcal{P}^z_N\, \overline{V}^M V^N \right) \text{(D.12)} \end{aligned}$$

Let us now consider the square of the gravitini shifts,

$$\begin{aligned} -12\, S^{AC}\, S_{BC} &= -3\,(\sigma^x \sigma^y)_B{}^A\, \mathcal{P}^x_M \mathcal{P}^y_N\, V^M \overline{V}^N \\ &= -3\, \mathcal{P}^x_M \mathcal{P}^x_N\, V^M \overline{V}^N + 3i\, \epsilon^{xyz}\, \mathcal{P}^y_M \mathcal{P}^z_N\, \overline{V}^M V^N (\sigma^x)_B{}^A\,. \quad \text{(D.13)} \end{aligned}$$

While the square of the hyperini shifts is given by

$$\begin{aligned} 2\, N_\alpha{}^A\, N^\alpha{}_A &= 8\,\mathcal{U}^{A\alpha}_u\, \mathcal{U}_{v\,B\alpha}\, k^u_M\, k^v_N\, \overline{V}^M V^N \\ &= 4\left(\delta^A_B h_{uv} + i\,(\sigma^x)_B{}^A\, K^x_{uv} \right) k^u_M\, k^v_N\, \overline{V}^M V^N\,, \quad \text{(D.14)} \end{aligned}$$

where we have used eq. (7.43).

Then, we can compute the left hand side of the Ward identity,

$$g_{i\bar{\jmath}}\,W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC} + 2\,N_\alpha{}^A\,N^\alpha{}_B - 12\,S^{AC}S_{BC} = \delta^A_B\,V(z,\bar{z},q) + i\,Z^x\,(\sigma^x)_B{}^A\,, \qquad \text{(D.15)}$$

where the general symplectic invariant expression of the scalar potential is given by

$$V(z,\bar{z},q) = (k^i_M k^{\bar{\jmath}}_N g_{i\bar{\jmath}} + 4\,h_{uv}k^u_M k^v_N)\overline{V}^M\,V^N + (U^{MN} - 3\,V^M\overline{V}^N)\mathcal{P}^x_N\mathcal{P}^x_M\,, \qquad \text{(D.16)}$$

and

$$Z^x = (-2\,X_{MN}{}^P\,\mathcal{P}^x_P + 2\,\epsilon^{xyz}\,\mathcal{P}^y_M\mathcal{P}^z_N + 4\,K^x_{uv}k^u_M\,k^v_N)\overline{V}^M V^N\,. \qquad \text{(D.17)}$$

In particular, from the equivariance condition (7.53),

$$2\,K^x_{uv}\,k^u_M\,k^v_N + \epsilon^{xyz}\,\mathcal{P}^y_M\,\mathcal{P}^z_N = X_{MN}{}^P\,\mathcal{P}^x_P\,, \qquad \text{(D.18)}$$

it follows that $Z^x = 0$, so that the Ward identity is proven.

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