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Author's version.
Published in: *Statistics & Probability Letters* 112 (2016), pp. 51–57.
doi: 10.1016/j.spl.2016.01.011,
URL: http://dx.doi.org/10.1016/j.spl.2016.01.011.

Abstract

A new property defined on the class of symmetric copulas is introduced and studied along this note. It is shown here that such a property can define a family of bivariate distribution functions satisfying all the characteristics listed in Kimeldorf and Sampson (1989) to be considered as a positive dependence notion. Applications, relationships with other positive dependence notions, further properties and the corresponding negative dependence notion are discussed as well.

Keywords: Positive Dependence, Copulas, Supermigrativity, Stochastic orders.

1 Preliminaries

In the last decades, the concept of positive dependence has played a significant role in many areas of applied probability and statistics, such as reliability or actuarial theory (see, e.g, the monographs by Joe (1997), Drouet-Mari and Kotz (2001), or Lai and Xie, 2006). Starting from the seminal papers by Kimeldorf and Sampson (1987) and (1989), many different notions of positive (or negative) dependence have been studied and applied in the literature to mathematically describe the different aspects and properties of this intuitive concept. Among others, recent theoretical and applied developments in this field are described in Genest and Verret (2002), Colangelo et al. (2005) and (2006), Durante et al. (2008), Cai and Wei (2012), Abhishek et al. (2015), and Bignozzi et al. (2015).

Since positive dependence involves several different aspects, Kimeldorf and Sampson in (1987) and in (1989) provided a unified framework for studying and relating three basic concepts of bivariate positive dependence: *positive dependence orderings* (i.e., comparisons based on dependence), *positive dependence notions* (i.e., classes of distributions satisfying some dependence properties) and *measures of positive dependence*. Dealing with positive dependence notions, a general statement for a bivariate property to be considered as a positive dependence notion, or for a class of bivariate distributions to be considered a positive dependence class, has been formulated in Kimeldorf and Sampson (1989) as follows. Here, $\mathcal{F}$ denotes the set of all bivariate distribution functions. Moreover, given any bivariate random vector (X, Y) having joint distribution F, and given $\mathcal{G} \subseteq \mathcal{F}$, the notation $(X, Y) \in \mathcal{G}$ will be sometimes used here and along the paper in place of $F \in \mathcal{G}$, whenever it will simplify the presentation.

Definition 1.1. *A subset $\mathcal{P}^+$ of the family $\mathcal{F}$ is a* positive dependence notion *if it satisfies the following seven conditions.*

(C1) $(X, Y) \in \mathcal{P}^+$ *implies* $P(X > x, Y > y) \geq P(X > x)P(Y > y)$ *for all* $x, y \in \mathbb{R}$.

(C2) $\mathcal{F}^+ \subseteq \mathcal{P}^+$, *where $\mathcal{F}^+$ denotes the set of upper Frèchet bounds, i.e., the set of bivariate distributions F such that* $F(x, y) = \min\big(F(x, +\infty), F(+\infty, y)\big)$.

(C3) *If (X, Y) is a pair of independent variables, then* $(X, Y) \in \mathcal{P}^+$.

(C4) $(X, Y) \in \mathcal{P}^+$ *implies* $(\phi(X), Y) \in \mathcal{P}^+$ *for all increasing functions* ϕ.

(C5) $(X, Y) \in \mathcal{P}^+$ *implies* $(Y, X) \in \mathcal{P}^+$.

(C6) $(X, Y) \in \mathcal{P}^+$ *implies* $(-X, -Y) \in \mathcal{P}^+$.

(C7) *Given the sequence $\{F_n, n \in N\}$, if $F_n \in \mathcal{P}^+, \forall n \in N$, and $F_n \to F$ in distribution as $n \to \infty$, then $F \in \mathcal{P}^+$.*

It should be pointed out that these seven conditions are logically independent, that is, if any six of them hold, then the seventh need not necessarily to hold (see Kimeldorf and Sampson, 1989).

Many of the most well–known positive dependence notions satisfy these axioms. This is for example the case of the *Positive Quadrant Dependence* (PQD) notion, of the *Totally Positive of order 2* (TP_2) notion and of the notion of *Association*, as shown in Kimeldorf and Sampson (1989) and the references therein.

Moreover, since all the monotone dependence properties based on the level of concordance between the components of a random vector (thus, based on rank invariant properties) are entirely described by its copula, whenever it exists and whenever it is unique, all the seven conditions described above can be translated in a more general setting in terms of families of copulas, without taking care of the marginal distributions (except assuming that they are continuous, to ensure a unique representation of the joint distribution with copulas). See, for example, Nelsen et al. (1997) or Fernandez Sanchez and Úbeda-Flores (2014) on relationships between copulas and positive dependence notions, indexes and orders. To provide the definition of positive dependence notion in these terms, we recall the definition of copula, and of survival copula.

Let (X, Y) be a random vector with the joint distribution function $F \in \mathcal{F}$ and marginal distributions F_1 and F_2. The function $C : [0,1]^2 \to [0,1]$ such that, for all $(x, y) \in \mathbb{R}^2$, satisfies

$$F(x, y) = C(F_1(x), F_2(y)).$$

is called *copula* of the vector (X, Y). In this case, it also holds

$$C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)),$$

for all $u, v \in [0,1]$, where F_i^{-1} denotes the quasi-inverse of F_i. Such a copula is a bivariate distribution function with margins uniformly distributed on $[0,1] \subset \mathbb{R}$, and is unique whenever F_1 and F_2 are continuous. For further details on copulas we refer the reader to the standard references Joe (1997) and Nelsen (2006). In a similar way is defined the survival copula, which is commonly considered in reliability analysis instead of the copula: given (X, Y) as above, and denoted $\overline{F}$, $\overline{F}_1$ and $\overline{F}_2$ its joint survival function and the marginal survival functions, its *survival copula* K is defined as

$$K(u, v) = \overline{F}(\overline{F}_1^{-1}(u), \overline{F}_2^{-1}(v)) = u + v - 1 + C(1 - u, 1 - v)$$

for all $u, v \in [0,1]$. See Nelsen (2006) for details.

Using copulas (or survival copulas), and recalling that the copula of a random pair is scale-invariant, thus that (C4) is automatically satisfied, restricting to continuous distribution functions one can restate the conditions described in Definition 1.1 as follows.

Definition 1.2. *Let $\mathcal{C}^+$ be a family of bivariate copulas, and let $\mathcal{P}^+$ be the subset of $\mathcal{F}$ of all continuous distribution functions F such that the copula associated with F belongs to $\mathcal{C}^+$. The family $\mathcal{P}^+$ is a* positive dependence notion *if $\mathcal{C}^+$ satisfies the following six conditions.*

(C1′) *$C(u, v) \geq uv$ for all $(u, v) \in [0,1]^2$ and for all $C \in \mathcal{C}^+$.*

(C2′) *The upper Frèchet copula C^+ defined as $C^+(u, v) = \min\big(u, v\big)$ belongs to $\mathcal{C}^+$.*

(C3′) *The independence copula $C^\perp$ defined as $C^\perp(u, v) = uv$ belongs to $\mathcal{C}^+$.*

(C5′) *If $C \in \mathcal{C}^+$ then $C^* \in \mathcal{C}^+$, where $C^*(u,v) = C(v,u),\ u,v \in [0,1]$.*

(C6′) *If $C \in \mathcal{C}^+$ then for the corresponding survival copula K it holds $K \in \mathcal{C}^+$.*

(C7′) *Given the sequence $\{C_n, n \in N\}$, if $C_n \in \mathcal{C}^+, \forall n \in N$, and $C_n \to C$ in distribution as $n \to \infty$, then $C \in \mathcal{C}^+$.*

The motivation of the present study is described now. It is an established fact that the common univariate comparisons among random lifetimes, such as the orders *st* (*usual stochastic order*), *hr* (*hazard rate order*) and *lr* (*likelihood ratio order*), are based on comparisons among the marginal distributions of the involved variables, without taking into consideration their dependence structure. Due to this reason, bivariate characterizations of the most well-known stochastic orders have been defined and studied by several authors, in order to be able to take into account their mutual dependence as well. These characterizations gave rise to new stochastic comparisons, commonly called *joint stochastic orders*, namely, *st:j* (*usual joint stochastic order*), *hr:j* (*joint hazard rate order*) and *lr:j* (*joint likelihood ratio order*), which are equivalent to the original ones under assumption of independence, but are different whenever the variables to be compared are dependent. Also, an alternative weaker version of the joint hazard rate order, namely, *hr:jw* (*weak joint hazard rate order*) has been defined, studied and applied in Belzunce et al. (2015). See Shaked and Shanthikumar (2007) for definitions and properties of all these orders (except for the hr:jw one), and Belzunce et al. (2015) and references therein for a list of applications in different fields.

Recently, relationships between standard stochastic orders and the corresponding joint ones have been investigated in Belzunce et al. (2015) and in Pellerey and Zalzadeh (2015), who provided conditions on the survival copula of a pair of absolutely continuous random lifetimes such that implications between some standard orders and the corresponding joint orders are satisfied, as described in the following propositions.

Proposition 1.1. *Let (X,Y) be a random vector having continuous marginal distributions and survival copula K. If K is symmetric, i.e., such that $K(u,v) = K(v,u)$ for every $(u,v) \in [0,1]^2$, and if it satisfies*

$$K(u,\gamma v) \geq\ [\leq]\ K(\gamma u, v) \tag{1.1}$$

for all $u \leq v$ and $\gamma \in (0,1)$, then $X \leq_{\text{hr}} Y \Rightarrow X \leq_{\text{hr:wj}} Y\ \ [X \leq_{\text{hr:wj}} Y \Rightarrow X \leq_{\text{hr}} Y]$.

Proposition 1.2. *Let (X,Y) be a random vector with absolutely continuous marginal distributions. Let its survival copula K be such that*

1. *K is symmetric;*
2. *K is twice differentiable and such that its mixed second partial derivative $k(u,v) = \frac{\partial^2}{\partial u \partial v} K(u,v)$ is non-increasing in u and non-decreasing in v for all $u \geq v$ [non-decreasing in u and non-increasing in v for all $u \geq v$].*

Then

$$X \geq_{lr} Y \Rightarrow X \geq_{lr:j} Y \quad \textit{and} \quad X \geq_{hr} Y \Rightarrow X \geq_{hr:j} Y$$
$$[X \geq_{lr:j} Y \Rightarrow X \geq_{lr} Y \quad \textit{and} \quad X \geq_{hr:j} Y \Rightarrow X \geq_{hr} Y]$$

It is interesting to observe that the property described in (1.1) has been firstly introduced by Bassan and Spizzichino (2005) in the study of bivariate notions of aging, and further studied and applied to dependence analysis by Durante and Ghiselli Ricci (2009), who called it *supermigrativity* [*submigrativity*] and showed that it generates a positive dependence property in the sense of Kimeldorf and Sampson (1989).

For this reason, motivated by the fact that the assumption on the survival copula required in Proposition 1.1 is actually a positive dependence notion, in this short note we consider in details the assumptions stated in Proposition 1.2, providing a more general property valid also for non absolutely continuous copulas, and showing that it satisfies the conditions described in Definition 1.2, i.e., that it is a positive dependence notion. In fact, roughly speaking, the assumption that should be satisfied by the survival copula in Proposition 1.2 has an immediate interpretation: it essentially means that for every point (u, v) below the diagonal and for any point $(\hat{u}, \hat{v})$ contained in the closed triangle with vertices (u, v), (u, u) and (v, v) it should holds $k(\hat{u}, \hat{v}) \geq k(u, v)$ (and similarly for points above the diagonal). In other words, the survival copula has probability mass mainly concentrated close to the diagonal.

In the next section we will also investigate its relationship with the supermigrativity notion, and some of its properties.

2 A new positive dependence property

Let $\mathcal{U} = \{(u, v) : 0 \leq u \leq v \leq 1\}$. To define a generalization of the property that should be satisfied by the survival copula in Proposition 1.2, we firstly need to define an order among points $(u, v) \in \mathcal{U}$ and an order among subsets of $\mathcal{U}$. Here, given two vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ of real numbers, the notation $\mathbf{x} \leq \mathbf{y}$ stands for $x_1 \leq y_1$ and $x_2 \leq y_2$.

Definition 2.1. *Given the points (u_1, v_1) and (u_2, v_2) in $\mathcal{U}$, we say that (u_1, v_1) precedes (u_2, v_2) according to the* right lower corner order *(shortly $(u_1, v_1) \succ_{rlc} (u_2, v_2)$) if $(u_1, v_1) \leq \varphi(u_2, v_2)$, where φ is the symmetry on the unit square given by $\varphi(u, v) = (u, 1 - v)$.*

Definition 2.2. *Given two Borel sets A_1 and A_2 contained in $\mathcal{U}$, we say that A_1 precedes A_2 according to the* right lower corner order *(shortly $A_1 \succ_{rlc} A_2$) if there exists a bijection $\psi : A_1 \longrightarrow A_2$ such that $(u, v) \succ_{rlc} \psi(u, v)$ for all $(u, v) \in A_1$.*

Now we can provide the definition of two new classes of copulas satisfying the statement of Proposition 1.2. For it, let us denote with $\lambda(A)$ the Lebesque measure of the set $A \subseteq \mathbb{R}^2$,

i.e., its area, and let μ_C be the measure induced by the copula C (see Durante and Sempi, 2015)).

Definition 2.3. *Let A_1 be any Borel set entirely contained in $\mathcal{U}$. Given a symmetric copula C, i.e. such that $C(u,v) = C(v,u)$, we say that C is* componentwise unimodal *[*componentwise bimodal*] if, and only if, $\mu_C(A_1) \geq \mu_C(A_2)$ [$\mu_C(A_1) \leq \mu_C(A_2)$] for any Borel set A_2 such that $\lambda(A_1) = \lambda(A_2)$ and $A_1 \succ_{rlc} A_2$.*

The componentwise unimodality of a copula C is equivalent to the condition stated in Proposition 1.2 whenever its derivative c exists. Thus, the componentwise unimodality is a generalization of such a condition to non absolutely continuous copulas. The same can be stated for its counterpart, componentwise bimodality.

The reason of the name for these two properties is as follows. Assume that the copula admits a density. Then fix one of the components (say, u), and look at the conditional density $c_u(v) = c(u,v)/\int_0^1 c(u,v)du$. If the inequality without the brackets is satisfied, then such a conditional density is unimodal, assuming maximum value in correspondence of $v = u$, i.e., in correspondence of the diagonal. Viceversa, assume that the inequality within the brackets is satisfied; then such a conditional density is bimodal, assuming local maximums in 0 and 1.

It should be mentioned that conditions for unimodality of copulas have been studied in Cuculescu and Theodorescu (2003). However, the concepts of multivariate unimodality studied there (central convex, block, and star unimodality) are different from the componentwise unimodality considered here, and no direct relationships between these notions exist.

Examples of copulas satisfying the property stated in Definition 2.3 are those in the Cuadras-Augé family, defined as $C(u,v) = (uv)^{1-\theta}[\min(u,v)]^{\theta}$, with $\theta \in [0,1]$, or those in the Frechét family, defined as mixtures between the independence copula and the upper Fréchet-Hoeffding copula (see Nelsen, 2006, for definitions of these copulas). Another class of copulas satisfying componentwise unimodality is a particular subset of those studied in Durante (2006), i.e., the family of copulas defined as $C_f(u,v) = \min(u,v)f(\max(u,v))$, with $u,v \in [0,1]$ for an increasing and continuous mapping f from $[0,1]$ into $[0,1]$ satisfying $f(1) = 1$ and such that $t \longmapsto \frac{f(t)}{t}$ is decreasing on $(0,1]$ (see also Durante et al., 2008, on this family). It can be seen that, wherever f is also concave, then the corresponding copula is componentwise unimodal.

Other families of copulas satisfying Definition 2.3 can be obtained by construction. For example, denoted with $\mathbf{1}_A(u,v)$ the indicator function of the set $A \subset [0,1]^2$, and considered the sets $A_1 = [0,\alpha] \times [0,\beta], A_2 = [1-\alpha,1] \times [1-\beta,1], B_1 = [0,\alpha] \times [1-\beta,1]$ and $B_2 = [1-\alpha,1] \times [0,\beta]$, the absolutely continuous copulas having densities $c_{\alpha,\beta,\delta}(u,v) = 1 + \delta \cdot \mathbf{1}_{A_1 \cup A_2}(u,v) - \delta \cdot \mathbf{1}_{B_1 \cup B_2}(u,v)$, with $\delta \in [0,1]$ and $\alpha, \beta \in [0,1/2)$, and all their mixtures, satisfy the componentwise unimodality property, as one can easily verify. Other examples may be found in Pellerey and Zalzadeh (2015).

Dealing with componentwise unimodality, first we show that it implies supermigrativity.

For it, observe that supermigrativity [submigrativity] is equivalent to

$$C(u', v') \geq \ [\leq] \ C(u, v) \tag{2.1}$$

for all $0 \leq v \leq v' \leq u' \leq u \leq 1$ such that $u'v' = uv = k$ for some $k \in (0, 1] \subseteq \mathbb{R}$.

Proposition 2.1. *Let C be a componentwise unimodal copula. Then C is supermigrative.*

Proof. Fix any $v \leq v' \leq u' \leq u$ such that $uv = u'v' = k$ for some $k \in (0, 1] \subseteq \mathbb{R}$, and consider the regions A, A', B, B', H, D and E shown in Figure 1, where $u'' = u - v' + v$. Note that $u'' \in [u, u']$ because of the convexity of $v(u) = k/u$, i.e., because $0 \leq v' - v \leq u - u'$. By (2.1), the statement is proved if from componentwise unimodality of the copula C follows that the total probability mass collected by C over the union $A' \cup B \cup B' \cup H$ is greater than the probability mass collected over the union $D \cup E$, being

$$C(u', v') = C(u', v) + \mu_C(A' \cup B \cup B' \cup H),$$

and

$$C(u, v) = C(u', v) + \mu_C(D \cup E).$$

First observe that, since $u'v' = uv = k$, it also holds $\lambda(D \cup E) = \lambda(A' \cup B \cup B' \cup H) = \lambda(A \cup B \cup B' \cup H)$.

Moreover, by construction, $\lambda(A) = \lambda(A') = \lambda(E)$, $\lambda(D) = \lambda(B \cup B' \cup H)$, and, by symmetry, $\lambda(D) = \lambda(H) + 2\lambda(B)$.

Let now $B^* \subseteq D$ be any region in D such that $\lambda(B^*) = \lambda(B)$. Observing that $A \succ_{rlc} E$, $B \succ_{rlc} B^*$ and $B \cup H \succ_{rlc} H - B^*$, from the componentwise unimodality assumption (see Definition 2.3) it holds

$$\mu_C(A') = \mu_C(A) \geq \mu_C(E) \tag{2.2}$$

and

$$\mu_C(B \cup B' \cup H) = \mu_C(B) + \big(\mu_C(B) + \mu_C(H)\big) \geq \mu_C(B^*) + \big(\mu_C(D) - \mu_C(B^*)\big) = \mu_C(D). \tag{2.3}$$

Thus, from (2.2) and (2.3) we obtain

$$\mu_C(B \cup B' \cup H) + \mu_C(A') \geq \mu_C(D) + \mu_C(E),$$

i.e., the assertion. □

The following example shows that a supermigrative copula (or survival copula) is not necessarily a componentwise unimodal copula (survival copula).

Example 2.1. *Let C be the Clayton copula, i.e., let it be defined as $C(u, v) = \max\{0, (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}\}$ for $\theta \in [0, 1]$, whose corresponding density is*

$$c(u, v) = (1 + \theta)(uv)^{\theta}(u^{\theta} + v^{\theta} - u^{\theta}v^{\theta})^{-(1+2\theta)/\theta}. \tag{2.4}$$

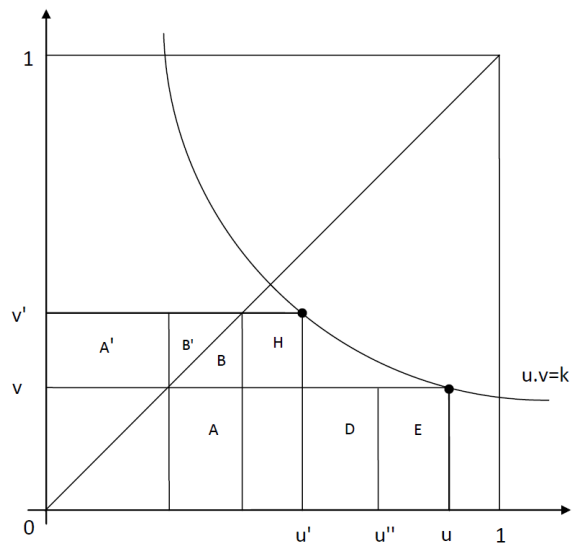

Figure 1: The regions defined in Proposition 2.1.

Such a copula satisfies the supermigrativity property for $\theta > 0$ (see Durante and Ghiselli Ricci, 2012), but it is not componenwise unimodal, being, for example, $c(0.125, 0.25) = 1.539 > c(0.25, 0.25) = 1.493$ when $\theta = 1$. Thus, for fixed $v = 0.25$ the maximum of its density is not reached in the diagonal.

As for Proposition 2.1, a relationship between componentwise bimodality and submigrativity may be stated as follow. The proof is similar to the proof of the above proposition, and therefore it is omitted.

Proposition 2.2. *Let C be a componentwise bimodal copula. Then C is submigrative.*

The following example shows that the implication described in Proposition 2.2 is strict.

Example 2.2. *Let C be defined as $C(u, v) = uv - uv(1 - u)(1 - v)$. It is easy to verify that it satisfies the submigrativity property, i.e., that the inequality $\leq$ holds in Equation (1.1). Moreover, it is an absolutely continuous copula, having density $c(u, v) = 2u(1 - v) + 2v(1 - u)$, which is clearly not componentwise unimodal (being monotone in v for any fixed $u \in [0, 1]$). Actually, this copula is a member of the Farlie-Gumbel-Morgenstern family, defined as $C_\theta(u, v) = uv - \theta uv(1 - u)(1 - v)$, with $\theta \in [-1, 1]$, and similar behaviors can be observed for all values in $[-1, 0)$ of the parameter θ.*

As mentioned in previous section, the property of componentwise unimodality can be used to introduce a new positive dependence property. To do it, let us denote with $\mathcal{C}_{cu}$ the class of copulas which are componentwise unimodal, and with $\mathcal{F}_{cu}$ the subset of all distribution functions $F \in \mathcal{F}$ such that the copula associated with F belongs to $\mathcal{C}_{cu}$.

The following proposition states that the class $\mathcal{C}_{cu}$ satisfies the conditions described in Definition 1.2, thus that $\mathcal{F}_{cu}$ defines a positive dependence notion in the sense of Kimeldorf and Sampson (1989).

Proposition 2.3. *The set $\mathcal{C}_{cu}$ is a class of copulas defining positive dependence in the sense of Definition 1.2.*

Proof. Property (C1′) follows from Proposition 2.1, observing that supermigrativity implies $C(u,v) \geq C(1,uv) = uv$ for all $u,v \in [0,1]$. Properties (C2′) and (C3′) can be easily obtained from the definition of componentwise unimodality. (C5′) follows by the fact that componentwise unimodal copulas are symmetric.

For the proof of (C6′), given a copula $C \in \mathcal{C}_{cu}$ denote with K its corresponding survival copula, and by μ_K the measure induced by K. Observe that, given any two Borel sets R_1 and R_2 contained in $\mathcal{U}$, one has $\mu_K(R_1) \geq \mu_K(R_2)$ if, and only if, $\mu_C(R_1') \geq \mu_C(R_2')$, where R_i' is obtained by applying to R_i the isometry $\phi(u,v) = (1-u, 1-v)$. Moreover, being the copula symmetric, $\mu_C(R_i') = \mu_C(R_i'')$, where R_i'' is the symmetric region of R_i' with respect to the diagonal $v = u$. Since the regions R_i''; $i = 1,2$, are contained in $\mathcal{U}$, and since $R_1'' \succ_{rlc} R_2''$ whenever $R_1 \succ_{rlc} R_2$, as one can easily verify, the assertion $\mu_K(R_1) \geq \mu_K(R_2)$ for any $R_1 \succ_{rlc} R_2$, with $R_i \subseteq \mathcal{U}$, immediately follows.

For the proof of (C7′), given $n \in \mathbb{N}$, let (U_n, V_n) be a vector with uniformly distributed margins such that C_n is its joint distribution function, with $C_n \in \mathcal{C}_{cu}$. Suppose that C_n converges to copula C in distribution, where C is the distribution function of a random vector (U,V). Then, for any A and A' contained in $\mathcal{U}$ and such that $A \succ_{rlc} A'$, one has $\mu_{C_n}(A) \leq \mu_{C_n}(A')$, or, equivalently,

$$P((U_n, V_n) \in A) \geq P((U_n, V_n) \in A').$$

By convergence in distribution it holds

$$P((U,V) \in A) = \lim_{n \longrightarrow \infty} P((U_n, V_n) \in A) \geq \lim_{n \longrightarrow \infty} P((U_n, V_n) \in A') = P((U,V) \in A'),$$

i.e., $\mu_C(A) \leq \mu_C(A')$, and the assertion follows. □

It should be pointed out that, viceversa, componentwise bimodality can be considered as a negative dependence notion, satisfying the Negative Quadrant Dependence (NQD) notion, i.e., implying the inequality $C(u,v) \leq uv$ for all $u,v \in [0,1]$. This assertion follows from Proposition 2.2 and from the fact that the submigrativity property for a copula C implies $C(u,v) \leq C(1,uv) = uv$.

The class $\mathcal{F}_{cu}$ of bivariate distribution function having componentwise unimodal copula satisfies an interesting property of positive dependence, namely, the mixture condition defined by Kimeldorf and Sampson (1989), recalled here.

Definition 2.4. *A set $\mathcal{P}$ of continuous distribution functions having the same pair of marginal distributions satisfies the* mixture condition *if*

$$F_1, F_2 \in \mathcal{P} \quad \textit{implies} \quad \alpha F_1 + (1-\alpha) F_2 \in \mathcal{P} \tag{2.5}$$

for all $\alpha \in [0,1]$

Note that the set $\mathcal{P}$ in Definition 2.4 contains only the bivariate distributions with the same marginals, while the set $\mathcal{F}_{cu}$ defined above contains bivariate distributions with both the same and different margins. However, the next statement shows that a linear combination of two distributions in the class $\mathcal{F}_{cu}$ having the same margins distribution satisfies the mixture condition (2.5).

Proposition 2.4. *The class $\mathcal{F}_{cu}$ satisfies the mixture condition.*

Proof. Let F and G be two bivariate distributions having the same margins F_X and F_Y and copulas C_F and C_G, respectively, both componentwise unimodal. It is easy to verify that the distribution H defined as $H = \alpha F + (1-\alpha)G$ has same margins F_X and F_Y. Observing that

$$\begin{aligned} H(x,y) &= \alpha F(x,y) + (1-\alpha)G(x,y) \\ &= \alpha C_F(F_X(x), F_Y(y)) + (1-\alpha)C_G(F_X(x), F_Y(y)), \end{aligned} \tag{2.6}$$

one gets

$$\begin{aligned} C_H(u,v) &= H(F_X^{-1}(u), F_Y^{-1}(v)) \\ &= \alpha C_F(F_X(F_X^{-1}(u)), F_Y(F_Y^{-1}(v))) + (1-\alpha)C_G(F_X(F_X^{-1}(u)), F_Y(F_Y^{-1}(v))), \end{aligned}$$

where $C_H(u,v)$ is the copula of H, that is,

$$C_H = \alpha C_F + (1-\alpha)C_G. \tag{2.7}$$

By using (2.7) and the fact that the integration with respect to a linear combination of measures is a linear combination of integrals with respect to each measure, it follows

$$\mu_{C_H}(A_1) = \alpha\mu_{C_F}(A_1) + (1-\alpha)\mu_{C_G}(A_1) \geq \alpha\alpha\mu_{C_F}(A_2) + (1-\alpha)\mu_{C_G}(A_2) = \mu_{C_H}(A_2),$$

for any A_1 and A_2 such that $A_1 \succ_{rlc} A_2$. Thus, $C_H \in \mathcal{C}_{cu}$, i.e., $H \in \mathcal{F}_{cu}$ □

Acknowledgment

We thank the referees for the careful reading of the manuscript and the insightful suggestions that greatly improved the presentation of the paper. The first author is grateful to the Semnan University for its support.

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