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Localization effect for a spectral problem in a perforated domain with fourier boundary conditions / CHIADO' PIAT, Valeria; Pankratova, I.; Piatnitski, A.. - In: SIAM JOURNAL ON MATHEMATICAL ANALYSIS. - ISSN 0036-1410. - STAMPA. - 45:3(2013), pp. 1302-1327. [10.1137/120868724]

*Availability:*

This version is available at: 11583/2624353 since: 2019-02-11T15:09:49Z

*Publisher:*

SIMA

*Published*

DOI:10.1137/120868724

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## LOCALIZATION EFFECT FOR A SPECTRAL PROBLEM IN A PERFORATED DOMAIN WITH FOURIER BOUNDARY CONDITIONS\*

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**Abstract.** This paper is aimed at homogenization of an elliptic spectral problem stated in a perforated domain, Fourier boundary conditions being imposed on the boundary of perforation. The presence of a locally periodic coefficient in the boundary operator gives rise to the effect of localization of the eigenfunctions. Moreover, the limit behavior of the lower part of the spectrum can be described in terms of an auxiliary harmonic oscillator operator. We describe the asymptotics of the eigenpairs and derive estimates for the rate of convergence.

**Key words.** homogenization, spectral problem, localization

**AMS subject classification.** 35B27

**DOI.** 10.1137/120868724

**1. Introduction.** This paper deals with a spectral problem for a second order divergence form elliptic operator in a periodically perforated bounded domain in  $\mathbb{R}^d$ . Assuming that on the perforation border a homogeneous Fourier boundary condition is stated and that the coefficient in the boundary operator is a function of “slow” argument, we arrive at the following eigenvalue problem:

$$(1.1) \quad \begin{cases} -\operatorname{div}(a(x/\varepsilon)\nabla u^\varepsilon(x)) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a(x/\varepsilon)\nabla u^\varepsilon(x) \cdot n = -q(x)u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega; \end{cases}$$

here  $\varepsilon$  is a small positive parameter defined as a microstructure period.

We impose some natural regularity and connectedness conditions on the perforated domain  $\Omega_\varepsilon$ , as well as usual periodicity and uniform ellipticity conditions on the matrix  $a(y)$ . These conditions are specified in detail in the next section.

Our crucial assumptions are as follows:

- $q \in C^2(\overline{\Omega})$ , and  $q(x) \geq q_0 > 0$  in  $\overline{\Omega}$ .
- The function  $q$  has only one global minimum point in  $\overline{\Omega}$ . The global minimum is attained at an interior point of  $\Omega$ .
- The Hessian matrix  $\partial^2 q / \partial x^2$  evaluated at the minimum point is positive definite.

Under the first two assumptions the localization phenomenon holds. Namely, for any  $k \in \mathbb{N}$  the  $k$ th eigenfunction of problem (1.1) is asymptotically localized, as  $\varepsilon \rightarrow 0$ , in a small neighborhood of the minimum point. In particular, the properly normalized principal eigenfunction converges to a  $\delta$ -function supported at the minimum point.

In this paper, assuming that all the above conditions are fulfilled, we construct the first two leading terms of the asymptotic expansions for the  $k$ th eigenpair,  $k = 1, 2, \dots$

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\*Received by the editors March 5, 2012; accepted for publication (in revised form) February 25, 2013; published electronically May 7, 2013.

<http://www.siam.org/journals/sima/45-3/86872.html>

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These asymptotic expansions have a number of interesting features. First, the expansions are in integer powers of  $\varepsilon^{1/4}$ . Then, the localization takes place in the scale  $\varepsilon^{1/4}$ . In this scale the leading term of the asymptotic expansion for the  $k$ th eigenfunction proved to be the  $k$ th eigenfunction of an auxiliary harmonic oscillator operator. If  $q \in C^3(\bar{\Omega})$ , then we also obtain estimates for the rate of convergence.

The localization in the scale  $\varepsilon^{1/4}$  is not standard and appears due to the specific scaling in the Fourier boundary condition. It is interesting to compare this asymptotic result with the results that hold for two different scalings, namely, for the cases when there is a factor  $\varepsilon$  and  $\varepsilon^{-1}$  in front of the function  $q(x)$  in the boundary condition. In the former case the surface integral in the variational formulation of the problem is of the same order of  $\varepsilon$  as the volume integral, and the classical homogenization methods apply. This scaling does not lead to the localization phenomenon (see Remark 2.2 and Theorem 2.4).

The case when  $\varepsilon^{-1}$  is presented in front of  $q(x)$  is similar to that studied in [5]: the ratio between the surface integral and the volume integral is of order  $\varepsilon^{-2}$ , and the localization is observed. Namely, the  $k$ th eigenfunction is asymptotically given as a product of a function periodically oscillating at the scale  $\varepsilon$  and an exponentially decaying function localized in an  $\varepsilon^{1/2}$ -neighborhood of the minimum point of  $q$ . This result is also discussed in Remark 2.2. We formulate the corresponding statement in Theorem 2.5 and then provide a sketch of the proof of this theorem.

We suppose that  $q$  does not oscillate just for presentation simplicity. The techniques developed in the present work also apply to the case of locally periodic coefficients  $q = q(x, x/\varepsilon)$ ,  $a = a(x, x/\varepsilon)$  with  $q(x, y)$  and  $a(x, y)$  being periodic in  $y$ ; see Remark 2.1 and Theorem 2.3.

Previously, the localization phenomenon in spectral problems has been observed in several mathematical works. In [5] the operator with a large locally periodic potential has been considered. It has been assumed that the first cell eigenvalue attains a unique minimum in the domain and at this point shows nondegenerate quadratic behavior. The authors prove that the  $k$ th original eigenfunction is asymptotically given as a product of a periodic rapidly oscillating function and a scaled exponentially decaying function; the former function is the first cell eigenfunction at the scale  $\varepsilon$ , and the latter is the  $k$ th eigenfunction of the harmonic oscillator type operator at the scale  $\sqrt{\varepsilon}$ . The localization appears due to the presence of a large factor in the potential and the fact that the operator coefficients depend on the slow variable.

In [3] the result of [5] has been generalized to the case of transport equation posed in a locally periodic heterogeneous domain. Under the assumption that the leading eigenvalue of an auxiliary periodic cell problem attains a unique minimum, the homogenization and localization have been proved. The effective problem appears to be a diffusion equation with quadratic potential stated in the whole space. This gives an interesting example of a nonelliptic PDE for which the localization phenomenon holds.

Localization phenomenon for a Schrödinger equation in a locally periodic medium has been considered in [4]. The authors show that there exists a localized solution which is asymptotically given as the product of a Bloch wave and of the solution of a homogenized Schrödinger equation with quadratic potential.

In [6] the Dirichlet spectral problem for the Laplacian in a thin two-dimensional strip of slowly varying thickness has been studied. Here the localization has been observed in the vicinity of the point of maximum thickness. The large parameter is the first eigenvalue of one-dimensional Laplacian in the cross-section.

In the mentioned works, under natural nondegeneracy conditions, the asymptotics of the eigenpairs was described in terms of the spectrum of an appropriate harmonic oscillator operator. However, the localization scale was of order  $\sqrt{\varepsilon}$  with  $\varepsilon$  being the microscopic length scale.

The localization in the scale  $\varepsilon^{1/4}$  that is observed in the present paper is not standard. It should also be noted that although the operators in (1.1) do not contain a large parameter, such a parameter is presented implicitly because  $(d-1)$ -dimensional volume of the perforation surface tends to infinity.

The homogenization of spectral problem (1.1) with constant or periodic functions  $a$  and  $q$  has been addressed in [12].

Spectral problems in perforated domains with Dirichlet and Neumann boundary condition at the perforation border are now well studied. There is a vast literature on the topic; see, for instance, [13], [11].

In the paper we combine asymptotic expansion techniques with various variational and compactness arguments and scaled trace and Poincaré type inequalities.

**2. Problem statement.** We start by describing the geometry of the domain. Let  $K = [0, 1)^d$ ,  $d \geq 2$ , and suppose that  $E \subset \mathbb{R}^d$  is a  $K$ -periodic, open, connected set with a Lipschitz boundary  $\Sigma$ ; the complement  $\mathbb{R}^d \setminus E$  is denoted by  $B$ . We also assume that  $K \cap E$  is a connected set, and  $K \cap B \Subset K$ , so that  $B = \mathbb{R}^d \setminus E$  consists of disjoint components. In what follows,  $Y = K \cap E$  denotes the periodicity cell and  $\Sigma^0 = K \cap \partial B = K \cap \Sigma$  the boundary of the inclusion. The symbols  $|Y|_d$  and  $|\Sigma^0|_{d-1}$  stand for the measures of  $Y$  and the  $(d-1)$ -dimensional surface measure of  $\Sigma^0$ , respectively.

For every  $i \in \mathbb{Z}^d$  we denote  $Y_\varepsilon^i = \varepsilon(i + Y)$ ,  $\Sigma_\varepsilon^i = \varepsilon \Sigma \cap Y_\varepsilon^i$ , and  $B_\varepsilon^i = \varepsilon B \cap Y_\varepsilon^i$ . Given  $\Omega$ , a bounded domain in  $\mathbb{R}^d$  with a Lipschitz boundary  $\partial\Omega$ , we introduce the perforated domain

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i \in I_\varepsilon} B_\varepsilon^i, \quad I_\varepsilon = \{i \in \mathbb{Z}^d : Y_\varepsilon^i \subset \Omega\}.$$

Notice that  $\Omega_\varepsilon$  remains connected, the perforation does not intersect the boundary  $\partial\Omega$ , and

$$\partial\Omega_\varepsilon = \partial\Omega \cup \Sigma_\varepsilon, \quad \Sigma_\varepsilon = \bigcup_{i \in I_\varepsilon} \Sigma_\varepsilon^i.$$

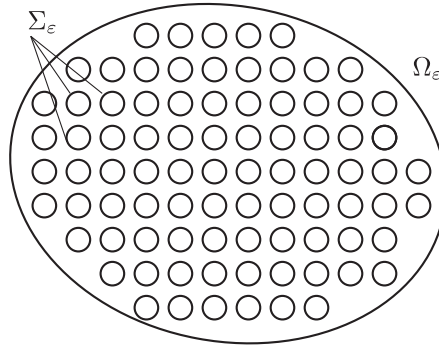


FIG. 2.1. Domain  $\Omega_\varepsilon$ .

In the perforated domain  $\Omega_\varepsilon$  we consider the following spectral problem:

$$(2.1) \quad \begin{cases} -\operatorname{div}(a^\varepsilon(x)\nabla u^\varepsilon(x)) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x)\nabla u^\varepsilon(x) \cdot n = -q(x)u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases}$$

Here  $\varepsilon$  is a small positive parameter,  $a^\varepsilon(x) = a(x/\varepsilon)$  with  $a(y)$  being a  $d \times d$  matrix, and  $n$  is an outward unit normal; the usual scalar product in  $\mathbb{R}^d$  is denoted by “ $\cdot$ ”.

In what follows we assume that the following conditions hold true:

- (H1)  $a(y)$  is a real symmetric  $d \times d$  matrix satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^d a_{ij}(y)\xi_i\xi_j \geq \Lambda |\xi|^2, \quad \xi \in \mathbb{R}^d,$$

for some  $\Lambda > 0$ .

- (H2) The coefficients  $a_{ij}(y)$  are  $Y$ -periodic and, moreover,  $a_{ij}(y) \in L^\infty(\mathbb{R}^d)$ .

- (H3) The function  $q(x) \in C^3(\mathbb{R}^d)$  is positive.

- (H4) The function  $q(x)$  has a unique global minimum attained at  $x = 0 \in \Omega$ .  
Moreover, in the vicinity of  $x = 0$

$$q(x) = q(0) + \frac{1}{2} x^T H(q) x + o(|x|^2)$$

with the positive definite Hessian matrix  $H(q)$ .

It is convenient to introduce the notation

$$H_0^1(\Omega_\varepsilon, \partial\Omega) = \{u \in H^1(\Omega_\varepsilon) : u = 0 \text{ on } \partial\Omega\}.$$

The weak formulation of spectral problem (2.1) reads as follows: find  $\lambda^\varepsilon \in \mathbb{C}$  (eigenvalues) and  $u^\varepsilon \in H_0^1(\Omega_\varepsilon, \partial\Omega)$ ,  $u^\varepsilon \neq 0$ , such that

$$(2.2) \quad \int_{\Omega_\varepsilon} a^\varepsilon \nabla u^\varepsilon \cdot \nabla v \, dx + \int_{\Sigma_\varepsilon} q u^\varepsilon v \, d\sigma = \lambda^\varepsilon \int_{\Omega_\varepsilon} u^\varepsilon v \, dx, \quad v \in H_0^1(\Omega).$$

LEMMA 2.1. *For any  $\varepsilon > 0$ , the spectrum of problem (2.2) is real and consists of a countable set of points*

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \dots \leq \lambda_j^\varepsilon \leq \dots \rightarrow +\infty.$$

*Every eigenvalue has a finite multiplicity. The corresponding eigenfunctions normalized by*

$$\int_{\Omega_\varepsilon} u_i^\varepsilon u_j^\varepsilon \, dx = \delta_{ij}$$

*form an orthonormal basis in  $L^2(\Omega_\varepsilon)$ .*

We omit the proof of Lemma 2.1, which is classical.

Under the assumptions (H1)–(H4) we study the asymptotic behavior of eigenpairs  $(\lambda^\varepsilon, u^\varepsilon)$ , as  $\varepsilon \rightarrow 0$ .

To avoid excessive technicalities for the moment, we state our main result in a slightly reduced form, without specifying the rate of convergence. For the detailed formulation of the main result see Theorem 3.17.

In what follows we assume that  $u_j^\varepsilon$  is extended to the whole domain  $\Omega$  in such a way that

$$(2.3) \quad \|u_j^\varepsilon\|_{H^1(\Omega)} \leq C \|u_j^\varepsilon\|_{H^1(\Omega_\varepsilon)};$$

the existence of such extension is justified in [1], and we keep the same notation for the extended function.

**THEOREM 2.2.** *Let conditions (H1)–(H4) be fulfilled. If  $(\lambda_j^\varepsilon, u_j^\varepsilon)$  stands for the  $j$ th eigenpair of problem (2.1), then for any  $j$ , the following representation takes place:*

$$\lambda_j^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + \frac{\mu_j^\varepsilon}{\sqrt{\varepsilon}}, \quad u_j^\varepsilon(x) = v_j^\varepsilon\left(\frac{x}{\varepsilon^{1/4}}\right),$$

where  $(\mu_j^\varepsilon, v_j^\varepsilon(z))$ , the  $j$ th eigenpair of the rescaled spectral problem, is such that

- $\mu_j^\varepsilon$  converges, as  $\varepsilon \rightarrow 0$ , to the  $j$ th eigenvalue  $\mu_j$  of the effective spectral problem

$$(2.4) \quad -\operatorname{div}(a^{\text{eff}} \nabla v(z)) + (z^T Q z) v(z) = \mu v(z), \quad v \in L^2(\mathbb{R}^d),$$

where  $a^{\text{eff}}$  is a positive definite matrix (see (3.19));  $Q$  is defined by

$$Q = \frac{1}{2} \frac{|\Sigma^0|_{d-1}}{|Y|_d} H(q)$$

with  $H(q)$  being the Hessian matrix of  $q$  at  $x = 0$ .

- If  $\mu_j$  is a simple eigenvalue, then, for small  $\varepsilon$ ,  $\mu_j^\varepsilon$  is also simple, and the convergence of the corresponding normalized eigenfunctions (extended to the whole  $\mathbb{R}^d$ ) holds:

$$\|\tilde{v}_j^\varepsilon - \tilde{v}_j\|_{L^2(\mathbb{R}^d)} \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

where

$$\tilde{v}_j^\varepsilon = \frac{v_j^\varepsilon}{\|v_j^\varepsilon\|_{L^2(\mathbb{R}^d)}}, \quad \tilde{v}_j = \frac{v_j}{\|v_j\|_{L^2(\mathbb{R}^d)}}.$$

*It should be emphasized that the homogenized spectral problem (2.4) has been obtained in the coordinates  $z = x/\varepsilon^{1/4}$ , and thus the eigenfunctions convergence holds in these rescaled coordinates.*

**Remark 2.1.** Theorem 2.2 can be generalized to the case of locally periodic coefficients in (2.1).

Namely, let us consider the following problem:

$$(2.5) \quad \begin{cases} -\operatorname{div}(a^\varepsilon(x) \nabla u^\varepsilon(x)) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x) \nabla u^\varepsilon(x) \cdot n = -q^\varepsilon(x) u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega, \end{cases}$$

with

$$a^\varepsilon(x) = a(x, x/\varepsilon), \quad q^\varepsilon(x) = q(x, x/\varepsilon).$$

Assume the following:

- $a_{ij}(x, y)$  and  $q(x, y)$  are  $Y$ -periodic in  $y$  functions such that  $a_{ij}(x, y), q(x, y) \in C^{2,\alpha}(\mathbb{R}^d; C^\alpha(\bar{Y}))$  with some  $\alpha > 0$ .
- The matrix  $a(x, y)$  satisfies the uniform ellipticity condition.
- The local average of  $q$  defined by

$$\bar{q}(x) = \frac{1}{|\Sigma^0|_{d-1}} \int_{\Sigma^0} q(x, y) d\sigma_y$$

admits its global minimum at  $x = 0$ .

- In the vicinity of  $x = 0$

$$\bar{q}(x) = \bar{q}(0) + \frac{1}{2} x^T H(\bar{q}) x + o(|x|^2)$$

with the positive definite Hessian matrix  $H(\bar{q})$ .

- $x = 0$  is the only global minimum point of  $\bar{q}$  in  $\bar{\Omega}$ .

Then the following convergence result holds.

**THEOREM 2.3.** *If  $(\lambda_j^\varepsilon, u_j^\varepsilon)$  stands for the  $j$ th eigenpair of problem (2.5), then for any  $j$ , the following representation takes place:*

$$\lambda_j^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \bar{q}(0) + \frac{\mu_j^\varepsilon}{\sqrt{\varepsilon}}, \quad u_j^\varepsilon(x) = v_j^\varepsilon\left(\frac{x}{\varepsilon^{1/4}}\right),$$

where  $(\mu_j^\varepsilon, v_j^\varepsilon(z))$  are such that

- $\mu_j^\varepsilon$  converges, as  $\varepsilon \rightarrow 0$ , to the  $j$ th eigenvalue  $\mu_j$  of the effective spectral problem

$$(2.6) \quad -\operatorname{div}(a^{\text{eff}} \nabla v) + (z^T P z) v = \mu v, \quad v \in L^2(\mathbb{R}^d),$$

where

$$P = \frac{1}{2} \frac{|\Sigma^0|_{d-1}}{|Y|_d} H(\bar{q})$$

and  $a^{\text{eff}}$  is a positive definite matrix defined by

$$a_{ij}^{\text{eff}} = \frac{1}{|Y|_d} \int_Y a_{ik}(0, \zeta) (\delta_{kj} + \partial_k N_j(\zeta)) d\zeta,$$

with the functions  $N_j$  solving auxiliary cell problems

$$\begin{cases} -\operatorname{div}_\zeta(a(0, \zeta) \nabla_\zeta N_k(\zeta)) = \operatorname{div}_\zeta a_{\cdot k}(0, \zeta), & k = 1, \dots, d, \quad \zeta \in Y, \\ a(0, \zeta) \nabla_\zeta N_k \cdot n = -a_{ik}(0, \zeta) n_i, & \zeta \in \Sigma^0, \\ N_k(\zeta) \in H_{\#}^1(Y), \end{cases}$$

- The normalized functions  $v_j^\varepsilon$  (extended to the whole  $\mathbb{R}^d$ ) converge, up to a subsequence, to the normalized eigenfunction  $v_j$  of (2.6) corresponding to  $\mu_j$ :

$$\|\tilde{v}_j^\varepsilon - \tilde{v}_j\|_{L^2(\mathbb{R}^d)} \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

where

$$\tilde{v}_j^\varepsilon = \frac{v_j^\varepsilon}{\|v_j^\varepsilon\|_{L^2(\mathbb{R}^d)}}, \quad \tilde{v}_j = \frac{v_j}{\|v_j\|_{L^2(\mathbb{R}^d)}}.$$

*Remark 2.2.* In the variational formulation (2.2) of the original spectral problem the ratio between the surface integral and the volume integral is of order  $\varepsilon^{-1}$  (see Lemma 4.1). Changing the scaling factor in front of the function  $q(x)$  in the boundary condition might change essentially the asymptotic behaviour of the eigenpairs. Namely, if we take the boundary condition

$$a^\varepsilon(x) \nabla u^\varepsilon(x) \cdot n = -\varepsilon q(x) u^\varepsilon(x), \quad x \in \Sigma_\varepsilon,$$

then the volume and surface integrals are of the same order of  $\varepsilon$ . In this case the problem admits classical homogenization, and there is no concentration effect.

On the other hand, if we consider the boundary condition with a large parameter  $1/\varepsilon$  in front of  $q(x)$ , that is

$$a^\varepsilon(x) \nabla u^\varepsilon(x) \cdot n = -\frac{1}{\varepsilon} q(x) u^\varepsilon(x), \quad x \in \Sigma_\varepsilon,$$

then the  $k$ th eigenvalue will be of order  $\varepsilon^{-2}$  and the  $k$ th eigenfunction will be the product of a rapidly oscillating function and a scaled exponentially decaying function.

The homogenization results for both cases are presented below. We recall that  $u_j^\varepsilon$  is extended to the whole domain  $\Omega$  in such a way that (2.3) holds.

**THEOREM 2.4.** *Let  $(\lambda_j^\varepsilon, u_j^\varepsilon)$  be the  $j$ th eigenpair of the problem*

$$(2.7) \quad \begin{cases} -\operatorname{div} \left( a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon(x) \right) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x) \nabla u^\varepsilon(x) \cdot n = -\varepsilon q(x) u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases}$$

*Under the assumptions (H1)–(H4), the following convergence result holds:*

- *For any  $j$ ,  $\lambda_j^\varepsilon$  converges to  $\lambda_j$ , as  $\varepsilon \rightarrow 0$ .*
- *The sequence  $u_j^\varepsilon$ , up to a subsequence, converges weakly in  $H_0^1(\Omega)$  to  $u_j$  being an eigenfunction of the effective spectral problem*

$$(2.8) \quad \begin{cases} -\operatorname{div}(a^{\text{eff}} \nabla u) + \frac{|\Sigma^0|_{d-1}}{|Y|_d} q u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

*that corresponds to  $\lambda_j$ . Here the homogenized matrix  $a^{\text{eff}}$  is given by*

$$a_{il}^{\text{eff}} = \frac{1}{|Y|_d} \int_Y a_{ik}(y) (\delta_{kl} + \partial_k N_l(y)) dy,$$

*where the periodic functions  $N_l$  are solutions of (3.17).*

*Proof.* We present just main ideas of the proof since it is classical.

Using the minimax principle for eigenvalues (see, for instance, [7]) one can see that for any  $j$ , the  $j$ th eigenvalue is bounded by some constant which depends on  $j$  but is independent of  $\varepsilon$ . Thus, up to a subsequence,  $\lambda_j^\varepsilon$  converges for a subsequence to some  $\lambda_j$ , as  $\varepsilon \rightarrow 0$ . We assume the following normalization condition for the eigenfunctions:  $\|u_j^\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$ . From the variational formulation corresponding to (2.7) it follows that  $\|u_j^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C(j)$ . By the classical two-scale compactness results (see [10], [2]),  $u_j^\varepsilon(x) \chi(x/\varepsilon)$  two-scale converges to  $u_j(x) \chi(y)$ , and  $\nabla u_j^\varepsilon(x) \chi(x/\varepsilon)$  two-scale converges to  $(\nabla u_j(x) + \nabla_y v_j(x, y)) \chi(y)$ , as  $\varepsilon \rightarrow 0$ , where  $\chi(y)$  is the characteristic function of the perforated cell  $Y$ .



Due to the boundedness in  $H^1(\Omega)$ ,  $u_j^\varepsilon$  converges to  $u_j$  strongly in  $L^2(\Omega)$  which, thanks to the normalization condition for  $u_j^\varepsilon$ , guarantees that the limit function  $u_j$  is not zero.

Choosing appropriate test functions in the weak formulation of the problem and passing to the limit yield the following representation for the functions  $v_j(x, y)$ :

$$v_j(x, y) = N_k(y) \partial_k u_j(x),$$

where  $N_k$  solve auxiliary cell problems (3.17). Then one can derive in the standard way that  $\lambda_j$  is the  $j$ th eigenvalue of the effective spectral problem (2.8), and function  $u_j$  belongs to the finite dimensional eigenspace corresponding to  $\lambda_j$ .  $\square$

In the presence of a large parameter in front of  $q(x)$  in the boundary condition, the following auxiliary cell eigenproblem plays an important role:

$$(2.9) \quad \begin{cases} -\operatorname{div}(a(y)\nabla p(y)) = \nu p, & y \in Y, \\ a\nabla p \cdot n + q(0)p = 0, & y \in \partial Y. \end{cases}$$

The spectrum of the last problem is discrete, the first eigenvalue  $\nu_1$  is simple, and the corresponding eigenfunction  $p(y)$  is Hölder continuous and can be chosen positive.

Now we formulate the homogenization result in this case.

**THEOREM 2.5.** *Let the assumptions (H1)–(H4) be fulfilled and let  $(\lambda_j^\varepsilon, u_j^\varepsilon)$  stand for the  $j$ th eigenpair of the problem*

$$(2.10) \quad \begin{cases} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(x)\right) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x)\nabla u^\varepsilon(x) \cdot n = -\frac{1}{\varepsilon}q(x)u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases}$$

The following representation takes place:

$$\lambda_j^\varepsilon = \frac{\nu_1}{\varepsilon^2} + \frac{\mu_j^\varepsilon}{\varepsilon}, \quad u_j^\varepsilon(x) = p\left(\frac{x}{\varepsilon}\right) v_j^\varepsilon\left(\frac{x}{\sqrt{\varepsilon}}\right),$$

where  $(\nu_1, p(y))$  is the first eigenpair of the cell spectral problem (2.9), and  $(\mu_j^\varepsilon, v_j^\varepsilon(z))$  are such that

- $\mu_j^\varepsilon$  converges, as  $\varepsilon \rightarrow 0$ , to the  $j$ th eigenvalue  $\mu_j$  of the effective spectral problem

$$(2.11) \quad -\operatorname{div}(a^{\text{eff}}\nabla v) + (z^T Dz)v = \mu v, \quad v \in L^2(\mathbb{R}^d),$$

with

$$D_{ij} = \frac{1}{2} \frac{\|p\|_{L^2(\Sigma^0)}^2}{\|p\|_{L^2(Y)}^2} \partial_i \partial_j q(0),$$

and  $a^{\text{eff}}$  is a positive definite matrix defined by

$$a_{ij}^{\text{eff}} = \frac{1}{\|p\|_{L^2(Y)}^2} \int_Y p(y)^2 a_{ik}(y)(\delta_{kj} + \partial_k N_j(y)) dy$$

with the periodic functions  $N_j$  solving auxiliary cell problems (3.17).

- Being normalized by  $\|u_j^\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$ , the corresponding eigenfunctions  $u_j^\varepsilon$ , for a subsequence, are approximated by  $p\left(\frac{x}{\varepsilon}\right) v_j\left(\frac{x}{\sqrt{\varepsilon}}\right)$ , that is,

$$\int_{\Omega_\varepsilon} \left| u_j^\varepsilon(x) - p\left(\frac{x}{\varepsilon}\right) v_j\left(\frac{x}{\sqrt{\varepsilon}}\right) \right|^2 dx \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

where  $p$  is the first eigenfunction of (2.9), and  $v_j$  stands for an eigenfunction of the homogenized spectral problem (2.11) corresponding to  $\mu_j$ .

*Proof.* The proof of the theorem follows the lines of Theorem 4.1 in [5], so we just present the main ideas.

Let  $(\lambda^\varepsilon, u^\varepsilon)$  be an eigenpair of (2.10). We perform the change of variables (so-called factorization)

$$u^\varepsilon(x) = p\left(\frac{x}{\varepsilon}\right) \tilde{u}^\varepsilon(x),$$

where  $p(y)$  is the first eigenfunction of the auxiliary cell spectral problem (2.9).

We obtain the following problem for the unknown function  $\tilde{u}^\varepsilon$ :

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right) p\left(\frac{x}{\varepsilon}\right)^2 \nabla \tilde{u}^\varepsilon(x)\right) = \left(\lambda^\varepsilon - \frac{\nu_1}{\varepsilon^2}\right) p\left(\frac{x}{\varepsilon}\right)^2 \tilde{u}^\varepsilon(x), & x \in \Omega^\varepsilon, \\ a\left(\frac{x}{\varepsilon}\right) p\left(\frac{x}{\varepsilon}\right)^2 \nabla \tilde{u}^\varepsilon(x) \cdot n + \frac{1}{\varepsilon}(q(x) - q(0)) p\left(\frac{x}{\varepsilon}\right)^2 \tilde{u}^\varepsilon(x) = 0, & x \in \Sigma_\varepsilon, \\ \tilde{u}^\varepsilon = 0, & x \in \partial\Omega. \end{cases}$$

In order to balance the volume and surface integrals in the weak formulation of this problem we rescale the variable by introducing

$$z = \frac{x}{\sqrt{\varepsilon}}, \quad v^\varepsilon(z) = \tilde{u}^\varepsilon\left(\frac{x}{\sqrt{\varepsilon}}\right).$$

Then the new unknown function  $v^\varepsilon$  satisfies the problem (2.12)

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{z}{\sqrt{\varepsilon}}\right) p\left(\frac{z}{\sqrt{\varepsilon}}\right)^2 \nabla v^\varepsilon(z)\right) = \mu^\varepsilon p\left(\frac{z}{\sqrt{\varepsilon}}\right)^2 v^\varepsilon(z), & z \in \varepsilon^{-\frac{1}{2}}\Omega^\varepsilon, \\ a\left(\frac{z}{\sqrt{\varepsilon}}\right) p\left(\frac{z}{\sqrt{\varepsilon}}\right)^2 \nabla v^\varepsilon(z) \cdot n + \frac{q(\sqrt{\varepsilon}z) - q(0)}{\sqrt{\varepsilon}} p\left(\frac{z}{\sqrt{\varepsilon}}\right)^2 v^\varepsilon(z) = 0, & z \in \varepsilon^{-\frac{1}{2}}\Sigma_\varepsilon, \\ v^\varepsilon(z) = 0, & z \in \varepsilon^{-\frac{1}{2}}\partial\Omega. \end{cases}$$

Here  $\mu^\varepsilon$  is given by

$$(2.13) \quad \mu^\varepsilon = \varepsilon \left( \lambda^\varepsilon - \frac{\nu_1}{\varepsilon^2} \right).$$

Notice that the domain  $\varepsilon^{-1/2}\Omega$  tends to the whole  $\mathbb{R}^d$ , as  $\varepsilon \rightarrow 0$ , and thanks to assumption (H4), in  $\varepsilon^{-1/2}\Omega^\varepsilon$

$$q(\sqrt{\varepsilon}z) - q(0) = \varepsilon z^T D z + o(|\sqrt{\varepsilon}z|^2), \quad D_{ij} = \frac{1}{2} \partial_j \partial_j q(0).$$

Passing to the limit in the weak formulation of (2.12) repeats the steps of the proof of Theorem 4.1 in [5], so we do not reproduce it here. As a result one gets the effective problem (2.11). Returning back to the original unknowns  $(\lambda^\varepsilon, u^\varepsilon)$  completes the proof.  $\square$

**3. Proof of Theorem 2.2.**

**3.1. Preliminaries. Estimates for  $\lambda_1^\varepsilon$ .** In this section we estimate the first eigenvalue  $\lambda_1^\varepsilon$  of problem (2.1). To this end we use the variational representation for  $\lambda_1^\varepsilon$ . Let us recall that, due to the classical min-max principle (see, for example, [8]),

$$(3.1) \quad \lambda_1^\varepsilon = \inf_{v \in H_0^1(\Omega_\varepsilon, \partial\Omega)} \frac{\int_{\Omega_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dx + \int_{\Sigma_\varepsilon} q(v)^2 \, d\sigma}{\|v\|_{L^2(\Omega_\varepsilon)}^2}.$$

LEMMA 3.1. *The first eigenvalue of the spectral problem (2.1) satisfies the estimate*

$$\frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(1) \leq \lambda_1^\varepsilon \leq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(\varepsilon^{-1/2}), \quad \varepsilon \rightarrow 0.$$

*Proof.* We start by proving the estimate from below. By (3.1),

$$\lambda_1^\varepsilon \geq \inf_{\substack{v \in H_0^1(\Omega_\varepsilon, \partial\Omega) \\ \|v\|_{L^2(\Omega_\varepsilon)} = 1}} \left\{ \int_{\Omega_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dx + q(0) \int_{\Sigma_\varepsilon} (v)^2 \, d\sigma \right\}.$$

The last infimum is attained on the first eigenfunction of the following spectral problem:

$$\begin{cases} -\operatorname{div}(a^\varepsilon(x) \nabla w^\varepsilon(x)) = \nu^\varepsilon w^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x) \nabla w^\varepsilon(x) \cdot n = -q(0) w^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ w^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases}$$

It has been proved in [12] that the first eigenvalue of this problem admits the following asymptotics:

$$\nu_1^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(1), \quad \varepsilon \rightarrow 0.$$

Thus,

$$\lambda_1^\varepsilon \geq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(1), \quad \varepsilon \rightarrow 0.$$

We proceed to the derivation of the upper bound for  $\lambda_1^\varepsilon$ . Choosing  $v \in C_0^\infty(\Omega)$  as a test function in (3.1), one can obtain a rough estimate

$$(3.2) \quad \lambda_1^\varepsilon \leq \tilde{C} \varepsilon^{-1}$$

with a constant  $\tilde{C}$  independent of  $\varepsilon$ . To specify  $\tilde{C}$  one should choose a “smarter” test function. Let us take  $v \in C_0^\infty(\mathbb{R}^d)$ ,  $\|v\|_{L^2(\mathbb{R}^d)} = 1$ , and choose  $v(x/\varepsilon^\alpha)$  as a test function in (3.1),  $0 < \alpha < 1/2$ . Note that if  $\operatorname{supp} v \subset B_R(0)$  for some  $R > 0$ , then  $\operatorname{supp} v(x/\varepsilon^\alpha) \subset B_{\varepsilon^\alpha R}(0)$ . Then we obtain

$$\lambda_1^\varepsilon \leq \frac{\int_{\Sigma_\varepsilon} q(x) \left| v\left(\frac{x}{\varepsilon^\alpha}\right) \right|^2 \, d\sigma + O(\varepsilon^{-2\alpha} \varepsilon^{d\alpha})}{\int_{\Omega_\varepsilon} \left| v\left(\frac{x}{\varepsilon^\alpha}\right) \right|^2 \, dx}.$$

Taking into account assumption (H4) and using Lemma 4.1, one has

$$\lambda_1^\varepsilon \leq \frac{\frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) \int_{\Omega_\varepsilon} \left| v\left(\frac{x}{\varepsilon^\alpha}\right) \right|^2 dx + O(\varepsilon^{2\alpha-1} \varepsilon^{d\alpha}) + O(\varepsilon^{-2\alpha} \varepsilon^{d\alpha})}{\int_{\Omega_\varepsilon} \left| v\left(\frac{x}{\varepsilon^\alpha}\right) \right|^2 dx}.$$

Notice that the best estimate is obtained for  $\alpha = 1/4$ . Finally,

$$(3.3) \quad \lambda_1^\varepsilon \leq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(\varepsilon^{-1/2}), \quad \varepsilon \rightarrow 0. \quad \square$$

*Remark 3.1.* When deriving the upper bound for  $\lambda_1^\varepsilon$ , we used a test function which is concentrated at  $x = 0$ , namely, the test function of the form  $v(\varepsilon^{-1/4}x)$ . This observation turns out to be very helpful for the construction of the asymptotics of eigenpairs  $(\lambda^\varepsilon, u^\varepsilon)$ .

The next definition explains the notion of concentration.

**DEFINITION 3.2.** *We say that a family  $\{w_\varepsilon(x)\}_{\varepsilon>0}$  with  $0 < c_1 \leq \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c_2$  is concentrated at  $x_0$ , as  $\varepsilon \rightarrow 0$ , if for any  $\gamma > 0$  there is  $\varepsilon_0 > 0$  such that*

$$\int_{\Omega_\varepsilon \setminus B_\gamma(x_0)} |w_\varepsilon|^2 dx < \gamma \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Here  $B_\gamma(x_0)$  is a ball of radius  $\gamma$  centered at  $x_0$ .

**LEMMA 3.3.** *The first eigenfunction  $u_1^\varepsilon$  of problem (2.1) is concentrated in the sense of Definition 3.2 at the minimum point of  $q(x)$ , that is, at  $x = 0$ .*

*Proof.* Assume that  $u_1^\varepsilon$ , normalized by  $\|u_1^\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$ , is not concentrated at  $x = 0$ . Then, there exists  $\gamma > 0$  such that for any  $\varepsilon_0$ , we have

$$(3.4) \quad \int_{\Omega_\varepsilon \setminus B_\gamma(0)} |u_1^\varepsilon|^2 dx > \gamma$$

for some  $\varepsilon < \varepsilon_0$ .

Estimate (3.2) together with (3.1) implies the estimate

$$\int_{\Omega_\varepsilon} |\nabla u_1^\varepsilon|^2 dx \leq C \varepsilon^{-1}.$$

Then, using Lemma 4.1, we obtain

$$\begin{aligned} \lambda_1^\varepsilon &= \int_{\Omega_\varepsilon} a^\varepsilon \nabla u_1^\varepsilon \cdot \nabla u_1^\varepsilon dx + \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \int_{\Omega_\varepsilon} q |u_1^\varepsilon|^2 dx + O(\varepsilon^{-1/2}) \\ &\geq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \min_{\Omega_\varepsilon \setminus B_\gamma(0)} q \int_{\Omega_\varepsilon \setminus B_\gamma(0)} |u_1^\varepsilon|^2 dx \\ &\quad + \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \left\{ q(0) \int_{\Omega_\varepsilon \cap B_\gamma(0)} |u_1^\varepsilon|^2 dx + \int_{\Omega_\varepsilon \cap B_\gamma(0)} (q(x) - q(0)) |u_1^\varepsilon|^2 dx \right\} \\ &\quad + O(\varepsilon^{-1/2}). \end{aligned}$$

Since  $x = 0$  is the global minimum point of  $q(x)$ ,

$$\lambda_1^\varepsilon \geq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \left\{ \min_{\Omega_\varepsilon \setminus B_\gamma(0)} q \int_{\Omega_\varepsilon \setminus B_\gamma(0)} |u_1^\varepsilon|^2 dx + q(0) \int_{\Omega_\varepsilon \cap B_\gamma(0)} |u_1^\varepsilon|^2 dx \right\} + O(\varepsilon^{-1/2}).$$

By (3.4),

$$(3.5) \quad \lambda_1^\varepsilon \geq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \left( \min_{\Omega_\varepsilon \setminus B_\gamma(0)} q - q(0) \right) \gamma + O(\varepsilon^{-1/2}),$$

which contradicts (3.3). The lemma is proved.  $\square$

*Remark 3.2.* The min-max principle allows us to compare the eigenvalues of Dirichlet, Neumann, and Fourier spectral problems. Namely, denote by  $\lambda_{D,k}^\varepsilon$  the  $k$ th eigenvalue of the Dirichlet problem ( $u^\varepsilon = 0$  on  $\Sigma_\varepsilon$ ) and by  $\lambda_{N,k}^\varepsilon$  the  $k$ th eigenvalue of the Neumann problem (the case  $q = 0$  in (2.1)). Then, one can see that

$$(3.6) \quad \lambda_{N,k}^\varepsilon \leq \lambda_k^\varepsilon \leq \lambda_{D,k}^\varepsilon, \quad k = 1, 2, \dots$$

It is well-known (see [13]) that  $\lambda_{N,k}^\varepsilon = O(1)$  and  $\lambda_{D,k}^\varepsilon = O(\varepsilon^{-2})$ ,  $\varepsilon \rightarrow 0$ . Lemma 3.1 specifies estimate (3.6) for the first eigenvalue  $\lambda_1^\varepsilon$ .

**3.2. Change of unknowns. Rescaled problem.** For brevity, we denote

$$\varkappa(x) = \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(x), \quad Q = \frac{1}{2} \frac{|\Sigma^0|_{d-1}}{|Y|_d} H(q),$$

where  $H(q)$  is the Hessian matrix of  $q$  at  $x = 0$ .

Note that Lemma 3.1 suggests studying the asymptotics of  $(\lambda_k^\varepsilon - \varepsilon^{-1} \varkappa(0))$ , rather than of  $\lambda_k^\varepsilon$  itself. On the other hand, when deriving the upper bound in Lemma 3.1, we used the test function  $v(x/\varepsilon^{1/4})$ , which allowed us to get the “optimal” estimate. Bearing in mind these two ideas, we first subtract  $\varepsilon^{-1} \varkappa(0) u^\varepsilon(x)$  from both sides of the equation in (2.1) and then make the change of variables  $z = \varepsilon^{-1/4} x$  in (2.1). Then, the rescaled problem is stated in the domain

$$\widetilde{\Omega}_\varepsilon = \varepsilon^{-1/4} \Omega_\varepsilon, \quad \widetilde{\Sigma}_\varepsilon = \varepsilon^{-1/4} \Sigma_\varepsilon,$$

and takes the form

$$(3.7) \quad \begin{cases} -\operatorname{div}(a^\varepsilon(z) \nabla v^\varepsilon(z)) - \frac{\varkappa(0)}{\sqrt{\varepsilon}} v^\varepsilon = \mu^\varepsilon v^\varepsilon(x), & z \in \widetilde{\Omega}_\varepsilon, \\ a^\varepsilon(z) \nabla v^\varepsilon(z) \cdot n = -\varepsilon^{1/4} q(\varepsilon^{1/4} z) v^\varepsilon(z), & z \in \widetilde{\Sigma}_\varepsilon, \\ v^\varepsilon(z) = 0, & z \in \varepsilon^{-1/4} \partial\Omega. \end{cases}$$

Here

$$(3.8) \quad v^\varepsilon(z) = u^\varepsilon(\varepsilon^{1/4} z), \quad a^\varepsilon(z) = a\left(\frac{z}{\varepsilon^{3/4}}\right), \quad \mu^\varepsilon = \sqrt{\varepsilon} \left( \lambda^\varepsilon - \frac{\varkappa(0)}{\varepsilon} \right).$$

The weak formulation of problem (3.4) reads as follows: find  $(\mu^\varepsilon, v^\varepsilon) \in \mathbb{R} \times H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4} \partial\Omega)$ ,  $v^\varepsilon \neq 0$ , such that

$$(3.9) \quad W^\varepsilon(v^\varepsilon, w) = \mu^\varepsilon (v^\varepsilon, w)_{L^2(\widetilde{\Omega}_\varepsilon)} \quad \text{for all } w \in H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4} \partial\Omega).$$

Here the bilinear form  $W^\varepsilon(u, v)$  is given by

$$(3.10) \quad W^\varepsilon(u, v) = \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla u \cdot \nabla v \, dz - \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} u v \, dz + \varepsilon^{1/4} \int_{\widetilde{\Sigma}_\varepsilon} q(\varepsilon^{1/4} z) u v \, d\sigma_z.$$

*Remark 3.3.* According to [1], for all sufficiently small  $\varepsilon$ , there exists an extension operator

$$P^\varepsilon : H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4}\partial\Omega) \rightarrow H_0^1(\varepsilon^{-1/4}\Omega)$$

such that

$$\|P^\varepsilon v\|_{L^2(\varepsilon^{-1/4}\Omega)} \leq C \|v\|_{L^2(\widetilde{\Omega}_\varepsilon)}, \quad \|\nabla(P^\varepsilon v)\|_{L^2(\varepsilon^{-1/4}\Omega)} \leq C \|\nabla v\|_{L^2(\widetilde{\Omega}_\varepsilon)},$$

where  $C$  is a constant independent of  $\varepsilon$ .

Letting  $v = 0$  in  $\mathbb{R}^d \setminus \Omega$ , we assume that the extended function (for which we keep the same notation) is defined in the whole  $\mathbb{R}^d$ .

**PROPOSITION 3.4.** *The spectrum of problem (3.9) is real and discrete and consists of a countable set of points*

$$0 < \mu_1^\varepsilon < \mu_2^\varepsilon \leq \dots \leq \mu_j^\varepsilon \leq \dots \rightarrow +\infty.$$

The corresponding eigenfunctions can be normalized by

$$(3.11) \quad W^\varepsilon(v_i^\varepsilon, v_j^\varepsilon) = \delta_{ij}$$

with  $W^\varepsilon(u, v)$  defined by (3.10).

*Proof.* For any fixed  $\varepsilon > 0$ , the bilinear form  $W^\varepsilon(\cdot, \cdot)$  defines an equivalent scalar product in  $H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4}\partial\Omega)$ . For brevity, we denote

$$(3.12) \quad H_{0,W}^1(\widetilde{\Omega}_\varepsilon) = \{w \in H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4}\partial\Omega) : \|w\|_{\varepsilon,W}^2 = W^\varepsilon(w, w) < \infty\}.$$

Let  $G^\varepsilon : L^2(\widetilde{\Omega}_\varepsilon) \rightarrow H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$  be the operator defined as follows:

$$(3.13) \quad W^\varepsilon(G^\varepsilon f, w) = (f, w)_{L^2(\widetilde{\Omega}_\varepsilon)}, \quad w \in H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4}\partial\Omega).$$

Obviously,  $G^\varepsilon$  is a positive, bounded (uniformly in  $\varepsilon$ ), self-adjoint operator. Since  $H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$ , for each fixed  $\varepsilon$ , is compactly embedded into  $L^2(\widetilde{\Omega})$ , then  $G^\varepsilon$  is compact as an operator from  $L^2(\widetilde{\Omega})$  ( $H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$ ) into itself.

Thus, the spectrum  $\sigma(G^\varepsilon)$  is a countable set of points in  $\mathbb{R}$  which does not have any accumulation points except for zero. Every nonzero eigenvalue has finite multiplicity. To complete the proof of the proposition, it is left to notice that in terms of the operator  $G^\varepsilon$  the eigenvalue problem (3.7) takes the form

$$G^\varepsilon v^\varepsilon = \frac{1}{\mu^\varepsilon} v^\varepsilon. \quad \square$$

We proceed with auxiliary technical results that will be useful in what follows. Define the following norms in  $H^1(\widetilde{\Omega}_\varepsilon)$ :

$$(3.14) \quad \begin{aligned} \|v\|_{\varepsilon,W}^2 &= \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dz - \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} |v|^2 \, dz + \varepsilon^{1/4} \int_{\widetilde{\Sigma}_\varepsilon} q(\varepsilon^{1/4}z) |v|^2 \, d\sigma_z; \\ \|v\|_{\varepsilon,\varkappa}^2 &= \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dz + \frac{1}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} (\varkappa(\varepsilon^{1/4}z) - \varkappa(0)) |v|^2 \, dz; \\ \|v\|_{\varepsilon,Q}^2 &= \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dz + \int_{\widetilde{\Omega}_\varepsilon} (z^T Q z) |v|^2 \, dz. \end{aligned}$$

LEMMA 3.5. *The norms  $\|\cdot\|_{\varepsilon,W}$ ,  $\|\cdot\|_{\varepsilon,\mathcal{X}}$ , and  $\|\cdot\|_{\varepsilon,Q}$  are equivalent. Moreover,*

$$(3.15) \quad \begin{aligned} C_1 \|v\|_{\varepsilon,\mathcal{X}}^2 &\leq \|v\|_{\varepsilon,W}^2 \leq C_2 \|v\|_{\varepsilon,\mathcal{X}}^2, \\ C_3 \|v\|_{\varepsilon,\mathcal{X}}^2 &\leq \|v\|_{\varepsilon,Q}^2 \leq C_4 \|v\|_{\varepsilon,\mathcal{X}}^2 \end{aligned}$$

with constants  $C_1, C_2, C_3$ , and  $C_4$  that do not depend on  $\varepsilon$ .

*Proof.* Indeed, by Lemma 4.2 and by the Poincaré inequality,

$$\| \|v\|_{\varepsilon,W}^2 - \|v\|_{\varepsilon,\mathcal{X}}^2 \| \leq C \varepsilon^{1/4} \|v\|_{L^2(\widetilde{\Omega}_\varepsilon)} \|\nabla v\|_{L^2(\widetilde{\Omega}_\varepsilon)} \leq C_1 \varepsilon^{1/4} \|v\|_{\varepsilon,\mathcal{X}}^2$$

and thus the first inequality in (3.15) holds for sufficiently small  $\varepsilon$ .

The second inequality follows easily from hypothesis (H4) and Lemma 4.3 of section 4.  $\square$

*Remark 3.4.* If  $v \in H^1(\widetilde{\Omega}_\varepsilon)$  decays exponentially, namely,

$$\|v\|_{L^2(\mathbb{R}^d \setminus B_R(0))} \leq M e^{-\gamma_0 R},$$

for some constant  $M$ , then the norms defined in Lemma 3.5 are asymptotically close. In particular, the following estimate holds:

$$\| \|v\|_{\varepsilon,W}^2 - \|v\|_{\varepsilon,Q}^2 \| \leq C \varepsilon^{1/4}$$

with the constant  $C = C(M, \gamma_0)$  independent of  $\varepsilon$ .

LEMMA 3.6. *Let  $\mu_1^\varepsilon$  be the first eigenvalue of the spectral problem (3.7). Then there exist two positive constants  $C_1$  and  $C_2$  such that*

$$C_1 \leq \mu_1^\varepsilon \leq C_2.$$

*Proof.* The upper bound follows from (3.8) and Lemma 3.1. The lower bound is the consequence of the boundedness of the operator  $G^\varepsilon$  (see the proof of Proposition 3.4).  $\square$

**3.2.1. Formal asymptotic expansion for the rescaled problem.** Following the classical asymptotic expansion method and bearing in mind Lemma 3.6, we seek a solution of problem (3.7) in the form of asymptotic series

$$(3.16) \quad \begin{aligned} \mu^\varepsilon &= \mu + \varepsilon^{1/4} \mu_{\frac{1}{4}} + \varepsilon^{1/2} \mu_{\frac{1}{2}} + \dots, \\ v^\varepsilon &= v(z) + \varepsilon^{1/4} v_{\frac{1}{4}}(z, \zeta) + \varepsilon^{1/2} v_{\frac{1}{2}}(z, \zeta) + \varepsilon^{3/4} v_{\frac{3}{4}}(z, \zeta) + \dots, \quad \zeta = \frac{z}{\varepsilon^{3/4}}, \end{aligned}$$

where the functions  $v_{\frac{k}{4}}(z, \zeta)$  are  $Y$ -periodic in  $\zeta$ ,  $k = 1, 2, \dots$

Substituting ansatz (3.16) into (3.7) and collecting the terms of order  $\varepsilon^{-5/4}$  and  $\varepsilon^{-1}$  in the equation and of order  $\varepsilon^{-1/2}$ ,  $\varepsilon^{-1/4}$  in the boundary condition, we see that the functions  $v_{\frac{1}{4}}$  and  $v_{\frac{1}{2}}$  do not depend on the fast variable  $\zeta$ . Then, collecting the terms of order  $\varepsilon^{-3/4}$ , we obtain that

$$v_{\frac{3}{4}}(z, \zeta) = N_k(\zeta) \partial_k v(z) + w_3(z),$$

where the vector function  $N(\zeta)$  solves the problem

$$(3.17) \quad \begin{cases} -\operatorname{div}_\zeta(a(\zeta)\nabla_\zeta N_k(\zeta)) = \operatorname{div}_\zeta a_{\cdot k}(\zeta), & k = 1, \dots, d, \quad \zeta \in Y, \\ a \nabla_\zeta N_k \cdot n = -a_{ik} n_i, & \zeta \in \Sigma^0, \\ N_k(\zeta) \in H^1_{\#}(Y). \end{cases}$$

The effective spectral problem comes out while collecting the terms of order  $\varepsilon^0$  and writing the compatibility condition for the resulting problem. It reads

$$(3.18) \quad -\operatorname{div}(a^{\text{eff}} \nabla v) + (z^T Q z) v = \mu v, \quad v \in L^2(\mathbb{R}^d),$$

where  $a^{\text{eff}}$  is given by

$$(3.19) \quad a_{ij}^{\text{eff}} = \frac{1}{|Y|_d} \int_Y a_{ik}(y) (\delta_{kj} + \partial_k N_j) dy.$$

The effective problem describes the eigenvalues and eigenfunctions of  $d$ -dimensional harmonic oscillator. In  $\mathbb{R}^1$  an explicit solution can be given in terms of Hermite polynomials. In the case under consideration the following statement characterizes the spectrum of problem (3.18).

LEMMA 3.7. *The spectrum of the effective problem (3.18) is real and discrete,*

$$0 < \mu_1 < \mu_2 \leq \dots \leq \mu_j \dots \rightarrow +\infty.$$

The corresponding eigenfunctions  $v_j(z)$  can be normalized by

$$(3.20) \quad (v_i, v_j)_Q \equiv \int_{\mathbb{R}^d} a^{\text{eff}} \nabla v_i \cdot \nabla v_j dz + \int_{\mathbb{R}^d} (z^T Q z) v_i v_j dz = \delta_{ij}.$$

We omit the proof of Lemma 3.7, which is classical.

It is well known that the eigenfunctions of the harmonic oscillator operator have the form

$$(3.21) \quad v_j(z) = P_{j-1}(z) e^{-z^T R z}, \quad R = \frac{\sqrt{2} Q^{1/2} (a^{\text{eff}})^{-1/2}}{2},$$

where  $P_k(z)$  is a polynomial of degree  $k$ .

To summarize, the formal asymptotic expansion for  $v^\varepsilon$  takes the form

$$v(z) + \varepsilon^{3/4} N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v(z),$$

where  $v$  is an eigenfunction of the limit spectral problem (3.18) and  $N$  is a periodic vector function solving (3.17).

Notice that we can neglect the summands  $v_{\frac{1}{4}}$  and  $v_{\frac{1}{2}}$  since they do not depend on the fast variable  $\zeta$ , and thus their  $H^1$ -norm is of order  $\varepsilon^{1/4}$ .

**3.2.2. Justification.** Denote  $J(j) = \min\{i \in \mathbb{Z}^+ : \mu_i = \mu_j\}$ , and let  $\kappa_j$  be the multiplicity of the  $j$ th eigenvalue  $\mu_j$  of the harmonic oscillator operator (3.18).

The main goal of this section is to prove the following theorem.

THEOREM 3.8. *Let hypotheses (H1)–(H4) be fulfilled. If  $(\mu_p^\varepsilon, v_p^\varepsilon)$  stands for the  $p$ th eigenpair of problem (3.7), then the following statements hold true:*

1. *For each  $j = 1, 2, \dots$ , there exist  $\varepsilon_j > 0$  and a constant  $c_j$  such that the eigenvalue  $\mu_j^\varepsilon$  of problem (3.7) satisfies the inequality*

$$|\mu_j^\varepsilon - \mu_j| \leq c_j \varepsilon^{1/4}, \quad \varepsilon \in (0, \varepsilon_j),$$

where  $\mu_j$  is an eigenvalue of the harmonic oscillator operator (3.18).



2. There exists a unitary  $\kappa_j \times \kappa_j$  matrix  $\beta^\varepsilon$  such that

$$(3.22) \quad \left\| v_p^\varepsilon - \sum_{k=J(j)}^{J(j)+\kappa_j-1} \beta_{pk}^\varepsilon \tilde{V}_k^\varepsilon \right\|_{\varepsilon, Q} \leq C_j \varepsilon^{1/4}, \quad p = J(j), \dots, J(j) + \kappa_j - 1,$$

where

$$(3.23) \quad \tilde{V}_k^\varepsilon = v_k(z) + \varepsilon^{3/4} N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_k(z).$$

Here the vector function  $N(\zeta)$  solves problem (3.17); eigenfunctions  $v_k(z)$  of the limit problem are defined in (3.18); the norm  $\|\cdot\|_{\varepsilon, Q}$  is defined just before Lemma 3.5.

Moreover, almost eigenfunctions  $\{\tilde{V}_k^\varepsilon\}$  satisfy the following almost orthonormality conditions:

$$(3.24) \quad \left| \int_{\tilde{\Omega}_\varepsilon} a^\varepsilon \nabla \tilde{V}_k^\varepsilon \cdot \nabla \tilde{V}_m^\varepsilon dz + \int_{\tilde{\Omega}_\varepsilon} (z^T Q z) \tilde{V}_k^\varepsilon \tilde{V}_m^\varepsilon dz - \delta_{km} \right| \leq C \varepsilon^{1/4}.$$

*Proof.* The justification procedure relies on Vishik’s lemma about almost eigenvalues and eigenfunctions (see, for example, [14] and [9, p. 319, Lemma 1.5]). For the reader’s convenience, we formulate the mentioned result.

LEMMA 3.9. *Given a self-adjoint operator  $\mathcal{K}^\varepsilon : \mathcal{H} \rightarrow \mathcal{H}$  with a discrete spectrum, let  $\nu \in \mathbb{R}$  and  $v \in \mathcal{H}$  be such that*

$$\|v\|_{\mathcal{H}} = 1, \quad \delta \equiv \|\mathcal{K}^\varepsilon v - \nu v\|_{\mathcal{H}} < |\nu|.$$

*Then there exists an eigenvalue  $\mu_l^\varepsilon$  of the operator  $\mathcal{K}^\varepsilon$  such that*

$$|\mu_l^\varepsilon - \nu| \leq \delta.$$

*Moreover, for any  $\delta_1 \in (\delta, |\nu|)$  there exist  $\{a_j^\varepsilon\} \in \mathbb{R}$  such that*

$$\|v - \sum a_j^\varepsilon u_j^\varepsilon\|_{\mathcal{H}} \leq 2 \frac{\delta}{\delta_1},$$

*where the sum is taken over the eigenvalues of the operator  $\mathcal{K}^\varepsilon$  on the segment  $[\nu - \delta_1, \nu + \delta_1]$ , and  $\{u_j^\varepsilon\}$  are the corresponding eigenfunctions. The coefficients  $a_j^\varepsilon$  are normalized so that  $\sum |a_j^\varepsilon|^2 = 1$ .*

Let  $\mu_j$  be an eigenvalue of the effective problem (3.18) of multiplicity  $\varkappa_j$ , that is, let  $\mu_j = \mu_{j+1} = \dots = \mu_{j+\varkappa_j-1}$  and  $\{v_p(z)\}$ ,  $p = j, \dots, j + \varkappa_j - 1$ , be the eigenfunctions corresponding to  $\mu_j$ . Denote

$$(3.25) \quad V_p^\varepsilon(z) = v_p(z) \chi_\varepsilon(z) + \varepsilon^{3/4} \chi_\varepsilon(z) N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z),$$

where  $v_p$  is the  $p$ th eigenfunction of the limit spectral problem (3.18),  $N$  is a solution of (3.17);  $\chi_\varepsilon(z)$  is a cut-off which is equal to 1 if  $|z| < \frac{\varepsilon^{-1/4}}{3} \text{dist}(0, \partial\Omega)$ , equal to 0 if  $|z| > \frac{\varepsilon^{-1/4}}{2} \text{dist}(0, \partial\Omega)$ , and is such that

$$(3.26) \quad 0 \leq \chi_\varepsilon(x) \leq 1, \quad |\nabla \chi_\varepsilon| \leq C \varepsilon^{1/4}.$$

We apply Lemma 3.9 to the operator  $G^\varepsilon : H_{0,W}^1(\widetilde{\Omega}_\varepsilon) \rightarrow H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$  constructed in Proposition 3.4 (see (3.13)). The normalized functions  $\mathcal{V}_p^\varepsilon \equiv V_p^\varepsilon / \|V_p^\varepsilon\|_{\varepsilon,W}$  and the numbers  $\mu_j$  will play the roles of  $v \in \mathcal{H}$  and  $\nu \in \mathbb{R}$  in Lemma 3.9. Notice that  $v_j$  need not be equal to zero on the boundary  $\varepsilon^{-1/4}\partial\Omega$ ; the cut-off function has been introduced in order to make approximate solution (3.25) belong to the space  $H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$  (see (3.12)).

LEMMA 3.10. *Almost eigenfunctions  $V_p^\varepsilon$  are almost orthonormal. Namely, the following inequalities hold:*

$$(3.27) \quad \begin{aligned} |W^\varepsilon(V_p^\varepsilon, V_q^\varepsilon) - \delta_{pq}| &\leq C \varepsilon^{1/4}, \\ |(V_p^\varepsilon, V_q^\varepsilon)_{\varepsilon,Q} - \delta_{pq}| &\leq C \varepsilon^{1/4}, \end{aligned}$$

where  $W^\varepsilon(u, v)$  and  $(\cdot, \cdot)_Q$  are defined by (3.10) and (3.20), respectively.

*Proof.* We calculate first the gradient of the function  $V_p^\varepsilon$ ,

$$\nabla V_p^\varepsilon = J_{1p}^\varepsilon(z) \chi_\varepsilon(z) + \varepsilon^{3/4} J_{2p}^\varepsilon(z) + J_{3p}^\varepsilon(z) \nabla \chi_\varepsilon(z),$$

where

$$\begin{aligned} J_{1p}^\varepsilon(z) &= \nabla v_p(z) + \nabla_\zeta(N(\zeta) \cdot v_p(z)) \Big|_{\zeta=z/\varepsilon^{3/4}}; \\ J_{2p}^\varepsilon(z) &= \chi_\varepsilon(z) \nabla_z(N(\zeta) \cdot \nabla v_p(z)); \\ J_{3p}^\varepsilon(z) &= v_p(z) + \varepsilon^{3/4} N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z). \end{aligned}$$

One can show that

$$\begin{aligned} &\left| W^\varepsilon(V_p^\varepsilon, V_q^\varepsilon) - \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon(z) (\chi_\varepsilon(z))^2 J_{1p}^\varepsilon \cdot J_{1q}^\varepsilon dz + \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} v_p(z) v_q(z) (\chi_\varepsilon(z))^2 dz \right. \\ &\quad \left. - \varepsilon^{1/4} \int_{\Sigma_\varepsilon} q(\varepsilon^{1/4}) v_p(z) v_q(z) (\chi_\varepsilon(z))^2 d\sigma_z \right| \leq C \varepsilon^{1/4}. \end{aligned}$$

On the other hand, using Lemma 4.5, exponential decay of the eigenfunctions  $v_p(z)$ , and the normalization condition (3.20), we can prove that

$$\begin{aligned} &\left| \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon(z) (\chi_\varepsilon(z))^2 J_{1p}^\varepsilon \cdot J_{1q}^\varepsilon dz - \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} v_p(z) v_q(z) (\chi_\varepsilon(z))^2 dz \right. \\ &\quad \left. + \varepsilon^{1/4} \int_{\Sigma_\varepsilon} q(\varepsilon^{1/4}) v_p(z) v_q(z) (\chi_\varepsilon(z))^2 d\sigma_z - \delta_{pq} \right| \leq C \varepsilon^{1/4}. \end{aligned}$$

Combining the last two estimates yields

$$|W^\varepsilon(V_p^\varepsilon, V_q^\varepsilon) - \delta_{pq}| \leq C \varepsilon^{1/4},$$

which is the first estimate in (3.27).

The second estimate in (3.27) follows from the first one and Remark 3.4.  $\square$

LEMMA 3.11. *Let  $\mathcal{V}_p^\varepsilon \equiv V_p^\varepsilon / \|V_p^\varepsilon\|_{\varepsilon,W}$  with  $V_p^\varepsilon$  defined by (3.25). Then the following estimate holds:*

$$(3.28) \quad \|G^\varepsilon \mathcal{V}_p^\varepsilon - (\mu_j)^{-1} \mathcal{V}_p^\varepsilon\|_{\varepsilon,W} \leq C_p \varepsilon^{1/4}, \quad p = i, \dots, i + \kappa_j - 1.$$

*Proof.* Simple transformations result in the following relations:

$$\|G^\varepsilon \mathcal{V}_p^\varepsilon - (\mu_j)^{-1} \mathcal{V}_p^\varepsilon\|_{\varepsilon, W} = \|V_p^\varepsilon\|_{\varepsilon, W}^{-1} \sup_{\|w\|_{\varepsilon, W}=1} W^\varepsilon (G^\varepsilon V_p^\varepsilon - (\mu_j)^{-1} V_p^\varepsilon, w).$$

By the definition of the operator  $G^\varepsilon$  in (3.13),

$$\begin{aligned} & \|G^\varepsilon \mathcal{V}_p^\varepsilon - (\mu_j)^{-1} \mathcal{V}_p^\varepsilon\|_{\varepsilon, W} \\ &= \frac{1}{\mu_p} \|V_p^\varepsilon\|_{\varepsilon, W}^{-1} \sup_{\|w\|_{\varepsilon, W}=1} \left\{ \mu_p (V_p^\varepsilon, w)_{L^2(\widetilde{\Omega}_\varepsilon)} - \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla V_p^\varepsilon \cdot \nabla w \, dz \right. \\ & \quad \left. + \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} V_p^\varepsilon w \, dz - \varepsilon^{1/4} \int_{\widetilde{\Sigma}_\varepsilon} q(\varepsilon^{1/4} z) V_p^\varepsilon w \, d\sigma_z \right\} \\ &= \frac{1}{\mu_p} \|V_p^\varepsilon\|_{\varepsilon, W}^{-1} \sup_{\|w\|_{\varepsilon, W}=1} \{I_1^\varepsilon + I_2^\varepsilon + \varepsilon^{3/4} I_3^\varepsilon\}. \end{aligned}$$

Here

$$\begin{aligned} I_1^\varepsilon &= \mu_p \int_{\widetilde{\Omega}_\varepsilon} \chi_\varepsilon(z) v_p(z) w(z) \, dz - \frac{1}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} (\varkappa(\varepsilon^{1/4} z) - \varkappa(0)) \chi_\varepsilon(z) v_p(z) w(z) \, dz \\ & \quad - \int_{\widetilde{\Omega}_\varepsilon} a(\zeta) (\nabla v_p(z) + \nabla_\zeta(N(\zeta) \cdot \nabla v_p(z))) \cdot \nabla w \chi_\varepsilon(z) \Big|_{\zeta=z/\varepsilon^{3/4}} \, dz; \\ I_2^\varepsilon &= \frac{1}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} \varkappa(\varepsilon^{1/4} z) v_p(z) \chi_\varepsilon(z) w(z) \, dz - \varepsilon^{1/4} \int_{\widetilde{\Sigma}_\varepsilon} q(\varepsilon^{1/4} z) v_p(z) \chi_\varepsilon(z) w(z) \, d\sigma; \\ I_3^\varepsilon &= \mu_p \int_{\widetilde{\Omega}_\varepsilon} \chi_\varepsilon(z) N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z) w(z) \, dz - \int_{\widetilde{\Omega}_\varepsilon} a(\zeta) \nabla \chi_\varepsilon(z) \cdot \nabla w v_p(z) \Big|_{\zeta=z/\varepsilon^{3/4}} \, dz \\ & \quad - \int_{\widetilde{\Omega}_\varepsilon} a(\zeta) \nabla_z (\chi_\varepsilon(z) N(\zeta) \cdot \nabla v_p(z)) \cdot \nabla w \Big|_{\zeta=z/\varepsilon^{3/4}} \, dz \\ & \quad + \varepsilon^{1/4} \varkappa(0) \int_{\widetilde{\Omega}_\varepsilon} \chi_\varepsilon(z) N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z) w(z) \, dz \\ & \quad - \varepsilon^{1/4} \int_{\widetilde{\Sigma}_\varepsilon} q(\varepsilon^{1/4} z) \chi_\varepsilon(z) N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z) w \, d\sigma. \end{aligned}$$

Integrating by parts in the last integral in  $I_1^\varepsilon$ , and taking into account (H4), (3.17), and (3.26), we obtain

$$\begin{aligned} I_1^\varepsilon &= \mu_p \int_{\widetilde{\Omega}_\varepsilon} \chi_\varepsilon(z) v_p(z) w(z) \, dz - \int_{\widetilde{\Omega}_\varepsilon} (z^T Q z) \chi_\varepsilon(z) v_p(z) w(z) \, dz \\ & \quad + \int_{\widetilde{\Omega}_\varepsilon} \operatorname{div}_z (a(\zeta) (I + \nabla_\zeta N(\zeta)) \nabla v_p(z)) \Big|_{\zeta=z/\varepsilon^{3/4}} w(z) \chi_\varepsilon(z) \, dz \\ & \quad + O(\varepsilon^{1/4}), \quad \varepsilon \rightarrow 0. \end{aligned}$$

Here we have also used Lemma 3.5 and the fact that  $\|w\|_{\varepsilon, W} = 1$ .

Bearing in mind the definition of the effective diffusion (3.19) and (3.18), by virtue of Lemma 4.5, one has

$$(3.29) \quad |I_1^\varepsilon| \leq C \varepsilon^{1/4} \|w\|_{H^1(\mathbb{R}^d)}.$$

By Lemma 4.2,

$$(3.30) \quad |I_2^\varepsilon| \leq C_2 \varepsilon^{1/4} \|v_p\|_{H^1(\mathbb{R}^d)} \|w\|_{H^1(\mathbb{R}^d)}.$$

Using the boundedness of  $a_{ij}$  and the regularity properties of  $N, v_p, \chi_\varepsilon$ , one can show that

$$(3.31) \quad |I_3^\varepsilon| \leq C_3 \|\nabla v_p\|_{L^2(\mathbb{R}^d)} \|w\|_{H^1(\mathbb{R}^d)}.$$

Using Lemma 3.10 we see that for small enough  $\varepsilon$ ,

$$(3.32) \quad \|V_p^\varepsilon\|_{\varepsilon, W}^2 \geq \frac{1}{2}.$$

Finally, combining (3.29)–(3.32) we obtain the desired estimate (3.28). Lemma 3.11 is proved.  $\square$

By Lemma 3.9, in view of the estimate obtained in Lemma 3.11, for any eigenvalue  $\mu_j$  of the effective problem (3.18) there exists an eigenvalue of the original problem such that

$$(3.33) \quad |\mu_{q(j)}^\varepsilon - \mu_j| \leq C_j \varepsilon^{1/4},$$

where  $q(j)$  might depend on  $\varepsilon$ .

Moreover, letting  $\delta_1$  in the statement of Lemma 3.9 be equal to  $\Theta_j \varepsilon^{1/4}$  (the constant  $\Theta_j$  will be chosen below), we conclude that there exists a  $K_j(\varepsilon) \times \kappa_j$  constant matrix  $\alpha^\varepsilon$  such that

$$(3.34) \quad \left\| \mathcal{V}_p^\varepsilon - \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon v_k^\varepsilon \right\|_{\varepsilon, W} \leq 2 \frac{C\varepsilon^{1/4}}{\delta_1} \leq C_j(\Theta_j)^{-1}, \quad p = j, \dots, j + \kappa_j - 1;$$

here  $\mu_k^\varepsilon, k = J_j(\varepsilon), \dots, J_j(\varepsilon) + K_j(\varepsilon) - 1$ , are all the eigenvalues of operator  $(G^\varepsilon)^{-1}$  which satisfy the estimate

$$(3.35) \quad |\mu_k^\varepsilon - \mu_j| \leq \Theta_j \varepsilon^{1/4}.$$

Since the eigenvalues  $\mu_j$  do not depend on  $\varepsilon$ , one can choose constants  $\varepsilon_j > 0$  so that the intervals  $(\mu_j - \Theta_j \varepsilon^{1/4}, \mu_j + \Theta_j \varepsilon^{1/4})$  and  $(\mu_i - \Theta_i \varepsilon^{1/4}, \mu_i + \Theta_i \varepsilon^{1/4})$  do not intersect if  $\mu_j \neq \mu_i$  and  $\varepsilon < \min(\varepsilon_j, \varepsilon_i)$ . Then the sets of eigenvalues  $\{\mu_k^\varepsilon\}$  related to different  $\mu_j$  in (3.35) do not intersect for sufficiently small  $\varepsilon$ .

In the following statement we prove that  $K_J(\varepsilon) \geq \kappa_j$ .

LEMMA 3.12. *The columns of the matrix  $\alpha^\varepsilon$ , that is, the vectors  $\{\alpha_{\cdot p}^\varepsilon\}_{p=J(j)}^{J(j)+\kappa_j-1}$  of length  $K_J(\varepsilon)$ , are linearly independent. As a consequence,  $K_J(\varepsilon) \geq \kappa_j$ .*

*Proof.* A simple transformation gives

$$\begin{aligned} W^\varepsilon(\mathcal{V}_p^\varepsilon, \mathcal{V}_q^\varepsilon) &= W^\varepsilon \left( \mathcal{V}_p^\varepsilon - \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon v_k^\varepsilon, \mathcal{V}_p^\varepsilon \right) \\ &\quad + W^\varepsilon \left( \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon v_k^{\varepsilon, \pm}, \mathcal{V}_q^\varepsilon - \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kq}^\varepsilon v_k^\varepsilon \right) \\ &\quad + \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon \alpha_{kq}^\varepsilon. \end{aligned}$$

Taking estimates (3.27) and (3.34) into account, we obtain

$$\left| \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon \alpha_{kq}^\varepsilon - \delta_{p,q} \right| \leq C \Theta_j^{-1}, \quad p, q = J(j), \dots, J(j) + \kappa_j - 1,$$

and, in other words,

$$(3.36) \quad |(\alpha^\varepsilon_p)^T \alpha^\varepsilon_q - \delta_{p,q}| \leq C \Theta_j^{-1}, \quad p, q = J(j), \dots, J(j) + \kappa_j - 1,$$

where  $\alpha^\varepsilon_p$  denotes a  $p$ th column in the matrix  $\alpha^\varepsilon$ . The last inequality means that the vectors  $\{\alpha^\varepsilon_p\}_{p=J(j)}^{J(j)+\kappa_j-1}$  are asymptotically orthonormal, as  $\Theta_j$  grows to infinity. This property implies the linear independence of the vectors  $\{\alpha^\varepsilon_p\}$  for sufficiently large  $\Theta_j$ . Indeed, assume that  $\{\alpha^\varepsilon_p\}_{p=J(j)}^{J(j)+\kappa_j-1}$  are not linearly independent. Then there exist constants  $c_{J(j)}, \dots, c_{J(j)+\kappa_j-1}$  such that

$$\sum_{k=J(j)}^{J(j)+\kappa_j-1} c_k \alpha^\varepsilon_k = 0.$$

Without loss of generality we assume that  $c_{J(j)} = 1 \geq \max_k |c_k|$ . Then

$$\alpha^\varepsilon_{J(j)} + \sum_{k>J(j)} c_k \alpha^\varepsilon_k = 0.$$

Multiplying the last equality by  $\alpha^\varepsilon_{J(j)}$  and using (3.36) we obtain the inequality

$$|(\alpha^\varepsilon_{J(j)})^T \alpha^\varepsilon_{J(j)}| \leq C_j \Theta_j^{-1},$$

which contradicts (3.36) if  $\Theta_j^{-1}$  is sufficiently small. Thus, the vectors  $\{\alpha^\varepsilon_p\}_{p=J(j)}^{J(j)+\kappa_j-1}$  of length  $K_J(\varepsilon)$  are linearly independent. Obviously, it is possible only in the case  $K_J(\varepsilon) \geq \kappa_j$ .  $\square$

LEMMA 3.13. For any  $q$ ,  $0 < m \leq \mu_q^\varepsilon \leq M_q$ .

*Proof.* The estimate from below is the immediate consequence of the boundedness of the operator  $G^\varepsilon$  constructed in Proposition 3.4.

To obtain an upper bound for  $\mu_q^\varepsilon$ , we recall estimate (3.33). For any  $j$ , there exists an eigenvalue of problem (3.7) converging to the  $j$ th eigenvalue of the effective problem. Namely, the estimate

$$|\mu_{q_\varepsilon(j)}^\varepsilon - \mu_{J(j)}| \leq C_j \varepsilon^{1/4}$$

holds, where  $J(j) = \min\{i \in \mathbb{Z}^+ : \mu_i = \mu_j\}$ . Obviously,  $q_\varepsilon(j) \geq J(j)$ . Thus,

$$\mu_{J(j)}^\varepsilon \leq \mu_{q_\varepsilon(j)}^\varepsilon \leq \mu_{J(j)} + C_j \varepsilon^{1/4},$$

which implies the desired bound.  $\square$

Our next goal is to prove that any accumulation point of the sequence  $\mu_q^\varepsilon$ , as  $\varepsilon \rightarrow 0$ , is an eigenvalue of (3.18).

LEMMA 3.14. If up to a subsequence,  $\mu_q^\varepsilon \rightarrow \mu^*$ , as  $\varepsilon \rightarrow 0$ , then  $\mu^*$  is an eigenvalue of the effective spectral problem (3.18).

*Proof.* Since  $\mu_q^\varepsilon$  is bounded, then

$$\|v_q^\varepsilon\|_{\varepsilon, W} \leq C_q$$

with  $\|\cdot\|_{\varepsilon, W}$  defined in (3.14). In view of Lemmata 3.5 and 4.4, the eigenfunction  $v_q^\varepsilon$  (extended to the whole  $\mathbb{R}^d$ ) converges weakly in  $H^1(\mathbb{R}^d)$  and strongly in  $L^2(\mathbb{R}^d)$  to some function  $v^*$ . To prove that  $(\mu^*, v^*)$  is an eigenpair of the effective problem

(3.18), we pass to the limit in the integral identity (3.9). Using standard two-scale convergence arguments we obtain

$$\int_{\mathbb{R}^d} a^{\text{eff}} \nabla v^* \cdot \nabla w \, dz + \int_{\mathbb{R}^d} (z^T Q z) v^* w \, dz = \mu^* \int_{\mathbb{R}^d} v^* w \, dz, \quad w \in H^1(\mathbb{R}^d).$$

The last equality is the weak formulation of (3.18). Since  $\mu_q^\varepsilon \rightarrow \mu^*$ , as  $\varepsilon \rightarrow 0$ , then considering (3.9) and (3.11) we conclude that  $\lim_{\varepsilon \rightarrow 0} \|v_q^\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)}^2 = \mu^*$ . Using the

strong convergence of  $v_q^\varepsilon$  in  $L^2(\mathbb{R}^d)$ , we see that  $\|v^*\|_{L^2(\mathbb{R}^d)}^2 \geq \mu^*$ . By Lemma 3.6 we have  $\mu^* > 0$ . Therefore,  $v^* \neq 0$ . This completes the proof.  $\square$

LEMMA 3.15. *Let  $\mu_j$  be the  $j$ th eigenvalue of problem (3.18) of multiplicity  $\kappa_j$ , that is,  $\mu_j = \mu_{j+1} = \dots = \mu_{j+\kappa_j-1}$ . Then there exist exactly  $\kappa_j$  eigenvalues of the original problem (2.1) converging to it.*

*Proof.* First, we prove that there are not more than  $\kappa_j$  eigenvalues of problem (3.7) converging to  $\mu_j$ . Assume that there exist  $\kappa_j + 1$  eigenvalues  $\mu_{J_k^\varepsilon(j)}$  such that

$$\mu_{J_k^\varepsilon(j)} \rightarrow \mu_j, \quad k = 1, \dots, \kappa_j + 1.$$

By Lemma 3.14, the corresponding eigenfunctions  $v_{J_k^\varepsilon(j)}$ , extended to the whole  $\mathbb{R}^d$ , converge weakly in  $H^1(\mathbb{R}^d)$  and strongly in  $L^2(\mathbb{R}^d)$  to the eigenfunctions  $v_k^*$  of the effective problem (3.18),  $k = 1, \dots, \kappa_j + 1$ . Passing to the limit in the normalization condition (3.11) yields

$$(v_i^*, v_k^*)_{L^2(\mathbb{R}^d)} = \frac{1}{|Y|_d} \mu_i \delta_{ik}, \quad i, k = 1, \dots, \kappa_j + 1.$$

Therefore, eigenfunctions  $\{v_k^*\}_{k=1}^{\kappa_j+1}$  corresponding to  $\mu_j$  are linearly independent. Recalling that the multiplicity of  $\mu_j$  is  $\kappa_j$ , we arrive at a contradiction. Thus, there are not more than  $\kappa_j$  eigenvalues of problem (3.7) converging to  $\mu_j$ .

On the other hand, by Lemma 3.12, there exist at least  $\kappa_j$  eigenvalues of (3.7) converging to  $\mu_j$  of multiplicity  $\kappa_j$ . Lemma 3.15 is proved.  $\square$

Combining Lemmata 3.13–3.15 completes the proof of the first statement of Theorem 3.8.

We turn to the proof of the second statement in Theorem 3.8.

First, let us notice that the orthogonality and normalization condition (3.24) follows directly from Lemma 3.10 and the exponential decay of  $v_k(z)$  as eigenfunctions of the harmonic oscillator.

In order to prove estimate (3.22), we recall the estimate obtained in Lemma 3.11 and apply the estimate in Lemma 3.9 with  $\delta_1 = c_j$ ,  $c_j$  being a sufficiently small constant. This estimate reads

$$\left\| \mathcal{V}_p^\varepsilon - \sum_{\mu_k^\varepsilon \in S(j, \varepsilon)} \alpha_{kp}^\varepsilon v_k^\varepsilon \right\|_{\varepsilon, W} \leq 2 \frac{C \varepsilon^{1/4}}{\delta_1} \leq C_j \varepsilon^{1/4}, \quad p = j, \dots, j + \kappa_j - 1,$$

where  $S(j, \varepsilon)$  is the set of eigenvalues  $\mu_k^\varepsilon$  satisfying the estimate

$$|\mu_k^\varepsilon - \mu_j| \leq c_j;$$

the constant matrix  $\alpha^\varepsilon$  is such that

$$(3.37) \quad |(\alpha_{p,q}^\varepsilon)^T \alpha_{p,q}^\varepsilon - \delta_{p,q}| \leq C_j \varepsilon^{1/4}, \quad p, q = J(j), \dots, J(j) + \kappa_j - 1.$$

From the first statement of Theorem 3.8 we deduce that the set  $S(j, \varepsilon)$  coincides with the set of eigenvalues  $\{\mu_k^\varepsilon\}_{k=J(j)}^{J(j)+\kappa_j-1}$  for sufficiently small  $\varepsilon$ . Therefore,

$$(3.38) \quad \left\| \mathcal{V}_p^\varepsilon - \sum_{k=J(j)}^{J(j)+\kappa_j-1} \alpha_{kp}^\varepsilon v_k^\varepsilon \right\|_{\varepsilon, W} \leq C_j \varepsilon^{1/4}, \quad p = j, \dots, j + \kappa_j - 1,$$

with a constant  $\kappa_j \times \kappa_j$  matrix  $\alpha^\varepsilon$  which satisfies inequality (3.37).

It remains to use the following simple statement.

LEMMA 3.16. *For any  $n \times n$  matrix  $A$  satisfying an equality*

$$\|A^T A - \mathbb{I}\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)} = \gamma \in (0, 1),$$

there exists a unitary matrix  $B$  such that

$$\|AB - \mathbb{I}\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)} \leq \gamma;$$

here  $\mathbb{I}$  is a unit matrix, and

$$\|D\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)} = \sup_{\substack{\xi \in \mathbb{R}^n \\ \|\xi\|=1}} \|D\xi\|.$$

We omit the proof of this lemma, which can be found in [9]. According to (3.37) and Lemma 3.16, there exists a unitary  $\kappa_j \times \kappa_j$  matrix  $\beta^\varepsilon$  such that

$$(3.39) \quad \|\alpha^\varepsilon \beta^\varepsilon - \mathbb{I}\|_{\mathcal{L}(\mathbb{R}^{\kappa_j}, \mathbb{R}^{\kappa_j})} \leq C_j \varepsilon^{1/4}.$$

Taking into account Lemma 3.10 and estimates (3.38), (3.39), one can show that

$$\left\| v_p^\varepsilon - \sum_{k=J(j)}^{J(j)+\kappa_j-1} \beta_{kp}^\varepsilon V_k^\varepsilon \right\|_{\varepsilon, W} \leq C_j \varepsilon^{1/4}, \quad p = J(j), \dots, J(j) + \kappa_j - 1.$$

Due to the exponential decay of the eigenfunctions  $v_k(z)$  defined in (3.18), one can replace  $V_k^\varepsilon$  defined by (3.25) with (3.23). Then, by Lemma 3.5, a similar estimate holds for  $\|\cdot\|_{\varepsilon, Q}$  norm. Theorem 3.8 is proved.  $\square$

Bearing in mind the result obtained in Theorem 3.8, we formulate the main result of the present paper characterizing the asymptotic behavior of eigenpairs  $(\lambda_j^\varepsilon, u_j^\varepsilon)$  of problem (2.1).

THEOREM 3.17. *Let conditions (H1)–(H4) be fulfilled. If  $(\lambda_j^\varepsilon, u_j^\varepsilon)$  stands for the  $j$ th eigenpair of problem (2.1), then for any  $j$ , the following representation takes place:*

$$\lambda_j^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + \frac{\mu_j^\varepsilon}{\sqrt{\varepsilon}}, \quad u_j^\varepsilon(x) = v_j^\varepsilon \left( \frac{x}{\varepsilon^{1/4}} \right),$$

where the eigenpairs  $(\mu_j^\varepsilon, v_j^\varepsilon(z))$  of problem (3.7) are such that the following hold:

1. For each  $j = 1, 2, \dots$ , there exist  $\varepsilon_j > 0$  and a constant  $c_j$  such that

$$|\mu_j^\varepsilon - \mu_j| \leq c_j \varepsilon^{1/4}, \quad \varepsilon \in (0, \varepsilon_j),$$

where  $\mu_j$  is an eigenvalue of the harmonic oscillator operator (3.18).

2. Let  $\mu_j$  be an eigenvalue of (3.18) of multiplicity  $\kappa_j$ , that is  $\mu_j = \dots = \mu_{j+\kappa_j-1}$ . Then, there exists a unitary  $\kappa_j \times \kappa_j$  matrix  $\beta^\varepsilon$  such that

$$\left\| v_p^\varepsilon - \sum_{k=J(j)}^{J(j)+\kappa_j-1} \beta_{kp}^\varepsilon \tilde{V}_k^\varepsilon \right\|_{\varepsilon, Q} \leq C_j \varepsilon^{1/4}, \quad p = J(j), \dots, J(j) + \kappa_j - 1,$$

where

$$\tilde{V}_k^\varepsilon = v_k(z) + \varepsilon^{3/4} N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_k(z).$$

Here the vector function  $N(\zeta)$  solves problem (3.17); eigenfunctions  $v_k(z)$  of the limit problem are defined in (3.18); the norm  $\|\cdot\|_{\varepsilon, Q}$  is defined in (3.14).

**4. Auxiliary results.**

LEMMA 4.1. For any  $w^\varepsilon(x) \in H_0^1(\Omega_\varepsilon, \partial\Omega)$  the following estimate holds:

$$\left| \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \int_{\Omega_\varepsilon} |w^\varepsilon|^2 dx - \int_{\Sigma_\varepsilon} |w^\varepsilon|^2 d\sigma \right| \leq C \|w^\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla w^\varepsilon\|_{L^2(\Omega_\varepsilon)}$$

with a constant  $C$  independent of  $\varepsilon$ .

*Proof.* Introduce a  $Y$ -periodic vector function  $\chi(y)$  as a solution of the following problem on the periodicity cell  $Y$ :

$$\begin{cases} -\operatorname{div}_y \chi = \frac{|\Sigma^0|_{d-1}}{|Y|_d}, & y \in Y, \\ \chi \cdot n = -1, & y \in \Sigma^0. \end{cases}$$

Notice that  $\chi$  is a smooth function. Then

$$-\varepsilon \operatorname{div}_x \chi\left(\frac{x}{\varepsilon}\right) = \frac{|\Sigma^0|_{d-1}}{|Y|_d}.$$

Multiplying the last equality by  $|w^\varepsilon|^2$  and integrating by parts over  $\Omega_\varepsilon$  yields

$$\frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \int_{\Omega_\varepsilon} |w^\varepsilon|^2 dx - \int_{\Sigma_\varepsilon} |w^\varepsilon|^2 d\sigma = \int_{\Omega_\varepsilon} \chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla |w^\varepsilon|^2 dx,$$

which easily implies the desired estimate. The lemma is proved.  $\square$

LEMMA 4.2. Let  $\tilde{\Omega}_\varepsilon = \varepsilon^{-\alpha} \Omega_\varepsilon$ ,  $\tilde{\Sigma}_\varepsilon = \varepsilon^{-\alpha} \Sigma_\varepsilon$ . Then, for  $\psi(z) \in H_0^1(\tilde{\Omega}_\varepsilon, \varepsilon^{-\alpha} \partial\Omega)$  and  $\varphi \in C^1(\mathbb{R}^d)$ , the following estimate holds:

$$\begin{aligned} & \left| \frac{1}{\varepsilon^{1-\alpha}} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \int_{\tilde{\Omega}_\varepsilon} \varphi(\varepsilon^\alpha z) |\psi(z)|^2 dx - \int_{\tilde{\Sigma}_\varepsilon} \varphi(\varepsilon^\alpha z) |\psi(z)|^2 d\sigma \right| \\ & \leq C \|\psi\|_{L^2(\tilde{\Omega}_\varepsilon)} \|\nabla \psi\|_{L^2(\tilde{\Omega}_\varepsilon)} \end{aligned}$$

with some constant  $C$  independent of  $\varepsilon$ .

Lemma 4.2 is proved in the same way as Lemma 4.1.

LEMMA 4.3. Suppose two nonnegative functions  $f_1, f_2 \in C^3(\bar{B})$ , defined on a bounded domain  $B$ , are such that  $x = 0$  is the global minimum point for both of them, and  $f_1(0) = f_2(0) = 0$ . Moreover, assume that

$$H(f_k)(0) \geq \alpha I, \quad \alpha > 0,$$

where  $H(f_k)$  is the Hessian matrix of  $f_k$ ,  $k = 1, 2$ .



Then there exists a constant  $C$  such that

$$C f_1 \leq f_2 \leq C^{-1} f_1.$$

*Proof.* Assume that there exists a sequence  $x_j \in B$  such that

$$\frac{f_1(x_j)}{f_2(x_j)} \rightarrow 0, \quad j \rightarrow \infty.$$

Since  $f_2$  is bounded,  $f_1(x_j) \rightarrow 0$ , as  $j \rightarrow \infty$ . And consequently,  $x_j \rightarrow 0$ , as  $j \rightarrow \infty$ . Recalling that  $H(f_1)(0)$  is bounded from below, we arrive at contradiction. The lemma is proved.  $\square$

LEMMA 4.4 (compactness result). *Denote*

$$H_Q^1(\mathbb{R}^d) = \left\{ w \in H^1(\mathbb{R}^d) : \|w\|_Q^2 = \int_{\mathbb{R}^d} |\nabla w|^2 dz + \int_{\mathbb{R}^d} (z^T Q z) |w|^2 dz < \infty \right\},$$

where  $Q$  is a positive definite symmetric matrix.

Then  $H_Q^1(\mathbb{R}^d)$  is compactly embedded into  $L^2(\mathbb{R}^d)$ . In other words, any  $\{v_n\} \subset H^1(\mathbb{R}^d)$  such that  $\|v_n\|_Q \leq C$  converges strongly along a subsequence in  $L^2(\mathbb{R}^d)$ .

*Proof.* Obviously,  $v_n$  up to a subsequence converges weakly in  $L^2(\mathbb{R}^d)$  to some function  $v^*$ ,  $n \rightarrow \infty$ . Let us prove that  $\|v_n\|_{L^2(\mathbb{R}^d)} \rightarrow \|v^*\|_{L^2(\mathbb{R}^d)}$ , as  $n \rightarrow \infty$ .

Since

$$\int_{\mathbb{R}^d} (z^T Q z) |v_n|^2 dz \leq C,$$

one can show that for any  $\delta > 0$ , there exists a ball  $B_{R(\delta)}(0)$  such that

$$\int_{\mathbb{R}^d \setminus B_{R(\delta)}(0)} |v_n|^2 dz \leq \delta.$$

Without loss of generality we assume that  $\|v_n\|_{L^2(\mathbb{R}^d)} = 1$ . Then

$$(4.1) \quad \|v_n\|_{L^2(B_{R(\delta)}(0))}^2 = 1 - \|v_n\|_{L^2(\mathbb{R}^d \setminus B_{R(\delta)}(0))}^2 \geq 1 - \delta^2.$$

Since  $\|v_n\|_{H^1(B_{R(\delta)}(0))} \leq C$ , then  $\|v_n - v^*\|_{L^2(B_{R(\delta)}(0))} \rightarrow 0$ , as  $n \rightarrow \infty$ . Passing to the limit in (4.1), we have

$$\|v^*\|_{L^2(\mathbb{R}^d)} \geq \|v^*\|_{L^2(B_{R(\delta)}(0))} \geq 1 - \delta^2.$$

On the other hand,

$$\|v^*\|_{L^2(\mathbb{R}^d)} \leq \liminf_{n \rightarrow \infty} \|v_n\|_{L^2(\mathbb{R}^d)} = 1.$$

Combining the last two inequalities yields  $\|v^*\|_{L^2(\mathbb{R}^d)} = 1$ . The lemma is proved.  $\square$

LEMMA 4.5 (mean-value theorem). *Let  $\Phi \in L^2(Y)$  be such that  $\int_Y \Phi dy = 0$  and  $V \in C^1(\mathbb{R}^d)$  satisfy the estimate*

$$(4.2) \quad |D^k V(z)| \leq C e^{-\gamma_0 |z|^2}, \quad \gamma > 0, \quad k = 0, 1.$$

*Denote by  $\chi(x)$  a cut-off which is equal to 1 if  $|x| < \frac{1}{3} \text{dist}(0, \partial\Omega)$ , equal to 0 if  $|x| > \frac{1}{2} \text{dist}(0, \partial\Omega)$ , and is such that*

$$(4.3) \quad 0 \leq \chi(x) \leq 1, \quad |\nabla \chi| \leq C.$$

Then, for any  $\alpha$  such that  $0 < \alpha < 1$ , the following estimate holds:

$$\left| \int_{\Omega_\varepsilon} \Phi\left(\frac{x}{\varepsilon}\right) V\left(\frac{x}{\varepsilon^\alpha}\right) \chi(x) W\left(\frac{x}{\varepsilon^\alpha}\right) dx \right| \leq C_\alpha \varepsilon^{1-\alpha} \varepsilon^{d\alpha} \|\Phi\|_{L^2(Y)} \|W\|_{H^1(\mathbb{R}^d)}$$

for all  $W \in H^1(\mathbb{R}^d)$ .

*Proof.* Since  $\int_Y \Phi dy = 0$ , there exists a periodic vector function  $\varphi(y)$  such that

$$\begin{cases} -\operatorname{div}_y \varphi(y) = \Phi(y), & y \in Y, \\ \varphi \cdot n = 0, & y \in \Sigma^0, \end{cases}$$

and  $\|\varphi\|_{L^2(Y)} \leq C \|\Phi\|_{L^2(Y)}$ . Changing the variables we have

$$-\varepsilon \operatorname{div} \varphi\left(\frac{x}{\varepsilon}\right) = \Phi\left(\frac{x}{\varepsilon}\right).$$

Multiplying the last equation by  $V\left(\frac{x}{\varepsilon^\alpha}\right) \chi_\varepsilon(x) W\left(\frac{x}{\varepsilon^\alpha}\right)$ , integrating by parts over  $\Omega_\varepsilon$ , and using (4.2), (4.3) we get

$$\begin{aligned} & \left| \int_{\Omega_\varepsilon} \Phi\left(\frac{x}{\varepsilon}\right) V\left(\frac{x}{\varepsilon^\alpha}\right) \chi(x) W\left(\frac{x}{\varepsilon^\alpha}\right) dx \right| \\ &= \varepsilon \left| \int_{\Omega_\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right) \cdot \nabla \left[ V\left(\frac{x}{\varepsilon^\alpha}\right) \chi(x) W\left(\frac{x}{\varepsilon^\alpha}\right) \right] dx \right| \\ &\leq C \varepsilon^{1-\alpha} \varepsilon^{d\alpha} \int_{\widetilde{\Omega}_\varepsilon} \left| \varphi\left(\frac{z}{\varepsilon^{1-\alpha}}\right) \right| e^{-\gamma_0|z|^2} [ |W| + |\nabla W| ] dz \\ &\leq C \varepsilon^{1-\alpha} \varepsilon^{d\alpha} \|W\|_{H^1(\mathbb{R}^d)} \left( \int_{\widetilde{\Omega}_\varepsilon} \left| \varphi\left(\frac{z}{\varepsilon^{1-\alpha}}\right) \right|^2 e^{-2\gamma_0|z|^2} dz \right)^{1/2} \\ &\leq C \varepsilon^{1-\alpha} \varepsilon^{d\alpha} \|W\|_{H^1(\mathbb{R}^d)} \|\varphi\|_{L^2(Y)} \left( \int_{\mathbb{R}^d} e^{-2\gamma_0|z|^2} dz \right)^{1/2}. \end{aligned}$$

The integral in the parentheses converges. Lemma 4.5 is proved.  $\square$

**Acknowledgments.** The present work was completed during the stay of the last two authors at Politecnico di Torino, the hospitality of which is kindly acknowledged. We would also like to thank the anonymous referees for their suggestions and comments, which have improved the paper.

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