Fast IRLS for sparse reconstruction based on gaussian mixtures

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***Abstract*—The theory of compressed sensing has demonstrated that sparse signals can be reconstructed from few linear measurements. In this work, we propose a new class of iteratively reweighted least squares (IRLS) for sparse recovery. The proposed methods use a two state Gaussian scale mixture as a proxy for the signal model and can be interpreted as an Expectation Maximization algorithm that attempts to perform the constrained maximization of the log-likelihood function. Under some conditions, standard in the compressed sensing theory, the sequences generated by these algorithms converge to the fixed points of the maps that rule their dynamics. A condition for exact sparse recovery, that is verifible a posteriori, is derived and the convergence is proved to be quadratically fast in a neighborhood of the desired solution. Numerical experiments show that these new reconstructions schemes outperform classical IRLS for ℓ_τ-minimization with $\tau \in (0,1]$ in terms of rate of convergence and accuracy.**

I. Sparse recovery via IRLS for ℓ_τ-minimization

The basic principle of the compressed sensing theory is that a k-sparse signal $x^\star \in \mathbb{R}^n$ (*i.e.*, it has at most k nonzero entries) can be recovered from a smaller number $m \ll n$ of linear measurements $y = Ax^\star \in \mathbb{R}^m$ than traditional sampling theory believed necessary [2]. The estimation of the sparsest signal, consistent with the observations, is an NP-hard problem. However, the constrained ℓ_τ-minimization with $\tau \in (0,1]$, which is the convex or nonconvex surrogate problem, has been proposed in [3] as an appealing alternative for sparse recovery. It consists in selecting the element which is compatible with the observations which has minimal ℓ_τ-norm with $\tau \in (0,1]$:

$$\min_{x\in\mathbb{R}^n} \|x\|_{\ell_\tau} \qquad \text{s.t. } y = Ax. \tag{1}$$

Under certain assumptions on the sensing matrix A, it is known that (1) has a unique solution and it provides the desired solution $x^\star$.

The minimization in (1) can be carried out by an iteratively rewighted least squares method (IRLS, [4]). More precisely, given an initial guess $x^{(0)}$, at each iteration the algorithm requires to solve a constrained weighted least-squares problem:

$$x^{(t+1)} = \underset{y=Ax}{\arg\min} \sum_{i=1}^{n} w_i^{(t)} x_i^2$$

with $w_i^{(t+1)} = ((\epsilon^{(t)})^2 + (x_i^{(t)})^2)^{\tau/2-1}$ and a suitable non-increasing sequence $\epsilon^{(t)}$. In particular, under certain assumptions, these methods have been proved to converge to $x^\star$ globally linearly fast when $\tau = 1$ and locally superlinearly fast with rate $2-\tau$ for $\tau \in (0,1)$.

Although classical IRLS algorithms appear very attractive for their simlplicity, theoretical results guarantee the superlinear convergence only in a neighborhood of the desired solution. In fact, numerical results point out that exact recovery is achieved when τ is not too small (*i.e.* $\tau > 1/2$) and tends to be trapped in local minima when $\tau < 1/2$ [5]. Heuristic techniques to avoid local minima are currently object of study.

II. GSM based IRLS

We derive a new class of IRLS procedures for sparse recovery which outperform the classical procedures. More precisely, we model the elements of the signal as a two state gaussian mixture (GSM, [1])

$$x_i^\star = z_i\sqrt{\alpha}u_i + (1-z_i)\sqrt{\beta}u_i$$

where u_i are identically and independently distributed (i.i.d.) zero mean Gaussians and z_i are i.i.d. Bernoulli variables with probability mass function $\mathbb{P}(z_i = 1) = 1-p$, $p = k/n$, $\alpha \approx 0$, and $\beta \gg 0$. The combination of the considered model, used as a proxy for the sparsity assumption, with the maximum log-likelihood estimation provides a new alternative to select the sparsest vector consistent to the data. More precisely, we want to minimize

$$\begin{aligned} L(x,z,\alpha,\beta,\epsilon) = \sum_{i=1}^{n} \Bigg[& \frac{z_i x_i^2 + \epsilon^2/n}{2\alpha} + \frac{z_i}{2}\log\frac{\alpha}{1-p} \\ & + \frac{(1-z_i)x_i^2 + \epsilon^2/n}{2\alpha} + \frac{(1-z_i)}{2}\log\frac{\beta}{p}\Bigg]. \end{aligned} \tag{2}$$

subject to the constraint $y = Ax$. We design three iterative techniques: ML-based IRLS, EM-based IRLS, and K-EM based IRLS. These strategies can be interpreted as instances of the Expectation Maximization algorithm. After choosing some initial values for the mixture parameters, two updates are alternated: in the E-step, we use the current values for the parameters to estimate the signal $x^\star$ and to evaluate the posterior distribution $\mathbb{P}(z_i = 1)$ of the signal coefficients; in M-step we use these probabilities to re-estimate the mixture parameters α and β.

Besides the design of the algorithms, we prove that, under suitable conditions, the sequence of provided estimations converges to a fixed point of the map that rules their dynamics. Moreover, we derive conditions for exact recovery that are verifible a posteriori. Finally, the algorithm turn out to be quadratically fast in a neighborhood of $x^\star$. Numerical simulations validate our claims and show that these new procedures avoid local minima, outperforming classical IRLS for sparse recovery in terms of rate of convergence and sparsity-undersampling tradeoff.

References

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