MARGINALLY OUTER TRAPPED SURFACES IN DE SITTER SPACE BY LOW-DIMENSIONAL GEOMETRIES

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Abstract. A marginally outer trapped surface (MOTS) in de Sitter spacetime is an oriented spacelike surface whose mean curvature vector is proportional to one of the two null sections of its normal bundle. Associated with a spacelike immersed surface there are two enveloping maps into Möbius space (the conformal 3-sphere), which correspond to the two future-directed null directions of the surface normal planes. We give a description of MOTSs based on the Möbius geometry of their envelopes. We distinguish three cases according to whether both, one, or none of the fundamental forms in the normal null directions vanish. Special attention is given to MOTSs with non-zero parallel mean curvature vector. Any such a surface is generically the central sphere congruence (conformal Gauss map) of a surface in Möbius space which is locally Möbius equivalent to a non-zero constant mean curvature surface in some space form subgeometry.

1. Introduction

A marginally outer trapped surface (MOTS) in a four dimensional (oriented and time-oriented) spacetime $(N, \langle\, ,\rangle)$ is a spacelike immersion $\gamma : M \to N$ of an oriented surface M whose mean curvature vector field is proportional to one of the two null sections of its normal bundle. The notion of a MOTS was introduced early in the development of the theory of spacetime singularities and black holes [19], [22], and has arisen in the work of Schoen and Yau in connection with their proof of the positive mass theorem [31]. For more recent mathematical developments of the theory we refer the reader to [3], [4], [17], and the references therein. MOTSs, both in de Sitter and Minkowski spacetime, also occur naturally in the study of Möbius and Laguerre surface geometry [1], [2], [30].

For a 2-dimensional manifold M and a spacelike immersion $\gamma : M \to S^4_1$ into de Sitter spacetime, each normal plane of γ at a point $p \in M$ is timelike and contains two future-directed linearly independent null vectors. These depend smoothly on $p \in M$ and determine two smooth maps of M into Möbius space S^3, the envelopes of γ. In this paper, we will give a description of MOTSs in de Sitter spacetime based on the Möbius and Euclidean geometry of their two envelopes.[1] With reference to

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[1]The description of MOTSs in Minkowski space by the Laguerre geometry of their envelopes will be given in [29].

the second fundamental forms of γ associated to the null normal directions, we will distinguish three cases according to whether: I) both forms vanish identically, II) one of the two forms vanishes identically, or III) none of the two forms vanishes (see Section 4). In Case I, γ is a fixed totally geodesic 2-sphere in S^4_1. In Case II, γ is either described as the congruence of the oriented tangent planes to an immersed surface in Euclidean space, or as the immersion in de Sitter spacetime of the two-dimensional manifold of orthonormal frames adapted to a curve in Euclidean space. In Case III, at least one of the envelopes of γ is immersed and γ can be described as the central sphere congruence (conformal Gauss map) of such an envelope. In Section 5, we then specify the discussion by imposing the additional condition that the mean curvature vector field of γ be parallel with respect to the normal connection. Accordingly, we have three cases, indicated by A, B and C, respectively. Case A is as Case I. In Case B, we find that γ can be interpreted as the tangent plane congruence of a parallel front of a minimal surface in $\mathbb{R}^3$. In case C, special attention is given to MOTSs with non-zero parallel mean curvature vector. We prove that any such a surface is the central sphere congruence of a conformal immersion in Möbius space which is locally Möbius equivalent to a non-zero constant mean curvature surface into some 3-dimensional space form embedded in Möbius space. The proof is based on the characterization of MOTSs with parallel mean curvature vector in Case III by the existence of a holomorphic quartic form, relative to the underlying complex structure of the surface. This is then used to discuss some aspects of the Möbius geometry of the envelopes in relation to the theory of isothermic surfaces and of their transformations. Ultimately, we find that MOTSs with non-zero parallel mean curvature vector are governed by a second order completely integrable (soliton) equation. The details of the discussion are given in Section 6. In Section 7, a special class of examples related to elastic curves in 2-dimensional space forms is discussed. Section 2 recalls some preliminary material about Möbius geometry. Section 3 develops the geometry of MOTSs and of their envelopes in the framework of Möbius geometry.

2. Preliminaries

2.1. **Basic material.** We start by recalling some preliminary material about the method of moving frames in Möbius geometry as developed in [9]. Let $\mathbb{R}^5_1$ be $\mathbb{R}^5$ with the Lorentz scalar product

$$\langle x, y\rangle = -(x^0y^4 + x^4y^0) + x^1y^1 + x^2y^2 + x^3y^3 = \sum_{i,j} \eta_{ij}x^iy^j, \tag{2.1}$$

where the orientation is defined by $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 > 0$ and the time-orientation is defined by the positive light cone

$$\mathcal{L}^+ = \{x \in \mathbb{R}^5_1 : \langle x, x\rangle = 0,\, x^0 + x^4 > 0\}.$$

The Möbius group is the identity component $G \cong \mathrm{SO}_0(4, 1)$ of the pseudo-orthogonal group of (2.1). It consists of elements $A = (A^i_j) \in \mathrm{GL}(5, \mathbb{R})$ such that

$$\det A = 1; \quad \langle Ax, Ay\rangle = \langle x, y\rangle; \quad A^0_j + A^4_j > 0,\ j = 0, 4.$$

A Möbius frame is a basis $(A_0, \dots, A_4)$ of $\mathbb{R}^5_1$ such that

$$\langle A_i, A_j\rangle = \eta_{ij}, \quad A_0, A_4 \in \mathcal{L}^+.$$

The group G acts simply transitively on the Möbius frames and, up to the choice of a reference frame, the manifold of all such frames may be identified with G. Let $\epsilon_0, \dots, \epsilon_4$ be the standard basis of $\mathbb{R}^5$, and for any $A \in G$ let $A_j = A\epsilon_j$ denote the jth column vector of A. Regarding the A_j's as $\mathbb{R}^5$-valued functions on G, there are unique left invariant 1-forms $\{\omega^i_j\}_{0\leq i,j\leq 4}$, such that

$$dA_i = \sum_j \omega^j_i A_j, \tag{2.2}$$

where ω^j_i are the components of the Maurer–Cartan form $\omega = A^{-1}dA$ of G. Exterior differentiation of (2.2) gives the the structure equations

$$d\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j, \tag{2.3}$$

while differentiating $\langle A_i, A_j\rangle = \eta_{ij}$ gives the symmetry relations

$$\sum_k \omega^k_i \eta_{kj} + \sum_k \omega^k_j \eta_{ik} = 0.$$

2.2. **Möbius geometry.** Classically, the Möbius space S^3 (the conformal 3-sphere) is realized as the projective quadric $\{[x] \in \mathbb{RP}^4 : \langle x, x\rangle = 0\}$. The Möbius group acts transitively on S^3 via $A[x] = [Ax]$ and the projection map $\pi_{S^3} : G \to S^3$, $\pi(A) = [A_0]$, makes G into a principal fiber bundle over S^3 with structure group

$$G_0 = \{A \in G \,:\, A\epsilon_0 = r^{-1}\epsilon_0, \text{ for some } r > 0\}.$$

The space of semibasic forms for the projection π_{S^3} is spanned by the 1-forms $\{\omega^1_0, \omega^2_0, \omega^3_0\}$. Moreover, the forms

$$\omega^1_0 \wedge \omega^2_0 \wedge \omega^3_0, \quad (\omega^1_0)^2 + (\omega^2_0)^2 + (\omega^3_0)^3$$

are well defined on S^3, up to a positive multiple, and therefore induce on S^3 a G-invariant oriented conformal structure. In particular, G acts on S^3 as group of orientation-preserving, conformal transformations.

2.2.1. *Space forms in Möbius space.* Let $M^3(\epsilon)$ be the 3-dimensional space form of constant sectional curvature ϵ, where $\epsilon \in \{-1, 0, 1\}$. The space $M^3(0)$ is the Euclidean space $x^1 = 1$ in $\mathbb{R}^4$, the space $M^3(1)$ is the unit 3-sphere in $\mathbb{R}^4$, and $M^3(-1)$ is the hyperbolic space

$$H^3 = \{x \in \mathbb{R}^4 \,:\, -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -1,\, x^0 \geq 1\}.$$

Conformal embeddings of space forms in Möbius space are given by

$$M^3(0) \ni \left(x^0, x^1, x^2, x^3\right)^T \mapsto \left[\left(1, \frac{x^1}{\sqrt{2}}, \frac{x^2}{\sqrt{2}}, \frac{x^3}{\sqrt{2}}, \frac{\|x\|^2}{4}\right)^T\right] \in S^3$$

$$M^3(1) \ni \left(x^0, x^1, x^2, x^3\right)^T \mapsto \left[\left(\frac{1+x^0}{2}, \frac{x^1}{\sqrt{2}}, \frac{x^2}{\sqrt{2}}, \frac{x^3}{\sqrt{2}}, \frac{1-x^0}{2}\right)^T\right] \in S^3$$

$$M^3(-1) \ni \left(x^0, x^1, x^2, x^3\right)^T \mapsto \left[\left(\frac{1+x^0}{2}, \frac{x^1}{\sqrt{2}}, \frac{x^2}{\sqrt{2}}, \frac{x^3}{\sqrt{2}}, \frac{x^0-1}{2}\right)^T\right] \in S^3$$

In particular, Euclidean space $M^3(0)$ can be viewed as S^3 minus the point at infinity $[\epsilon_4] \in S^3$, and the stereographic projection is given by

$$S^3 \ni \left[(y^0, y^1, y^2, y^3, y^4)^T\right] \mapsto \sqrt{2}\left(\frac{y^1}{y^0}, \frac{y^2}{y^0}, \frac{y^3}{y^0}\right)^T \in \mathbb{R}^3.$$

Let K_ϵ be the group of orientation preserving isometries of $M^3(\epsilon)$, that is, $K_0 = \mathbb{E}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$, $K_1 = \mathrm{SO}(4)$, and $K_{-1} = \mathrm{SO}(3,1)$. These groups can be realized as subgroups of G by the faithful representations:

$$\rho_0 : K_0 \ni \begin{bmatrix} 1 & 0 \\ p & A \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ \frac{p}{\sqrt{2}} & A & 0 \\ \frac{p^T p}{4} & \frac{p^T A}{\sqrt{2}} & 1 \end{bmatrix} \in G, \tag{2.4}$$

$p = (p^1, p^2, p^3)^T \in \mathbb{R}^3$, $A = (A^i_j) \in \mathrm{SO}(3)$;

$$\rho_{\pm 1} : K_{\pm 1} \ni \begin{bmatrix} C^0_0 & C^0_i \\ C^i_0 & C^i_j \end{bmatrix} \longmapsto \begin{bmatrix} \frac{1+C^0_0}{2} & \frac{C^0_j}{\sqrt{2}} & \frac{\pm(1-C^0_0)}{2} \\ \frac{C^i_0}{\sqrt{2}} & C^i_j & \frac{\mp C^i_0}{\sqrt{2}} \\ \frac{\pm(1-C^0_0)}{2} & \frac{\mp C^0_j}{\sqrt{2}} & \frac{1+C^0_0}{2} \end{bmatrix} \in G, \tag{2.5}$$

where $1 \leq i, j \leq 3$.

2.2.2. *The space of round 2-spheres.* The space of round, oriented 2-spheres in S^3 can be parametrized by the points of the hyperquadric

$$S^4_1 = \{x \in \mathbb{R}^5_1 \,:\, \langle x, x\rangle = 1\}$$

(see, for example, [9], [21]). More precisely, the oriented sphere $\sigma(p,r) \subset \mathbb{R}^3$ with center p and signed radius $r \in \mathbb{R}^*$ is identified with

$$y(p,r) = \left(\frac{\sqrt{2}}{r}, \frac{p^1}{r}, \frac{p^2}{r}, \frac{p^3}{r}, \frac{p^T p - r^2}{2\sqrt{2}r}\right)^T \in S^4_1$$

and the oriented plane $\pi(n,h)$ with normal $n \in S^2$ and equation $n^T p = \sqrt{2}h$ is identified with

$$y(n,h) = \left(0, n^1, n^2, n^3, h\right)^T \in S^4_1.$$

Conversely, if $y \in S^4_1$ and $y^\perp = \{x \in \mathbb{R}^5_1 \,:\, \langle x, y\rangle = 0\}$, then the stereographic projection of $y^\perp \cap S^3$ into $\mathbb{R}^3$ is an oriented sphere in $\mathbb{R}^3$ if $y^0 \neq 0$, and an oriented plane if $y^0 = 0$.

The hyperquadric S^4_1, endowed with the G-invariant Lorentzian metric induced from $\mathbb{R}^5_1$, becomes a complete, simply connected, 4-dimensional Lorentz manifold of constant curvature 1, which identifies with de Sitter spacetime. The Möbius group G acts transitively on S^4_1. This action defines a K-principal bundle

$$\varpi : G \to S^4_1 \cong G/K, \quad A \in G \mapsto A\epsilon_3 = A_3 \in S^4_1,$$

where K, the isotropy subgroup at ϵ_3, is isomorphic to identity component of $\mathrm{SO}(3,1)$, $K \cong \mathrm{SO}_0(3,1)$.

2.3. **Surface theory in Möbius space.** Let M be an oriented surface and let $f : M \to S^3$ be a smooth conformal immersion. Let U be a open subset of M. A *Möbius frame field* along f is a map $B : U \to G$ such that $f(p) = [B_0(p)]$, for all $p \in U$. Let $\beta = (\beta^i_j) = B^*\omega$ denote the pullback by B of the Maurer–Cartan form $\omega = (\omega^i_j)$ of G. Any other Möbius frame field on U is given by $\tilde{B} = BX$, where $X : U \to G_0$. Under this change of frame, the pullback of ω transforms by $\tilde{B}^*\omega = X^{-1}BX + X^{-1}dX$.

A *first order* frame field along f is a Möbius frame field B along f for which

$$\beta^3_0 = 0, \quad \beta^1_0 \wedge \beta^2_0 > 0.$$

First order frame fields exist locally about any point of M.

Exterior differentiation of $\beta^3_0 = 0$ and use of Cartan's Lemma yield

$$\beta^3_1 = \ell_{11}\beta^1_0 + \ell_{12}\beta^2_0, \quad \beta^3_2 = \ell_{12}\beta^1_0 + \ell_{22}\beta^2_0,$$

for smooth functions ℓ_{11}, ℓ_{12} and ℓ_{22}. The 2-form given by

$$\Omega_f = \left(\frac{1}{4}(\ell_{11} - \ell_{22})^2 + \ell^2_{12}\right)\beta^1_0 \wedge \beta^2_0$$

is independent of first order frame field B along f, and hence globally defined on M. The 2-form Ω_f is called the *conformal area element* of f. A point $p \in M$ is an *umbilic point* of f if $\Omega_f(p) = 0$ (see [9]) .

A Möbius frame field B along f is a *central frame field* if there exist smooth functions p_1, p_2, p_3, q_1, $q_2 : U \to \mathbb{R}$ such that $\beta = (\beta^i_j) = B^{-1}dB$ takes the form

$$\begin{bmatrix} -2q_2\beta^1_0 + 2q_1\beta^2_0 & p_1\beta^1_0 + p_2\beta^2_0 & -p_2\beta^1_0 + p_3\beta^2_0 & 0 & 0 \\ \beta^1_0 & 0 & -q_1\beta^1_0 - q_2\beta^2_0 & -\beta^1_0 & p_1\beta^1_0 + p_2\beta^2_0 \\ \beta^2_0 & q_1\beta^1_0 + q_2\beta^2_0 & 0 & \beta^2_0 & -p_2\beta^1_0 + p_3\beta^2_0 \\ 0 & \beta^1_0 & -\beta^2_0 & 0 & 0 \\ 0 & \beta^1_0 & \beta^2_0 & 0 & 2q_2\beta^1_0 - 2q_1\beta^2_0 \end{bmatrix}$$

with $\beta^1_0 \wedge \beta^2_0 > 0$; (β^1_0, β^2_0) is called the *central coframe* of f. The existence of a central frame field along f was proved in [9], under the assumption that f is free of umbilic points. The smooth functions q_1, q_2, p_1, p_2, p_3 form a complete system of conformal invariants for f and satisfy the following structure equations:

$$d\beta^1_0 = -q_1\beta^1_0 \wedge \beta^2_0, \qquad d\beta^2_0 = -q_2\beta^1_0 \wedge \beta^2_0, \tag{2.6}$$

$$dq_1 \wedge \beta^1_0 + dq_2 \wedge \beta^2_0 = (1 + p_1 + p_3 + {q_1}^2 + {q_2}^2)\beta^1_0 \wedge \beta^2_0, \tag{2.7}$$

$$dq_2 \wedge \beta^1_0 - dq_1 \wedge \beta^2_0 = -p_2\beta^1_0 \wedge \beta^2_0, \tag{2.8}$$

$$dp_1 \wedge \beta^1_0 + dp_2 \wedge \beta^2_0 = (4q_2p_2 + q_1(3p_1 + p_3))\beta^1_0 \wedge \beta^2_0, \tag{2.9}$$

$$dp_2 \wedge \beta^1_0 - dp_3 \wedge \beta^2_0 = (4q_1p_2 - q_2(p_1 + 3p_3))\beta^1_0 \wedge \beta^2_0. \tag{2.10}$$

If B is a central frame field along f and U is connected, the only other central frame field on U is given by $\tilde{B} = (B_0, -B_1, -B_2, B_3, B_4)$. Under this frame change, the functions p_1, p_2, p_3, q_1, q_2 transform by

$$\tilde{q}_1 = -q_1, \quad \tilde{q}_2 = -q_2, \quad \tilde{p}_1 = p_1, \quad \tilde{p}_2 = p_2, \quad \tilde{p}_3 = p_3.$$

Thus, there are well defined global functions J, M, $\mathrm{W} : M \to \mathbb{R}$ such that locally

$$\mathrm{J} = \frac{1}{2}(p_1 + p_3), \quad \mathrm{M} = p_2, \quad \mathrm{W} = \frac{1}{2}(p_1 - p_3).$$

Remark 2.1. According to [9], a conformal immersion $f : M \to S^3$ without umbilic points is a *Willmore* surface, that is, is a critical point of the conformal area functional

$$\mathcal{W}(f) = \int_M \Omega_f,$$

if and only if $\mathrm{W} = 0$ on M.

An umbilic free conformal immersion $f : M \to S^3$ is *isothermic*, i.e., admits conformal curvature line coordinates $z = x + iy$, if and only if $\mathrm{M} = 0$ on M (see [24] and Remark 6.1). In this case, the central coframe (β^1_0, β^2_0) takes the form

$$\beta^1_0 = e^u dx, \quad \beta^2_0 = e^u dy,$$

for a smooth function u. The function $\Phi = e^u$ is called the *Calapso potential* of f (see, for instance, [28], [5]). Accordingly, from (2.6) and (2.7) it follows that

$$q_1 = e^{-u} u_y, \qquad q_2 = -e^{-u} u_x, \tag{2.11}$$

$$p_1 + p_3 = -(1 + e^{-2u}\Delta u). \tag{2.12}$$

Substituting into (2.9) and (2.10) yields

$$\begin{aligned} d\left(e^{2u}(p_1 - p_3)\right) = &-e^{2u}\left\{\left(e^{-2u}\Delta u\right)_x + 4u_x\left(1 + e^{-2u}\Delta u\right)\right\} dx \\ &+ e^{2u}\left\{(e^{-2u}\Delta u)_y + 4u_y(1 + e^{-2u}\Delta u)\right\} dy. \end{aligned} \tag{2.13}$$

The integrability condition of (2.13) is the so-called Calapso equation,

$$\Delta\left(e^{-u}(e^u)_{xy}\right) + 2\left(e^{2u}\right)_{xy} = 0,$$

which has been proved to be a completely integrable (soliton) equation [14].

2.4. **Euclidean and Möbius frames.**

2.4.1. *Frames adapted to a surface.* For a smooth immersion $x : M \to \mathbb{R}^3 \subset S^3$ of a two-dimensional oriented manifold M into Euclidean space, let $\mathbf{e} = (x; e_1, e_2, e_3) : M \to \mathbb{E}(3)$ be an adapted frame field along x, that is, a lift of x to $\mathbb{E}(3)$, defined by the requirements that e_1, e_2 are tangent to $x(M)$, e_3 is orthogonal to $x(M)$, and (e_1, e_2, e_3) is a positive basis of $\mathbb{R}^3$. There are then unique 1-forms θ's such that

$$\begin{aligned} dx &= \theta^1_0 e_1 + \theta^2_0 e_2 \\ de_1 &= \theta^2_1 e_2 + \theta^3_1 e_3 \\ de_2 &= -\theta^2_1 e_1 + \theta^3_2 e_3 \\ de_3 &= -\theta^3_1 e_1 - \theta^3_2 e_2, \end{aligned} \tag{2.14}$$

satisfying the structure equations

$$\begin{aligned} &d\theta^1_0 = \theta^2_1 \wedge \theta^2_0, \quad d\theta^2_0 = -\theta^2_1 \wedge \theta^1_0, \quad 0 = \theta^3_1 \wedge \theta^1_0 + \theta^3_2 \wedge \theta^2_0, \\ &d\theta^2_1 = -\theta^3_1 \wedge \theta^3_2, \quad d\theta^3_1 = \theta^2_1 \wedge \theta^3_2, \quad d\theta^3_2 = -\theta^2_1 \wedge \theta^3_1. \end{aligned} \tag{2.15}$$

By (2.4), the adapted Euclidean frame field $\mathbf{e} : M \to \mathbb{E}(3)$ determines a well-defined Möbius frame field along x, that is, a smooth map $A(x; e_1, e_2, e_3) = (A_0, \dots, A_4) :$

$M \to G$, such that

$$
\begin{aligned}
dA_0 &= \frac{\theta_0^1}{\sqrt{2}} A_1 + \frac{\theta_0^2}{\sqrt{2}} A_2 \\
dA_1 &= \theta_1^2 A_2 + \theta_1^3 A_3 + \frac{\theta_0^1}{\sqrt{2}} A_4 \\
dA_2 &= -\theta_1^2 A_1 + \theta_2^3 A_3 + \frac{\theta_0^2}{\sqrt{2}} A_4 \\
dA_3 &= -\theta_1^3 A_1 - \theta_2^3 A_2, \quad dA_4 = 0.
\end{aligned} \tag{2.16}
$$

Here, the null vector $A_0(p)$ represents $x(p)$ in S^3, the spacelike vectors $A_i(p)$, $i = 1, 2, 3$, represent the oriented planes through $x(p)$ orthogonal to $e_i(p)$, while $A_4(p)$ is the constant vector ϵ_4, for each $p \in M$.

2.4.2. *The bundle of orthonormal frames adapted to a curve.* Let $c : I \to \mathbb{R}^3$ be a curve, i.e., an immersion of an interval $I \subset \mathbb{R}$ in $\mathbb{R}^3$. Let $\mathcal{O}_c(I)$ be the manifold of all $(t; e_1, e_2, e_3)$, where $t \in I$ and e_1, e_2, e_3 is an orthonormal basis of $\mathbb{R}^3$ such that $e_1 = c'(t)/\|c'(t)\|$. Let $\pi : \mathcal{O}(3) \to \mathbb{R}^3$ be the bundle of orthonormal frames of $\mathbb{R}^3$. Then the mapping $\hat{c} : \mathcal{O}_c(I) \to \mathcal{O}(3)$ given by

$$
\hat{c}(t; e_1, e_2, e_3) = (c(t); e_1, e_2, e_3)
$$

is an immersion of $\mathcal{O}_c(I)$ into $\mathcal{O}(3)$ and $\pi \circ \hat{c} = c \circ \pi_c$, where π_c is the projection $\pi_c : \mathcal{O}_c(I) \to I$, $(t; e_1, e_2, e_3) \mapsto t$. The e_i and the map $c \circ \pi_c$ can be regarded as vector valued functions on the two dimensional manifold $\mathcal{O}_c(I)$. Thus, there are unique 1-forms such that

$$
\begin{aligned}
d(c \circ \pi_c) &= \theta_0^1 e_1 \\
de_1 &= \theta_1^2 e_2 + \theta_1^3 e_3 \\
de_2 &= -\theta_1^2 e_1 + \theta_2^3 e_3 \\
de_3 &= -\theta_1^3 e_1 - \theta_2^3 e_2,
\end{aligned} \tag{2.17}
$$

satisfying the structure equations

$$
\begin{aligned}
&d\theta_0^1 = 0, \quad 0 = \theta_1^2 \wedge \theta_0^1, \quad 0 = \theta_1^3 \wedge \theta_0^1, \\
&d\theta_1^2 = -\theta_1^3 \wedge \theta_2^3, \quad d\theta_1^3 = \theta_1^2 \wedge \theta_2^3, \quad d\theta_2^3 = -\theta_1^2 \wedge \theta_1^3.
\end{aligned} \tag{2.18}
$$

On the corresponding space of adapted Möbius frames $A = (A_0, \ldots, A_4)$ we have the equations

$$
\begin{aligned}
dA_0 &= \frac{\theta_0^1}{\sqrt{2}} A_1 \\
dA_1 &= \theta_1^2 A_2 + \theta_1^3 A_3 + \frac{\theta_0^1}{\sqrt{2}} A_4 \\
dA_2 &= -\theta_1^2 A_1 + \theta_2^3 A_3 \\
dA_3 &= -\theta_1^3 A_1 - \theta_2^3 A_2, \quad dA_4 = 0.
\end{aligned} \tag{2.19}
$$

3. Marginally outer trapped surfaces

Given a 2-dimensional connected oriented manifold M, let $\gamma : M \to S^4_1$ be a spacelike immersion of M into de Sitter spacetime and denote by $g_\gamma = \gamma^*\langle\, ,\, \rangle$ the Riemannian metric induced on M by γ. If TS^4_1 denotes the tangent bundle of S^4_1, the bundle $\gamma^*(TS^4_1)$ induced by γ over M splits into the direct sum

$$\gamma^*(TS^4_1) = \mathcal{T}(\gamma) \oplus \mathcal{N}(\gamma),$$

where $\mathcal{T}(\gamma) = d\gamma(TM)$ is the tangent bundle of γ and $\mathcal{N}(\gamma)$ its normal bundle. For each $p \in M$, the two-dimensional normal space $\mathcal{N}(\gamma)_p$ is timelike and admits two future-directed null direction orthogonal to M. Actually, by possibly passing to a suitable covering of M, we may assume that there is a basis $\{L_0, L_4\}$ of smooth future-directed null sections of $\mathcal{N}(\gamma)$. Thus,

$$\mathcal{N}(\gamma) = \mathcal{N}_0 \oplus \mathcal{N}_4,$$

where $\mathcal{N}_0$ and $\mathcal{N}_4$ are null subbundle, trivialized by L_0 and L_4, respectively.

Remark 3.1. From the viewpoint of Möbius geometry, $\gamma : M \to S^4_1$ can be interpreted as a 2-parameter family of oriented spheres in S^3, known in the classical literature as a *sphere congruence*.

Definition 3.2. An *adapted frame field along* a sphere congruence γ is a smooth map $A : U \subset M \to G$, defined on an open subset $U \subset M$, such that

(1) $\gamma = \varpi \circ A = A_3$;
(2) $dA_{3|p} \equiv 0 \mod (A_1(p), A_2(p))$, for all $p \in U$.

In particular, A_0 and A_4 are local sections of $\mathcal{N}_0$ and $\mathcal{N}_4$, respectively.

Adapted frame fields can be defined about every point of M. For any adapted frame field $A : U \to G$, we let $\alpha = (\alpha^i_j) = A^*\omega$. It follows from the definition that

$$\alpha^3_0 = \alpha^0_3 = 0. \tag{3.1}$$

Thus

$$\alpha = \begin{bmatrix} \alpha^0_0 & \alpha^1_4 & \alpha^2_4 & 0 & 0 \\ \alpha^1_0 & 0 & -\alpha^2_1 & -\alpha^3_1 & \alpha^1_4 \\ \alpha^2_0 & \alpha^2_1 & 0 & -\alpha^3_2 & \alpha^2_4 \\ 0 & \alpha^3_1 & \alpha^3_2 & 0 & 0 \\ 0 & \alpha^1_0 & \alpha^2_0 & 0 & -\alpha^0_0 \end{bmatrix}. \tag{3.2}$$

For an adapted frame A, the induced metric

$$g_\gamma = (\alpha^3_1)^2 + (\alpha^3_2)^2,$$

so that α^3_1, α^3_2 is an orthonormal coframe field for g_γ on M. From the structure equations, we have

$$d\alpha^3_1 = \alpha^2_1 \wedge \alpha^3_2, \quad d\alpha^3_2 = -\alpha^2_1 \wedge \alpha^3_1.$$

Thus α^2_1 is the Levi-Civita connection form of g_γ with respect to α^3_1, α^3_2.

Any other adapted frame field along γ on U is given by

$$\hat{A} = AL(r, a),$$

where $L(r,a) : U \to K_1 \subset G_0$ is any smooth map into the subgroup

$$K_1 = \left\{ L(r,a) = \begin{bmatrix} r^{-1} & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \end{bmatrix} \,:\, r > 0,\, a = (a^i_j) \in \mathrm{SO}(2) \right\}.$$

Then

$$\hat{\alpha} = L(r^{-1}, a^T)\, \alpha\, L(r,a) + L(r^{-1}, a^T)\, dL(r,a),$$

so that,

$$\begin{aligned} \hat{\alpha}^0_0 &= \alpha^0_0 - r^{-1} dr \\ \begin{bmatrix} \hat{\alpha}^3_1 \\ \hat{\alpha}^3_2 \end{bmatrix} &= a^T \begin{bmatrix} \alpha^3_1 \\ \alpha^3_2 \end{bmatrix} \\ \begin{bmatrix} \hat{\alpha}^1_0 \\ \hat{\alpha}^2_0 \end{bmatrix} &= \frac{1}{r} a^T \begin{bmatrix} \alpha^1_0 \\ \alpha^2_0 \end{bmatrix} \\ \begin{bmatrix} \hat{\alpha}^1_4 \\ \hat{\alpha}^2_4 \end{bmatrix} &= r a^T \begin{bmatrix} \alpha^1_4 \\ \alpha^2_4 \end{bmatrix} \end{aligned} \tag{3.3}$$

In particular,

$$\hat{\alpha}^3_1 \wedge \hat{\alpha}^3_2 = \alpha^3_1 \wedge \alpha^3_2.$$

Thus M inherits from S^4_1 an orientation defined by the volume form Ω_M, locally given by $\alpha^3_1 \wedge \alpha^3_2$.

Definition 3.3. A smooth map $f : M \to S^3$ is an *envelope* of a spacelike immersion (sphere congruence) $\gamma : M \to S^4_1$ if, for any lift $F : M \to \mathcal{L}^+ \subset \mathbb{R}^5_1$, such that $f = [F]$, we have

$$\langle F, \gamma \rangle = 0, \quad \langle dF, \gamma \rangle = 0,$$

at every point of M.

Remark 3.4. This amounts to saying that γ assigns to each point $p \in M$ an oriented 2-sphere of S^3 which has first order contact with f at $f(p)$ (cf. [21] for more details).

If $\gamma : M \to S^4_1$ is a sphere congruence and $A : M \to G$ is an adapted frame field along γ, then the two maps defined by

$$\begin{aligned} f : M \to S^3, &\quad p \mapsto [A_0(p)], \\ \tilde{f} : M \to S^3, &\quad p \mapsto [A_4(p)], \end{aligned}$$

are two envelopes of γ. In fact, A_0 and A_4 are lifts of f and $\tilde{f}$, respectively. Moreover, $dA_0 = \alpha^0_0 A_0 + \alpha^1_0 A_1 + \alpha^2_0 A_2$, since $\alpha^3_0 = 0$, and $dA_4 = \alpha^1_4 A_1 + \alpha^2_4 A_2 - \alpha^0_0 A_4$, since $\alpha^3_4 = \alpha^0_3 = 0$, and therefore

$$\begin{aligned} \langle A_0, \gamma \rangle = 0, &\quad \langle dA_0, \gamma \rangle = 0, \\ \langle A_4, \gamma \rangle = 0, &\quad \langle dA_4, \gamma \rangle = 0. \end{aligned}$$

The second fundamental form of the sphere congruence γ is the normal bundle valued symmetric 2-form given by

$$\mathrm{II} = \mathrm{II}_0 \otimes A_4 + \mathrm{II}_4 \otimes A_0,$$

where

$$\mathrm{II}_\nu = -\langle d\gamma, dA_\nu \rangle = \alpha^1_\nu \alpha^3_1 + \alpha^2_\nu \alpha^3_2, \quad \nu = 0, 4,$$

is the second fundamental form of the congruence γ in the direction A_ν.

Remark 3.5. Note that Π is well-defined on M, since, according to (3.3),

$$\hat{\Pi}_0 = \frac{1}{r}\Pi_0, \quad \hat{\Pi}_4 = r\Pi_4.$$

Taking the exterior derivative of (3.1), we have

$$\alpha^1_0 \wedge \alpha^3_1 + \alpha^2_0 \wedge \alpha^3_2 = 0,$$
$$\alpha^1_4 \wedge \alpha^3_1 + \alpha^2_4 \wedge \alpha^3_2 = 0,$$

and by Cartan's Lemma,

$$\begin{pmatrix}\alpha^1_0\\ \alpha^2_0\end{pmatrix} = \begin{pmatrix}h^0_{11} & h^0_{12}\\ h^0_{12} & h^0_{22}\end{pmatrix}\begin{pmatrix}\alpha^3_1\\ \alpha^3_2\end{pmatrix}, \quad \begin{pmatrix}\alpha^1_4\\ \alpha^2_4\end{pmatrix} = \begin{pmatrix}h^4_{11} & h^4_{12}\\ h^4_{12} & h^4_{22}\end{pmatrix}\begin{pmatrix}\alpha^3_1\\ \alpha^3_2\end{pmatrix}. \tag{3.4}$$

for smooth functions $h^\nu_{ij} = h^\nu_{ji}$, $i,j = 1,2$, $\nu = 0,4$. Thus

$$\Pi_\nu = \sum_{i,j=1,2} h^\nu_{ij}\alpha^3_i\alpha^3_j, \quad \nu = 0,4$$

and

$$\Pi = \sum_{i,j=1,2} h^0_{ij}\alpha^3_i\alpha^3_j \otimes A_4 + \sum_{i,j=1,2} h^4_{ij}\alpha^3_i\alpha^3_j \otimes A_0.$$

The 2×2 symmetric matrices

$$h^0 = (h^0_{ij}), \quad h^4 = (h^4_{ij}) \tag{3.5}$$

are the *shape operator matrices* of γ in the directions A_ν, $\nu = 0,4$, relative to the orthonormal coframe α^3_1, α^3_2. If $\hat{A} = AL(r,a)$, by (3.3),

$$\hat{h}^0 = r^{-1}a^T h^0 a, \quad \hat{h}^4 = ra^T h^4 a. \tag{3.6}$$

The mean curvature vector of γ is half the trace of Π with respect to g_γ,

$$2\mathbf{H} := \operatorname{tr}\Pi = \operatorname{tr} h^0\, A_4 + \operatorname{tr} h^4\, A_0. \tag{3.7}$$

With respect to the null frame field $\{A_0, A_4\}$, the normal connection $\nabla^\perp$ in the normal bundle $N(\gamma)$ of γ is given by

$$\nabla^\perp A_0 = \alpha^0_0 \otimes A_0, \quad \nabla^\perp A_4 = -\alpha^0_0 \otimes A_4.$$

In particular, we have

$$\nabla^\perp \mathbf{H} = \left[d\left(\operatorname{tr} h^0\right) - \operatorname{tr} h^0\, \alpha^0_0\right] A_4 + \left[d\left(\operatorname{tr} h^4\right) + \operatorname{tr} h^4\, \alpha^0_0\right] A_0. \tag{3.8}$$

Definition 3.6. A spacelike immersion $\gamma : M \to S^4_1$ is a *marginally outer trapped surface* (MOTS) if $\mathbf{H}$ is a section of either $\mathcal{N}_0$ or $\mathcal{N}_4$, that is, if $\mathbf{H}$ is proportional to one of the elements of a basis of future-directed null sections of the normal bundle.

According to (3.7), a spacelike immersion γ is a MOTS if either $\operatorname{tr} h^0{}_{|p} = 0$, or $\operatorname{tr} h^4{}_{|p} = 0$. In the following, we will assume that γ is of *constant type*, that is,

$$\operatorname{rank} h^\nu{}_{|p} = \text{const}, \quad \nu = 0,4.$$

Remark 3.7. Let $\gamma : M \to S^4_1$ be a sphere congruence with envelopes $f = [A_0]$ and $\tilde{f} = [A_4]$. Then γ can be interpreted as a spacelike normal field of the lifts $A_0, A_4 : M \to \mathcal{L}^+$ as maps into Minkowski 5-space $\mathbb{R}^5_1$. Let

$$\Pi_3 = -\langle dA_0, d\gamma\rangle, \quad \tilde{\Pi}_3 = -\langle dA_4, d\gamma\rangle$$

denote, respectively, the second fundamental forms of A_0 and A_4 in the spacelike normal direction $\gamma = A_3$. If both envelopes $f, \tilde{f}$ are immersions, the matrices of Π_3

and $\tilde{\Pi}_3$, calculated with respect to the coframes α_0^1, α_0^2 and α_4^1, α_4^2, respectively, are given by

$$h^3 = (h^0)^{-1}, \quad \tilde{h}^3 = (h^4)^{-1}.$$

Definition 3.8. (see [6], [21]) Let $f = [F] : M \to S^3$ be a conformal immersion. The *central sphere congruence* of f is the only sphere congruence (spacelike immersion) $\zeta : M \to S_1^4$ such that the shape operator of F in the normal direction ζ has trace zero.

Let $\gamma : M \to S_1^4$ be a sphere congruence and $A : M \to G$ an adapted frame field along γ and assume that the two envelopes $f, \tilde{f}$ are immersed. From the above discussion, it follows that γ is the central sphere congruence of $f = [A_0]$ if and only if

$$\operatorname{tr} h^3 = 0 \iff \operatorname{tr} \left(h^3\right)^{-1} = 0 \iff \operatorname{tr} h^0 = 0.$$

Analogously, γ is the central sphere congruence of $\tilde{f} = [A_4]$ if and only if

$$\operatorname{tr} \tilde{h}^3 = 0 \iff \operatorname{tr} \left(\tilde{h}^3\right)^{-1} = 0 \iff \operatorname{tr} h^4 = 0.$$

Moreover, γ is the central sphere congruence for both the (immersed) envelopes $f, \tilde{f}$ if and only if it is minimal in S_1^4, i.e., $\mathbf{H} = 0$ (see [6], [21]).

Remark 3.9. If $f : M \to S^3$ is an umbilic free conformal immersion and $B = (B_0, \dots, B_4)$ denotes a central frame field along f, it follows from the above discussion that the map $B_3 : M \to S_1^4$ is the central sphere congruence of f.

Example 3.10 (The central sphere congruence of surfaces in space forms)**.** Let $f : M \to M^3(\epsilon)$ be an umbilic free immersion and consider a principal orthonormal frame field $\mathbf{e} : M \to K_\epsilon$ along f with Maurer–Cartan form $\eta = \mathbf{e}^{-1}d\mathbf{e}$. Let a and c be the two principal curvatures and let $H = \frac{1}{2}(a + c)$ denote the mean curvature. For any function g, let g_1 and g_2 be the functions defined by

$$dg = R(g_1\eta_0^1 + g_2\eta_0^2),$$

where $R = \frac{1}{2}(a - c)$.

According to Section 2.2.1, the frame field $\mathbf{e}$ gives rise to a Möbius frame field $E = (E_0, \dots, E_4) : M \to G$ along $f : M \to M^3(\epsilon) \subset S^3$. Let $B = (B_0, \dots, B_4)$ be the Möbius frame field along f defined by posing

$$\begin{aligned}
B_0 &= \sqrt{2}\, R E_0, \quad B_1 = E_1 - H_1\sqrt{2}\, E_0, \\
B_2 &= E_2 + H_2\sqrt{2}\, E_0, \quad B_3 = E_3 + \sqrt{2}\, H E_0, \\
B_4 &= \frac{1}{R}\left\{\frac{1}{\sqrt{2}}\left(H^2 + H_1^2 + H_2^2\right) E_0 - H_1 E_1 + H_2 E_2 + H E_3 + \frac{E_4}{\sqrt{2}}\right\}.
\end{aligned}$$

A direct calculation yields

$$\beta_0^1 = R\eta_0^1, \quad \beta_0^2 = R\eta_0^2, \quad \beta_0^3 = 0, \qquad \beta_1^3 = \beta_0^1, \quad \beta_2^3 = -\beta_0^2, \quad \beta_3^0 = 0$$

$$\beta_0^0 = \frac{1}{R}\left[a_1\beta_0^1 - c_2\beta_0^2\right], \qquad \beta_1^2 = -\frac{1}{2R}\left[c_2\beta_0^1 + a_1\beta_0^2\right],$$

$$\begin{aligned}
\beta_1^0 = \frac{1}{R}\Bigg\{&\left[\frac{1}{R}\left(\frac{H^2}{2} - aH - \frac{\epsilon}{2}\right) - (H_1)_1 - \frac{H_2 a_2}{2R} - \frac{H_1^2 - H_2^2}{2R}\right]\beta_0^1 \\
&+ \left[\frac{H_1 H_2}{R} - (H_1)_2 - \frac{H_2 c_1}{2R}\right]\beta_0^2\Bigg\},
\end{aligned}$$

$$\beta^0_2 = \frac{1}{R}\left\{\left[\frac{H_1H_2}{R} + (H_2)_1 - \frac{H_1a_2}{2R}\right]\beta^1_0 + \left[\frac{1}{R}\left(\frac{H^2}{2} - cH - \frac{\epsilon}{2}\right) + (H_2)_2 - \frac{H_1c_1}{2R} + \frac{H_1^2 - H_2^2}{2R}\right]\beta^2_0\right\},$$

from which follows that B is a central frame field along $f : M \to M^3(\epsilon) \subset S^3$; observe, in particular, that exterior differentiation of $dH = R(H_1\eta^1_0 + H_2\eta^2_0)$ gives

$$\frac{1}{R}\left[\frac{H_1H_2}{R} - (H_1)_2 - \frac{H_2c_1}{2R}\right] + \frac{1}{R}\left[\frac{H_1H_2}{R} + (H_2)_1 - \frac{H_1a_2}{2R}\right] = 0.$$

This allows to express the central sphere congruence $\gamma = B_3$, as well as the set of Möbius invariants functions q_1, q_2, p_1, p_2, p_3, in terms of the metric invariants of f.

4. Classification of marginally outer trapped surfaces

Let $\gamma : M \to S^4_1$ be a MOTS and let $f,\ \tilde{f} : M \to S^3$ be the two envelopes of γ. As above, let h^0 and h^4 denote the shape operator matrices of γ. According to whether: I) both second fundamental forms of γ vanish identically; II) one of the two vanishes identically; or III) none of the two vanish identically and, say, $\operatorname{tr} h^0 = 0$, we obtain a classification of marginally outer trapped surfaces in terms of the Möbius and the Euclidean geometry of their envelopes.

Case I: $h^0 = h^4 = 0$.
This means that γ is a fixed totally geodesic 2-sphere in S^4_1.

Case II: $h^0 \neq 0$ and $h^4 = 0$.
There are two subcases to consider:

Subcase IIa: $\operatorname{rank} h^0 = 2$.
Here, from the second equation of (3.4) and the structure equations, we obtain that $d\alpha^0_0 = 0$. Thus, about each $p \in M$, there is a simply connected open set $U \subset M$ and a smooth function $u : U \to \mathbb{R}$, so that $\alpha^0_0 = du$. Under a change of adapted frame field of the form $\hat{A} = AL$, where $L = L(e^u, I_2)$, we can make $\hat{\alpha}^0_0 = 0$. Therefore, we can always assume that there exists an adapted frame field A along γ such that

$$\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha^1_0 & 0 & -\alpha^2_1 & -\alpha^3_1 & 0 \\ \alpha^2_0 & \alpha^2_1 & 0 & -\alpha^3_2 & 0 \\ 0 & \alpha^3_1 & \alpha^3_2 & 0 & 0 \\ 0 & \alpha^1_0 & \alpha^2_0 & 0 & 0 \end{bmatrix}.$$

According to the embedding (2.4) of the Euclidean group in the Möbius group, the Maurer–Cartan equations of α amount to the structure equations of a Euclidean frame adapted to a surface immersed in Euclidean space (see Section 2.4). *The envelope f can be seen as an immersion in Euclidean space and γ can be interpreted as the congruence of the oriented tangent planes to f.*

Subcase IIb: $\operatorname{rank} h^0 = 1$.

In this case, under a change of adapted frame field as above, the connection form can be put in the form

$$\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha_0^1 & 0 & -\alpha_1^2 & -\alpha_1^3 & 0 \\ 0 & \alpha_1^2 & 0 & -\alpha_2^3 & 0 \\ 0 & \alpha_1^3 & \alpha_2^3 & 0 & 0 \\ 0 & \alpha_0^1 & 0 & 0 & 0 \end{bmatrix}.$$

The Maurer–Cartan equations of α amount to the structure equations of the bundle of adapted frames to a curve in Euclidean space (see Section 2.4). The envelope f results in a space curve, that is, an immersion of an interval, I, in Euclidean 3-space. If $\mathcal{O}_f(I)$ denote the two-dimensional manifold of orthonormal frames adapted to the curve f, then γ *can be interpreted as the spacelike immersion of* $\mathcal{O}_f(I)$ *in de Sitter spacetime*. The manifold of frames $\mathcal{O}_f(I)$ can be visualized as a tube about f: about each point of the curve there is a unit circle in the plane normal to the tangent line corresponding to the possible choices of A_2.

Case III: h^0, $h^4 \neq 0$ and, say, $\operatorname{tr} h^0 = 0$, i.e. $\mathbf{H}$ is a section of $\mathcal{N}_0$.

In this case, we must have $\operatorname{rank} h^0 = 2$. Therefore, *the envelope* f *is immersed and then*, according to Remark 3.7, *the marginally outer trapped surface* γ *is the central sphere congruence of* f. Using (3.6), we can locally construct an adapted frame field A along γ, so that

$$h^0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

that is, $\alpha_1^3 = -\alpha_0^1$ and $\alpha_2^3 = \alpha_0^2$. Moreover, there are smooth functions p_1, p_2, p_3, q_1, $q_2 : U \to \mathbb{R}$ such that the connection form of the adapted frame becomes

$$\begin{bmatrix} -2q_2\alpha_0^1 + 2q_1\alpha_0^2 & p_1\alpha_0^1 + p_2\alpha_0^2 & -p_2\alpha_0^1 + p_3\alpha_0^2 & 0 & 0 \\ \alpha_0^1 & 0 & -q_1\alpha_0^1 - q_2\alpha_0^2 & -\alpha_0^1 & p_1\alpha_0^1 + p_2\alpha_0^2 \\ \alpha_0^2 & q_1\alpha_0^1 + q_2\alpha_0^2 & 0 & \alpha_0^2 & -p_2\alpha_0^1 + p_3\alpha_0^2 \\ 0 & \alpha_0^1 & -\alpha_0^2 & 0 & 0 \\ 0 & \alpha_0^1 & \alpha_0^2 & 0 & 2q_2\alpha_0^1 - 2q_1\alpha_0^2 \end{bmatrix}.$$

If we take the viewpoint in which the envelope f is the primary object of interest, we observe that A is a *central frame field* along the immersion $f = [A_0] : M \to S^3$ (see Section 2.3). In particular, the volume form induced by γ can be written locally as

$$\Omega_M = \alpha_1^3 \wedge \alpha_2^3 = \frac{1}{\det(h^0)}\alpha_0^1 \wedge \alpha_0^2 = -\alpha_0^1 \wedge \alpha_0^2 = -\Omega_f.$$

Moreover,

$$h^4 = \begin{pmatrix} -p_1 & p_2 \\ p_2 & p_3 \end{pmatrix},$$

from which

$$2\mathbf{H} = (p_3 - p_1)A_0.$$

Observe that the vanishing of $\mathbf{H}$ amounts to the Willmore condition $p_1 = p_3$ for the envelope f (see Remark 2.1).

5. MOTS with parallel mean curvature vector

In this section we give a classification of marginally outer trapped surfaces in de Sitter spacetime under the additional condition that the mean curvature vector field is parallel with respect to the normal connection. The corresponding cases are indicated by A, B, and C, respectively.

Case A: $h^0 = h^4 = 0$.
As in the Case I above, γ is a fixed totally geodesic 2-sphere in S^4_1.

Case B: $h^0 \neq 0$ and $h^4 = 0$.
There are two subcases to consider:

Subcase Ba: $\operatorname{rank} h^0 = 2$.
We know already that the envelope f of γ is an immersed surface in Euclidean space and that γ is the congruence of tangent planes of such a surface. By (3.8), the parallel condition $\nabla^{\perp}\mathbf{H} = 0$ reduces to

$$\operatorname{tr} h^0 = h^0_{11} + h^0_{22} = \text{const.}$$

Now, from (3.4),

$$\begin{pmatrix} \alpha^3_1 \\ \alpha^3_2 \end{pmatrix} = \frac{1}{h^0_{11}h^0_{22} - (h^0_{12})^2} \begin{pmatrix} h^0_{22} & -h^0_{12} \\ -h^0_{12} & h^0_{11} \end{pmatrix} \begin{pmatrix} \alpha^1_0 \\ \alpha^2_0 \end{pmatrix},$$

so that the envelope f has Gauss curvature

$$K = \frac{1}{h^0_{11}h^0_{22} - (h^0_{12})^2}$$

and mean curvature

$$H = \frac{h^0_{11} + h^0_{22}}{h^0_{11}h^0_{22} - (h^0_{12})^2}.$$

This implies that

$$\frac{H}{K} = h^0_{11} + h^0_{22} = \text{const.}$$

Therefore, γ *can be interpreted as the tangent plane congruence of a surface immersed in Euclidean space with constant ratio H/K of mean curvature to Gauss curvature.* The parallel surfaces of a minimal surface in Euclidean space are surfaces for which H/K is constant.

Subcase Bb: $\operatorname{rank} h^0 = 1$.
We show that this case cannot occur. In fact, (3.4) becomes

$$\begin{pmatrix} \alpha^1_0 \\ \alpha^2_0 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha^3_1 \\ \alpha^3_2 \end{pmatrix},$$

$$\alpha^1_0 = k\alpha^3_1, \quad k \in \mathbb{R} \setminus \{0\}. \tag{5.1}$$

Moreover, up a a change of frame field, we can consider an adapted frame field so that its connection form is

$$\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha^1_0 & 0 & -\alpha^2_1 & -\alpha^3_1 & 0 \\ 0 & \alpha^2_1 & 0 & -\alpha^3_2 & 0 \\ 0 & \alpha^3_1 & \alpha^3_2 & 0 & 0 \\ 0 & \alpha^1_0 & 0 & 0 & 0 \end{bmatrix}.$$

The structure equations yield $d\alpha_0^1 = 0$ and then by (5.1),

$$d\alpha_1^3 = 0 = -\alpha_2^3 \wedge \alpha_1^2. \tag{5.2}$$

On the other hand, differentiation of $\alpha_0^2 = 0$ yields $\alpha_1^2 \wedge \alpha_0^1 = 0$, which implies $\alpha_1^2 = q_1^2\alpha_1^3$, for some function q_1^2. Substituting into (5.2), we would have $q_1^2 = 0$, since $\alpha_2^3 \wedge \alpha_1^3 \neq 0$, and hence $\alpha_1^2 = 0$. But this is in contrast with the fact that $d\alpha_1^2 = \alpha_2^3 \wedge \alpha_1^3 \neq 0$.

Case C: $h^0, h^4 \neq 0$ and, say, $\operatorname{tr} h^0 = 0$, i.e. $\mathbf{H}$ is a section of $\mathcal{N}_0$.

By the discussion in Case III, γ is the central sphere congruence (conformal Gauss map) of its immersed envelope $f = [A_0] : M \to S^3$ and the adapted frame field along γ can be suitably reduced to become the central frame field along f. Accordingly, the condition $\nabla^\perp \mathbf{H} = 0$ takes the form

$$d(p_1 - p_3) + 2(p_1 - p_3)(-q_2\alpha_0^1 + q_1\alpha_0^2) = 0. \tag{5.3}$$

We claim that *when γ is a marginally outer trapped surfaces with non-zero parallel mean curvature vector field, then f is locally Möbius equivalent to a non-zero constant mean curvature surface in some space form embedded in the conformal 3-sphere.*

The claim will follow from the discussion in the next section.

6. Conformal Gauss map and holomorphic differentials

Let γ be a marginally outer trapped surface as in Case III. Then γ is the central sphere congruence of its envelope $f : M \to S^3$, which is an umbilic free immersion into Möbius space. Moreover, an adapted frame field A along γ can be chosen so that A be the central frame field along f.

We start by making some general consideration about f. The metric $I = (\alpha_0^1)^2 + (\alpha_0^2)^2$ and the area element $\alpha_0^1 \wedge \alpha_0^2$ induced by A determine on M an oriented conformal structure and hence, by the existence of isothermal coordinates, a unique compatible complex structure which makes M into a Riemann surface. In terms of the central frame field A, the complex structure is characterized by the property that the complex-valued 1-form

$$\omega = \alpha_0^1 + i\alpha_0^2 \tag{6.1}$$

is of type $(1,0)$. Moreover, the differential form

$$\mathcal{H} = (\alpha_0^1 + i\alpha_0^2)^2$$

is a Möbius-invariant quadratic form of type $(2,0)$.

Remark 6.1. We recall that a conformal immersion $f : M \to S^3$ is isothermic if there exist a holomorphic quadratic differential $\mathcal{D}$ on M and a real-valued smooth function $r : M \to \mathbb{R}$ such that $\mathcal{H} = r\mathcal{D}$. It is known that f is isothermic if and only if p_2 vanishes identically on M (see [24]).

Definition 6.2. As above, let $f : M \to S^3$ be a conformal immersion free of umbilic points. The complex-valued quartic differential form given by

$$\mathcal{Q}_f = F\omega^4, \quad F := \frac{1}{2}(p_1 + p_3) - ip_2, \tag{6.2}$$

and the complex-valued quadratic differential form given by

$$\mathcal{P}_f = G\omega^2, \quad G := p_1 - p_3 \tag{6.3}$$

are globally defined on the Riemann surface M.

Remark 6.3. The quartic form $\mathcal{Q}_f$ was considered by Bryant [9] for Willmore immersions and by Chern [15], [16] for minimal immersions in S^n.

Lemma 6.4. *The quartic differential $\mathcal{Q}_f$ is holomorphic if and only if*

$$dF \wedge \omega = -4(\alpha_0^0 - i\alpha_1^2)F \wedge \omega. \tag{6.4}$$

Proof. Taking the exterior derivative of (6.1) and using the structure equations give

$$d\omega = (\alpha_0^0 - i\alpha_1^2) \wedge \omega. \tag{6.5}$$

Let z be a local complex coordinate on M, so that

$$\omega = \lambda dz, \quad \lambda \neq 0. \tag{6.6}$$

Then, locally,

$$\mathcal{Q}_f = F\lambda^4 (dz)^4.$$

Exterior differentiation of (6.6) and use of (6.5) give

$$\big(d\lambda - \lambda(\alpha_0^0 - i\alpha_1^2)\big) \wedge \omega = 0. \tag{6.7}$$

By (6.7), it is easily seen that condition (6.4) holds if and only if

$$d(F\lambda^4) \wedge \omega = \lambda^4 \left[dF + 4(\alpha_0^0 - i\alpha_1^2)F\right] \wedge \omega = 0,$$

that is, if and only if $\frac{\partial}{\partial \bar z}(F\lambda^4) = 0$. □

We are now in a position to prove the following.

Theorem 6.5. *In Case III, a marginally outer trapped surface $\gamma : M \to S_1^4$ has parallel mean curvature vector field if and only if the quartic form $\mathcal{Q}_f$ is holomorphic.*

Proof. It suffices to prove that (6.4) is equivalent to the **H**-parallel condition (5.3). Writing out the left and right hand side of (6.4) using the structure equations (2.6)-(2.10), we get

$$\begin{aligned} dF \wedge \omega = &-\frac{1}{2}(dp_1 - dp_3) \wedge \alpha_0^1 + \frac{i}{2}(dp_1 - dp_3) \wedge \alpha_0^2 \\ &+ (4q_2p_2 + q_1(3p_1 + p_3))\alpha_0^1 \wedge \alpha_0^2 - i(4q_1p_2 - q_2(p_1 + 3p_3))\alpha_0^1 \wedge \alpha_0^2 \end{aligned}$$

and

$$-4(\alpha_0^0 - i\alpha_1^2)F \wedge \omega = \left[2q_1(p_1 + p_3) + 4q_2p_2 + i(2q_2(p_1 + p_3) - 4q_1p_2)\right] \alpha_0^1 \wedge \alpha_0^2.$$

Thus, (6.4) is equivalent to

$$\begin{aligned} -\frac{1}{2}(dp_1 - dp_3) \wedge \alpha_0^1 + (q_1p_1 - q_1p_3)\alpha_0^1 \wedge \alpha_0^2 &= 0, \\ \frac{1}{2}(dp_1 - dp_3) \wedge \alpha_0^2 + (q_2p_3 - q_2p_1)\alpha_0^1 \wedge \alpha_0^2 &= 0, \end{aligned}$$

which in turn is equivalent to the condition $\nabla^{\perp}\mathbf{H} = 0$,

$$d(p_1 - p_3) + 2(p_1 - p_3)(-q_2\alpha_0^1 + q_1\alpha_0^2) = 0,$$

as claimed. □

As a particular case, we obtain a well-known result of Bryant [9].

Corollary 6.6. *In Case III, if a marginally outer trapped surface $\gamma : M \to S^4_1$ has zero mean curvature vector, then the envelope f is a Willmore surface and the quartic form $\mathcal{Q}_f$ is holomorphic.*

The holomorphicity of $\mathcal{Q}_f$ implies that of $\mathcal{P}_f$.

Lemma 6.7. *If the quartic differential $\mathcal{Q}_f$ is holomorphic, then the quadratic differential $\mathcal{P}_f$ defined in* (6.3) *is also holomorphic.*

Proof. First, observe that the exterior derivative of ω can be written as

$$d\omega = (q_1\alpha^2_0 - q_2\alpha^1_0) \wedge \omega. \tag{6.8}$$

By reasoning as in the proof of Lemma 6.4, the quadratic differential $\mathcal{P}_f$ is holomorphic if and only if

$$dG \wedge \omega = -2(q_1\alpha^2_0 - q_2\alpha^1_0)G \wedge \omega. \tag{6.9}$$

The claim follows from the condition $dG + 2G(-q_2\alpha^1_0 + q_1\alpha^2_0) = 0$, which expresses the holomorphicity of $\mathcal{Q}_f$. □

Next, we collect two additional useful results.

Lemma 6.8. *In case III, if $\gamma : M \to S^4_1$ is a marginally outer trapped surface with non-zero parallel mean curvature vector, then its enveloping surface $f : M \to S^3$ is isothermic.*

Proof. Taking the exterior derivative of (5.3) and using the structure equations, we get

$$0 = 2p_2(p_1 - p_3)\alpha^1_0 \wedge \alpha^2_0,$$

which implies $p_2 = 0$, since $p_1 - p_3 \neq 0$ by hypothesis. □

Lemma 6.9. *If $\mathcal{Q}_f = F\omega^4$ is holomorphic and $\mathcal{P}_f = G\omega^2 \neq 0$, then*

$$F = cG^2,$$

for a real constant c.

Proof. Under the given hypotheses, it follows from Lemma 6.8 that $p_2 = 0$ and then that condition (6.4) can be written

$$dF + 4\mu F \equiv 0, \quad \mathrm{mod}\ \omega,$$

where $\mu = q_1\alpha^2_0 - q_2\alpha^1_0$ and $d\omega = \mu \wedge \omega$. Moreover, condition (6.9) expressing that $\mathcal{P}_f$ is holomorphic can be written

$$dG + 2\mu G \equiv 0, \quad \mathrm{mod}\ \omega.$$

Actually, $dG + 2\mu G = 0$. It then follows that

$$d\left(\frac{F}{G^2}\right) \equiv 0, \quad \mathrm{mod}\ \omega.$$

This proves that the real-valued function F/G^2 is holomorphic, and hence a constant function, as claimed. □

6.1. **MOTS, CMC surfaces in space forms, and T-transforms.** The previous results are now applied to prove our claim about marginally outer trapped surfaces with non-zero mean curvature vector.

Theorem 6.10. *In Case III, if $\gamma : M \to S^4_1$ is a MOTS with non-zero parallel mean curvature vector, then its enveloping surface $f : M \to S^3$ is isothermic and its Calapso potential, $\Phi = e^u$, satisfies*

$$\Delta u = ce^{-2u} - e^{2u} \quad (c \in \mathbb{R}) \tag{6.10}$$

with $\mathcal{P}_f = kdz^2$, where $k \in \mathbb{R} \setminus \{0\}$ and $z = x + iy$ is an isothermic chart on M.

Proof. By Lemma 6.8, the envelope f is an isothermic immersion. Let $z = x + iy : U \subset M \to \mathbb{C}$ be an isothermic chart, so that the central coframe (α^1_0, α^2_0) takes the form $\alpha^1_0 = \Phi dx$ and $\alpha^2_0 = \Phi dy$, where $\Phi = e^u$ is the Calapso potential (see Remark 2.1). From (2.11) and (2.12) we get

$$\mathcal{Q}_f = \frac{1}{2}(p_1 + p_3)\omega^4 = \mathrm{J}e^{4u}(dz)^4 = -\frac{1}{2}(1 + e^{-2u}\Delta u)e^{4u}(dz)^4.$$

Since $\mathcal{Q}_f$ is holomorphic,

$$(1 + e^{-2u}\Delta u)e^{4u} = c,$$

for a constant $c \in \mathbb{R}$, that is

$$\Delta u = ce^{-2u} - e^{2u}.$$

Moreover, since the quadratic differential

$$\mathcal{P}_{f\,|U} = (p_1 - p_3)\omega^2 = (p_1 - p_3)e^{2u}dz^2$$

is also holomorphic, $(p_1 - p_3)e^{2u} = k$, for a non-zero constant k. □

Remark 6.11. Let $f : M \to S^3$ be an umbilic free isothermic immersion with Calapso potential $\Phi = e^{2u}$ and let B be a central frame field along f. By the previous discussion, we know that the map $\gamma := B_3 : M \to S^4_1$ is a spacelike immersion into de Sitter spacetime which coincides with the central sphere congruence of f. The mean curvature vector field of γ is $\mathbf{H} = (p_3 - p_1)B_0$, so that γ is a MOTS.

If we now assume that $\Phi = e^{2u}$ satisfy the equation (6.10), then the right hand side of (2.13) vanishes identically, which implies that $p_1 - p_3 = ke^{-2u}$, for a constant $k \in \mathbb{R}$. A direct computation shows that $p_1 - p_3 = ke^{-2u}$ satisfies the equation

$$d(p_1 - p_3) + 2(p_1 - p_3)(-q_2\alpha^1_0 + q_1\alpha^2_0) = 0.$$

This expresses the fact that the central sphere congruence of f, $\gamma = B_3$, has parallel mean curvature vector, or equivalently, that the quartic form Q_f is holomorphic. If $k \neq 0$, the central sphere congruence γ is a MOTS with non-zero parallel mean curvature vector. If $k = 0$, f is Willmore and the mean curvature vector field of γ vanishes identically.

Definition 6.12. According to [11] (see also [20]), two isothermic immersions $f, \tilde{f}$ which are not Möbius equivalent are said to be *T-transforms* of each other if they have the same Calapso potential. In [24], it has been shown that T-transforms may be viewed as second order conformal deformation in the sense of Cartan.

Let $M^3(\epsilon)$, $\epsilon = -1, 0, 1$, denote the 3-dimensional space form of constant sectional curvature ϵ, conformally embedded into Möbius space S^3.

Theorem 6.13 ([11], [12], [20]). *Let $f : M \to S^3$ be an umbilic free, conformal, isothermic immersion. Then the Calapso potential of f, $\Phi = e^u$, satisfies the equation*

$$\Delta u = ce^{-2u} - e^{2u} \quad (c \in \mathbb{R})$$

if and only if it is locally Möbius equivalent to a constant mean curvature (CMC) immersion into some $M^3(\epsilon)$.

A classical result of G. Thomsen [32] asserts that a conformal immersion into S^3 is Willmore and isothermic if and only if it is locally Möbius equivalent to a minimal immersion into some 3-dimensional space form in Möbius space S^3. Therefore, CMC surfaces in space forms are T-transforms of minimal surfaces.

This, combined with Theorem 6.10 and with Theorem 3.7 of [12], gives the following.

Corollary 6.14. *In Case III, if $\gamma : M \to S^4_1$ is a marginally outer trapped surfaces with non-zero parallel mean curvature vector, then its enveloping surface $f : M \to S^3$ is the T-transform of an isothermic Willmore immersion. Moreover, if $z = x + iy : U \to \mathbb{C}$ is a local isothermic chart and $\Phi = e^u$ is the Calapso potential of f, which satisfies*

$$\Delta u = ce^{-2u} - e^{2u}, \quad s \in \mathbb{R}, \quad \textit{and} \quad \mathcal{P}_f = kdz^2, \quad k \in \mathbb{R} \setminus \{0\},$$

then:

(1) *if $c - k^2 > 0$, f is Möbius equivalent to a CMC isothermic immersion $f' : M \to M^3(1)$ whose fundamental forms are given by*

$$I_{f'} = (c - k^2)e^{-2u}dzd\bar{z}, \quad II_{f'} = \sqrt{c - k^2}\left(-ke^{-2u}dzd\bar{z} + \frac{1}{2}(dz^2 + d\bar{z}^2)\right).$$

(2) *if $c - k^2 < 0$, f is Möbius equivalent to a CMC isothermic immersion $f' : M \to M^3(-1)$ whose fundamental forms are given by*

$$I_{f'} = (k^2 - c)e^{-2u}dzd\bar{z}, \quad II_{f'} = \sqrt{k^2 - c}\left(-ke^{-2u}dzd\bar{z} + \frac{1}{2}(dz^2 + d\bar{z}^2)\right).$$

(3) *if $c - k^2 = 0$, f is Möbius equivalent to a CMC isothermic immersion $f' : M \to M^3(0)$ whose fundamental forms are given by*

$$I_{f'} = 4L^2e^{-2u}dzd\bar{z}, \quad II_{f'} = 2L\left(-ke^{-2u}dzd\bar{z} + \frac{1}{2}(dz^2 + d\bar{z}^2)\right),$$

for a positive constant $L > 0$.

This proves the claim in Case C.

Remark 6.15. As a by-product of the above discussion, it follows that the quartic differential Q_f of an umbilic free conformal immersion $f : M \to S^3$ is holomorphic is and only if f is locally Möbius equivalent to a Willmore surface or to a CMC surface in some space form. According to Bohle–Peters [7], [8], this result was pointed out by K. Voss in a talk at Oberwolfach.

7. MOTSs from CMC canal surfaces in space forms

According to Remark 6.11, examples of MOTSs with parallel mean curvature vector can be obtained as central sphere congruences of umbilic free isothermic surfaces whose Calapso potential satisfies equation (6.10). In this section we discuss a significant class of such isothermic surfaces, namely the class of CMC canal surfaces in space forms. Their relationship with elastic curves in 2-dimensional space forms is also discussed.

7.1. **Isothermic canal surfaces.** Let $f : M \to S^3$ be an umbilic free isothermic immersion with Calapso potential $\Phi = e^u$. According to [27] (see also [28], [5]), if $z = x + iy$ is an isothermic chart, then f is called *canal* if $\Phi_x \equiv 0$ (or $\Phi_y \equiv 0$). If Φ is constant, then f is called a *Dupin surface*.

If f is an isothermic canal surface, from the structure equations (2.9) and (2.10), it follows that

$$\mathrm{w} = e^{-2u}\left(\frac{1}{2}\ddot{u} + \frac{1}{2}\dot{u}^2 + m + e^{2u}\right), \tag{7.1}$$

for a constant $m \in \mathbb{R}$, where $\dot{u}$ denotes the derivative with respect to y.

It is a classical result of Darboux and Vessiot that isothermic canal surfaces in Möbius space are Möbius equivalent to surfaces of revolution, cones, or cylinders in Euclidean space [33], [21]. More precisely, by arguing as in [27], one can prove the following.

1. If $m = 0$, then f is Möbius equivalent to a cylinder

$$(x, y) \mapsto \alpha(y) + xn \in \mathbb{R}^3$$

where n is a unit vector and α is a curve in the plane through the origin orthogonal to n. The arclength and the (geodesic) curvature of α are given by

$$s = \sqrt{2}y, \quad \kappa(s) = -\sqrt{2}\,\Phi(s/\sqrt{2}).$$

2. If $m > 0$, then f is Möbius equivalent to a cone

$$(x, y) \mapsto e^{-\sqrt{2m}x}\alpha(y) \in \mathbb{R}^3$$

with vertex in the origin and directrix curve α which takes values in the unit 2-sphere S^2. The arclength and the (geodesic) curvature of α are given by

$$s = \sqrt{2m}y, \quad \kappa(s) = \sqrt{2/m}\,\Phi(s/\sqrt{2m}).$$

3. If $m < 0$, then f is Möbius equivalent to a surface of revolution

$$(x, y) \mapsto (-a(y)\sin\sqrt{-2m}x, -b(y)\cos\sqrt{-2m}x, b(y))^T \in \mathbb{R}^3,$$

where the profile curve $\alpha : y \to (0, a(y), b(y))^T$, $a > 0$, takes values in the half plane

$$H^2 = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \,:\, x^1 = 0, x_2 > 0\}.$$

If H^2 is endowed with the hyperbolic metric $ds^2 = (x_2)^{-2}(dx_2^2 + dx_3^2)$, the arclength and the (geodesic) curvature of α are given by

$$s = \sqrt{-2m}y, \quad \kappa(s) = \sqrt{-2/m}\,\Phi(-s/\sqrt{-2m}).$$

7.2. **Elastic curves in 2-dimensional space forms.** Following [23], we briefly recall the notion of an elastic curve. Let $M^2(\epsilon)$, $\epsilon = -1,\ 0,\ 1$, be a complete, simply connected, 2-dimensional space form of constant sectional curvature ϵ. On the space of immersed curves in $M^2(\epsilon)$, consider the functional

$$\alpha \mapsto \int (\kappa^2 + \lambda) ds, \tag{7.2}$$

where s denotes the arclength, κ the geodesic curvature of the curve α, and λ is a constant. The Euler-Lagrange equation of (7.2) is

$$2\kappa_{ss} + \kappa^3 + 2\kappa\epsilon - \lambda\kappa = 0. \tag{7.3}$$

A unit-speed curve α is called an *elastica* (or an *elastic curve*) if it satisfies equation (7.3) for some value of λ, and a *free elastica* (or a *free elastic curve*) if it satisfies (7.3) with $\lambda = 0$. Multiplication of (7.3) by $2\kappa_s$ and integration yields

$$(\kappa_s)^2 + \frac{1}{4}\kappa^4 + (\epsilon - \lambda/2)\kappa^2 = A, \quad (A \text{ undetermined constant}), \tag{7.4}$$

Making the change of variable $u = \kappa^2$, one obtains an equation of the form $(u_s)^2 = P(u)$, P a polynomial of degree three, which can be solved by standard techniques in terms of elliptic functions.

Remark 7.1 (Elasticae, motion of curves, and the mKdV equation). Let $\mathbb{R}^3(\epsilon)$, $\epsilon = -1, 0, 1$, be $\mathbb{R}^3$ with the scalar product

$$\langle x, y\rangle_\epsilon = \epsilon x^1 y^1 + x^2 y^2 + x^3 y^3 = x^T g^\epsilon y, \tag{7.5}$$

where the orientation is defined by requiring $dx^1 \wedge dx^2 \wedge dx^3 > 0$. If $\epsilon = -1$, we fix a time-orientation by saying that a timelike or lightlike vector x is future-directed if $x^1 > 0$. Let G^ϵ be the identity component of the pseudo-orthogonal group of (7.5). The group G^ϵ can be identified with the manifold of all oriented basis (e_1, e_2, e_3) of $\mathbb{R}^3(\epsilon)$ such that $\langle e_i, e_j\rangle_\epsilon = g^\epsilon_{ij}$. If $\epsilon = -1$, e_1 is required to be future-directed.

Let $M^2(\epsilon)$, $\epsilon = -1, 0, 1$, be the 2-dimensional Riemannian space forms of constant sectional curvature ϵ, defined by

$$\begin{aligned} M^2(-1) &= \left\{x \in \mathbb{R}^3(-1) \,:\, \langle x, x\rangle_{-1} = -1,\ x^1 > 0\right\}, \\ M^2(0) &= \left\{x \in \mathbb{R}^3(0) \,:\, x^1 = 1\right\}, \\ M^2(-1) &= \left\{x \in \mathbb{R}^3(1) \,:\, \langle x, x\rangle_1 = 1\right\}, \end{aligned}$$

Let $\alpha : I \to M^2(\epsilon)$ be a unit-speed regular curve defined on an open interval $I \subset \mathbb{R}$. For each $s \in I$, let $\mathbf{t}(s) = \alpha'(s)$ and let $\mathbf{n}(s)$ be the unique unit vector of $\mathbb{R}^3(\epsilon)$ such that $(\alpha(s), \mathbf{t}(s), \mathbf{n}(s)) \in G^\epsilon$. The Frenet frame along α is the map

$$\Gamma : I \to G^\epsilon, s \mapsto (\alpha(s), \mathbf{t}(s), \mathbf{n}(s)),$$

which satisfies the Frenet–Serret equation

$$(\alpha', \mathbf{t}', \mathbf{n}') = (\alpha, \mathbf{t}, \mathbf{n}) \begin{pmatrix} 0 & -\epsilon & 0 \\ 1 & 0 & -\kappa \\ 0 & \kappa & 0 \end{pmatrix},$$

where $\kappa : I \to \mathbb{R}$ is the geodesic curvature of α.

As shown by Goldstein and Petrich in [18] (see also [25] for the generalization to 2-dimensional space forms), the modified Korteweg–de Vries (mKdV) equation

$$\kappa_t + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0 \tag{7.6}$$

can be interpreted as the evolution equation of the geodesic curvature of a curve propagating in $M^2(\epsilon)$ according to the equation

$$\frac{\partial}{\partial t}\alpha(s,t) = (\epsilon - \kappa^2/2)\,\mathbf{t} - \kappa_s\,\mathbf{n}. \tag{7.7}$$

In fact, if $\Gamma(\,\cdot,t) : I \to G^\epsilon$ is the Frenet frame of the curve $\alpha(\,\cdot,t) : I \to M^2(\epsilon)$, writing $\Theta = \Gamma^{-1}d\Gamma$, we have

$$\Theta = \begin{pmatrix} 0 & -\epsilon & 0 \\ 1 & 0 & -\kappa \\ 0 & \kappa & 0 \end{pmatrix} ds + \begin{pmatrix} 0 & -\epsilon(\epsilon-\kappa^2/2) & \epsilon\kappa_s \\ \epsilon - \kappa^2/2 & 0 & \kappa_{ss} - \kappa(\epsilon - \kappa^2/2) \\ -\kappa_s & -\kappa_{ss} + \kappa(\epsilon - \kappa^2/2) & 0 \end{pmatrix} dt.$$

The integrability condition $d\Theta + \Theta\wedge\Theta = 0$ amounts to (7.6). Conversely, if $\kappa(s,t)$ is a solution of (7.6) and Θ is defined as above, then $d\Theta+\Theta\wedge\Theta = 0$, which implies the existence a curve motion satisfying (7.7).

The curves that move under (7.7) retaining their shape are called *congruence curves*. They correspond to the solutions of (7.6) which are in the form of a traveling wave, so $\kappa(s,t) = \kappa(x)$, where $x = s + (\epsilon - v)t$, for some constant v, and

$$\kappa''' + \frac{3}{2}\kappa^2\kappa' + (\epsilon - v)\kappa' = 0.$$

Integrating twice, we find

$$(\kappa')^2 + \frac{1}{4}\kappa^4 + (\epsilon - v)\kappa^2 + B\kappa = A,$$

where A, B are two constants of integration. Comparing with (7.4), we see that, for $B = 0$ and $v = \lambda/2$, the congruence curves of the motion (7.7) are elastic curves in $M^2(\epsilon)$.

7.3. **CMC canal surfaces in space forms.** By the above cited Theorem 6.13, an umbilic free, isothermic, canal surface $f : M \to S^3$ is Möbius equivalent to a CMC immersion in some space form $M^3(\epsilon) \subset S^3$ if and only if the Calapso potential of f, $\Phi = e^u$, satisfies the equation

$$\ddot{u} = ce^{-2u} - e^{2u}, \tag{7.8}$$

for a constant $c \in \mathbb{R}$. From (7.1) and (7.8), we have

$$\mathrm{W} = e^{-2u}\left(\frac{1}{2}\dot{u}^2 + \frac{c}{2}e^{-2u} + \frac{1}{2}e^{2u} + m\right). \tag{7.9}$$

On the other hand, using the condition (7.8), it follows from the structure equation (2.13) that $\mathrm{W} = ke^{-2u}$, $k \in \mathbb{R}$. This, combined with (7.9), gives

$$\frac{1}{2}\dot{u}^2 + \frac{c}{2}e^{-2u} + \frac{1}{2}e^{2u} = h, \quad (h \text{ a constant}), \tag{7.10}$$

where h, k and m are related by $k = h+m$. This implies that the Calapso potential $\Phi = e^u$ satisfies the equation

$$\dot{\Phi}^2 + \Phi^4 - 2h\Phi^2 + c = 0. \tag{7.11}$$

We have then the following classification of CMC canal surfaces:

1. <u>If $m = 0$</u>, then f is Möbius equivalent to a cylinder

$$(x,y) \mapsto \alpha(y) + xn \in \mathbb{R}^3$$

where n is a unit vector and α is a curve in the plane through the origin orthogonal to n. Using the fact that

$$s = \sqrt{2}y, \quad \kappa(s) = -\sqrt{2}\,\Phi(s/\sqrt{2}),$$

equation (7.11) becomes

$$(\kappa_s)^2 + \frac{1}{4}\kappa^4 - k\kappa^2 = -c, \quad (c \text{ constant}),$$

which implies that α is an elastic curve in $\mathbb{R}^2$, possibly a free elastic one, with $\lambda = 2k$, $A = -c$.

2. If $m > 0$, then f is Möbius equivalent to a cone

$$(x, y) \mapsto e^{-\sqrt{2m}x}\alpha(y) \in \mathbb{R}^3$$

with vertex in the origin and directrix curve α which takes values in the unit 2-sphere S^2. Using the fact that

$$s = \sqrt{2m}y, \quad \kappa(s) = \sqrt{2/m}\,\Phi(s/\sqrt{2m}),$$

equation (7.11) becomes

$$(\kappa_s)^2 + \frac{1}{4}\kappa^4 - \frac{(k-m)}{m}\kappa^2 = -\frac{c}{m^2}, \quad (c \text{ constant}),$$

which implies that α is an elastic curve in S^2, possibly a free elastic one, with $\lambda = 2k/m$, $A = -c/m^2$.

3. If $m < 0$, then f is Möbius equivalent to a surface of revolution

$$(x, y) \mapsto (-a(y)\sin\sqrt{-2m}x, -b(y)\cos\sqrt{-2m}x, b(y))^T \in \mathbb{R}^3$$

where the profile curve $\alpha : y \to (0, a(y), b(y))^T$, $a > 0$, takes values in the hyperbolic plane H^2. Using the fact that

$$s = \sqrt{-2m}y, \quad \kappa(s) = \sqrt{-2/m}\,\Phi(-s/\sqrt{-2m}),$$

equation (7.11) becomes

$$(\kappa_s)^2 + \frac{1}{4}\kappa^4 + \frac{(k-m)}{m}\kappa^2 = -\frac{c}{m^2}, \quad (c \text{ constant}),$$

which implies that α is an elastic curve in H^2, possibly a free elastic one, with $\lambda = -2k/m$, $A = -c/m^2$.

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