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Conditional Kaniadakis Entropy: a Preliminary Discussion

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Abstract

Conditional entropies are fundamental for evaluating the mutual information of random variables. These entropies must be properly defined in the case of nonadditive entropies. Here, we propose the conditional entropy for one of them, the Kaniadakis entropy. Keywords: Mutual Information, Entropy, Kaniadakis Entropy, Generalized additivity.

Article body

An entropy is additive when the entropy of the union of two independent subsystems, X and Y, is the sum of the subsystems entropies, that is $S(X,Y)=S(X)+S(Y)$. Among the generalized entropies, there are some for which this additivity does not hold. Two nonadditive entropies are the Tsallis and the Kaniadakis entropies [1-3]. The use of them is quite interesting for applications, because they have entropic indices that can be used to tune behaviour of the different contributing variables (see for instance their use in image processing [4]). Tsallis entropy is a generalization of the standard Boltzmann–Gibbs entropy, introduced in 1988 as a basis for generalizing the standard statistical mechanics, whereas the Kaniadakis entropy, also known as κ-entropy, emerged in the context of the special relativity. Both entropies possess a generalized sum [5].

When nonadditive entropies are involved, in the calculus of the mutual information, the conditional entropies must be properly defined. Let us remember that the mutual information $I(X;Y)$ of two random variables X,Y is providing a measure of the mutual dependence of the variables [6]. If X and Y are independent, knowing X does not give any information about Y and vice versa: the mutual information is zero.

The conditional entropy (or equivocation) is quantifying the information needed to describe the outcome of a random variable Y if the value of another random variable X is known: it is written as $H(Y|X)$. Let us assume the joint entropy $H(X,Y)$ for the combined system determined by two random variables X and Y. We need $H(X,Y)$ "bits of information" to describe its exact state [7]. If we first learn the value of X, we have gained $H(X)$ bits of information. "Once X is known, we only need $H(X,Y)-H(X)$ bits to describe the state of the whole system" [7]. This quantity is exactly $H(Y|X)$, which gives the chain rule of conditional entropy: $H(Y|X)=H(X,Y)-H(X)$. The mutual information is then given as $I(X;Y)=H(X)+H(Y)-H(X,Y)$, with the following properties, $I(X;Y)=I(Y;X)$ and $I(X;X)=H(X)$ [7]. When H is the Shannon entropy S and X,Y are independent, we have that $S(X,Y)=S(X)+S(Y)$, and therefore $I(X;Y)=0$.
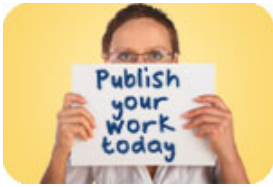
Let us to investigate the mutual information with the generalized entropies, in particular with the nonadditive entropies. In the case of the generalized entropies, it is defined by the so-called Tsallis mutual entropy [8]:

(1) $MT(X;Y)=T(X)-T(X|Y)=T(Y)-T(Y|X)$

In (1), T is referring to the Tsallis entropy. According to [8], $T(X,Y)=T(X)+T(Y|X)$ and $T(Y,X)=T(Y)+T(X|Y)$. Let us remember that Tsallis entropy and Rényi entropy [9] are linked:

$$(2)\quad \text{Tsallis Entropy}: T = T_q = \frac{1}{q-1}\left(1-\sum_i p_i^q\right)$$

$$(3)\quad \text{Rényi entropy}: R = \frac{1}{1-q}\ln\sum_i p_i^q = \overline{T}$$

$$(4)\quad \overline{T} = \frac{\ln[1+(1-q)T]}{1-q}$$

Here q is the entropic index. We have the probabilities $\{p_i\}$, where index i is running from 1 to the total number of configurations. As q approaches 1, the Tsallis entropy becomes the Shannon entropy.

If X,Y are independent, we must have a mutual information equal to zero. Is it possible to write $I(X;Y)=T(X)+T(Y)-T(X,Y)$, as for the Shannon entropy, and have it equal to zero? Let us calculate:

(5) $T(X)+T(Y)-T(X,Y)=(1-q)T(X)T(Y)$

Since:

(6) $T(X,Y)=T(X)+T(Y)+(1-q)T(X)T(Y)$

Therefore I(X;Y) defined in this manner is different from zero. In his paper, Tsallis is discussing the problem of correlated systems too [1]. He used the Rényi entropy for correlated systems:

$$(7)\qquad \Gamma = \overline{T}(X,Y) - \overline{T}(X) - \overline{T}(Y)$$

It is easy to see that, if X,Y are independent, Γ is equal to zero. Let us note that it is function Γ which seems working as the mutual information. However, a quite useful formula was given in [10], by S. Abe and A.K. Rajagopal,. In this reference, it is the nonadditive conditional entropy, which is defined, so that:

$$(8)\qquad MT(X;Y) = T(X) - T(X\,|\,Y) = T(Y) - T(Y\,|\,X) = T(Y) - \frac{T(X,Y)-T(X)}{1+(1-q)T(X)}$$

For X,Y independent variables:

$$(9)\quad MT(X;Y) = \frac{T(Y)(1+(1-q)T(X)) - T(X) - T(Y) - (1-q)T(X)T(Y) + T(X)}{1+(1-q)T(X)} = 0$$

Let us consider, instead of the Tsallis entropy T, the Kaniadakis entropy $K=S_\kappa$ (κ is the entropic index):

$$(10)\quad K = K_\kappa = -\sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa}\ ;\quad \Im = \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2}$$

$$\lim_{\kappa\to 0} K = S_{Shannon}\ ;\quad \lim_{\kappa\to 0}\Im = 1\ \text{ and }: K(X,Y)=K(X)\Im(Y)+K(Y)\Im(X)$$

We have the probabilities $\{p_i\}$, where index i is running from 1 to the total number of configurations. The generalized additivity is discussed in [11] and [5]. As in the case of the Tsallis entropy, we have to be careful because of its generalized additivity. Following the approach of Ref.10, here we propose, for Kaniadakis entropy, a nonadditive conditional

entropy and a mutual entropy as in the following formulas:

$$(11)\quad MK(X;Y) = K(X) - K(X\mid Y) = K(X) - \frac{K(X,Y) - K(Y)\Im(X)}{\Im(Y)}$$

$$= K(Y) - K(Y\mid X) = K(Y) - \frac{K(X,Y) - K(X)\Im(Y)}{\Im(X)}$$

When X,Y are independent:

$$(12)\quad MK(X;Y) = \frac{K(Y)\Im(X) - K(X)\Im(Y) - K(Y)\Im(X) + K(X)\Im(Y)}{\Im(X)} = 0$$

From (11) and (12), it is more clear the role of the auxiliary function which is necessary for the Kaniadakis generalized additivity (we have also discussed this function is [12]). Further studies are in progress on this conditional entropy and for evaluating the conditional Kaniadakis entropies for multivariate problems.

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