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Author's version.
Published in: *STATISTICS* 50 (2016), pp. 649–666.
DOI: 10.1080/02331888.2015.1086350,
URL:
http://www.tandfonline.com/doi/full/10.1080/02331888.2015.1086350.

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The frailty approach is commonly used in reliability theory and survival analysis to model the dependence between lifetimes of individuals or components subject to common risk factors; according to this model the frailty (an unobservable random vector that describes environmental conditions) acts simultaneously on the hazard functions of the lifetimes. Some interesting conditions for stochastic comparisons between random vectors defined in accordance with these models have been described in the literature; in particular, comparisons between frailty models have been studied by assuming independence for the baseline survival functions and the corresponding environmental parameters. In this paper, a generalization of these models is developed, which assumes conditional dependence between the components of the random vector, and some conditions for stochastic comparisons are provided. Some examples of frailty models satisfying these conditions are also described.

Keywords: Bivariate Lifetimes, Dependence Notions, Multivariate Stochastic Orders, Survival Copulas.

AMS Subject Classification: 62N99, 60E15

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1. Introduction

The frailty approach, introduced in [1], provides a convenient tool in survival analysis to model dependence between lifetimes of individuals, or between components, due to common environmental conditions; according to this model, the frailty (an unobservable random variable or vector that describes common risk factors) acts simultaneously on the hazard functions of the lifetimes. In fact, frailty models have been considered in a variety of contexts, such as medicine, biology, engineering, economics and demography, and a number of monographs are entirely devoted to their definitions, properties and applications in all these fields (see [2–6]).

We recall here the basic ideas of this approach and, for the sake of simplicity, we restrict ourselves to the case of bivariate vectors of random lifetimes. Generalizations and results in any dimension can be directly obtained, and hence are omitted throughout the paper. Given the vector $\mathbf{X} = (X_1, X_2)$, it is said to be described by a *bivariate correlated frailty model* if its joint survival function is defined as

$$\bar{F}_{\mathbf{X}}(x_1, x_2) = \mathbf{E}\left[\prod_{i=1}^{2} \bar{G}_i^{V_i}(x_i)\right], \ x_i \in \mathbb{R}^+, \tag{1.1}$$

where $\mathbf{V} = (V_1, V_2)$ is a frailty random vector taking values in $\mathbb{R}^2_+$, while $\bar{G}_i$, for $i = 1, 2$, is any survival function, commonly called *the baseline survival function* of X_i (and, of course, different from the survival function of X_i unless $V_i = 1$ a.s.). Note that this model is based on the assumption that the components in the vectors are independent given the vector of frailties $\mathbf{V}$.

Suppose that the frailty random vector $\mathbf{V}$ has a joint distribution function given by $\mathbf{H}$ with marginal distributions H_1 and H_2. From (1.1), the vector $\mathbf{X}$ has a joint distribution function given by

$$\bar{F}_{\mathbf{X}}(x_1, x_2) = \int_0^{+\infty}\int_0^{+\infty} \bar{G}_1^{v_1}(x_1)\bar{G}_2^{v_2}(x_2)d\mathbf{H}(v_1, v_2), \tag{1.2}$$

and a probability density function given by

$$f_{\mathbf{X}}(x_1, x_2) = \int_0^{+\infty}\int_0^{+\infty} v_1 v_2 g_1(x_1) g_2(x_2)\bar{G}_1^{v_1-1}(x_1)\bar{G}_2^{v_2-1}(x_2)d\mathbf{H}(v_1, v_2). \tag{1.3}$$

When the frailty is common to both components in vector $\mathbf{X}$, that is, when V is a univariate random variable, then it is called a *shared frailty model* (see, e.g., [2]). In this case, the joint distribution function is given by

$$\bar{F}_{\mathbf{X}}(x_1, x_2) = \mathbf{E}\left[\prod_{i=1}^{2} \bar{G}_i^{V}(x_i)\right], \ x_i \in \mathbb{R}^+. \tag{1.4}$$

Different conditions for stochastic comparisons between shared frailty models and correlated frailty models have been provided in recent literature. Misra et al. [7] compared, by using stochastic ordering, multivariate shared frailty models arising from different choices of frailty distribution, generalizing the results given in [8, 9] for the univariate frailty models. Khaledy and Shaked [10] simplified the proof of some partial results given by Misra et al. [7] as particular cases of comparison between multivariate mixtures. Mulero et al. [11] used the shared frailty model to provide sufficient conditions for the stochastic comparisons of residual lifetimes by assuming independence between the baseline survival functions. Several dependence properties of correlated frailty models have been studied by Khaledy and Kochar [12] and later by Li and Da [13], who also investigated stochastic comparisons in these models.

In the references cited above the conditional independence between the components given the frailties is always assumed. However, this assumption could provide an overestimation problem. For example, Hua [14] studied a tail order of copulas to describe the strength of the dependence in tails of a joint distribution (see below for definition of copula). Particularly, a copula based on a scale mixture with a generalized Gamma random variable was used for modeling asymmetric tail negative dependence. In that paper, a dataset for aggregate loss modeling of a medical expenditure panel survey is studied. The empirical analysis suggested that, when the upper tail appears to be negatively dependent, a misspecified independence model that is often used in aggregate loss modeling may overestimate the aggregate loss. Other examples where the non-independence between the components appears in a natural way when the environmental factors are fixed can be read in [15] and [16]. In such cases, the dependence structure is modeled by a Gaussian copula whose parameters can be considered as the corresponding frailties parameters.

Due to these considerations, frailty models with the conditional dependence hypothesis, and univariate frailty, have recently been defined and studied in [17, 18]. Here, a more general model is studied, defined by considering the conditional dependence hypothesis and a bivariate frailty. A number of the results presented in this paper extend properties and statements about the shared frailty model given in [17], several stochastic comparisons results dealing with multivariate conditionally independent frailty models provided in [7], and some results given in [19].

Certain preliminary definitions and results should be recalled in order to describe the main statements presented in the following sections.

First, we recall that the *copula* of a random vector $\mathbf{X} = (X_1, X_2)$, which is a common tool to describe the structure of dependence between its components, is the function $C : [0,1]^2 \to [0,1]$ defined as

$$C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)), \quad u, v \in [0,1],$$

where F is the joint distribution function of $\mathbf{X}$; F_i, for $i = 1, 2$, is the cumulative distribution function of X_i; and $F_i^{-1}(u) = \sup\{x : F_i(x) \leq u\}, u \in [0,1]$, is the right continuous version of the inverse of F_i. The copula is unique whenever the marginal distributions F_i are continuous. We also recall the notion of survival copula, which similarly describes the dependence structure between the components of the random vector, but considers the survival function $\bar{F}_i$ of the marginal X_i instead of its cumulative distribution F_i. Given the vector $\mathbf{X} = (X_1, X_2)$ as above, the function $C_S : [0,1]^2 \to [0,1]$, defined as

$$C_S(u, v) = \bar{F}(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)), \quad u, v \in [0,1],$$

is called *the survival copula*. Further details, properties and applications of these two notions may be found in [20].

Two properties of scalar functions, recalled here, will also be used. Let $\wedge$ and $\vee$ denote the coordinatewise minimum and maximum, respectively, i.e., given the vectors $\mathbf{x} =$

$(x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{R}^n$, let $\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, .., \min\{x_n, y_n\})$ and $\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, ..., \max\{x_2, y_2\})$. A function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be *multivariate totally positive of order 2* (abbreviated to MTP_2) if

$$\phi(\mathbf{x})\phi(\mathbf{y}) \leq \phi(\mathbf{x} \wedge \mathbf{y})\phi(\mathbf{x} \vee \mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

while it is said to be *supermodular* if, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\phi(\mathbf{x}) + \phi(\mathbf{y}) \leq \phi(\mathbf{x} \wedge \mathbf{y}) + \phi(\mathbf{x} \vee \mathbf{y}).$$

The definitions of certain multivariate stochastic orders considered throughout the paper are now recalled. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be random vectors with corresponding distributions $F(\mathbf{x})$, $G(\mathbf{x})$, survival functions $\bar{F}(\mathbf{x})$, $\bar{G}(\mathbf{x})$, and density functions $f(\mathbf{x}), g(\mathbf{x})$. $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the

i) *usual stochastic order* (denoted by $\mathbf{X} \leq_{st} \mathbf{Y}$), if the inequality $E[\psi(\mathbf{X})] \leq E[\psi(\mathbf{Y})]$ is satisfied for any increasing function $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that the expectations exist;

ii) *increasing [componentwise] concave order* (denoted by $\mathbf{X} \leq_{icv} [\leq_{iccv}]\mathbf{Y}$), if $E[\psi(\mathbf{X})] \leq E[\psi(\mathbf{Y})]$ for any increasing and [componentwise] concave function $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$;

iii) *supermodular order* (denoted by $\mathbf{X} \leq_{sm} \mathbf{Y}$), if $E[\psi(\mathbf{X})] \leq E[\psi(\mathbf{Y})]$ for any supermodular function $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$;

iv) *upper [lower] orthant order* (denoted by $\mathbf{X} \leq_{uo\ [lo]} \mathbf{Y}$), if $\bar{F}(\mathbf{x}) \leq \bar{G}(\mathbf{x})$ $[F(\mathbf{x}) \geq G(\mathbf{x})]$ for all $\mathbf{x} \in \mathbb{R}^2$;

v) *hazard rate order* (denoted by $\mathbf{X} \leq_{hr} \mathbf{Y}$), if $\bar{F}(\mathbf{x})\bar{G}(\mathbf{y}) \leq \bar{F}(\mathbf{x} \wedge \mathbf{y})\bar{G}(\mathbf{x} \vee \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$;

vi) *weak multivariate hazard rate order* (denoted by $\mathbf{X} \leq_{whr} \mathbf{Y}$), if

$$\frac{\overline{G}(\mathbf{x})}{\overline{F}(\mathbf{x})} \text{ is increasing in } \mathbf{x} \in \{\mathbf{x} : \overline{G}(\mathbf{x}) > 0\};$$

vii) *likelihood ratio order* (denoted by $\mathbf{X} \leq_{lr} \mathbf{Y}$), if $f(\mathbf{x})g(\mathbf{y}) \leq f(\mathbf{x} \wedge \mathbf{y})g(\mathbf{x} \vee \mathbf{y})$ for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$.

All the multivariate stochastic orders defined above are well-known, and have been applied to compare random vectors in a variety of contexts, such as reliability, risk theory, and actuarial sciences. A list of equivalent definitions, properties and applications may be found in [21]. The purpose of this paper is to study the relationships between stochastic comparisons of the frailty vectors and the corresponding comparisons of the random lifetime vectors. For example, it is intuitive to believe that the more stochastically the frailty increases, the less stochastically the random lifetime vectors are.

Some conventions used in this paper are now clarified. By "increasing" and "decreasing" we mean "non-decreasing" and "non-increasing", respectively. We also denote $=_{ST}$ to indicate equality in law. For any random vector $\mathbf{X}$, or random variable, we denote $[\mathbf{X}|A]$ as a random vector, or random variable, whose distribution is the conditional distribution of $\mathbf{X}$ given A. Finally, by considering a vector with the distribution defined as in (1.1), $\tilde{X}_i$ denotes the random variable whose survival function is the baseline survival function $\bar{G}_i$, for $i = 1, 2$.

The paper is organized as follows. In Section 2, the *generalized bivariate frailty model* is defined, and the hazard rates of the corresponding distribution are described and characterized. Moreover, two particular frailty models are defined through different interpretations. In Section 3, several conditions for stochastic comparisons between generalized bivariate frailty models are provided. In the last section, certain examples dealing with the Gamma, the Lognormal and other frailty models, are proposed.

2. Generalized bivariate frailty models

A natural generalization of model (1.2) can be obtained by removing the assumption of independence among the frailties V_1 and V_2. To this end, let us consider the absolutely continuous joint survival function $\bar{F}(t_1, t_2|v_1, v_2)$ parametrized by a frailty vector (v_1, v_2).

Definition 2.1 *The vector* $\mathbf{X} = (X_1, X_2)$ *is said to be described by a* generalized bivariate frailty model *if its joint survival function is defined as*

$$\bar{F}_{\mathbf{X}}(x_1, x_2) = \int_0^{+\infty} \int_0^{+\infty} \bar{F}(x_1, x_2|v_1, v_2) d\mathbf{H}(v_1, v_2), \; x_i \in \mathbb{R}_+, \tag{2.1}$$

where $\mathbf{H}$ *is the joint distribution function of a frailty random vector* $\mathbf{V} = (V_1, V_2)$ *that takes values in* $\mathbb{R}_+^2$.

Note that the conditional independence hypothesis is not needed in the generalized bivariate frailty model, and that it can be considered as a bivariate generalization of the model provided in [17].

Two particular cases of generalized bivariate frailty models are extensively examined below. The first one is the *bivariate exponential frailty model*, which is a generalization of the model considered in Example 2.1 in [17], and defined by assigning,

$$\bar{F}(x_1, x_2|v_1, v_2) = e^{-v_1\phi_1(x_1,x_2)-v_2\phi_2(x_1,x_2)}, \; x_i \in \mathbb{R}_+ \tag{2.2}$$

where the functions $\phi_i(x_1, x_2)$, $i = 1, 2$, are proper non-negative differentiable functions in both arguments, such that $\bar{F}(x_1, x_2|v_1, v_2)$ is a survival function. Among other properties, $\phi_i(0, 0) = 0$, $\phi_i(+\infty, +\infty) = +\infty$ and $\frac{\partial\phi_i(x_1,x_2)}{\partial x_j} > 0$, for $i, j = 1, 2$ should be verified. Note that if $\phi_1 = \phi_2 = -\ln \bar{F}$ holds in (2.2), then this model coincides with the multiplicative model studied in Section 4 in [17] and in Section 5 in [18].

The second particular case is based on a representation of $\bar{F}(x_1, x_2|v_1, v_2)$ in terms of the copula. In fact, representing it as $C_S(\bar{G}_1^{v_1}(x_1), \bar{G}_2^{v_2}(x_2))$, where the $\bar{G}_i$ are baseline survival functions and C_S is a survival copula, one can obtain the model

$$\begin{aligned} \bar{F}_{\mathbf{X}}(x_1, x_2) &= \mathbf{E}\left[C_S(\bar{G}_1^{V_1}(x_1), \bar{G}_2^{V_2}(x_2))\right] \\ &= \int_0^{+\infty} \int_0^{+\infty} C_S(\overline{G}_1^{v_1}(x_1), \bar{G}_2^{v_2}(x_2)) d\mathbf{H}(v_1, v_2), \; x_i \in \mathbb{R}_+, \end{aligned} \tag{2.3}$$

where $\mathbf{H}$ is the joint distribution function of the frailty vector $\mathbf{V} = (V_1, V_2)$. We will refer to this model as the *Bivariate Model with Marginal Frailty Distributions (BMMFD)*.

Note that when C_S is the independence copula, that is when $C_S(\bar{G}_1^{V_1}(x_1), \bar{G}_2^{V_2}(x_2)) = \bar{G}_1^{V_1}(x_1)\bar{G}_2^{V_2}(x_2)$ for all x_1, x_2, then the survival functions given in (1.1) and (2.3) are the same. In addition, if $V_1 = V_2$ a.s., then the shared frailty model defined in (1.4) is obtained.

It is worth mentioning that, in the model defined in (2.3), the marginal survival function of the i-th component is $\bar{F}_i(x) = \int_0^\infty \bar{G}_i^{v_i} dH_i(v_i)$, therefore, $\bar{G}_i(x) = exp[-\phi_i^{-1} \circ \bar{F}_i(x)]$, where ϕ_i^{-1} is the inverse Laplace transform of H_i. Thus, the model defined in (2.3) coincides with the family of multivariate distributions generated by the mixture described in [22].

A characterization of the hazard gradient of a vector, which is described by a generalized bivariate frailty model, is now studied. To this end, recall that, given vector $\mathbf{X} = (X_1, X_2)$ with survival function $\bar{F}_{\mathbf{X}}(x_1, x_2)$, a well-known bivariate generalization of the failure rate function is the hazard gradient

$$\lambda(x_1, x_2) = (\lambda^{(1)}(x_1, x_2), \lambda^{(2)}(x_1, x_2))$$

where

$$\lambda^{(i)}(x_1, x_2) = -\frac{\partial}{\partial x_i} \ln \bar{F}_{\mathbf{X}}(x_1, x_2), \quad i = 1, 2.$$

As pointed out in [23] and references therein, the hazard gradient uniquely determines the distribution $\bar{F}_{\mathbf{X}}$.

Let

$$\lambda^{(i)}(x_1, x_2|v_1, v_2) = -\frac{\partial}{\partial x_i} \ln \bar{F}(x_1, x_2|v_1, v_2)$$

denote the failure rate function of the i-th unit with the j-th ($i \neq j$) unit surviving until time x_j, conditioned on $\mathbf{V} = \mathbf{v}$, with $\mathbf{v} = (v_1, v_2)$. The following result is the bivariate version of Theorem 2.1 in [19] and shows that the components $\lambda^{(i)}(x_1, x_2)$ of the hazard gradient of $\mathbf{X}$ are the averages of the conditional hazard components. In order to simplify the proof of results given in the next section, we use a slight modification in the notation used in [19]. It is assumed that h is the corresponding density function of $\mathbf{H}$ and that the conditional distribution of $\mathbf{V}$, given $X_1 > x_1, X_2 > x_2$, is absolutely continuous, such that the conditional density exists.

Theorem 2.1 *Let the vector $\mathbf{X}$ have the survival function as defined in (2.1). Let the joint distribution function $\mathbf{H}$ and the conditional distribution of $\mathbf{V}$, given $X_1 > x_1, X_2 > x_2$, be absolutely continuous. Therefore the population failure rate function of the ith component of vector $\mathbf{X}$ with the $j-$th component of fixed age x_j is the expected value of $\lambda^{(i)}(x_1, x_2|v_1, v_2)$ with respect to the conditional distribution of the frailty effect $\mathbf{V}$, given $X_1 > x_1$ and $X_2 > x_2$. That is,*

$$\lambda^{(i)}(x_1, x_2) = E[\lambda^{(i)}(x_1, x_2|\widetilde{\mathbf{V}}_{(x_1,x_2)})],$$

where $\widetilde{\mathbf{V}}_{(x_1,x_2)} =_{st} [\mathbf{V}|X_1 > x_1, X_2 > x_2]$.

Example 2.1 *Assume that the vector $\mathbf{X}$ is described by a bivariate exponential frailty model, i.e., it has a distribution as described in (2.2). Given that $\lambda^{(i)}(x_1, x_2|v_1, v_2) =$*

$-\frac{\partial}{\partial x_i} \ln \bar{F}(x_1, x_2|v_1, v_2)$, *it follows that*

$$\begin{aligned}\lambda^{(i)}(x_1, x_2|v_1, v_2) &= \frac{\partial}{\partial x_i}(v_1\phi_1(x_1, x_2) + v_2\phi_2(x_1, x_2)) \\ &= v_1\frac{\partial}{\partial x_i}\phi_1(x_1, x_2) + v_2\frac{\partial}{\partial x_i}\phi_2(x_1, x_2).\end{aligned}$$

Therefore, for the bivariate exponential frailty model, one has

$$\begin{aligned}\lambda^{(i)}(x_1, x_2) &= \int_{\mathbb{R}^2}\left[v_1\frac{\partial}{\partial x_i}\phi_1(x_1, x_2) + v_2\frac{\partial}{\partial x_i}\phi_2(x_1, x_2)\right] h(v_1, v_2|X_1 > x_1, X_2 > x_2)dv_1dv_2 \\ &= \frac{\partial}{\partial x_i}\phi_1(x_1, x_2)E[V_1|X_1 > x_1, X_2 > x_2] + \frac{\partial}{\partial x_i}\phi_2(x_1, x_2)E[V_2|X_1 > x_1, X_2 > x_2] \\ &= \lambda_{10}^{(i)}(x_1, x_2)E[V_1|X_1 > x_1, X_2 > x_2] + \lambda_{20}^{(i)}(x_1, x_2)E[V_2|X_1 > x_1, X_2 > x_2],\end{aligned}$$

where $\lambda_{10}^{(i)}(x_1, x_2)$, *for* $i = 1, 2$, *is the* i*-th component of the hazard gradient without incorporating the frailty effect* v_1 *and where* $v_2 = 0$*; and where* $\lambda_{20}^{(i)}(x_1, x_2)$, *for* $i = 1, 2$, *is the* i*-th component of the hazard gradient without incorporating the frailty effect* v_2 *and where* $v_1 = 0$.

3. Stochastic comparisons of generalized bivariate frailty models

In this section, some stochastic comparisons between generalized frailty models are presented. From now on, unless stated otherwise, we assume that $\mathbf{X}_1$ and $\mathbf{X}_2$ are two bivariate random vectors that have survival functions defined as in (2.1), that is,

$$\bar{F}_{\mathbf{X}_k}(x_1, x_2) = \int_0^{+\infty}\int_0^{+\infty}\bar{F}(x_1, x_2|v_1, v_2)d\mathbf{H}_k(v_1, v_2),\ x_i \in \mathbb{R}_+ \text{ for } k = 1, 2, \quad (3.1)$$

where $\mathbf{H}_k$ is the joint distribution function of the vector frailty $\mathbf{V}_k = (V_{k1}, V_{k2})$, for $k = 1, 2$. Note that $[\mathbf{X}_1|\mathbf{V}_1 = (v_1, v_2)]$ and $[\mathbf{X}_2|\mathbf{V}_2 = (v_1, v_2)]$ have the same survival function for all (v_1, v_2) in $\mathbb{R}^2$. From now on, it is said that $[\mathbf{X}_1|\mathbf{V}_1 = (v_1, v_2)] =_{st} [\mathbf{X}_2|\mathbf{V}_2 = (v_1, v_2)] =_{st} [\mathbf{X}|\mathbf{V} = (v_1, v_2)]$.

The first two statements deal with comparisons in the usual stochastic order and the likelihood ratio order. The proof of the former immediately follows from Theorem 3.3 in in [24], while the proof of latter follows from Theorem 2.4 in [25]. For this reason, both these proofs are omitted.

THEOREM 3.1 *Let* $\mathbf{X}_k = (X_{k1}, X_{k2})$, *with* $k = 1, 2$, *be two random vectors that have survival functions defined as in (3.1). If* $[\mathbf{X}|\,\boldsymbol{V} = (v_1, v_2)]$ *is increasing [decreasing] in the stochastic order in* (v_1, v_2) *and if* $\mathbf{V}_1 \leq_{st} \mathbf{V}_2$, *then* $\mathbf{X}_1 \leq_{st} [\geq_{st}]\mathbf{X}_2$.

THEOREM 3.2 *Let* $\mathbf{X}_k = (X_{k1}, X_{k2})$, *with* $k = 1, 2$, *be two absolutely continuous bivariate random vectors that have survival functions defined as in (3.1). Let* $f(x_1, x_2|v_1, v_2)$ *denote the density corresponding to the distribution* $F(x_1, x_2|v_1, v_2)$. *If* $\mathbf{V}_2 \leq_{lr} \mathbf{V}_1$ *and if* $f(x_1, x_2|v_1, v_2)$ *is MTP2 in* (x_1, x_2, v_1, v_2), *then* $\mathbf{X}_1 \leq_{lr} \mathbf{X}_2$.

Further details, and examples of distributions that satisfy the assumptions of these two statements, may be found in [24, 25] and references therein.

The next statements describe conditions for the weak hazard order between generalized frailty models. Particularly, Theorem 3.3 and 3.4. can be considered generalizations of Theorem 3.1 in [17].

THEOREM 3.3 *Let* $\mathbf{X}_k = (X_{k1}, X_{k2})$, $k = 1, 2$, *be two bivariate random vectors that have survival functions defined as in (3.1). If*

a) $[\mathbf{V}_2|X_1 > x_1, X_2 > x_2] \leq_{st} [\mathbf{V}_1|X_1 > x_1, X_2 > x_2]$ *for all* $(x_1, x_2) \in \mathbb{R}_+^2$;
b) $\lambda^{(i)}(x_1, x_2|v_1, v_2)$, *for* $i = 1, 2$, *is an increasing function in* (v_1, v_2) *for all* $(x_1, x_2) \in \mathbb{R}_+^2$,

then $\mathbf{X}_1 \leq_{whr} \mathbf{X}_2$.

Proof. From Theorem 6.D.2 in [21], it is sufficient to prove that

$$\lambda^{(i)}_{\mathbf{X}_1}(x_1, x_2) \geq \lambda^{(i)}_{\mathbf{X}_2}(x_1, x_2), \quad \text{for} \quad i = 1, 2, \quad (x_1, x_2) \in \mathbb{R}_+^2.$$

Let $\Delta^{(i)}_{\mathbf{X}_1,\mathbf{X}_2} = \lambda^{(i)}_{\mathbf{X}_1}(x_1, x_2) - \lambda^{(i)}_{\mathbf{X}_2}(x_1, x_2)$, $i = 1, 2$. From Theorem 2.1, it follows that

$$\Delta^{(i)}_{\mathbf{X}_1,\mathbf{X}_2} = E[\lambda^{(i)}(x_1, x_2|\widetilde{\mathbf{V}}_{1(x_1,x_2)})] - E[\lambda^{(i)}(x_1, x_2|\widetilde{\mathbf{V}}_{2(x_1,x_2)})],$$

where $\widetilde{\mathbf{V}}_{i(x_1,x_2)} =_{st} [\mathbf{V}_i|X_1 > x_1, X_2 > x_2]$. Hence, from (a) and (b), $\Delta^{(i)}_{\mathbf{X}_1,\mathbf{X}_2} \geq 0$ holds for $i = 1, 2$. Thus, the result is obtained. ■

Note that assuming conditionally independence hypothesis, Theorem 3.3 above and Theorem 3.11 in [24] are similar. However, the conditions in Theorem 3.3 are weaker, as well as the conclusion, since the multivariate *hr* order implies the multivariate *whr* order

A preliminary result is needed for the following result, which provides alternative conditions for the weak hazard order. To this end, let h_k, with $k = 1, 2$, denote the density function of $\mathbf{V}_k$, let $h_{V_{kj}}$, $j = 1, 2$, denote the density function of the *j-th* marginal of $\mathbf{V}_k$, and let $h_{[V_{ki}|V_{kj}=v_{kj}]}$ denote the density function of $[V_{ki}|V_{kj} = v_{kj}]$.

LEMMA 3.1 *Let* $\mathbf{V}_1 = (V_{11}, V_{12})$ *and* $\mathbf{V}_2 = (V_{21}, V_{22})$ *be two bivariate frailty random vectors such that* $\mathbf{V}_2 \leq_{lr} \mathbf{V}_1$. *Therefore,* $[V_{21}|X_1 > x_1, X_2 > x_2, V_{22} = v_2] \leq_{lr} [V_{11}|X_1 > x_1, X_2 > x_2, V_{12} = v_2]$ *for all* $(x_1, x_2) \in \mathbb{R}_+^2$ *and* $v_2 \in \mathbb{R}_+$.

Proof. First, observe that whenever $\mathbf{V}_1 = (V_{11}, V_{12})$ and $\mathbf{V}_2 = (V_{21}, V_{22})$ are two bivariate random vectors such that $\mathbf{V}_2 \leq_{lr} \mathbf{V}_1$, then

$$[V_{2i}|V_{2j} = t] \leq_{lr} [V_{1i}|V_{1j} = t] \text{ for } i \neq j \text{ and for all } t \in \mathbb{R}_+. \tag{3.2}$$

In fact, $\mathbf{V}_2 \leq_{lr} \mathbf{V}_1$ if $h_2(\mathbf{x})h_1(\mathbf{y}) \leq h_2(\mathbf{x} \wedge \mathbf{y})h_1(\mathbf{x} \vee \mathbf{y})$ for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}_+^2$. In particular, for $\mathbf{x} = (x, t)$ and $\mathbf{y} = (y, t)$, it follows that

$$h_2(x, t)h_1(y, t) \leq h_2(x \wedge y, t)h_1(x \vee y, t). \tag{3.3}$$

By dividing by $h_{V_{22}}(t)h_{V_{12}}(t)$ in both sides of equation (3.3), it follows that

$$h_{[V_{21}|V_{22}=t]}(x)\ h_{[V_{11}|V_{12}=t]}(y) \leq h_{[V_{21}|V_{22}=t]}(x \wedge y)\ h_{[V_{11}|V_{12}=t]}(x \vee y).$$

Thus, $[V_{21}|V_{22}=t] \leq_{lr} [V_{11}|V_{12}=t]$ for all $t \in \mathbb{R}_+$, i.e., (3.2) holds for $i=1$ and $j=2$ (and the proof for $i=2$ and $j=1$ is similar).

Now, from (3.2), the ratio $h_{[V_{21}|V_{22}=v_2]}(w_1)/h_{[V_{11}|V_{12}=v_2]}(w_1) = g(w_1|v_2)$ is a decreasing function in w_1, for all $v_2 \in \mathbb{R}_+$. Moreover, for $(x_1,x_2) \in \mathbb{R}^2_+$, we have

$$H_{[V_{21}|X_1>x_1,X_2>x_2,V_{22}=v_2]}(u_1) = \frac{\int_0^{u_1} \bar{F}(x_1,x_2|w_1,v_2)h_{[V_{21}|V_{22}=v_2]}(w_1)dw_1}{\int_0^{\infty} \bar{F}(x_1,x_2)|w_1,v_2)h_{[V_{21}|V_{22}=v_2]}(w_1)dw_1}$$
$$= \frac{\int_0^{u_1} \bar{F}((x_1,x_2|w_1,v_2)g(w_1|v_2)h_{[V_{11}|V_{22}=v_2]}(w_1)dw_1}{\int_0^{\infty} \bar{F}(x_1,x_2)|w_1,v_2)g(w_1|v_2)h_{[V_{11}|V_{22}=v_2]}(w_1)dw_1}.$$

From the monotonicity of $g(w_1|v_2)$, and by using a similar development to the proof in Theorem 2.4 in [17], the result is obtained. ■

Note that, in general,

$$\begin{aligned}\lambda^{(1)}(x_1,x_2|v_1,v_2) &= -\frac{\partial\bar{F}(x_1,x_2|v_1,v_2)/\partial x_1}{\bar{F}(x_1,x_2|v_1,v_2)} \\ &= -\frac{\bar{F}_{X_2}(x_2|v_1,v_2)\partial\bar{F}_{X_1}(x_1|v_1,v_2,X_2>x_2)/\partial x_1}{\bar{F}_{X_2}(x_2|v_1,v_2)\bar{F}_{X_1}(x_1|v_1,v_2,X_2>x_2)} \\ &= \lambda^{(1)}(x_1|v_1,v_2,X_2>x_2). \end{aligned} \tag{3.4}$$

Similarly, it can be proved that $\lambda^{(2)}(x_1,x_2|v_1,v_2) = \lambda^{(2)}(x_2|v_1,v_2,X_1>x_1)$.

For simplicity, we denote, for $i,j=1,2$, with $i \neq j$,

$$\lambda_k^{(i)}(x_i|v_j,X_j>x_j) = \int_0^{\infty} \lambda^{(i)}(x_i|v_1,v_2,X_j>x_j)h_{[V_{ki}|X_1>x_1,X_2>x_2,V_{kj}=v_j]}(v_i)dv_i, \tag{3.5}$$

where $h_{[V_{ki}|X_1>x_1,X_2>x_2,V_{kj}=v_j]}(v_i)$ is the marginal density function of $[V_{ki}|X_1>x_1,X_2>x_2,V_{kj}=v_j]$.

The following result can now be proved.

Theorem 3.4 *Let* $\mathbf{X}_k=(X_{k1},X_{k2})$, $k=1,2$, *be bivariate random vectors that have survival functions defined as in (3.1). Let*

a) $\mathbf{V}_2 \leq_{lr} \mathbf{V}_1$;
b) $\lambda^{(i)}(x_1,x_2|v_1,v_2), i=1,2$ *be an increasing function in* (v_1,v_2) *for all* $(x_1,x_2)\in\mathbb{R}^2_+$;
c) $\lambda_1^{(i)}(x_i|v_j,X_j>x_j), i=1,2$ *and* $j\neq i$ *be increasing in* v_j *or* $\lambda_2^{(i)}(x_i|v_j,X_j>x_j), i=1,2$, *be increasing in* v_j.

Then $\mathbf{X}_1 \leq_{whr} \mathbf{X}_2$.

Proof. From Theorem 6.D.2 in [21], it is sufficient to prove that $\lambda^{(i)}_{\mathbf{X}_1}(x_1,x_2) \geq \lambda^{(i)}_{\mathbf{X}_2}(x_1,x_2)$, for $i=1,2$. The proof for $i=1$ is given, the proof for $i=2$ is similar and is therefore omitted.

Let $\Delta^{(1)}_{\mathbf{X}_1,\mathbf{X}_2} = \lambda^{(1)}_{\mathbf{X}_1}(x_1,x_2) - \lambda^{(1)}_{\mathbf{X}_2}(x_1,x_2)$. It follows that

$$\Delta^{(1)}_{\mathbf{X}_1,\mathbf{X}_2} = \int_0^\infty\int_0^\infty \lambda^{(1)}(x_1,x_2|v_1,v_2)[h_1(v_1,v_2|x_1,x_2) - h_2(v_1,v_2|x_1,x_2)]dv_1dv_2$$

$$= \int_0^\infty\int_0^\infty \lambda^{(1)}(x_1|v_1,v_2,X_2>x_2)h_{[V_{11}|X_1>x_1,X_2>x_2,V_{12}=v_2]}(v_1)h_{[V_{12}|X_1>x_1,X_2>x_2]}(v_2)dv_1dv_2$$

$$-\int_0^\infty\int_0^\infty \lambda^{(1)}(x_1|v_1,v_2,X_2>x_2)h_{[V_{21}|X_1>x_1,X_2>x_2,V_{22}=u_2]}(v_1)h_{[V_{22}|X_1>x_1,X_2>x_2]}(v_2)dv_1dv_2$$

$$= \int_0^\infty \lambda^{(1)}_1(x_1|v_2,X_2>x_2)h_{[V_{12}|X_1>x_1,X_2>x_2]}(v_2)dv_2 - \int_0^\infty \lambda^{(1)}_2(x_1|v_2,X_2>x_2)h_{[V_{22}|X_1>x_1,X_2>x_2]}(v_2)dv_2.$$

Moreover, assume that in assumption *c)* the hazard rate $\lambda^{(1)}_1(x_1|v_2,X_2>x_2)$ is increasing (note that if $\lambda^{(1)}_2(x_1|v_2,X_2>x_2)$ is increasing, then the proof is similarly obtained by interchanging the corresponding roles of these functions). By adding and subtracting $\int_0^\infty \lambda^{(1)}_1(x_1|v_2,X_2>x_2)h_{V_{22}|\{X_1>x_1,X_2>x_2\}}(v_2)dv_2$, it follows that

$$\Delta^{(1)}_{\mathbf{X}_1,\mathbf{X}_2} = \int_0^\infty \lambda^{(1)}_1(x_1|v_2,X_2>x_2)[h_{[V_{12}|X_1>x_1,X_2>x_2]}(v_2) - h_{[V_{22}|\{X_1>x_1,X_2>x_2]}(v_2)]dv_2$$
$$+\int_0^\infty [\lambda^{(1)}_1(x_1|v_2,X_2>x_2) - \lambda^{(1)}_2(x_1|v_2,X_2>x_2)]h_{[V_{22}|X_1>x_1,X_2>x_2]}(v_2)dv_2. \quad (3.6)$$

The first and second terms on the right-hand side of (3.6) are denoted by A and B, respectively, that is

$$A = E[\lambda^{(1)}_1(x_1|\widetilde{V}_{12},X_2>x_2)] - E[\lambda^{(1)}_1(x_1|\widetilde{V}_{22},X_2>x_2)],$$

where $\widetilde{V}_{12} =_{st} [V_{12}|X_1>x_1,X_2>x_2]$ and $\widetilde{V}_{22} =_{st} [V_{22}|X_1>x_1,X_2>x_2]$, and

$$B = \int_0^\infty [\lambda^{(1)}_1(x_1|v_2,X_2>x_2) - \lambda^{(1)}_2(x_1|v_2,X_2>x_2)]h_{V_{22}|\{X_1>x_1,X_2>x_2\}}(v_2)dv_2.$$

From assumption *a)*, and given that the multivariate likelihood ratio order is closed under marginalization (see Theorem 6.E.4.b in [21]), it follows that $V_{22} \leq_{lr} V_{12}$. Moreover, from Theorem 2.4 in [17] and given that the $\leq_{lr}$ order implies the multivariate $\leq_{st}$ order, it follows that $[V_{22}|X_1>x_1,X_2>x_2] \leq_{st} [V_{12}|X_1>x_1,X_2>x_2]$. Therefore, $A \geq 0$, given that from hypothesis *c)*, $\lambda^{(1)}_1(x_1|v_2,X_2>x_2)$ is increasing in v_2 for all (x_1,x_2).

It can now be proved that the second term on the right-hand side of (3.6) is also non-negative. Denote $\Delta^{(1)}_{12} = \lambda^{(1)}_1(x_1|v_2,X_2>x_2) - \lambda^{(1)}_2(x_1|v_2,X_2>x_2)$. From (3.4) and (3.5),

$$\Delta^{(1)}_{12} = \int_0^\infty \lambda^{(1)}(x_1,x_2|v_1,v_2)[h_{[V_{11}|X_1>x_1,X_2>x_2,V_{12}=v_2]}(v_1) - h_{[V_{21}|X_1>x_1,X_2>x_2,V_{22}=v_2]}(v_1)]dv_1$$
$$= E[\lambda^{(1)}(x_1,x_2|\widetilde{V}_{11},v_2)] - E[\lambda^{(1)}(x_1,x_2|\widetilde{V}_{21},v_2)], \quad (3.7)$$

where $\widetilde{V}_{11} =_{st} [V_{11}|X_1 > x_1, X_2 > x_2, V_{12} = v_2]$ and $\widetilde{V}_{21} =_{st} [V_{21}|X_1 > x_1, X_2 > x_2, V_{12} = v_2]$. From *a)*, Lemma 3.1 and *b)*, it follows that (3.7) is non-negative and, therefore, $B \geq 0$. Thus, the result is obtained. ■

As a particular case, if the vectors $\mathbf{X}_k = (X_{k1}, X_{k2})$, $k = 1, 2$, are defined according to the bivariate exponential frailty model, i.e., if $\bar{F}(x_1, x_2|v_1, v_2)$ is defined as in (2.2), then a simpler statement can be provided.

Theorem 3.5 *Let $\mathbf{X}_k = (X_{k1}, X_{k2})$, $k = 1, 2$ be two bivariate random vectors that have survival functions defined as in (3.1), and let $\bar{F}(x_1, x_2|v_1, v_2)$ be defined as in (2.2). If $\mathbf{V}_2 \leq_{lr} \mathbf{V}_1$, then $\mathbf{X}_1 \leq_{whr} \mathbf{X}_2$.*

Proof. If $\bar{F}(x_1, x_2|v_1, v_2)$ is defined as in (2.2), then

$$\lambda_k^{(i)}(x_1, x_2) = \frac{\partial}{\partial x_i}\phi_1(x_1, x_2) E[V_{k1}|X_1 > x_1, X_2 > x_2] + \frac{\partial}{\partial x_i}\phi_2(x_1, x_2) E[V_{k2}|X_1 > x_1, X_2 > x_2].$$

Therefore,

$$\begin{aligned}\Delta^{(i)}_{\mathbf{X}_1,\mathbf{X}_2} &= \frac{\partial}{\partial x_i}\phi_1(x_1, x_2)\big[E[V_{11}|X_1 > x_1, X_2 > x_2] - E[V_{21}|X_1 > x_1, X_2 > x_2]\big] \\ &+ \frac{\partial}{\partial x_i}\phi_2(x_1, x_2)\big[E[V_{12}|X_1 > x_1, X_2 > x_2] - E[V_{22}|X_1 > x_1, X_2 > x_2]\big].\end{aligned}$$

Moreover, given that the order $\leq_{lr}$ is closed under marginalization, and by using Theorem 2.4 from [17], $[V_{2i}|X_1 > x_1, X_2 > x_2] \leq_{st} [V_{1i}|X_1 > x_1, X_2 > x_2]$ holds. Thus, since $\frac{\partial \phi_j(x_1,x_2)}{\partial x_i} \geq 0$ for all $i, j = 1, 2$, it follows that $\Delta^{(i)}_{\mathbf{X}_1,\mathbf{X}_2} \geq 0$, and the result is obtained. ■

For the same model as above, the upper orthant order (which is weaker than the usual stochastic order) can be obtained under mild conditions, as stated in the following result.

Theorem 3.6 *Let $\mathbf{X}_k = (X_{k1}, X_{k2})$, $k = 1, 2$, be two bivariate random vectors that have survival functions defined as in (3.1) and $\bar{F}(x_1, x_2|v_1, v_2)$ defined as in (2.2). If $\mathbf{V}_2 \leq_{st} \mathbf{V}_1$, then $\mathbf{X}_1 \leq_{uo} \mathbf{X}_2$.*

Proof. Since $g_{(x_1,x_2)}(v_1, v_2) = -e^{-v_1\phi_1(x_1,x_2)-v_2\phi_2(x_1,x_2)}$ is an increasing function in (v_1, v_2), for every $(x_1, x_2) \in \mathbb{R}^2_+$, then by inequality $\mathbf{V}_2 \leq_{st} \mathbf{V}_1$, it immediately follows that

$$\begin{aligned}\bar{F}_{\mathbf{X}_1}(x_1, x_2) &= \int_0^\infty\int_0^\infty e^{-v_1\phi_1(x_1,x_2)-v_2\phi_2(x_1,x_2)} h_1(v_1, v_2)dv_1dv_2 \\ &= -E[g_{(x_1,x_2)}(\mathbf{V}_1)] \\ &\leq -E[g_{(x_1,x_2)}(\mathbf{V}_2)] \\ &= \int_0^\infty\int_0^\infty e^{-v_1\phi_1(x_1,x_2)-v_2\phi_2(x_1,x_2)} h_2(v_1, v_2)dv_1dv_2 \\ &= \bar{F}_{\mathbf{X}_2}(x_1, x_2),\end{aligned}$$

which constitutes the assertion. ■

The following statements deal with the BMMFD, i.e., the model of survival functions defined as in (2.3). Recall that $\tilde{X}_{ki}$ denotes the random variable whose survival function is the baseline survival function $\bar{G}_{ki}$, for $k = 1, 2$ and $i = 1, 2$.

Theorem 3.7 *Let $\mathbf{X}_k = (X_{k1}, X_{k2})$, for $k = 1, 2$, be two bivariate random vectors with survival functions given by a BMMFD. If*

a) $\mathbf{V}_2 \leq_{st} \mathbf{V}_1$*;*
b) $\tilde{X}_{1i} \leq_{st} \tilde{X}_{2i}, i = 1, 2$*;*

then $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$*.*

Proof. Let $\mathbf{X}_k$, with $k = 1, 2$, be two vectors with the survival function defined as in (2.3). Now, consider a new random vector $\mathbf{Y}$ having survival function given by

$$\bar{F}_{\mathbf{Y}}(x_1, x_2) = \int_0^{+\infty} \int_0^{+\infty} C_S(\bar{G}_{11}^{v_1}(x_1), \bar{G}_{12}^{v_2}(x_2)) d\mathbf{H}_2(v_1, v_2).$$

First, we prove that $\mathbf{Y} \leq_{st} \mathbf{X}_2$. For fixed (v_1, v_2) and $k = 1, 2$, let $\mathbf{X}_k^{(v_1,v_2)}$ be a random vector with survival function defined as

$$\bar{F}_{\mathbf{X}_k^{(v_1,v_2)}}(x_1, x_2) = C_S(\bar{G}_{k1}^{v_1}(x_1), \bar{G}_{k2}^{v_2}(x_2)).$$

Observe that, for any fixed (v_1, v_2), the vectors $\mathbf{X}_1^{(v_1,v_2)}$ and $\mathbf{X}_2^{(v_1,v_2)}$ have the same survival copula $C_S^{(v_1,v_2)} = C_S(u_1^{v_1}, u_2^{v_2})$. Moreover, by hypothesis (b), $\mathbf{X}_1^{(v_1,v_2)}$ and $\mathbf{X}_2^{(v_1,v_2)}$ have ordered univariate margins. Thus, from Theorem 4.1 in [26], $\mathbf{X}_1^{(v_1,v_2)} \leq_{st} \mathbf{X}_2^{(v_1,v_2)}$. By the closure under mixture of the *st* order, and given that $\mathbf{X}_2$ is the mixture of $\mathbf{X}_2^{(v_1,v_2)}$ with respect to $\mathbf{V}_2$ and that $\mathbf{Y}$ is the mixture of $\mathbf{X}_1^{(v_1,v_2)}$ with respect to $\mathbf{V}_2$, it follows that

$$\mathbf{Y} \leq_{st} \mathbf{X}_2. \tag{3.8}$$

It will now be proved that $\mathbf{X}_1 \leq_{st} \mathbf{Y}$. Let ϕ be an increasing function. Observe that

$$\begin{aligned} E[\phi(\mathbf{X}_k)] &= \int_{\mathbb{R}^2} \phi(\mathbf{t}) d\bar{F}_{\mathbf{X}_k}(\mathbf{t}) \\ &= \int_{\mathbb{R}^2} \phi(\mathbf{t}) d\left[\int_0^{+\infty} \int_0^{+\infty} C_S(\bar{G}_{k1}^{v_1}(t_1), \bar{G}_{k2}^{v_2}(t_2)) d\mathbf{H}_k(v_1, v_2)\right] \\ &= \int_0^{+\infty} \int_0^{+\infty} \left[\int_{\mathbb{R}^2} \phi(\mathbf{t}) dC_S(\bar{G}_{k1}^{v_1}(t_1), \bar{G}_{k2}^{v2}(t_2))\right] d\mathbf{H}_k(v_1, v_2) \\ &= \int_0^{+\infty} \int_0^{+\infty} \left[\int_{[0,1]^2} \phi(\bar{G}_{k1}^{-1}(u_1^{1/v_1}), \bar{G}_{k2}^{-1}(u_2^{1/v_2})) dC_S(u_1, u_2)\right] d\mathbf{H}_k(v_1, v_2) \\ &= -E[\psi_k(\mathbf{V}_k)] \end{aligned} \tag{3.9}$$

where

$$\psi_k(v_1, v_2) = -\int_{[0,1]^2} \phi(\bar{G}_{k1}^{-1}(u_1^{1/v_1}), \bar{G}_{k2}^{-1}(u_2^{1/v_2}))dC_S(u_1, u_2).$$

Therefore, $E[\phi(\mathbf{X}_1)] = -E[\psi_1(\mathbf{V}_1)]$ and, analogously, $E[\phi(\mathbf{Y})] = -E[\psi_1(\mathbf{V}_2)]$. Now, given that ϕ is increasing, then ψ_1 is also increasing. Thus, from hypothesis (a), it follows that $E[\phi(\mathbf{X}_1)] = -E[\psi_1(\mathbf{V}_1)] \leq -E[\psi_1(\mathbf{V}_2)] = E[\phi(\mathbf{Y})]$, i.e.,

$$\mathbf{X}_1 \leq_{st} \mathbf{Y}. \tag{3.10}$$

Therefore, the assertion follows from (3.8) and (3.10). ■

Remark 3.1 *Note that the proof of Theorem 3.7 is immediately obtained assuming that* $\tilde{X}_{1i} =_{st} \tilde{X}_{2i}, i = 1, 2,$ *since for any increasing function* ϕ,

$$E[\phi(\mathbf{X}_k)|\, \boldsymbol{V}] = \int_{[0,1]^2} \phi(\bar{G}_{k1}^{-1}(u_1^{1/v_1}), \bar{G}_{k2}^{-1}(u_2^{1/v_2}))dC_S(u_1, u_2)$$

is decreasing in (v_1, v_2). *That is,* $[\mathbf{X}|\mathbf{V} = (v_1, v_2)]$ *is stochastically decreasing in* (v_1, v_2). *Thus, from Theorem 3.1, the corresponding multivariate usual stochastic order between* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *holds.*

Remark 3.2 *Consider the survival functions* $\bar{F}_{\mathbf{X}_k^{(v_1,v_2)}}(x_1, x_2) = C_{S,k}(\bar{G}_{k1}^{v_1}(x_1), \bar{G}_{k2}^{v_2}(x_2))$ *for* $k = 1, 2$, *where* $C_{S,1}(u, v) \leq C_{S,2}(u, v)$ *for all* $u, v \in [0, 1]$. *That is, assume that the copulas* $C_{S,1}$ *and* $C_{S,2}$ *are ordered in concordance sense (see [20], for details on the concordance order). If a) and b) in Theorem 3.7 are verified, then it is easy to show that* $\mathbf{X}_1 \leq_{uo} \mathbf{X}_2$.

Remark 3.3 *Note that, when* C_S *is the independence copula, then Theorem 3.7 reduces, as a particular case, to Theorem 2.4 in [7].*

The following two statements describe suitable conditions for the orthant orders between two BMMFDs.

Theorem 3.8 *Let* $\mathbf{X}_k = (X_{k1}, X_{k2})$, $k = 1, 2$ *be two bivariate random vectors with survival functions given by a BMMFD. Assume that the survival copula* $C_S(u, v)$ *is convex in each component. If*

a) $\mathbf{V}_2 \leq_{iccv} \mathbf{V}_1$;
b) $\tilde{X}_{1i} \leq_{st} \tilde{X}_{2i}, i = 1, 2$;

then $\mathbf{X}_1 \leq_{uo} \mathbf{X}_2$ *and* $\mathbf{X}_2 \leq_{lo} \mathbf{X}_1$.

Proof. First, the statement for the $\leq_{uo}$ is shown. Observe that, since C_S is a copula and $\bar{G}_{2i}^{v_i}(x_i)$ is a decreasing function in v_i, it follows that $-C_S(\bar{G}_{21}^{v_1}(x_1), \bar{G}_{22}^{v_2}(x_2))$ is an increasing function in $\mathbf{v} = (v_1, v_2)$. Furthermore, observe that, for fixed $\mathbf{x} \in \mathbb{R}_+^2$, the equality $\bar{F}_{\mathbf{X}_2}(\mathbf{x}) = -E[\psi_2(\mathbf{V}_2)]$ holds, where $\psi_2(\mathbf{v}) = -C_S(\bar{G}_{21}^{v_1}(x_1), \bar{G}_{22}^{v_2}(x_2))$. Note that the function $\psi_2(\mathbf{v})$ is a componentwise concave function. In fact, for $\lambda \in (0, 1), v_1, w_1 \in \mathbb{R}$

and fixed v_2, it is obtained that

$$\begin{aligned}\psi_2(\lambda v_1 + (1-\lambda)w_1, v_2) &= -C_S(\bar{G}_{21}^{v_1+(1-\lambda)w_1}(x_1), \bar{G}_{22}^{v_2}(x_2)) \\ &\geq -C_S(\lambda \bar{G}^{v_1}(x_1) + (1-\lambda)\bar{G}^{w_1}(x_1), \bar{G}_{22}^{v_2}(x_2))) \quad (3.11)\\ &\geq -\lambda C_S(\bar{G}^{v_1}(x_1), \bar{G}_{22}^{v_2}(x_2)) - (1-\lambda)C_S(\bar{G}^{w_1}(x_1), \bar{G}_{22}^{v_2}(x_2)), \quad (3.12)\end{aligned}$$

where (3.11) follows from the convexity in v_1 of $\bar{G}^{v_1}(x_1)$, while (3.12) follows from the componentwise convexity of C_S.

Therefore, ψ_2 is concave in the first argument when the second argument is held fixed. Analogously, it can be shown that ψ_2 is concave in the second argument when the first one is held fixed.

Thus, ψ_2 is an increasing and componentwise concave function and, from assumption *a)*, it follows that $-E[\psi_2(\mathbf{V}_2)] \geq -E[\psi_2(\mathbf{V}_1)]$.

Finally, from *b)*, it follows that, for all $\mathbf{v}$,

$$\begin{aligned}\psi_2(\mathbf{v}) &= -C_S(\bar{G}_{21}^{v_1}(x_1), \bar{G}_{22}^{v_2}(x_2)) \\ &\leq -C_S(\bar{G}_{11}^{v_1}(x_1), \bar{G}_{12}^{v_2}(x_2)) \\ &= \psi_1(\mathbf{v}).\end{aligned}$$

Thus, $E[\psi_2(\mathbf{V}_1)] \leq E[\psi_1(\mathbf{V}_1)]$, and therefore $\bar{F}_{\mathbf{X}_2}(\mathbf{x}) = -E[\psi_2(\mathbf{V}_2)] \geq -E[\psi_2(\mathbf{V}_1)] \geq -E[\psi_1(\mathbf{V}_1)] = \bar{F}_{\mathbf{X}_1}(\mathbf{x})$ for all $\mathbf{x}$, and the result is obtained.

The proof for the $\leq_{lo}$ order is now given. That is, it should be shown that $F_2(x_1, x_2) \geq F_1(x_1, x_2)$ for all (x_1, x_2) in $\mathbb{R}_+^2$. Let (x_1, x_2) be a fixed point in $\mathbb{R}_+^2$. It is well-known that

$$F_i(x_1, x_2) = 1 - F_{i1}(x_1) - F_{i2}(x_2) + \bar{F}_i(x_1, x_2) \text{ for } i = 1, 2.$$

Therefore, it will be sufficient to show that $F_{11}(x_1) \geq F_{21}(x_1)$ and $F_{12}(x_2) \geq F_{22}(x_2)$. Thus,

$$\begin{aligned}-\bar{F}_{11}(x_1) &= \int_{\mathbb{R}^2} -\bar{F}_1(x_1, 0|v_1, v_2)dH_1(v_1, v_2) \\ &\geq \int_{\mathbb{R}^2} -\bar{F}_2(x_1, 0|v_1, v_2)dH_1(v_1, v_2),\end{aligned}$$

where the last inequality follows from hypothesis *b)*. By taking into account that $\bar{F}_2(x_1, 0|v_1, v_2) = \bar{G}_{21}^{v_1}(x_1)$ is a decreasing and convex function in v_1, it is obtained by applying a) that

$$\begin{aligned}\int_{\mathbb{R}^2} -\bar{F}_2(x_1, 0|v_1, v_2)dH_1(v_1, v_2) &\geq \int_{\mathbb{R}^2} -\bar{F}_2(x_1, 0|v_1, v_2)dH_2(v_1, v_2) \\ &= -\bar{F}_{12}(x_1).\end{aligned}$$

Consequently, $-\bar{F}_{11}(x_1) \geq -\bar{F}_{12}(x_1)$, i.e., $F_{11}(x_1) \geq F_{12}(x_1)$. The inequality $F_{21}(x_2) \geq F_{22}(x_2)$ can be proved similarly. Thus, the result is obtained. ■

Examples of survival copulas C_S satisfying the assumptions in Theorem 3.8 are those in the Farlie-Gumbel-Morgenstern family, defined as

$$C_S(u,v) = uv(1+\beta(1-u)(1-v)), \text{ with } -1 \leq \beta \leq 1$$

(see [20], for details on Farlie-Gumbel-Morgenstern copulas). In fact, it can be easily verified that these copulas are componentwise convex whenever the parameter $\beta < 0$. Note also that componentwise convexity is satisfied whenever the copula is convex. However, as described in [20], page 102, the only copula satisfying general convexity is the Fréchet-Hoeffding lower bound copula.

Recall that the increasing concave order is weaker than the increasing componentwise concave order (see Theorem 7.A.22 in [21]). It is interesting to observe that, for the case of independence copula, the previous result also holds if the order $\leq_{iccv}$ between the $\mathbf{V}_1$ and $\mathbf{V}_2$ is replaced by $\leq_{icv}$. This fact, which generalizes Theorem 2.5 in [7], is proved in the following statement.

THEOREM 3.9 *Let the bivariate vectors $\mathbf{X}_k$, with $k = 1, 2$, have survival functions defined as in (1.1). If*

a) $\mathbf{V}_2 \leq_{icv} \mathbf{V}_1$*;*
b) $\tilde{X}_{1i} \leq_{st} \tilde{X}_{2i}$ *for any* $i = 1, 2$*;*

then $\mathbf{X}_1 \leq_{uo} \mathbf{X}_2$ *and* $\mathbf{X}_2 \leq_{lo} \mathbf{X}_1$*.*

Proof. First, $\mathbf{X}_1 \leq_{uo} \mathbf{X}_2$ is proved. For any fixed $\mathbf{x} \in \mathbb{R}^2_+$, we can write $\bar{F}_{\mathbf{X}_2}(\mathbf{x}) = -E\big[\psi_2(\mathbf{V}_2)\big]$, where $\psi_2(\mathbf{v}) = -[\prod_{i=1}^2 \bar{F}_{2i}(x_i)^{v_i}]$. Given that $\mathbf{V}_2 \leq_{icv} \mathbf{V}_1$ and that ψ_2 is an increasing and concave function of $(v_1, v_2) \in \mathbb{R}^2_+$, it follows that

$$E[\psi_2(\mathbf{V}_2)] \leq E[\psi_2(\mathbf{V}_1)]. \tag{3.13}$$

Now, from $\tilde{X}_{1i} \leq_{st} \tilde{X}_{2i}$ it follows that $-[\prod_{i=1}^2 \bar{F}_{2i}(x_i)^{v_i}] \leq -[\prod_{i=1}^2 \bar{F}_{1i}(x_i)^{v_i}]$, and, therefore, that

$$E[\psi_2(\mathbf{V}_1)] \leq E[\psi_1(\mathbf{V}_1)], \tag{3.14}$$

where $\psi_1(\mathbf{v}) = -[\prod_{i=1}^2 \bar{F}_{1i}(x_i)^{v_i}]$. From (3.13) and (3.14), it follows that $E[\psi_2(\mathbf{V}_2)] \leq E[\psi_1(\mathbf{V}_1)]$, that is, $\bar{F}_{\mathbf{X}_1}(\mathbf{x}) \leq \bar{F}_{\mathbf{X}_2}(\mathbf{x})$, and therefore $\mathbf{X}_1 \leq_{uo} \mathbf{X}_2$.

Finally, $\mathbf{X}_2 \leq_{lo} \mathbf{X}_1$ is similarly shown as in Theorem 3.8. ■

The last result provides conditions to compare two BMMFDs, with marginal distributions defined by univariate frailties, in the supermodular order sense.

THEOREM 3.10 *Let $\mathbf{X}_k = (X_{k1}, X_{k2})$, $k = 1, 2$ be two bivariate random vectors with survival functions given by a BMMFD. If*

a) $\mathbf{V}_1 \leq_{sm} \mathbf{V}_2$*;*
b) $\tilde{X}_{2i} =_{st} \tilde{X}_{1i}, i = 1, 2$*;*

then $\mathbf{X}_1 \leq_{sm} \mathbf{X}_2$*.*

Proof. First, observe that if $\varphi(x, y, \mathbf{u})$ is a supermodular function in (x, y) for every fixed

$\mathbf{u} \in \mathbb{R}^n$, $\mu-$integrable in $\mathbf{u} \in \mathbb{R}^n$ (where μ is a measure in $\mathbb{R}^n$), then

$$\Phi(x, y) = \int \varphi(x, y, \mathbf{u}) d\mu(\mathbf{u}) \tag{3.15}$$

is also a supermodular function in (x, y). In fact, on considering $x_1 < x_2$ and $y_1 < y_2$, through assumptions on $\varphi(x, y, \mathbf{u})$, then

$$\varphi(x_1, y_1, \mathbf{u}) + \varphi(x_2, y_2, \mathbf{u}) \geq \varphi(x_1, y_2, \mathbf{u}) + \varphi(x_2, y_1, \mathbf{u}) \text{ for all } \mathbf{u},$$

and hence

$$\begin{aligned}
\Phi(x_1, y_1) + \Phi(x_2, y_2) &= \int [\varphi(x_1, y_1, \mathbf{u}) + \varphi(x_2, y_2, \mathbf{u})] d\mu\mathbf{u} \\
&\geq \int [\varphi(x_2, y_1, \mathbf{u}) + \varphi(x_1, y_2, \mathbf{u})] d\mu\mathbf{u} \\
&= \Phi(x_2, y_1) + \Phi(x_1, y_2).
\end{aligned}$$

Now, let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a supermodular function. If ψ is defined as

$$\psi(v_1, v_2) = \int_{[0,1]^2} \phi(\bar{G}_{k1}^{-1}(u_1^{1/v_1}), \bar{G}_{k2}^{-1}(u_2)^{1/v_2}) dCs(u_1, u_2), \tag{3.16}$$

then $E[\phi(\mathbf{X}_k)] = E[\psi(\mathbf{V}_k)]$ holds. Note that the right-hand side in (3.16) does not depend on k, since $\tilde{X}_{2i} =_{st} \tilde{X}_{1i}$. It is easy to verify that, for fixed $\mathbf{u}$, $h_i(v_i) = \bar{G}_{ki}^{-1}(u_i^{1/v_i})$ is decreasing in v_i. Thus, since ϕ is supermodular, for every fixed $\mathbf{u}$, the function $\tilde{\phi}(v_1, v_2, \mathbf{u}) = \phi(\bar{G}_{k1}^{1}(u_1^{1/v_1}), \bar{G}_{k2}^{1}(u_2^{1/v_2}))$ is supermodular in (v_1, v_2). From the supermodularity of (3.15), it follows that $\psi(v_1, v_2)$ is supermodular, and hence

$$E[\phi(\mathbf{X}_1)] = E[\psi(\mathbf{V}_1)] \leq E[\psi(\mathbf{V}_2)] = E[\phi(\mathbf{X}_2)].$$

Therefore, the assertion is obtained. ■

4. Examples

4.1. *Bivariate correlated gamma proportional hazard frailty model*

Yashin and Iachine [27] showed how methods and ideas from different research areas could be merged in a new approach based on a bivariate survival model of correlated frailty. Using Danish twin survival data, they showed how that model could be used for genetic analysis of important demographic characteristics such as human mortality and longevity. Analogous to the bivariate correlated gamma frailty model due to [27], and to its reversed version studied by Li and Da [13], we propose here an example of stochastic comparison for a similar model. Consider the frailty vector $\mathbf{V}_1 = (V_{11}, V_{12})$, such that $V_{11} = Y_{01} + Y_1$ and $V_{12} = \alpha(Y_{01} + Y_2)$, where the variables Y_{01}, Y_1 and Y_2 are independent, and such that $Y_{01} \sim_{st} \Gamma(k_{01}, \lambda_1)$, $Y_1 \sim_{st} \Gamma(k_1, \gamma_1)$ and $Y_2 \sim_{st} \Gamma(k_2, \gamma_2)$. Therefore, V_{11} and V_{12} are gamma distributed and correlated.

Assume that ρ_1 is the correlation coefficient between V_{11} and V_{12}. By straightforward computations

$$\rho_1 = \left[1 + \frac{k_1 \lambda_1^2}{k_{01}\gamma_1^2}\right]^{-1/2} \left[1 + \frac{k_2 \lambda_1^2}{k_{01}\gamma_2^2}\right]^{-1/2}.$$

Similarly, let $\mathbf{V}_2 = (V_{21}, V_{22})$ be a frailty vector, such that $V_{21} = Y_{02} + Y_1$ and $V_{22} = \alpha(Y_{02} + Y_2)$ where $Y_{02} \sim_{st} \Gamma(k_{02}, \lambda_2)$, and ρ_2 is the corresponding correlation coefficient.

Note that, $\frac{K_{01}}{\lambda_1^2} \leq \frac{K_{02}}{\lambda_2^2}$ if, and only if, $\rho_1 \leq \rho_2$.

Assume that, for $k = 1, 2$, the bivariate vector $\mathbf{X}_k$ has the survival function defined as in (2.1) and $\bar{F}_k(x_1, x_2|\nu_1, \nu_2) = \bar{F}(x_1, x_2|\nu_1, \nu_2)$ is defined as in (2.2), for $\phi_i(x_1, x_2) = -\ln \bar{G}_i(x_1, x_2)$, $i = 1, 2$, where $\bar{G}_i$ is a joint baseline survival function. Note that in this case,

$$\begin{aligned}\bar{F}(x_1, x_2|\nu_1, \nu_2) &= e^{\nu_1 \ln \bar{G}_1(x_1,x_2) + \nu_2 \ln \bar{G}_2(x_1,x_2)} \\ &= \bar{G}_1(x_1, x_2)^{\nu_1} \bar{G}_2(x_1, x_2)^{\nu_2}\end{aligned}$$

can be considered as a generalization of the multiplicative model given, for example, in [17]. The following statement holds.

THEOREM 4.1 *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be defined as described above.*

i) If $k_{01} \leq k_{02}$ and $\lambda_1 \geq \lambda_2$, then $\mathbf{X}_2 \leq_{uo} \mathbf{X}_1$.
ii) If $k_{01} \leq k_{02}$, $k_1 \leq k_2$ and $\lambda_1 \geq \lambda_2$, then $\mathbf{X}_2 \leq_{whr} \mathbf{X}_1$.

Proof. i) From Example 3.1 in [28], $\lambda_1 Y_{01} \leq_{disp} \lambda_2 Y_{02}$ is verified. Thus, by applying Theorem 3.B.11 and Theorem 3.B.13 in [21], $Y_{01} \leq_{st} Y_{02}$ holds. If $\mathbf{Z}_k = (Y_{0k}, Y_1, Y_2)$ for $k = 1, 2$, and by taking into account that Y_{0k}, Y_1, and Y_2 are independent, then $\mathbf{Z}_1$ and $\mathbf{Z}_2$, have the same copula. Consequently, from Theorem 6.B.14 in [21], $\mathbf{Z}_1 \leq_{st} \mathbf{Z}_2$. Therefore, by using Theorem 6.B.16 a) in [21], it follows that $\mathbf{V}_1 \leq_{st} \mathbf{V}_2$. Finally, by applying Theorem 3.6, it is obtained that $\mathbf{X}_2 \leq_{uo} \mathbf{X}_1$.

ii) If $k_{01} \leq k_{02}$, $k_1 \leq k_2$ and $\lambda_1 \geq \lambda_2$, then $\mathbf{V}_1 \leq_{lr} \mathbf{V}_2$ (similar to the proof of Proposition 5.1 (iii) in [13]). Therefore, from Theorem 3.5, $\mathbf{X}_2 \leq_{whr} \mathbf{X}_1$ holds.

■

Assume now that, for $k = 1, 2$, the bivariate vector $\mathbf{X}_k$ has the survival function as defined in (1.1), that is,

$$\bar{F}_{\mathbf{X}_k}(x_1, x_2) = \mathbf{E}\left[\prod_{i=1}^{2} \bar{G}_{ki}^{V_{ki}}(x_i)\right], \ x_i \in \mathbb{R}^+.$$

THEOREM 4.2 *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be defined as described above. If*

i) $k_{01} < k_{02}$, $\lambda_1 < \lambda_2$, and $\frac{\lambda_1}{k_{01}} = \frac{\lambda_2}{k_{02}}$;
ii) $\tilde{X}_{2i} \leq_{st} \tilde{X}_{1i}$ for any $i = 1, 2$;

then $\mathbf{X}_2 \leq_{uo} \mathbf{X}_1$.

Proof. By using *i)* and Proposition A.9 in [29], $Y_{01} \geq_{cx} Y_{02}$ holds. Now, by applying Theorem 4.A.34 in [21], $Y_{01} \leq_{icv} Y_{02}$ is verified. Consequently, by taking into account a statement similar to Theorem 7.A.6 in [21] with respect to the icv order, and from a) in Theorem 7.A.5 in [21], it is easily obtained that $\mathbf{V}_1 \leq_{icv} \mathbf{V}_2$. Thus, by Theorem 3.9, $\mathbf{X}_2 \leq_{uo} \mathbf{X}_1$. ∎

4.2. *Bivariate log-normal frailty model*

The log-normal distribution is, along with the gamma distribution, one of the most commonly used distributions to model variables with necessarily positive range. In particular, this distribution has been used to model the distribution of the univariate frailty model as well the correlated frailty model (see, among others, [3, 5, 30]; and references therein).

In this example, we consider two frailty vectors $\mathbf{V}_k = (V_{k1}, V_{k2}), k = 1, 2$, with a bivariate log-normal distribution. That is, for $k = 1, 2$, assume

$$\mathbf{V}_k \sim_{st} LnN(\mu, \Sigma_k),$$

where LnN denotes the bivariate log-normal distribution, $\mu = (m_1, m_2)$ and $\Sigma_k = \begin{pmatrix} s^2 & r_k s^2 \\ r_k s^2 & s^2 \end{pmatrix}$, where m_i, s^2 and r_k are the mean, variance and correlation of the respective normal distributions. The mean, variance and correlation of the frailties are related to these parameters as follows:

$$\mu_i = E[V_{ki}] = e^{m_i + \frac{s^2}{2}},$$

$$\sigma_{ki}^2 = Var(V_{ki}) = e^{2m_i + s^2}(e^{s^2} - 1),$$

$$\rho_k = Corr(V_{k1}, V_{k2}) = \frac{e^{r_k s^2} - 1}{e^{s^2} - 1}.$$

Assume that, for $k = 1, 2$, the bivariate vectors $\mathbf{X}_k$ have survival functions given by a BMMFD, with the same baseline survival functions $\bar{G}_{ki} = \bar{G}_i$, for $i = 1, 2$, but with different frailties $\mathbf{V}_k$, as above. The following result gives conditions for the comparison of $\mathbf{X}_1$ and $\mathbf{X}_2$ in the supermodular order in terms of the order between the correlation of the frailty vectors $\mathbf{V}_1$ and $\mathbf{V}_2$.

THEOREM 4.3 *Let the vectors* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be defined as described above. If* $r_1 \leq r_2$, *then* $\mathbf{X}_1 \leq_{sm} \mathbf{X}_2$.

Proof. It is known that two normally distributed bivariate vectors are ordered in the PQD order (see page 388 in [21]) if their covariance matrices are ordered (see Example 9.A.8, in [21]). Moreover, from Theorem 9.A.9 in [21], it follows that the PQD order is closed under increasing transformations of the components of the vectors. Therefore, if $r_1 \leq r_2$, then $\mathbf{V}_1 \leq_{PQD} \mathbf{V}_2$.

Furthermore, observe that the two vectors have the same marginal distributions. Thus, given that the PQD and *sm* order are equivalent in the bivariate case whenever the compared vectors have the same margins (see page 395 in [21]), it follows that $\mathbf{V}_1 \leq_{sm} \mathbf{V}_2$. From Theorem 3.10 the inequality $\mathbf{X}_1 \leq_{sm} \mathbf{X}_2$ is obtained. ■

4.3. *A bivariate additive Gamma frailty model*

Assume that, for $k = 1, 2$, the bivariate vectors $\mathbf{X}_k$ have survival functions given by a BMMFD, with the same baseline survival functions $\bar{G}_{ki} = \bar{G}_i$, for $i = 1, 2$. Let $\mathbf{V}_i = (V_{i1}, V_{i2})$ be frailty vectors such that $V_{i1} = Y_{0i} + Y_{1i}$ and $V_{i2} = Y_{0i} + Y_{2i}$, where the variables Y_{0i}, Y_1 and Y_2 are independent and such that $Y_{0i} \sim \Gamma(k_{0i}, \lambda)$, $Y_{1i} \sim \Gamma(k_{1i}, \lambda)$ and $Y_{2i} \sim \Gamma(k_{2i}, \lambda)$. As follows from Proposition 3.1. in [31], if $k_{02} \leq k_{01}$, and $k_{02} - k_{01} = k_{11} - k_{12} = k_{21} - k_{22}$, then $\mathbf{V}_2 \leq_{sm} \mathbf{V}_1$. Therefore, by applying Theorem 3.10, $\mathbf{X}_2 \leq_{sm} \mathbf{X}_1$.

5. Conclusions

The frailty approach is commonly used in reliability theory and survival analysis to model the dependence between lifetimes of individuals or components subjected to common risk factors; according to this model the frailty (an unobservable random vector that describes environmental conditions) acts simultaneously on the hazard functions of the lifetimes. Several interesting conditions have been described in this paper for stochastic comparisons between random vectors that are defined according to these models by assuming conditional dependence hypothesis for the frailty vector.

Two particular cases of generalized bivariate frailty models have been considered. The first one is the *bivariate exponential frailty model* and the second particular case is based on a representation of $\bar{F}(x_1, x_2|v_1, v_2)$ in terms of copula. The multivariate upper and lower orthant orders for the populations are obtained in the first model by comparing the corresponding frailty vectors in the usual multivariate stochastic order. For this model, we also investigate the weak hazard rate order for the populations in terms of the multivariate likelihood ratio order of the frailty vectors. However, for the second model, the populations are ordered in the usual multivariate stochastic order if the random frailty vectors are also ordered in the same order. For this model, the upper orthant order and the supermodular order of the populations are obtained when the random frailty vectors are ordered in the increasing componentwise concave order and the supermodular order, respectively. The examples used herein are common in the literature on multivariate frailty.

Acknowledgements

The authors wish to thank the Editor and the two anonymous referees whose comments helped to improve a previous version of this paper. This work was partly supported by a grant from Junta de Andalucía (Spain) for research group (FQM-328).

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