BV CONTINUOUS SWEEPING PROCESSES

VINCENZO RECUPERO

Abstract. We consider a large class of continuous sweeping processes and we prove that they are well posed with respect to the BV strict metric.

1. Introduction

Let $\mathcal{X}$ be real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{C}(t) \subseteq \mathcal{X}$ be a family of nonempty closed convex sets parametrized by the time variable $t \in [0, T]$, where $T > 0$. A *sweeping process* is the following evolution differential inclusion in the unknown $\xi : [0, T] \longrightarrow \mathcal{X}$:

$$- \xi'(t) \in N_{\mathcal{C}(t)}(\xi(t)), \qquad \text{for a.e. } t \in [0, T], \tag{1.1}$$

$$\xi(0) = \xi^0, \tag{1.2}$$

where $\xi^0 \in \mathcal{C}(0)$ is a prescribed initial datum and

$$N_{\mathcal{K}}(x_0) := \{\nu \in \mathcal{X} \ : \ \langle \nu, x_0 - w \rangle \geq 0 \ \forall w \in \mathcal{K}\} \tag{1.3}$$

is the exterior normal cone to a closed convex set $\mathcal{K} \subseteq \mathcal{X}$ at the point $x_0 \in \mathcal{K}$. Notice that it is implicitly assumed that

$$\xi(t) \in \mathcal{C}(t) \qquad \forall t \in [0, T]. \tag{1.4}$$

Sweeping processes were introduced by J.J. Moreau in the fundamental paper [21] and originated a research which is still active: see, e.g., the monograph [20], the expository papers [17, 29], and the references therein.

In the present paper we continue the analysis of [27], where we studied some continuity properties of the solution operator $\mathcal{C} \longmapsto \xi$ of the sweeping processes by setting it in the wider framework of *rate independent operators*, indeed problem (1.1)–(1.2) has the following property, called *rate independence*: if $\phi : [0, T] \longrightarrow [0, T]$ is an increasing surjective reparametrization of time and ξ is the solution associated to $\mathcal{C}(t)$, then $\xi(\phi(t))$ is the solution corresponding to $\mathcal{C}(\phi(t))$. Rate independent evolution problems are strictly connected to elasto-plasticity and hysteresis and have been deeply studied from the mathematical point of view in the monographs[12, 30, 6, 13, 19]. The study of continuity properties with respect to various topologies has been recently performed also in, e.g., [17, 5, 16, 31] and these properties are important since they ensure robustness of the model.

Here we address the sweeping process in the following formulation provided in [5]: a Banach space $\mathcal{Y}$, two functions $u : [0, T] \longrightarrow \mathcal{X}$, $r : [0, T] \longrightarrow \mathcal{Y}$, and a family of closed convex sets $\mathcal{Z}(r) \subseteq \mathcal{X}$ parametrized by $r \in \mathcal{Y}$ are given, and we have to find a function $\xi : [0, T] \longrightarrow \mathcal{X}$ such that

$$\langle u(t) - \xi(t) - z, \xi'(t) \rangle \geq 0, \qquad \text{for a.e. } t \in [0, T], \quad \forall z \in \mathcal{Z}(r(t)), \tag{1.5}$$

$$u(0) - \xi(0) = x^0. \tag{1.6}$$

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Again it is implicitly assumed that $u(t) - \xi(t) \in \mathcal{Z}(r(t))$ for all $t \in [0,T]$ (all the precise definitions, assumptions and formulations will be given in the next Sections 2 and 3).

Note that (1.5)–(1.6) is actually a reformulation of (1.1)–(1.2), indeed, as observed in [5], one can reduce (1.5)–(1.6) to (1.1)–(1.2) by setting $u(t) = 0$, $r(t) = t$, $x^0 = -\xi^0$, $\mathcal{C} = -\mathcal{Z}$; vice versa with the position $\mathcal{C}(t) = u(t) - \mathcal{Z}(r(t))$, $\xi^0 = u(0) - x^0$ one can reduce the first problem to the second. However formulation (1.5)–(1.6) introduces the new parameters $u(t), r(t)$ that are relevant in applications, so that it is useful to study the properties of the sweeping process with respect to u and r. This analysis is performed in [5] where it is shown that the solution operator $\mathsf{S} : (u,r) \longrightarrow \xi$ of (1.5)–(1.6) is continuous with respect to the $W^{1,1}$-topology (or the strong BV topology, see (2.9)), i.e. if $u_n \to u$ in $W^{1,1}(0,T;\mathcal{X})$ and $r_n \to r$ in $W^{1,1}(0,T;\mathcal{Y})$, then $\mathsf{S}(u_n, r_n) \to \mathsf{S}(u,r)$ in $W^{1,1}(0,T;\mathcal{X})$. This property is essentially proved under some geometrical assumptions on $\mathcal{Z}(r)$ (cf. Assumption 3.1) which however turn out to be not so restrictive for applications.

In [21, 15, 16] the BV-generalization of (1.5)–(1.6) is considered: $\mathcal{Z}(r)$ is given as above, but u and r are with bounded variation, and one has to find a continuous function $\xi : [0,T] \longrightarrow \mathcal{X}$ of bounded variation such that (1.6) holds together with the condition

$$\begin{aligned} &\int_0^T \langle u(t) - \xi(t) - z(t), \mathrm{d}\mathrm{D}\xi(t) \rangle \geq 0, \\ &\qquad \forall z \in BV([0,T];\mathcal{X}), \quad z(t) \in \mathcal{Z}(r(t)) \quad \forall t \in [0,T], \end{aligned} \tag{1.7}$$

where the integral is meant in the sense of the Stieltjes or differential measures (see [21, 16]). In [16] it is proved that also in this case the corresponding solution operator $\overline{\mathsf{S}} : (u,r) \longmapsto \xi$ is continuous with respect to the BV-norm.

Here instead we prove that the well posedness of (1.7)–(1.6) (and (1.5)–(1.6)) with respect to the BV *strict metric* (cf. (2.10)) when u and r are continuous in time (for non-continuous data the BV-strict discontinuity is proved in [26] when $\mathcal{Z}(r) = \mathcal{Z}$ for every r, $\mathcal{Z}$ belonging to wide class of constant convex sets). The strict metric is very natural, especially when one deals with approximation procedures (see [1]): indeed given a function of bounded variation v, by means of the classical convolution operation one can find a sequence of regular functions v_n converging strictly to v. The geometric meaning is clear, two curves u and v are near with respect to the strict metric if they are near in the L^1-norm and if their lengths are near.

In connection with rate independent problems the strict metric has been studied for instance in [7, 30, 13, 22, 24, 25]. In particular, concerning the specific sweeping process when the data are continuous and $\mathcal{Z}(r(t)) = \mathcal{Z}$, a fixed closed convex subset of $\mathcal{X}$, in [13] it is proved its continuity with respect to the strict metric provided the boundary $\mathcal{Z}$ satisfies certain smoothness assumptions. This requirement was completely removed in [26]. Since in the present paper we address the more general case (1.7)–(1.6), where the product $\mathcal{X} \times \mathcal{Y}$ of a Hilbert and a Banach space is involved, the Hilbert technique used in [26] does not apply due to some uniform convexity issues (see Remark 4.2).

A byproduct of our result is that only the analysis of the sweeping process for Lipschitz data is needed: then the analogous results for the continuous BV case are a straightworward consequence of standard measure theory arguments.

We conclude this introduction with a brief plan of the paper. In the next section we recall all the necessary rigorous and precise preliminaries. In Section 3 we state the main theorems of the paper and in Section 4 we prove them. Finally in the Appendix we prove some technical properties about the strict convergence of sequences of Banach valued functions of bounded variation.

2. Preliminaries

If $\mathcal{B}$ is a real Banach space with norm $\|\cdot\|_{\mathcal{B}}$, then $\mathcal{B}^*$ will denote its topological dual space and ${}_{\mathcal{B}^*}\langle\cdot,\cdot\rangle_{\mathcal{B}}$ the duality between $\mathcal{B}$ and $\mathcal{B}^*$. We use the notation $B_\rho(v_0) := \{v \in \mathcal{B} \ : \ \|v-v_0\|_{\mathcal{B}} < \rho\}$ for open balls with center $v_0 \in \mathcal{B}$ and radius $\rho > 0$. The topological interior of a set $\mathcal{S}$ is indicated by $\mathrm{int}(\mathcal{S})$. If $v, v_n \in \mathcal{B}$ for every $n \in \mathbb{N}$ and v_n converges weakly to v as $n \to \infty$, we will write $v_n \rightharpoonup v$ *in* $\mathcal{B}$ *as* $n \to \infty$. We also set

$$\mathscr{C}_{\mathcal{B}} := \{\mathcal{K} \subseteq \mathcal{B} \ : \ \mathcal{K} \text{ nonempty, closed and convex}\}. \tag{2.1}$$

If

$$\mathcal{K} \in \mathscr{C}_{\mathcal{B}} \text{ is bounded and } 0 \in \mathrm{int}(\mathcal{K}), \tag{2.2}$$

we recall that the *Minkowski functional associated with* $\mathcal{K}$ is the function $M_{\mathcal{K}} : \mathcal{B} \longrightarrow [0,\infty[$ defined by

$$M_{\mathcal{K}}(v) := \inf\left\{\lambda > 0 \ : \ \frac{v}{\lambda} \in \mathcal{K}\right\}, \qquad v \in \mathcal{B}. \tag{2.3}$$

Here are some properties of the Minkowski functional that will be implicitly used in the sequel (cf., e.g., [28, Theorems 1.34–1.36] and recall that (2.2) holds):

$$M(x+y) \le M(x) + M(y), \quad M(\lambda x) = \lambda M(x) \qquad \forall x, y \in \mathcal{B},\ \forall \lambda \ge 0, \tag{2.4}$$

$$M \text{ is continuous}, \tag{2.5}$$

$$\mathcal{K} = \{x \in \mathcal{K} \ : \ M(x) \le 1\}, \tag{2.6}$$

$$M(x) = 0 \iff x = 0. \tag{2.7}$$

In the sequel $T > 0$ will be a fixed positive number denoting the final time of the sweeping process. If $\mathcal{L}^1$ is the one-dimensional Lebesgue measure and if $p \in [1,\infty]$, then the space of $\mathcal{B}$-valued Lebesgue functions which are integrable on $[0,T]$ with respect to $\mathcal{L}^1$ will be denoted by $L^p(0,T;\mathcal{B})$ (see [18, Chapter III]).

For a function $v : [0,T] \longrightarrow \mathcal{B}$ we set $\|v\|_\infty := \sup_{t\in[0,T]} \|v(t)\|_{\mathcal{B}}$. Moreover if $J \subseteq [0,T]$ is an interval, the *variation of* v *on* J is the real extended number $\mathrm{V}(v,J)$ defined by

$$\mathrm{V}(v,J) := \sup\left\{\sum_{j=1}^{m} \|v(t_j) - v(t_{j-1})\|_{\mathcal{B}} \ : \ m \in \mathbb{N},\ t_j \in J,\ t_1 < \ldots < t_m\right\}, \tag{2.8}$$

and we say that v *is of bounded variation on* J if $\mathrm{V}(v,J) < \infty$. We set

$$BV([0,T];\mathcal{B}) := \big\{v : [0,T] \longrightarrow \mathcal{B} \ : \ \mathrm{V}(v,[0,T]) < \infty\big\}.$$

Let us recall two natural topologies in BV: the *strong topology* induced by the semimetric

$$d_{BV}(u,v) := \|u-v\|_{L^1(0,T;\mathcal{B})} + |\mathrm{V}(u-v,[0,T])|, \qquad u, v \in BV([0,T];\mathcal{B}), \tag{2.9}$$

and the *strict topology*, induced by the strict semimetric

$$d_s(u,v) := \|u-v\|_{L^1(0,T;\mathcal{B})} + |\mathrm{V}(u,[0,T]) - \mathrm{V}(v,[0,T])|, \qquad u, v \in BV([0,T];\mathcal{B}). \tag{2.10}$$

When we restrict to continuous functions, then d_{BV} and d_s are actually metrics. If $v, v_n \in BV([0,T];\mathcal{B})$, we say that $v_n \to v$ *strictly on* $[0,T]$ if $d_s(v_n, v) \to 0$ as $n \to \infty$. Geometrically this means that $v_n \to v$ in L^1 and the lengths of the curves v_n converge to the length of v.

If $p \in [1,\infty]$ we denote by $W^{1,p}(0,T;\mathcal{B})$ the Sobolev spaces of $\mathcal{B}$-valued function: we recall that $v \in W^{1,p}(0,T;\mathcal{B})$ if and only if there exists $w \in L^p(0,T;\mathcal{B})$ such that $v(t) = v(0) + \int_0^t w(s)\,\mathrm{d}s$ for every $t \in [0,T]$, in other words w is the distributional derivative of v. If $v \in W^{1,p}(0,T;\mathcal{B})$ then we have that v is differentiable $\mathcal{L}^1$-a.e. and any representative of v' is the distributional derivative of v, moreover $v \in BV([0,T];\mathcal{B})$ and $\mathrm{V}(v,[0,T]) = \int_0^T \|v'(t)\|_{\mathcal{B}}\,\mathrm{d}t$. If $1 \le p \le q \le \infty$ we obviously have that $W^{1,q}([0,T];\mathcal{B}) \subseteq W^{1,p}([0,T];\mathcal{B}) \subseteq C([0,T];\mathcal{B})$, the space of $\mathcal{B}$-valued continuous functions. For any $v : [0,T] \longrightarrow \mathcal{B}$ we set $\mathrm{Lip}(v) := \sup_{t\neq s} \|v(t)-$

$v(s)\|_{\mathcal{B}}/|t-s|$ and $Lip([0,T];\mathcal{B}) := \{v : [0,T] \longrightarrow \mathcal{B} \ : \ \mathrm{Lip}(v) < \infty\}$. Clearly $W^{1,\infty}(0,T;\mathcal{B}) \subseteq Lip([0,T];\mathcal{B})$. If $\mathcal{B}$ is reflexive then $W^{1,\infty}(0,T;\mathcal{B}) = Lip([0,T];\mathcal{B})$ (we refer to [3, Appendix] for vector valued Sobolev spaces).

3. Main results

In the sequel of the paper we will assume that

$$\mathcal{X} \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle \text{ and norm } \|\cdot\|_{\mathcal{X}} = \langle\cdot,\cdot\rangle^{1/2}, \tag{3.1}$$

$$\mathcal{Y} \text{ is a reflexive real Banach space with norm } \|\cdot\|_{\mathcal{Y}}, \tag{3.2}$$

$$\mathcal{R} \in \mathscr{C}_{\mathcal{Y}} \text{ and } \mathrm{int}(\mathcal{R}) \neq \varnothing. \tag{3.3}$$

There will be given a multivalued map

$$\mathcal{Z} : \mathcal{R} \longrightarrow \mathscr{C}_{\mathcal{X}} \tag{3.4}$$

and the functional $M : \mathcal{X} \times \mathcal{Y} \longrightarrow [0,\infty]$ defined by

$$M(x,r) := M_{\mathcal{Z}(r)}(x), \qquad (x,r) \in \mathcal{X} \times \mathcal{R}. \tag{3.5}$$

the Minkowski functional of $\mathcal{Z}(r)$.

Now we can state the problem defining the sweeping process in the absolutely continuous framework.

Problem 3.1. Assume that $\mathcal{Z} : \mathcal{R} \longrightarrow \mathscr{C}_{\mathcal{X}}$, $u \in W^{1,1}(0,T;\mathcal{X})$, $r \in W^{1,1}(0,T;\mathcal{Y})$, and $x^0 \in \mathcal{Z}(r(0))$ are given such that $r([0,T]) \subseteq \mathcal{R}$. Find $\xi \in W^{1,1}(0,T;\mathcal{X})$ such that

$$u(t) - \xi(t) \in \mathcal{Z}(r(t)) \qquad \forall t \in [0,T], \tag{3.6}$$

$$u(0) - \xi(0) = x^0, \tag{3.7}$$

$$\langle u(t) - \xi(t) - z, \xi'(t)\rangle \geq 0, \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T], \quad \forall z \in \mathcal{Z}(r(t)). \tag{3.8}$$

We need the following set of assumptions (cf. [5]).

Assumption 3.1. *There exists $C > 0$ such that*

$$0 \in \mathcal{Z}(r) \subseteq B_C(0) \qquad \forall r \in \mathcal{R}. \tag{3.9}$$

There exist the partial (Fréchet) derivatives $\partial_x M(x,r) \in \mathcal{X}$, $\partial_r M(x,r) \in \mathcal{Y}^*$ for every $(x,r) \in \mathcal{X} \times \mathcal{R}$, and there are positive constants K_0, C_J, C_K such that the maps $J : (\mathcal{X} \smallsetminus \{0\}) \times \mathrm{int}(\mathcal{R}) \longrightarrow \mathcal{X}$, $K : (\mathcal{X} \smallsetminus \{0\}) \times \mathrm{int}(\mathcal{R}) \longrightarrow \mathcal{Y}^*$ defined by

$$J(x,r) := M(x,r)\partial_x M(x,r), \qquad (x,r) \in (\mathcal{X} \smallsetminus \{0\}) \times \mathrm{int}(\mathcal{R}), \tag{3.10}$$

$$K(x,r) := M(x,r)\partial_r M(x,r), \qquad (x,r) \in (\mathcal{X} \smallsetminus \{0\}) \times \mathrm{int}(\mathcal{R}), \tag{3.11}$$

can be continuously extended to $(0,r) \in \mathcal{X} \times \mathcal{R}$ for any $r \in \mathcal{R}$, and

$$\|J(x_1,r_1) - J(x_2,r_2)\|_{\mathcal{X}} \leq C_J(\|x_1 - x_2\|_{\mathcal{X}} + \|r_1 - r_2\|_{\mathcal{Y}}), \tag{3.12}$$

$$\|K(x_1,r_1) - K(x_2,r_2)\|_{\mathcal{Y}^*} \leq C_K(\|x_1 - x_2\|_{\mathcal{X}} + \|r_1 - r_2\|_{\mathcal{Y}}), \tag{3.13}$$

$$\|K(x,r)\|_{\mathcal{Y}^*} \leq K_0. \tag{3.14}$$

for every $x_1, x_2 \in B_C(0)$ and $r_1, r_2 \in \mathcal{R}$.

Remark 3.1. The map J can be seen as the partial derivative with respect to x of the function $(x,r) \longmapsto (M(x,r))^2/2$, i.e. J associates to every (x,r) the vector $\partial_x M(x,r)$ multiplied by the scalar $M(x,r)$. A similar remark holds for K.

Let us recall two consequences of Assumption 3.1. In [16, Lemma 2.3] it is proved that there exists $c \in]0, C[$ such that

$$B_c(0) \subseteq \mathcal{Z}(r) \qquad \forall r \in \mathcal{R}. \tag{3.15}$$

Moreover if $r \in \mathcal{R}$ then (cf. [5, Lemma 3.1])

$$J(x,r) \neq 0, \qquad N_{\mathcal{Z}(r)}(x) = \left\{\lambda \frac{J(x,r)}{\|J(x,r)\|_{\mathcal{X}}} \ : \ \lambda \geq 0\right\} \qquad \forall r \in \mathcal{R},\ \forall x \in \partial\mathcal{Z}(r) \tag{3.16}$$

where $N_{\mathcal{Z}(r)}(x) := \{\nu \in \mathcal{X} \ : \ \langle \nu, x_0 - w\rangle \geq 0 \ \forall w \in \mathcal{K}\}$ is the normal cone of convex analysis. In other words the normal cone to $\mathcal{Z}(r)$ at x is a half-line whose direction is $J(x,r)/\|J(x,r)\|_{\mathcal{X}}$.

Observe that condition (3.9) assumed here and in [5] is not very restrictive for applications, indeed the function $u(t)$ allows a translation of the moving convex set $\mathcal{C}(t)$ of (1.2), whereas (3.9) and (3.15) require that $\mathcal{C}(t)$ remains uniformly bounded and does not shrink to a point.

In [5, Proposition 4.1, Theorem 7.1] the following theorem is proved.

Theorem 3.1. *Let us assume that Assumption 3.1 holds. Then Problem 3.1 admits a unique solution. Let*

$$D := \left\{(u, r, x^0) \in W^{1,1}(0,T;\mathcal{X}) \times W^{1,1}(0,T;\mathcal{Y}) \times \mathcal{X} \ : \ r([0,T]) \subseteq \mathcal{R},\ x^0 \in \mathcal{Z}(r(0))\right\} \tag{3.17}$$

and let $\mathsf{S} : D \longrightarrow W^{1,1}(0,T;\mathcal{X})$ *be the operator assigning to each* $(r, u, x^0) \in D$ *the unique* $\xi \in W^{1,1}(0,T;\mathcal{X})$ *satisfying* (3.6)–(3.8). *Then* S *is continuous with respect to the* $W^{1,1}$*-topology, in the following sense: if* $(u, r, x^0), (u_n, r_n, x_n^0) \in D$ *for every* $n \in \mathbb{N}$ *and*

$$u_n \to u \ \text{in}\ W^{1,1}(0,T;\mathcal{X}), \tag{3.18}$$

$$r_n \to r \ \text{in}\ W^{1,1}(0,T;\mathcal{Y}), \tag{3.19}$$

$$x_n^0 \to x^0 \ \text{in}\ \mathcal{X} \tag{3.20}$$

as $n \to \infty$, *then* $\mathsf{S}(u_n, r_n, x_n^0) \to \mathsf{S}(u, r, x^0)$ *in* $W^{1,1}(0,T;\mathcal{X})$.

A key tool in our arguments will rely on the following proposition whose proof is straightforward. Its content is described by saying that Problem 3.1 (or the operator S) is *rate independent.*

Proposition 3.1. *Let* $\mathsf{S} : D \longrightarrow W^{1,1}(0,T;\mathcal{X})$ *be the operator defined by Theorem 3.1. If* $\phi : [0,T] \longrightarrow [0,T]$ *is absolutely continuous and increasing, then*

$$\mathsf{S}(u \circ \phi, r \circ \phi, x^0) = \mathsf{S}(u, r, x^0) \circ \phi \tag{3.21}$$

for every $(u, r, x^0) \in D$.

Remark 3.2. In the previous proposition the function ϕ may have some constancy intervals.

In [16] it is considered the following BV version of the sweeping processes (analogous to the BV-version in [21]):

Problem 3.2. Assume that $\mathcal{Z} : \mathcal{Y} \longrightarrow \mathscr{C}_{\mathcal{X}}$, $u \in BV([0,T];\mathcal{Y}) \cap C([0,T];\mathcal{Y})$, $r \in BV([0,T];\mathcal{Y}) \cap C([0,T];\mathcal{Y})$, and $x^0 \in \mathcal{Z}(r(0))$ are given such that $r([0,T]) \subseteq \mathcal{R}$. Find $\xi \in BV([0,T];\mathcal{X}) \cap C([0,T];\mathcal{X})$ such that

$$u(t) - \xi(t) \in \mathcal{Z}(r(t)) \qquad \forall t \in [0,T], \tag{3.22}$$

$$u(0) - \xi(0) = x^0, \tag{3.23}$$

$$\int_0^T \langle u(t) - \xi(t) - z(t), \mathrm{d}\mathrm{D}\xi(t)\rangle \geq 0,$$

$$\forall z \in BV([0,T];\mathcal{X}), \quad z(t) \in \mathcal{Z}(r(t)) \quad \forall t \in [0,T], \tag{3.24}$$

where the integral in (3.24) is meant in the Riemann-Stieltjes sense (cf., e.g., [18, Chapter 10]) or equivalently in the ordinary Lebesgue sense with respect to the Stieltjes vector measure $\mathrm{D}\xi$, the function ξ being continuous (see [8, Section III.17] or [26, Section 2]).

Remark 3.3. In the reference [16], the integral of (3.24) is considered in the sense of Kurzweil or Young (cf. [14, 15]). However in [26] it is proved that when ξ is left continuous and with bounded variation, then these integrals coincide with the ordinary Lebesgue integral with respect to the differential measure $\mathrm{D}\xi$. Moreover in [16] the test functions of (3.24) are allowed to belong to $Reg([0,T];\mathcal{Y})$, the space of *regulated functions on* $[0,T]$, i.e. those functions $v : [0,T] \longrightarrow \mathcal{Y}$ such that there exist the left and right limits $u(t-), u(t+)$ in $\mathcal{Y}$ at any point $t \in [0,T]$, with the convention that $u(0-) = u(0)$ and $u(T+) = u(T)$. Actually this more restrictive condition is implied by (3.24), indeed it is enough to approximate any $z \in Reg([0,T];\mathcal{X})$ with a uniformly convergent sequence $z_n \in BV([0,T];\mathcal{X})$ (cf. [2, Section II.1.3]) and pass to the limit in (3.2) where z is replaced by z_n (see also [14, Theorem 3.9]).

In [15] it is shown that Problem 3.2 admits a unique solution by means of an *approximation-a priori estimates-limit procedure.* In Theorem 4.1 below we will give a different short proof of this result making use of basic measure theory tools. This proof will provide a sort of representation formula for the solution that will allow to prove our main result, i.e. that Problem 3.2 is well-posed with respect to he strict metric. Here is the precise formulation.

Theorem 3.2. *Let us assume that Assumption 3.1 holds. Let*

$$\overline{D} := \Big\{(r,u,x^0) \in \big[BV([0,T];\mathcal{X}) \cap C([0,T];\mathcal{X})\big] \times \big[BV([0,T];\mathcal{Y}) \cap C([0,T];\mathcal{Y})\big] \times \mathcal{X} \ : \\ r([0,T]) \subseteq \mathcal{R},\ x^0 \in \mathcal{Z}(r(0))\Big\}. \tag{3.25}$$

For every $(r,u,x^0) \in \overline{D}$ *there exists a unique* $\xi =: \overline{\mathsf{S}}(r,u,x^0) \in BV([0,T];\mathcal{X}) \cap C([0,T];\mathcal{X})$ *satisfiying* (3.22)–(3.24). *The resulting solution operator* $\overline{\mathsf{S}} : \overline{D} \longrightarrow BV([0,T];\mathcal{X}) \cap C([0,T];\mathcal{X})$ *is continuous with respect to the strict metric, in the following sense: if* $(u,r,x^0), (u_n,r_n,x_n^0) \in \overline{D}$ *for every* $n \in \mathbb{N}$, *and*

$$u_n \to u \textit{ strictly on } [0,T], \tag{3.26}$$

$$r_n \to r \textit{ strictly on } [0,T], \tag{3.27}$$

$$x_n^0 \to x^0 \textit{ in } \mathcal{X} \tag{3.28}$$

as $n \to \infty$, *then* $\overline{\mathsf{S}}(u_n,r_n,x_n^0) \to \overline{\mathsf{S}}(u,r,x^0)$ *strictly on* $[0,T]$.

4. Proofs

In general, for a real Banach space $\mathcal{B}$ and a function $v \in BV(0,T;\mathcal{B}) \cap C([0,T];\mathcal{B})$, we can define the following increasing (continuous) surjective arc length function $\ell_v : [0,T] \longrightarrow [0,T]$ by setting

$$\ell_v(t) := \begin{cases} \dfrac{T}{\mathrm{V}(v,[0,T])}\mathrm{V}(v,[0,t]) & \text{if } \mathrm{V}(v,[0,T]) \neq 0 \\ 0 & \text{if } \mathrm{V}(v,[0,T]) = 0 \end{cases} \tag{4.1}$$

(the only difference with the usual arc length function is given by a multiplicative factor allowing the range of ℓ_v to be $[0,T]$). Arguing as in [11, Section 2.5.16, p. 109] we infer that there exists a unique $\widetilde{v} \in Lip([0,T];\mathcal{B})$ such that

$$v(t) = \widetilde{v}(\ell_v(t)) \qquad \forall t \in [0,T], \tag{4.2}$$

$$\|\widetilde{v}'\|_{L^\infty(0,T;\mathcal{B})} \leq \frac{\mathrm{V}(v,[0,T])}{T}. \tag{4.3}$$

The function $\widetilde{v}$ is the *reparametrization of* v *by the arc length* ℓ_v. Clearly we have

$$\mathrm{V}(\widetilde{v},[0,T]) = \mathrm{V}(v,[0,T]). \tag{4.4}$$

In the sequel we will set

$$\mathcal{B} := \mathcal{X} \times \mathcal{Y} \tag{4.5}$$

endowed with the norm

$$\|(x,y)\|_{\mathcal{B}} := \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}, \qquad (x,y) \in \mathcal{B}. \tag{4.6}$$

Note that with this norm the space $\mathcal{B}$ is not uniformly convex because $\mathbb{R}^2$ is not uniformly convex with the 1-norm. This fact prevents from applying the Hilbert techniques used in [26] (cf. Remark 4.2 below). Nevertheless $\mathcal{B}$ is reflexive, due to the reflexivity of $\mathcal{X}$ and $\mathcal{Y}$ and to Kakutani's theorem (cf., e.g., [4, Theorem 3.17]). In this case if

$$v = (v_x, v_y) : [0,T] \longrightarrow \mathcal{B}, \tag{4.7}$$

from (2.8), (4.5) and (4.6) we immediately infer that

$$\mathrm{V}(v,[0,T]) = \mathrm{V}(v_x,[0,T]) + \mathrm{V}(v_y,[0,T]). \tag{4.8}$$

Therefore if $v_x \in BV([0,T];\mathcal{X}) \cap C([0,T];\mathcal{X})$ and $v_y \in BV([0,T];\mathcal{Y}) \cap C([0,T];\mathcal{Y})$ then $v = (v_x, v_y) \in BV([0,T];\mathcal{B}) \cap C([0,T];\mathcal{B})$ and there exist $\overline{v}_x \in Lip([0,T];\mathcal{X})$, $\overline{v}_y \in Lip([0,T];\mathcal{Y})$ such that

$$\widetilde{v} = (\overline{v}_x, \overline{v}_y) : [0,T] \longrightarrow \mathcal{B} \tag{4.9}$$

and

$$(v_x(t), v_y(t)) = v(t) = \widetilde{v}(\ell_v(t)) = (\overline{v}_x(\ell_v(t)), \overline{v}_y(\ell_v(t))) \qquad \forall t \in [0,T]. \tag{4.10}$$

By Proposition 3.1 we immediately have that

$$\mathsf{S}(u,r,x^0) = \mathsf{S}(\overline{u},\overline{r},x^0) \circ \ell_v \qquad \forall (u,r,x^0) \in D. \tag{4.11}$$

We start by showing that such formula also holds for BV-solutions. The following theorem also provides an alternative proof for the existence of Problem 3.2.

Theorem 4.1. *If* $(u,v,x^0) \in \overline{D}$ *then*

$$\overline{\mathsf{S}}(u,r,x^0) = \mathsf{S}(\overline{u},\overline{r},x^0) \circ \ell_v \tag{4.12}$$

is the unique solution of Problem 3.2.

Proof. The uniqueness of a solution for Problem 3.2 is standard and we refer to [15]. Now we prove formula (4.12). We set $v := (u,r) \in BV([0,T];\mathcal{B}) \cap C([0,T];\mathcal{B})$ and we prove that

$$\xi := \mathsf{S}(\overline{u},\overline{r},x^0) \circ \ell_v$$

solves Problem 3.2. Formulas (3.22), (3.23) are obvious. In order to check (3.24) let $z \in Reg([0,T];\mathcal{Y})$ be such that $z(t) \in \mathcal{Z}(r(t))$ for every $t \in [0,T]$. Then by a change of variable in the Stieltjes integral (cf. [23, Lemma 5.1]) we have

$$\begin{aligned} &\int_0^T \langle u(t) - \xi(t) - z(t), \mathrm{d}\mathrm{D}\xi(t)\rangle \qquad (4.13)\\ &= \int_0^T \Big\langle \overline{u}(\ell_v(t)) - (\mathsf{S}(\overline{u},\overline{r},x^0)\circ\ell_v)(t) - z(t), \mathrm{d}(\mathsf{S}(\overline{u},\overline{r},x^0)\circ\ell_v)(t)\Big\rangle \\ &= \int_0^T \Big\langle \overline{u}(\ell_v(t)) - \mathsf{S}(\overline{u},\overline{r},x^0)((\ell_v)(t)) - z(t), (\mathsf{S}(\overline{u},\overline{r},x^0))'(\ell_v(t))\Big\rangle \,\mathrm{d}\mathrm{D}\ell_v(t) \qquad (4.14)\end{aligned}$$

Now let

$$A = \big\{\sigma \in [0,T] \ : \ \Big\langle \overline{u}(\sigma) - \mathsf{S}(\overline{u},\overline{r},x^0)(\sigma) - z, \big(\mathsf{S}(\overline{u},\overline{r},x^0)\big)'(\sigma)\Big\rangle < 0 \ \forall z \in \mathcal{Z}(\overline{r}(\sigma))\big\}.$$

From (3.8) it follows that A has Lebesgue measure zero, hence $\mathrm{D}\ell_v(\ell_v^{-1}(A)) = 0$ (cf. [23, Proposition 2.2]) and, since $z(t) \in \mathcal{Z}(r(t)) = \mathcal{Z}(\overline{r}(\ell_v(t)))$, we find that

$$\begin{aligned} &\mathrm{D}\ell_v \left(\left\{t \in [0,T] \ : \ \left\langle \overline{u}(\ell_v(t)) - \mathsf{S}(\overline{u},\overline{r},x^0)(\ell_v(t)) - z(t), \left(\mathsf{S}(\overline{u},\overline{r},x^0)\right)'(\ell_v(t))\right\rangle < 0\right\}\right) \\ &\leq \mathrm{D}\ell_v\left(\{s \in [0,t] \ : \ \ell_v(s) \in A\}\right) = 0 \end{aligned}$$

Consequently from (4.14) we infer that $\int_0^T \langle u(t) - \xi(t) - z(t), \mathrm{d}\mathrm{D}\xi(t)\rangle \geq 0$ and we are done. $\square$

Remark 4.1. Let us observe that Theorem 4.1 provides a proof for the existence/uniqueness of Problem 3.2 which allows to reduce to the Lipschitz case by means of basic measure theoretical facts. The same argument shows that the operator $\overline{\mathsf{S}}$ is rate independent.

Proposition 4.1. *Assume that* $u, u_n \in BV([0,T];\mathcal{X}) \cap C([0,T];\mathcal{X})$ *and* $r, r_n \in BV([0,T];\mathcal{Y}) \cap C([0,T];\mathcal{Y})$ *for every* $n \in \mathbb{N}$ *and set*

$$v := (u,r) : [0,T] \longrightarrow \mathcal{B}, \tag{4.15}$$

$$v_n := (u_n,r_n) : [0,T] \longrightarrow \mathcal{B}, \qquad n \in \mathbb{N}. \tag{4.16}$$

If $u_n \to u$ *and* $r_n \to r$ *strictly as* $n \to \infty$, *then*

$$\widetilde{v}_n \to \widetilde{v} \ \textit{strictly on} \ [0,T], \tag{4.17}$$

where $\widetilde{v}_n$ *and* $\widetilde{v}$ *are the arc length reparametrizations defined above in* (4.2)–(4.3). *Moreover if* $\widetilde{v} := (\overline{u},\overline{r})$ *and* $\widetilde{v}_n := (\overline{u}_n,\overline{r}_n)$, *then*

$$\overline{u}_n \to \overline{u} \ \textit{uniformly on} \ [0,T], \tag{4.18}$$

$$\overline{r}_n \to \overline{r} \ \textit{uniformly on} \ [0,T]. \tag{4.19}$$

Proof. From the continuity of the functions involved and from (4.8), it follows that $v, v_n \in BV([0,T];\mathcal{B}) \cap C([0,T];\mathcal{B})$ for every $n \in \mathbb{N}$ and

$$v_n \to v \qquad \text{strictly in } BV([0,T];\mathcal{B}) \tag{4.20}$$

as $n \to \infty$. Moreover $\overline{u}, \overline{u}_n \in Lip([0,T];\mathcal{X})$, $\overline{r}, \overline{r}_n \in Lip([0,T];\mathcal{Y})$ and

$$u(t) = \overline{u}(\ell_v(t)), \qquad r(t) = \overline{r}(\ell_v(t)), \tag{4.21}$$

$$u_n(t) = \overline{u}_n(\ell_{v_n}(t)), \qquad r_n(t) = \overline{r}_n(\ell_{v_n}(t)) \tag{4.22}$$

for every $t \in [0,T]$ and every $n \in \mathbb{N}$.

If $s \in [0,T]$ and $n \in \mathbb{N}$ we have that

$$\|\widetilde{v}_n(s)\|_{\mathcal{B}} \leq \|\widetilde{v}_n(0)\|_{\mathcal{B}} + \mathrm{V}(\widetilde{v}_n,[0,T]) = \|v_n(0)\|_{\mathcal{B}} + \mathrm{V}(v_n,[0,T]),$$

therefore from (4.2), (4.20) and Lemma 5.4 of the Appendix we infer that

$$\|\widetilde{v}_n\|_{L^\infty(0,T;\mathcal{B})} \leq C_1 \tag{4.23}$$

for some $C_1 > 0$ independent of $n \in \mathbb{N}$. Moreover by (4.3) we have $\|\widetilde{v}_n'\|_\infty \leq \mathrm{V}(v_n,[0,T])/T$ for every $n \in \mathbb{N}$, hence there exists $C_2 > 0$ such that

$$\|\widetilde{v}_n'\|_\infty \leq C_2 \tag{4.24}$$

for all $n \in \mathbb{N}$. It follows that $\widetilde{v}_n$ is bounded in $W^{1,p}(0,T;\mathcal{B})$ for every $p \in [1,\infty]$. The reflexivity of $L^p(0,T;\mathcal{B})$ for $p \in \,]1,\infty[$ (cf. [10, Theorem 8.20.5, p. 607]) and a standard Sobolev spaces argument imply that there exists $\widehat{v} \in W^{1,1}(0,T;\mathcal{B})$ such that, at least for a subsequence that we do not relabel,

$$\widetilde{v}_n \rightharpoonup \widehat{v} \quad \text{in } W^{1,p}(0,T;\mathcal{B}) \qquad \forall p \in \,]1,\infty[\,. \tag{4.25}$$

Now let us fix $\sigma \in [0,T]$ and for every $x^* \in \mathcal{B}^*$ let us consider the linear functional $\phi^\sigma_{x^*} : W^{1,p}(0,T;\mathcal{B}) \longrightarrow \mathbb{R} : v \longmapsto {}_{\mathcal{B}^*}\langle x^*, v(\sigma)\rangle_{\mathcal{B}}$. Since $W^{1,p}(0,T;\mathcal{B})$ is continuously embedded in

$C([0,T];\mathcal{B})$ (cf. Corollary 5.2), we have that $\phi^\sigma_{x^*}$ is also continuous, thus from (4.25) we infer that

$$\lim_{n\to\infty} {}_{\mathcal{B}^*}\langle x^*, \widetilde{v}_n(\sigma)\rangle_{\mathcal{B}} = \lim_{n\to\infty} \phi^\sigma_{x^*}(\widetilde{v}_n) = \phi^\sigma_{x^*}(\widehat{v}) = {}_{\mathcal{B}^*}\langle x^*, \widehat{v}(\sigma)\rangle_{\mathcal{B}},$$

i.e.

$$\widetilde{v}_n(\sigma) \rightharpoonup \widehat{v}(\sigma) \quad \text{in } \mathcal{B} \qquad \forall \sigma \in [0,T] \tag{4.26}$$

as $n \to \infty$. Now for every $x^* \in \mathcal{B}^*$ and every $n \in \mathbb{N}$ let us define the functions $f_n^{x^*} : [0,T] \longrightarrow \mathbb{R}$ and $f^{x^*} : [0,T] \longrightarrow \mathbb{R}$ by

$$f_n^{x^*}(\sigma) := {}_{\mathcal{B}^*}\langle x^*, \widetilde{v}_n(\sigma)\rangle_{\mathcal{B}}, \qquad f^{x^*}(\sigma) := {}_{\mathcal{B}^*}\langle x^*, \widehat{v}(\sigma)\rangle_{\mathcal{B}}, \qquad \sigma \in [0,T]. \tag{4.27}$$

From the continuity of $\widetilde{v}_n$ and $\widetilde{v}$ we infer that $f_n^{x^*}$ and f^{x^*} are continuous, moreover from (4.29) it follows that $f_n^{x^*} \to f^{x^*}$ pointwise in $[0,T]$. Moreover if $\sigma, \tau \in [0,T]$ we have, thanks to (4.24), that

$$|f_n^{x^*}(\sigma) - f_n^{x^*}(\tau)| \le \|x^*\| \|\widetilde{v}_n(\sigma) - \widetilde{v}_n(\tau)\| \le \|x^*\| \|\widetilde{v}_n'\|_{L^\infty(0,T;\mathcal{B})} |\tau - \sigma| \le C\|x^*\| |\sigma - \tau|,$$

thus $(f_n^{x^*})_n$ is equicontinuous and $f_n^{x^*} \to f^{x^*}$ uniformly on $[0,T]$ for every $x^* \in \mathcal{B}^*$. But $\ell_{v_n}(t) \to \ell_v(t)$ pointwise on $[0,T]$ by Lemma 5.2, hence $f_n^{x^*}(\ell_{v_n}(t)) \to f^{x^*}(\ell_v(t))$ for every $t \in [0,T]$, i.e.

$$\widetilde{v}_n(\ell_{v_n}(t)) \rightharpoonup \widehat{v}(\ell_v(t)) \quad \text{in } \mathcal{B} \quad \forall t \in [0,T]. \tag{4.28}$$

On the other hand by the strict convergence of v_n and by Lemma 5.4 we have that

$$\lim_{n\to\infty} \widetilde{v}_n(\ell_{v_n}(t)) = \lim_{n\to\infty} v_n(t) = v(t) = \widetilde{v}(\ell_v(t)) \qquad \forall t \in [0,T],$$

hence, as ℓ_v is surjective, we get that $\widehat{v} = \widetilde{v}$. Hence from (4.25)–(4.26) we infer that

$$\widetilde{v}_n(\sigma) \rightharpoonup \widetilde{v}(\sigma) \quad \text{in } \mathcal{B} \qquad \forall \sigma \in [0,T] \tag{4.29}$$

and

$$\widetilde{v}_n \rightharpoonup \widetilde{v} \quad \text{in } W^{1,p}(0,T;\mathcal{B}) \qquad \forall p \in \,]1,\infty[\,. \tag{4.30}$$

If $\sigma \in [0,T]$ is fixed, then for every $n \in \mathbb{N}$ there exists $t_n \in [0,T]$ such that

$$\widetilde{v}_n(\sigma) = \widetilde{v}_n(\ell_{v_n}(t_n)) = v_n(t_n). \tag{4.31}$$

Passing to a subsequence, not relabeled, we have that $t_n \to t_*$ for some $t_* \in [0,T]$. Hence, thanks to the uniform convergence of v_n, $v_n(t_n) \to v(t_*)$ as $n \to \infty$. It follows, as $v(t_*) = \widetilde{v}(\ell_v(t_*))$, that

$$\widetilde{v}_n(\sigma) \to \widetilde{v}(\ell_v(t_*)) \tag{4.32}$$

as $n \to \infty$. From (4.29) we get that

$$\widetilde{v}_n(\sigma) \to \widetilde{v}(\sigma) \quad \text{in } \mathcal{B} \qquad \forall \sigma \in [0,T] \tag{4.33}$$

and the whole sequence is converging by the uniqueness of the limit. Hence, taking into account (4.23), we can apply the dominated convergence theorem and infer that $\widetilde{v}_n \to \widetilde{v}$ in $L^1(0,T;\mathcal{B})$. Since it is clear that $\mathrm{V}(\widetilde{v}_n, [0,T]) \to \mathrm{V}(\widetilde{v}, [0,T])$, we have that $\widetilde{v}_n \to \widetilde{v}$ strictly on $[0,T]$. Therefore, by Proposition 5.1, we get that $\widetilde{v}_n \to \widetilde{v}$ uniformly on $[0,T]$ and (4.18)–(4.19) follow. $\square$

Lemma 4.1. *Assume that $(u, r, x^0), (u_n, r_n, x_n^0) \in \overline{D}$ for every $n \in \mathbb{N}$, $u_n \to u$, $r_n \to r$ strictly on $[0,T]$, and $x_n^0 \to x^0$ in $\mathcal{X}$, as $n \to \infty$. With the same notations of Proposition 4.1, we have that $\mathsf{S}(\overline{u}_n, \overline{r}_n, x_n^0) \to \mathsf{S}(\overline{u}, \overline{r}, x^0)$ strictly on $[0,T]$.*

Proof. Let us set

$$\overline{\xi} := \mathsf{S}(\overline{u}, \overline{r}, x^0), \qquad \overline{\xi}_n := \mathsf{S}(\overline{u}_n, \overline{r}_n, x^0_n)$$

and

$$\overline{x} := \overline{u} - \overline{\xi}, \qquad \overline{x}_n := \overline{u}_n - \overline{\xi}_n$$

for every $n \in \mathbb{N}$. Observe that from (4.24) we get

$$\max\{\|\overline{u}_n'\|_\infty, \|\overline{r}_n'\|_\infty\} \le \|\widetilde{v}_n'\|_\infty \le C_2. \tag{4.34}$$

Since $\overline{u}, \overline{r}, \overline{u}_n, \overline{r}_n$ are Lipschitz continuous, the following basic estimate holds (cf. [17, Theorem 4] or [5, Formulas (36)–(38), (46)]):

$$\begin{aligned} \|\overline{\xi}(t) - \overline{\xi}_n(t)\|_{\mathcal{X}} \le {} & (\|x^0 - x^0_n\|_{\mathcal{X}} + \|\overline{u}(0) - \overline{u}_n(0)\|_{\mathcal{X}})^2 \\ & + L_n \int_0^t (\|\overline{u}(s) - \overline{u}_n(s)\|_{\mathcal{X}} + C^3 K_0 \|\overline{r}(s) - \overline{r}_n(s)\|_{\mathcal{Y}}) \,\mathrm{d}s \end{aligned} \tag{4.35}$$

where

$$L_n := 2\big(\|\overline{u}'\|_\infty + \|\overline{u}_n'\|_\infty + C^3 K_0 (\|\overline{r}'\|_\infty + \|\overline{r}_n'\|_\infty)\big). \tag{4.36}$$

The sequence L_n is bounded by virtue of (4.34), therefore from (4.18)–(4.19) and from (4.35)–(4.36) we infer that

$$\overline{\xi}_n \to \overline{\xi} \ \text{ uniformly on } [0,T], \tag{4.37}$$

which together with (4.18) yields

$$\overline{x}_n \to \overline{x} \ \text{ uniformly on } [0,T]. \tag{4.38}$$

Therefore from (3.12), (4.38), and (4.19) we infer that

$$J(\overline{x}_n(t), \overline{r}_n(t)) \to J(\overline{x}(t), \overline{r}(t)) \qquad \forall t \in [0,T]. \tag{4.39}$$

as $n \to \infty$. If $(v, \rho, z^0) \in D$, $\eta := \mathsf{S}(v, \rho, z^0)$, and $y := v - \eta$, then [5, Lemma 5.2] yields the following implication:

$$\eta'(t) \ne 0 \implies \begin{cases} y(t) \in \partial \mathcal{Z}(\rho(t)) \\[2ex] \|\eta'(t)\|_{\mathcal{X}} = \left\langle \eta'(t), \dfrac{J(y(t), \rho(t))}{\|J(y(t), \rho(t))\|_{\mathcal{X}}} \right\rangle \end{cases} \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T]. \tag{4.40}$$

Let us define $H : \mathcal{X} \times \mathcal{R} \longrightarrow \mathcal{X}$ by

$$H(y,\rho) := \begin{cases} M(y,\rho) \dfrac{J\left(\dfrac{y}{M(y,\rho)}, \rho(t)\right)}{\left\| J\left(\dfrac{y}{M(y,\rho)}, \rho(t)\right)\right\|_{\mathcal{X}}} & \text{if } x \ne 0 \\[4ex] 0 & \text{if } x = 0 \end{cases} \tag{4.41}$$

The map H is well-defined thanks to (3.16) and to the fact that $y/M(y,\rho) \in \partial\mathcal{Z}(\rho)$, therefore we have that

$$\|\eta'(t)\|_{\mathcal{X}} = \langle \eta'(t), H(y(t), \rho(t)) \rangle \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T]. \tag{4.42}$$

Moreover, since $M(x) \to 0$ as $x \to 0$, from (3.5) and Assumption 3.1 we infer that H is continuous, hence $H(\overline{x}_n(t), \overline{r}_n(t)) \to H(\overline{x}(t), \overline{r}(t))$ for every $t \in [0,T]$. On the other hand the sequence $H(\overline{x}_n(\cdot), \overline{r}_n(\cdot))$ is uniformly bounded, thus by the dominated convergence theorem

$$H(\overline{x}_n(\cdot), \overline{r}_n(\cdot)) \to H(\overline{x}(\cdot), \overline{r}(\cdot)) \qquad \text{in } L^q(0,T;\mathcal{X}) \qquad \forall q \in \,]1,\infty[\,. \tag{4.43}$$

Observe that (cf. [5, Formula 50])

$$\|\overline{\xi}_n'(t)\|_{\mathcal{X}} \le \|\overline{u}_n'(t)\|_{\mathcal{X}} + C K_0 \|\overline{r}_n'(t)\|_{\mathcal{Y}} \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T], \tag{4.44}$$

hence, thanks to (4.34), $\overline{\xi}_n$ is bounded in $W^{1,p}(0,T;\mathcal{X})$ for every $p \in]1,\infty[$, and (4.37) implies that

$$\overline{\xi}_n \rightharpoonup \overline{\xi} \qquad \text{in } W^{1,p}(0,T;\mathcal{X}) \text{ for every } p \in]1,\infty[\,. \tag{4.45}$$

Thus from (4.42), (4.43), and (4.45) we get that

$$\begin{aligned}\lim_{n\to\infty} \mathrm{V}(\overline{\xi}_n,[0,T]) &= \lim_{n\to\infty}\int_0^T \|\overline{\xi}_n'(t)\|_{\mathcal{X}}\,\mathrm{d}t = \lim_{n\to\infty}\int_0^T \left\langle \overline{\xi}_n'(t), H(\overline{x}_n(t),\overline{r}_n(t))\right\rangle \mathrm{d}t \\ &= \int_0^T \left\langle \overline{\xi}'(t), H(\overline{x}(t),\overline{r}(t))\right\rangle \mathrm{d}t = \int_0^T \|\overline{\xi}'(t)\|_{\mathcal{X}}\,\mathrm{d}t = \mathrm{V}(\overline{\xi},[0,T]),\end{aligned} \tag{4.46}$$

which together with (4.37) proves the lemma. □

Proof of Theorem 3.2. We are left to prove the continuity property. to this aim let (u,r,x^0), $(u_n,r_n,x_n^0) \in \overline{D}$ be such that $u_n \to u$, $r_n \to r$ strictly on $[0,T]$ and $x_n^0 \to x^0$ in $\mathcal{X}$. If $v=(u,r)$ and $v_n=(u_n,r_n)$ then by Lemma 5.2 we have that

$$\ell_{v_n}(t) \to \ell_v(t) \qquad \forall t \in [0,T]. \tag{4.47}$$

Observe that by Theorem 4.1 we have

$$\mathsf{S}(u,r,x^0)(t) = \mathsf{S}(\overline{u}\circ\ell_v,\overline{r}\circ\ell_v,x^0)(t) = \mathsf{S}(\overline{u},\overline{r},x^0)(\ell_v(t)), \tag{4.48}$$

$$\mathsf{S}(u_n,r_n,x_n^0)(t) = \mathsf{S}(\overline{u}_n\circ\ell_{v_n},\overline{r}_n\circ\ell_{v_n},x_n^0)(t) = \mathsf{S}(\overline{u}_n,\overline{r}_n,x_n^0)(\ell_v(t)). \tag{4.49}$$

Moreover from Lemma 4.1 we get that

$$\mathsf{S}(\overline{u}_n,\overline{r}_n,x_n^0) \to \mathsf{S}(\overline{u},\overline{r},x^0) \quad \text{strictly on } [0,T], \tag{4.50}$$

in particular $\mathsf{S}(\overline{u}_n,\overline{r}_n,x_n^0) \to \mathsf{S}(\overline{u},\overline{r},x^0)$ uniformly on $[0,T]$ thanks to Proposition 5.1, therefore from (4.47) we get

$$\begin{aligned}\lim_{n\to\infty}\mathsf{S}(u_n,r_n,x_n^0)(t) &= \lim_{n\to\infty}\mathsf{S}(\overline{u}_n\circ\ell_{v_n},\overline{r}_n\circ\ell_{v_n},x_n^0)(t)\\ &= \lim_{n\to\infty}\mathsf{S}(\overline{u}_n,\overline{r}_n,x_n^0)(\ell_{v_n}(t))\\ &= \mathsf{S}(\overline{u},\overline{r},x^0)(\ell_v(t))\\ &= \mathsf{S}(\overline{u}\circ\ell_v,\overline{r}\circ\ell_v,x^0)(t)\\ &= \mathsf{S}(u,r,x^0)(t)\end{aligned} \tag{4.51}$$

Now $\|\mathsf{S}(u_n,r_n,x_n^0)\|_\infty = \|\mathsf{S}(\overline{u}_n,\overline{r}_n,x_n^0)\circ\ell_{v_n}\|_\infty = \|\mathsf{S}(\overline{u}_n,\overline{r}_n,x_n^0)\|_\infty$, thus $\mathsf{S}(u_n,r_n,x_n^0)$ is uniformly bounded because of the strict convergence of $\mathsf{S}(\overline{u}_n,\overline{r}_n,x_n^0)$, and by the dominated convergence theorem we infer that

$$\mathsf{S}(u_n,r_n,x_n^0) \to \mathsf{S}(u,r,x^0) \qquad \text{in } L^1(0,T;\mathcal{X}). \tag{4.52}$$

Finally we have to prove the convergence of the variations. From (4.48)–(4.49) and from the continuity of ℓ_v we have that

$$\mathrm{V}(\mathsf{S}(u_n,r_n,x_n^0),[0,T]) = \mathrm{V}(\mathsf{S}(\overline{u}_n,\overline{r}_n,x_n^0),[0,T]), \tag{4.53}$$

$$\mathrm{V}(\mathsf{S}(u,r,x^0),[0,T]) = \mathrm{V}(\mathsf{S}(\overline{u},\overline{r},x^0),[0,T]), \tag{4.54}$$

moreover (4.50) yields

$$\lim_{n\to\infty}\mathrm{V}(\mathsf{S}(\overline{u}_n,\overline{r}_n,x_0^n),[0,T]) = \mathrm{V}(\mathsf{S}(\overline{u},\overline{r},x_0),[0,T])$$

and the theorem is completely proved. □

Remark 4.2. As we mentioned in the Introduction, when $\mathscr{Z}(r(t)) = \mathscr{Z}$, a fixed closed convex subset of the Hilbert space $\mathcal{X}$, the solution operator S is actually acting on $BV([0,T];\mathcal{X}) \cap C([0,T];\mathcal{X})$ only, and its strict continuity was deduced in [26] by applying the general implication

$$\mathsf{R}\ d_{BV}\text{-continuous} \implies \mathsf{R}\ d_s\text{-continuous}, \tag{4.55}$$

holding for a rate independent operator $\mathsf{R} : BV([0,T];\mathcal{X}) \cap C([0,T];\mathcal{X}) \longrightarrow BV([0,T];\mathcal{X}) \cap C([0,T];\mathcal{X})$. Property (4.55) (proved in [26, Theorem 3.4]) cannot be applied in our new framework where $\mathcal{B} = \mathcal{X} \times \mathcal{Y}$ replaces $\mathcal{X}$ in the domain of R, because the norm (4.6) is not uniformly convex, and property (4.55) does not hold in the non-uniformly convex case, even if $\mathcal{B}$ is reflexive. Let us show this fact with a counterexample by considering the space $\mathcal{B}_1 = \mathbb{R}^2$ endowed with the 1-norm $\|(x,y)\|_{\mathcal{B}_1} := |x| + |y|$, $(x,y) \in \mathbb{R}^2$. Notice that $\mathcal{B}_1$ is reflexive but is not uniformly convex. By $\mathcal{B}_2$ we denote the space $\mathbb{R}^2$ endowed with the euclidean norm $\|(x,y)\|_{\mathcal{B}_2} := (|x|^2 + |y|^2)^{1/2}$, $(x,y) \in \mathbb{R}^2$. If an interval $J \subseteq [0,T]$ and $v : [0,T] \longrightarrow \mathbb{R}^2$ are given, for $k = 1,2$ we denote by $\mathrm{V}_k(u,J)$ the variation of u on J with respect to the norm $\|\cdot\|_{\mathcal{B}_k}$, and we also set $\mathrm{V}_k(u)(t) := \mathrm{V}_k(u,[0,t])$, $t \in [0,T]$. Accordingly we denote by d_{BV}^k and by d_s^k the distances defined in (2.9) and in (2.10) with $\mathcal{B} = \mathcal{B}_k$, $k = 1,2$, while d_{BV} will be used for the case $\mathcal{B} = \mathbb{R}$. Observe that the metrics d_{BV}^1 and d_{BV}^2 are equivalent, hence they generate the same topology. Let us define $\mathsf{R} : BV([0,T];\mathcal{B}_1) \cap C([0,T];\mathcal{B}_1) \longrightarrow BV([0,T];\mathcal{B}_1) \cap C([0,T];\mathcal{B}_1)$ by

$$\mathsf{R}(u)(t) := (\mathrm{V}_1(u)(t), \mathrm{V}_2(u)(t)), \qquad u \in BV([0,T];\mathcal{B}_1) \cap C([0,T];\mathcal{B}_1)$$

(we could take $\mathrm{V}_2(u)$ in both components, but we prefer to keep them distinct). Clearly R is rate independent. In order to prove that it is d_{BV}^1-continuous, assume that $d_{BV}^1(u_n,u) \to 0$, thus $d_{BV}^2(u_n,u) \to 0$ as well. Since $\mathrm{V}_k(u_n)$ and $\mathrm{V}_k(u)$ are increasing functions, a straightforward computation shows that

$$\mathrm{V}(\mathrm{V}_k(v) - \mathrm{V}_k(w), J) = \mathrm{V}_k(u - w, J) \tag{4.56}$$

for every v, w, and J, therefore, using also the inequality $|\mathrm{V}_k(v,J) - \mathrm{V}_k(w,J)| \le \mathrm{V}_k(v-w,J)$, we have

$$\begin{aligned} d_{BV}(\mathrm{V}_k(u_n), \mathrm{V}_k(u)) &= \|\mathrm{V}_k(u_n) - \mathrm{V}_k(u)\|_{L^1(0,T;\mathbb{R})} + |\mathrm{V}(\mathrm{V}_k(u_n) - \mathrm{V}_k(u)), [0,T])| \\ &= \int_0^T |\mathrm{V}_k(u_n)(t) - \mathrm{V}_k(u)(t)|\,\mathrm{d}t + \mathrm{V}_k(u_n - u, [0,T]) \\ &\le \int_0^T \mathrm{V}_k(u_n - u, [0,t])\,\mathrm{d}t + \mathrm{V}_k(u_n - u, [0,T]) \\ &\le (T+1)\,\mathrm{V}_k(u_n - u, [0,T]), \end{aligned}$$

hence $d_{BV}(\mathrm{V}_k(u_n), \mathrm{V}_k(u)) \to 0$ as $n \to \infty$ for $k = 1,2$, and this implies that $d_{BV}^1(\mathsf{R}(u_n), \mathsf{R}(u)) \to 0$ as $n \to \infty$, and R is d_{BV}^1-continuous. Now we show that R is *not* d_s^1-continuous. To this aim we consider a sequence of Lipschitz curves u_n whose trace is a kind of "staircase with n steps" laid upon the line $y = x$, going from the origin to the point $(1,1)$. More precisely, for every $n \in \mathbb{N}$ we split $[0,1]$ into n subintervals $[(j-1)/2^{n-1}, j/2^{n-1}]$, $j = 1,\dots,2^{n-1}$, and let $u_n : [0,1] \longrightarrow \mathbb{R}^2$ be the unique Lipschitz curve such that

$$u_n(t) = \begin{cases} ((j-1)/2^{n-1}, g_n(t)) & \text{if } t \in [(j-1)/2^{n-1}, (2j-1)/2^n] \\ (h_n(t), j/2^{n-1}) & \text{if } t \in [(2j-1)/2^n, j/2^{n-1}] \end{cases}, \qquad j = 1,\dots,2^{n-1},$$

where $g_n : [(j-1)/2^{n-1}, (2j-1)/2^n] \longrightarrow [(j-1)/2^{n-1}, j/2^{n-1}]$ and $h_n : [(2j-1)/2^n, j/2^{n-1}] \longrightarrow [(j-1)/2^{n-1}, j/2^{n-1}]$ are affine increasing surjective functions. If $u : [0,1] \longrightarrow \mathbb{R}^2$ is defined by $u(t) := (t,t)$, then we have $\|u_n - u\|_{L^1(0,1;\mathcal{B}_k)} \to 0$ for $k = 1,2$. Since $\mathrm{V}_1(u_n,[0,1]) = \mathrm{V}_2(u_n,[0,1]) = 2$ for every $n \in \mathbb{N}$, by (4.56) we have that $\mathrm{V}_1(\mathsf{R}(u_n),[0,1]) = \mathrm{V}(\mathrm{V}_1(u_n),[0,1]) +$

$\mathrm{V}(\mathrm{V}_2(u_n),[0,1]) = 2+2 = 4$. On the other hand $\mathrm{V}_1(u,[0,1]) = 2$ and $\mathrm{V}_2(u,[0,1]) = \sqrt{2}$, therefore $\mathrm{V}_1(\mathsf{R}(u),[0,1]) = \mathrm{V}(\mathrm{V}_1(u),[0,1]) + \mathrm{V}(\mathrm{V}_2(u),[0,1]) = 2+\sqrt{2}$, hence $\mathsf{R}(u_n)$ is not d_s^1-convergent and R is not d_s^1-continuous.

Remark 4.3. In the simpler case $\mathcal{Z}(r(t)) = \mathcal{Z}$ the strict continuity of the solution operator S was deduced in [26] without any smoothness assumption on $\mathcal{Z}$. Therefore it seems natural to wonder if Assumption 3.1 can be relaxed in the present framework. This question is still open.

5. Appendix

In this section we show some properties about the strict convergence in $BV([0,T];\mathcal{B})$.

Lemma 5.1. *Assume that $v_n, v \in BV([0,T];\mathcal{B}) \cap C([0,T];\mathcal{B})$ and let $J \subseteq [0,T]$ be an interval. If $v_n(t) \to v(t)$ for a.e. $t \in J$, then $\mathrm{V}(v,J) \le \liminf_{n\to\infty} \mathrm{V}(v_n,J)$.*

Proof. Let $0 = s_0 < \cdots < s_m = T$ be such that

$$\mathrm{V}(v,J) < \varepsilon/2 + \sum_{j=0}^{m} \|v(s_j) - v(s_{j-1})\|_{\mathcal{B}}.$$

The set $E := \{t \in [0,T] \ : \ v_n(t) \to v(t) \text{ as } n \to \infty\}$ has full measure in $[0,T]$, therefore we can find points $t_j \in E$, $j = 1,\ldots,m$ such that $0 < t_1 < \cdots < t_m = T$ and $\|v(t_j) - v(s_j)\|_{\mathcal{B}} < m\varepsilon/4$ for $j = 1,\ldots,m$, and we have

$$\begin{aligned}
\mathrm{V}(u,[0,T]) &< \varepsilon/2 + \sum_{j=0}^{m} \|v(s_j) - v(s_{j-1})\|_{\mathcal{B}} \\
&\le \varepsilon/2 + \sum_{j=0}^{m} (\|v(s_j) - v(t_j)\|_{\mathcal{B}} + \|v(t_j) - v(t_{j-1})\|_{\mathcal{B}} + \|v(t_{j-1}) - v(s_{j-1})\|_{\mathcal{B}}) \\
&< \varepsilon + \sum_{j=0}^{m} \|v(t_j) - v(t_{j-1})\|_{\mathcal{B}}.
\end{aligned}$$

For every $n \in \mathbb{N}$ we have

$$\mathrm{V}(v_n,[0,T]) \ge \sum_{j=0}^{m} \|v_n(t_j) - v_n(t_{j-1})\|_{\mathcal{B}}, \tag{5.1}$$

therefore taking the lower limit we get

$$\begin{aligned}
\liminf_{n\to\infty} \mathrm{V}(v_n,[0,T]) &\ge \liminf_{n\to\infty} \sum_{j=0}^{m} \|v_n(t_j) - v_n(t_{j-1})\|_{\mathcal{B}} \\
&\ge \sum_{j=0}^{m} \liminf_{n\to\infty} \|v_n(t_j) - v_n(t_{j-1})\|_{\mathcal{B}} \\
&= \sum_{j=0}^{m} \|v(t_j) - v(t_{j-1})\|_{\mathcal{B}} > \mathrm{V}(v,[0,T]) - \varepsilon.
\end{aligned}$$

and the statement follows from the arbitrariness of ε. □

Corollary 5.1. *Let $v, v_n \in BV([0,T];\mathcal{B}) \cap C([0,T];\mathcal{B})$ be such that $v_n \to v$ strictly on $[0,T]$ as $n \to \infty$. Let $J \subseteq [0,T]$ be an interval. Then*

$$\mathrm{V}(v,J) \le \liminf_{n\to\infty} \mathrm{V}(v_n,J).$$

Proof. Let $(n_k)_k$ be a sequence of positive integers such that $n_k \to \infty$ and $\mathrm{V}(v_{n_k}, I) \to \ell$ as $k \to \infty$ for some $\ell \geq 0$. By the strict convergence it follows that there is a further subsequence n_{k_h} such that $v_{n_{k_h}} \to u$ almost everywhere. Hence by Lemma 5.1 $\mathrm{V}(v, J) \leq \ell$ and we are done. ☐

Lemma 5.2. *Assume that $v_n, v \in BV([0,T];\mathcal{B}) \cap C([0,T];\mathcal{B})$ for every $n \in \mathbb{N}$. If $v_n \to v$ strictly on $[0,T]$ as $n \to \infty$, then $\mathrm{V}(v_n, [s,t]) \to \mathrm{V}(v, [s,t])$ for every $s, t \in [0,T]$, $s < t$.*

Proof. Thanks to Corollary 5.1 we have that

$$\mathrm{V}(v, [s,t]) = \liminf_{n\to\infty} \mathrm{V}(v_n, [s,t]).$$

On the other hand, using again Corollary 5.1 and the strict convergence, we infer that

$$\begin{aligned}\limsup_{n\to\infty} \mathrm{V}(v_n, [s,t]) &= \limsup_{n\to\infty} (\mathrm{V}(v_n, [0,T]) - \mathrm{V}(v_n, [0,s]) - \mathrm{V}(v_n, [t,T])) \\ &\leq \mathrm{V}(v, [0,T]) - \mathrm{V}(v, [0,s]) - \mathrm{V}(v, [t,T]) = \mathrm{V}(v, [s,t]).\end{aligned}$$

☐

Lemma 5.3. *Assume that $v_n, v \in BV([0,T];\mathcal{B}) \cap C([0,T];\mathcal{B})$ for every $n \in \mathbb{N}$. If $v_n \to v$ strictly as $n \to \infty$, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $c, d \in [0,T]$ we have*

$$0 < d - c < \delta \quad \Longrightarrow \quad \sup_{n\in\mathbb{N}} \mathrm{V}(v_n, [c,d]) < \varepsilon. \tag{5.2}$$

Proof. Thanks to Lemma 5.2, the sequence of real functions $V_n : [0,T] \longrightarrow \mathbb{R} : t \longmapsto V(v_n, [0,t])$ is pointwise converging to the continuous function $V : [0,T] \longrightarrow \mathbb{R} : t \longmapsto V(v, [0,t])$. Moreover V_n is a monotone function for every $n \in \mathbb{N}$, therefore from the Polya Lemma (cf. [9, Theorem 10, p. 166]) we deduce that $V_n \to V$ uniformly on $[0,T]$, hence for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup_{n\in\mathbb{N}} |V(d) - V(c)| < \varepsilon$ whenever $0 < d - c < \delta$, $c, d \in [0,T]$. This is what we wanted to prove. ☐

Lemma 5.4. *Assume that $v_n, v \in BV([0,T];\mathcal{B}) \cap C([0,T];\mathcal{B})$ for every $n \in \mathbb{N}$. If $v_n \to v$ strictly as $n \to \infty$, then $v_n(t) \to v(t)$ as $n \to \infty$ for every $t \in [0,T]$.*

Proof. If $t \in [0,T]$ is fixed and a subsequence $v_{n'}(t)$ of $v_n(t)$ is given, we can extract a further subsequence $(n'_k)_k$ such that $v_{n'_k} \to v$ a.e. in $[0,T]$. If $\varepsilon > 0$ there exists $\delta > 0$ such that (5.2) holds. We can find a point t_0 such that $0 \leq t - t_0 < \delta$ and $v_{n'_k}(t_0) \to v(t_0)$. Hence we get

$$\begin{aligned}\|v_{n'_k}(t), v(t)\|_{\mathcal{B}} &\leq \|v_{n'_k}(t_0) - v(t_0)\|_{\mathcal{B}} + \|v_{n'_k}(t) - v_{n'_k}(t_0)\|_{\mathcal{B}} + \|v_{n'_k}(t) - v(t_0)\|_{\mathcal{B}} \\ &\leq \|v_{n'_k}(t_0) - v(t_0)\|_{\mathcal{B}} + \mathrm{V}(v_{n'_k}, [t_0,t]) + \mathrm{V}(v_, [t_0,t]) \leq 3\varepsilon,\end{aligned}$$

provided k is large enough. The thesis follows. ☐

Proposition 5.1. *Assume $v, v_n \in BV([0,T];\mathcal{B}) \cap C([0,T];\mathcal{B})$ and $v_n \to v$ strictly as $n \to \infty$. Then $v_n \to v$ uniformly on $[0,T]$.*

Proof. It is enough to apply the Ascoli theorem for $\mathcal{B}$ valued functions (cf. [18, Theorem 3.1, p. 57]). The pointwise convergence of v_n is proved in Lemma 5.4, the equicontinuity follows immediately from Lemma 5.3. ☐

Notice that as a consequence of Proposition 5.1 we can also obtain the following

Corollary 5.2. *$W^{1,1}([0,T];\mathcal{B})$ is continuously embedded in $C([0,T];\mathcal{B})$.*

References

[1] L. Ambrosio, N. Fusco and D. Pallara, "Functions of Bounded Variation and Free Discontinuity Problems", Oxford Mathematical Monographs, Clarendon Press, Oxford, 2000.

[2] N. Bourbaki "Éléments de mathématique. Fonctions d'une variable réelle. Théorie élémentaire", Springer-Verlag, Berlin, 2007.

[3] H. Brezis, "Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert", North-Holland Mathematical Studies, Vol. 5, North-Holland Publishing Company, Amsterdam, 1973.

[4] H. Brezis, "Analyse Fonctionelle - Théorie et applications", Masson, Paris, 1983.

[5] M. Brokate, P. Krejčí and H. Schnabel, *On uniqueness in evolution quasivariational inequalities*, J. Convex Anal. **11** (2004), 111–130.

[6] M. Brokate and J. Sprekels, "Hysteresis and Phase Transitions", Applied Mathematical Sciences, **121**, Springer-Verlag, New York, 1996.

[7] M. Brokate and V. Visintin, *Properties of the Preisach model for hysteresis*, J. reine angew. Math. **402** (1989) 1-40.

[8] N. Dinculeanu, "Vector Measures", International Series of Monographs in Pure and Applied Mathematics, Vol. 95, Pergamon Press, Berlin, 1967.

[9] J. L. Doob, "Measure Theory", Springer-Verlag, New York, 1994.

[10] R. E. Edwards, "Functional Analysis", Holt, Rinehart and Winston, New York, 1965.

[11] H. Federer, "Geometric Measure Theory", Springer-Verlag, Berlin-Heidelberg, 1969.

[12] M. A. Krasnosel'skiĭ and A. V. Pokrovskiĭ, "Systems with Hysteresis", Springer-Verlag, Berlin Heidelberg, 1989.

[13] P. Krejčí, "Hysteresis, Convexity and Dissipation in Hyperbolic Equations", Gakuto International Series Mathematical Sciences and Applications, Vol. 8, Gakkōtosho, Tokyo, 1997.

[14] P. Krejčí and P. Laurençot, *Generalized variational inequalities*, J. Convex Anal., **9** (2002), 159–183.

[15] P. Krejčí and M. Liero, *Rate independent Kurzweil processes*, Appl. Math., **54** (2009), 117–145.

[16] P. Krejčí and T. Roche, *Lipschitz continuous data dependence of sweeping processes in BV spaces*, Discrete Contin. Dyn. Syst. Ser. B, **15** (2011), 637650.

[17] M. Kunze and M. D. P. Monteiro Marques, *An introduction to Moreau's sweeping processes*, Impact in Mechanical Systems - Analysis and Modelling, B. Brogliato (Ed.) Lecture Notes in Physics **551**, Springer (2000), 1–60

[18] S. Lang, "Real and Functional Analysis - Third Edition", Graduate Text in Mathematics, Vol. 142, Springer Verlag, New York, 1993.

[19] A. Mielke, *Evolution in rate-independent systems*, In "Handbook of Differential Equations, Evolutionary Equations, vol. 2", C. Dafermos, E. Feireisl editors, Elsevier, 2005, 461–559.

[20] M. D. P. Monteiro Marques, "Differential Inclusions in Nonsmooth Mechanical Problems - Shocks and Dry Friction", Birkhauser Verlag, Basel, 1993.

[21] J. J. Moreau, *Evolution problem associated with a moving convex set in a Hilbert space*, J. Differential Equations **26** (1977), 347–374.

[22] V. Recupero, *On locally isotone rate independent operators*, Appl. Math. Letters, **20** (2007), 1156–1160.

[23] V. Recupero, *The play operator on the rectifiable curves in a Hilbert space*, Math. Methods Appl. Sci., **31** (2008), 1283–1295.

[24] V. Recupero, *BV-extension of rate independent operators*, Math. Nachr., **282** (2009), 86–98.

[25] V. Recupero, *Sobolev and strict continuity of general hysteresis operators*, Math. Methods Appl. Sci., **32** (2009), 2003–2018.

[26] V. Recupero, *BV solutions of rate independent variational inequalities*, Ann. Sc. Norm. Super. Pisa Cl. Sc. (5), **10** (2011), 269–315.

[27] V. Recupero, *A continuity method for sweeping processes*, J. Differential Equations **251** (2011), 2125–2142.

[28] W. Rudin, "Functional Analysis", McGraw Hill, New York, 1973.

[29] A. H. Siddiqi, P. Manchanda and M. Brokate, *On some recent developments concerning Moreau's sweeping process*, Trends in industrial and applied mathematics (Amritsar, 2001), 339354, Appl. Optim. **72**, Kluwer Acad. Publ., Dordrecht, 2002.

[30] A. Visintin, "Differential Models of Hysteresis", Applied Mathematical Sciences, Vol. 111, Springer-Verlag, Berlin Heidelberg, 1994.

[31] A. Vladimirov, *Equicontinuous sweeping processes*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013), 565–573.

Vincenzo Recupero, Dipartimento di Scienze Matematiche, Politecnico di Torino, C.so Duca degli Abruzzi, 24, I 10129 Torino, Italy.
E-mail address: `vincenzo.recupero@polito.it`