A unifying analysis of error exponents for MIMO channels with application to multiple-scattering

# A unifying analysis of error exponents for MIMO channels with application to multiple-scattering

G. Alfano
DET
Politecnico di Torino
alfano@tlc.polito.it

C.-F. Chiasserini
DET
Politecnico di Torino
chiasserini@polito.it

A. Nordio
IEIIT CNR
alessandro.nordio@ieiit.cnr.it

S. Zhou
DET
Politecnico di Torino
siyuan.zhou@polito.it

***Abstract*—Expressions for the Gallager's Random Coding Error Exponent (RCEE) and the corresponding Expurgated Error Exponent (EEE) are derived in a unifying framework, as functions only of the squared singular values of the channel matrix. The results encompass spatially Kronecker-correlated Rayleigh channels (whose error exponents expressions are already present in the literature), Line of Sight MIMO systems, multiple-scattering channels, multi-hop Amplify and Forward MIMO channels with non-noisy relays and noisy destination. As an instance of application of our framework, we evaluate closed-form expressions for both RCEE and EEE for multiple-scattering Rayleigh MIMO channels, with an arbitrary but finite number of scattering stages.**

## I. INTRODUCTION

The error exponent of a channel code is an important metric to understand the performance of a communication system. Indeed, it gives expression to the trade-off that exists between the average block-error probability (corresponding to the optimum code) and the required coding length at a prescribed rate below the channel capacity.

Owing to the difficulty in evaluating it, bounds have been proposed since early stage of information-theoretic analysis of communication systems. Among these, the largely adopted Random Coding Error Exponent (RCEE), proposed by Gallager [1], is based on random selection of the codewords with equal weight. Its refined version (see again [1]) assumes that bad codewords are expurgated from the actual set of codewords in order to decrease the error probability. In the case of MIMO channels, evaluation of the error exponent hinges upon the exploitation of Harish-Chandra-Itzykon-Zuber (HCIZ)-like integrals on matrix spaces, and it has been carried out in some particular scenarios with Rayleigh fading[1]. In particular, error exponents of MIMO block-fading channels have been already evaluated for spatially correlated Rayleigh fading in [3], and for the case of single and multi-keyholes in [4] and [5], respectively. Rayleigh-product channel exponents have been evaluated in [4], too, assuming the channel matrix to be the product of two rectangular matrices, with i.i.d. complex standard Gaussian entries.

In this paper, we first provide general expressions for both RCEE and EEE in the MIMO case, assuming that Channel State Information (CSI) is available at the receiver only. These expressions only depend on the joint law of the squared non-zero, singular values of the channel matrix. They can be further elaborated to express the error exponent bound in nice closed form, for most channel models of interest. We then specialize our findings to a MIMO system whose channel matrix is the product of an arbitrary number of independent rectangular matrices with standard Gaussian i.i.d. entries. This case models both a Rayleigh-faded, multiple-scattering channel with uncorrelated scatterers, as well as a multi-hop MIMO relay channel with Uniform Power Allocation (UPA) at each relay stage, non-noisy relays and noisy received signal.

We remark that, without CSI at either link ends, the unique available result on the error exponent in MIMO systems is for Rayleigh channels and is derived in [6].

The rest of the paper is organized as follows. Section II describes the system model under study. Section III provides the expressions for the error exponent in the general case of a MIMO system with block-memoryless fading channel. Such expressions are then evaluated in the case of multiple-scattering in Section IV. Concluding remarks are provided in Section V, while proofs of main statements are relegated to the Appendices.

## II. SYSTEM MODEL

Let the input-output relationship[2]

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{N} \tag{1}$$

model a block-memoryless fading MIMO channel with $n_t$ transmit and $n_r$ receive antennas and coherence time equal to $n_c$ channel uses[3]. In (1), the $n_r \times n_c$ matrix $\mathbf{Y}$ represents the output, the $n_t \times n_c$ matrix $\mathbf{X}$ is the channel input, the entries of the $n_r \times n_t$ matrix $\mathbf{H}$ represent the channel gains, and $\mathbf{N}$ represents AWGN noise. We assume that information

[1]In [2], where first the problem was set down, there is no final analytic expression for the error exponent.

[2]Uppercase and lowercase boldface letters denote matrices and vectors, respectively and the identity matrix is denoted by $\mathbf{I}$. For notation simplicity we denote the pdf of a random matrix $\mathbf{A}$, $p_{\mathbf{A}}(\mathbf{A})$, simply by $p(\mathbf{A})$. $\mathbb{E}[\cdot]$ represents statistical expectation, $(\cdot)^{\mathsf{H}}$ indicates the conjugate transpose operator. Also $\mathsf{Tr}\{\mathbf{A}\}$ and $|\mathbf{A}|$ denote, respectively the trace and the determinant of the square matrix $\mathbf{A}$. Moreover $\|\mathbf{A}\|^2 = \mathsf{Tr}\{\mathbf{A}\mathbf{A}^{\mathsf{H}}\}$ stands for the Euclidean norm of $\mathbf{A}$.

[3]This is tantamount to assume that the channel remains constant for $n_c$ symbol periods and changes independently to a new value every successive $n_c$ symbols.

is encoded in such a way that each codeword spans over $n_b n_c$ channel uses, i.e., we collect $n_b$ independent realizations of (1).

Under the assumption of CSI available at the receiver only, one can bound the average error probability achievable by a code of rate $R$ with maximum likelihood decoding as [1, Ch. 7]:

$$P_e \leq \left(\frac{2\mathrm{e}^{r\delta}}{\chi}\right)^2 \exp\left(-n_b n_c E(p(\mathbf{X}), R, n_c)\right), \tag{2}$$

where $r, \delta > 0$, $\chi \approx \frac{\delta}{\sqrt{2\pi n_b \sigma_\chi^2}}$,

$$\sigma_\chi^2 = \mathbb{E}_{\mathbf{X}}\left[\left(\|\mathbf{X}\|^2 - n_c\mathcal{P}\right)^2\right]. \tag{3}$$

and $E(p(\mathbf{X}), R, n_c)$ is the RCEE. In (3), $\mathcal{P}$ denotes the average input-power constraint, i.e.,

$$\mathbb{E}_{\mathbf{X}}[\|\mathbf{X}\|^2] \leq n_c\mathcal{P} \tag{4}$$

for a given distribution of the input matrix $p(\mathbf{X})$. The RCEE in (2) is given by

$$E(p(\mathbf{X}), R, n_c) = \max_{0\leq\rho\leq1}\left\{\max_{r\geq0} -\frac{\ln\mathcal{E}}{n_c} - \rho R\right\}, \tag{5}$$

where $\mathcal{E}$ is defined as

$$\mathcal{E} = \mathbb{E}_{\mathbf{H}}\left[\int_{\mathbb{C}^{n_r\times n_c}} \mathbb{E}_{\mathbf{X}}\left[p(\mathbf{Y}|\mathbf{X},\mathbf{H})^{\frac{1}{1+\rho}} \mathrm{e}^{r(\|\mathbf{X}\|^2 - n_c\mathcal{P})}\right]^{1+\rho} \mathrm{d}\mathbf{Y}\right]. \tag{6}$$

Rigorously, random coding equally weights both good as well as bad codewords. An improved bound for the average error probability can be obtained by expurgating bad codewords from the code ensemble (see, e.g., [1]). Such an expurgating procedure leads to the following error probability upper bound

$$P_e \leq \exp\left(-n_b n_c E_e(p(\mathbf{X}), R, n_c) + o(1)\right), \tag{7}$$

where

$$E_e(p(\mathbf{X}), R, n_c) = \max_{0\leq\rho\leq1}\left\{\max_{r\geq0} -\frac{\ln\mathcal{E}_e}{n_c} - \rho R\right\}, \tag{8}$$

is the EEE, with $\mathcal{E}_e$ denoting the matrix integral

$$\mathcal{E} = \mathbb{E}_{\mathbf{H}}\left[\mathbb{E}_{\mathbf{X},\mathbf{X}'}\left[\mathrm{e}^{r(\|\mathbf{X}\|^2 + \|\mathbf{X}'\|^2 - 2n_c\mathcal{P})} w(\mathbf{X},\mathbf{X}',\mathbf{H})^{\frac{1}{\rho}}\right]^{\rho}\right] \tag{9}$$

and

$$w(\mathbf{X},\mathbf{X}',\mathbf{H}) = \int_{\mathbb{C}^{n_r\times n_c}} \sqrt{p(\mathbf{Y}|\mathbf{X},\mathbf{H})p(\mathbf{Y}|\mathbf{X}',\mathbf{H})}\,\mathrm{d}\mathbf{Y}. \tag{10}$$

In (9) $\mathbf{X}'$ shares the same distribution as $\mathbf{X}$ and represents the input signal of good codewords.

Note that, without CSI at the receiver, the expressions in (6) and (9) would depend on $p(\mathbf{Y}|\mathbf{X})$ and $p(\mathbf{Y}|\mathbf{X}')$ rather than on $p(\mathbf{Y}|\mathbf{X},\mathbf{H})$ and $p(\mathbf{Y}|\mathbf{X}',\mathbf{H})$. However, $p(\mathbf{Y}|\mathbf{X})$ and $p(\mathbf{Y}|\mathbf{X}')$ are difficult to evaluate, but for the case of uncorrelated Rayleigh fading. The case where CSI is not available at the receiver is currently under investigation and will not be reported in this work.

We further observe that the optimal distribution $p(\mathbf{X})$ is the one that maximizes the error exponents $E(p(\mathbf{X}), R, n_c)$ and $E_e(p(\mathbf{X}), R, n_c)$. However, in the following, as usually done in the literature (see, e.g., [3]–[5] and references therein) we assume that $p(\mathbf{X})$ follows the Gaussian distribution, i.e,

$$p(\mathbf{X}) = \mathrm{e}^{-\mathsf{Tr}\{\mathbf{Q}^{-1}\mathbf{X}\mathbf{X}^{\mathsf{H}}\}}|\pi\mathbf{Q}|^{-n_c}, \tag{11}$$

where the covariance matrix $\mathbf{Q}$ should satisfy the average power constraint

$$\mathbb{E}_{\mathbf{X}}[\|\mathbf{X}\|^2] = n_c\mathsf{Tr}\{\mathbf{Q}\} \leq n_c\mathcal{P}.$$

This assumption simplifies the evaluation of the error exponent. Also, the Gaussian law for $\mathbf{X}$ is optimal if the rate $R$ approaches channel capacity.

When $R$ is close to the capacity and CSI is available at the receiver but not at the transmitter, UPA across transmit antennas yields optimal performance. Under UPA, the covariance matrix of the channel input is scalar. That is, it can be written as $\mathbf{Q} = \gamma\mathbf{I}$ where $\gamma = \frac{\mathcal{P}}{n_t}$ represents the per-antenna transmit power, assuming the average power constraint is met with equality.

## III. Error Exponent Analysis

In this section, we state two theorems providing the expression of the error exponents. More specifically, in Section III-A we provide the expression of the random coding error exponent in (6), while in Section III-B we obtain an analytic expression of the expurgated error exponent given in (9).

### A. RCEE evaluation

*Theorem 3.1:* Consider a block-fading channel as in (1) where the channel pdf, $p(\mathbf{H})$, depends on $\mathbf{H}$ through $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ only. Then the RCEE related to an observation window of $n_b$ independent fading blocks of $n_c$ channel uses each, can be expressed as per (5) with[4]

$$\mathcal{E} = k\int_{\mathbb{R}^{+m}} p(\mathbf{\Lambda})\prod_{\ell=1}^{m}\left(\alpha + \frac{\gamma\lambda_\ell}{1+\rho}\right)^{-n_c\rho}\mathrm{d}\mathbf{\Lambda}, \tag{12}$$

where $k = \frac{\mathrm{e}^{-rn_c\mathcal{P}(1+\rho)}}{\alpha^{n_c(n_t+\rho(n_t-m))}}$, $\alpha = 1-\gamma r$, $m = \min\{n_t, n_r\}$, and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ denotes the diagonal matrix of the eigenvalues $\mathbf{H}\mathbf{H}^{\mathsf{H}}$.

**Proof:** The proof is given in Appendix A.

### B. EEE evaluation

*Theorem 3.2:* Consider a block-fading channel as in (1) where the channel pdf, $p(\mathbf{H})$, depends on $\mathbf{H}$ through $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ only. Then the EEE related to an observation window of $n_b$ independent fading blocks of $n_c$ channel uses each, can be expressed as per (8) with

$$\mathcal{E}_e = k_e\int_{\mathbb{R}^{+m}} p(\mathbf{\Lambda})\prod_{\ell=1}^{m}\left(\alpha^2 + \frac{\alpha\gamma\lambda_\ell}{2\rho}\right)^{-n_c\rho}\mathrm{d}\mathbf{\Lambda} \tag{13}$$

where $k_e = \mathrm{e}^{-2rn_c\mathcal{P}\rho}/\alpha^{2n_c\rho(n_t-m)}$ and $m$, $\mathbf{\Lambda}$, and $r$ are defined as in Theorem 3.1.

**Proof:** The proof is given in Appendix A.

[4] We have to remark that our derivation, albeit slightly more compact, partially overlaps with contents of [3, Appendix], where however no general considerations on the channel matrix spectrum are made. Therein, the analysis is focused on some instances of Rayleigh-faded channels.

## IV. Application to Multiple-Scattering Channels

As an instance of application of our Theorems 3.1 and 3.2, we now provide closed-form expressions for RCEE and EEE in case where the channel $\mathbf{H}$ is subject to multiple scattering and where the channel between any two successive scattering stages, included the link between the transmitter and the first scatter and the link between the last scatter and the receiver are subject to Rayleigh fading. This is tantamount to assume that the channel matrix $\mathbf{H}$ is the product of an arbitrary number of independent, rectangular matrix-variate factors. In the case of $M-1$ successive scattering stages, the matrix $\mathbf{H}$ is given by

$$\mathbf{H} = \mathbf{H}_M \ldots \mathbf{H}_i \ldots \mathbf{H}_1 \,, \tag{14}$$

where matrix $\mathbf{H}_i$ has size $n_i \times n_{i-1}$, for $i = 1, \ldots, M$, $n_0 = n_t$, and $n_M = n_r$. If the entries of $\mathbf{H}_i$ are independent, Gaussian distributed, circularly-complex with zero mean and unit variance, the pdf of $\mathbf{H}_i$ is given by

$$p(\mathbf{H}_i) = \mathrm{e}^{-\mathsf{Tr}\{\mathbf{H}_i \mathbf{H}_i^{\mathsf{H}}\}} \pi^{-n_i n_{i-1}} \,.$$

This scenario can be thought as an extension of the model in [7] to an arbitrary but finite number of Rayleigh factors with uncorrelated sensors and scatterers.

*Proposition 4.1:* For a block-fading channel as in (1) and in case of $M-1$ Rayleigh scattering stages and absence of spatial correlation at either link ends, the RCEE error exponent is given by (5) where

$$\mathcal{E} = \frac{m! k}{\mathcal{Z}_m} |\mathbf{Z}| \tag{15}$$

$k$ is given after (12) and $\mathcal{Z}_m$ is a normalization constant. The elements of the $m \times m$ matrix $\mathbf{Z}$ are given by

$$(\mathbf{Z})_{i,j} = \left[ \frac{\alpha(1+\rho)}{\gamma} \right]^j \frac{\alpha^{-n_c \rho}}{\Gamma(n_c \rho)} \times \tag{16}$$
$$G^{M+1,1}_{1,M+1} \left( \begin{matrix} 1-j \\ n_c\rho - j, n_M, \ldots, n_2, n_1 + i - 1 \end{matrix} \middle| \frac{\alpha(1+\rho)}{\gamma} \right)$$

where $G(\cdot)$ is the the Meijer G function [11, Ch. 8]. Similarly, the EEE is given by (13) where

$$\mathcal{E}_e = \frac{m! k_e}{\mathcal{Z}_m} |\mathbf{Z}_e| \,, \tag{17}$$

and where $k_e$ is defined after (13). The elements of the $m \times m$ matrix $\mathbf{Z}_e$ are given by

$$(\mathbf{Z}_e)_{i,j} = \frac{G^{M+1,1}_{1,M+1} \left( \begin{matrix} 1-j \\ n_c\rho - j, n_M, \ldots, n_1 + i - 1 \end{matrix} \middle| \frac{2\alpha\rho}{\gamma} \right)}{\Gamma(n_c\rho)(2\rho)^{-j}\gamma^j \alpha^{2n_c\rho - j}} \tag{18}$$

**Proof:** The proof is given in Appendix B.

## V. Conclusion

We presented a unifying framework to evaluate in closed form the error exponent bounds for MIMO block-fading channels when CSI is available at the receiver, and under widely used constraints on the channel matrix distribution. Within this setting, RCEE and EEE are evaluated for the case where the channel is composed of a cascade of $M$ MIMO Rayleigh fading links, with a finite number of antennas and uncorrelated scatterers. This analysis paves the way for the optimal design of multi-hop MIMO relay channels, which will be the subject of our future work.

## Appendix A
## Proof of Theorems 3.1 and 3.2

Under the assumption of UPA, i.e., $\mathbf{Q} = \gamma \mathbf{I}$, the density of the input specified in (11) is given by

$$p(\mathbf{X}) = \exp\left(-\|\mathbf{X}\|^2/\gamma\right) (\pi\gamma)^{-n_c n_t} \,. \tag{19}$$

The conditional law $p(\mathbf{Y}|\mathbf{X},\mathbf{H})$ appearing in (6) is Gaussian non-central matrix-variate and can be written as

$$p(\mathbf{Y}|\mathbf{X},\mathbf{H}) = \exp\left(-\|\mathbf{Y} - \mathbf{H}\mathbf{X}\|^2\right) \pi^{-n_r n_c} \,. \tag{20}$$

Let $t(\mathbf{Y},\mathbf{H})$ be the average over $\mathbf{X}$ appearing in (6). Then

$$\begin{aligned} t(\mathbf{Y},\mathbf{H}) &= \mathbb{E}_{\mathbf{X}} \left[ p(\mathbf{Y}|\mathbf{X},\mathbf{H})^{\frac{1}{1+\rho}} \mathrm{e}^{r(\|\mathbf{X}\|^2 - n_c \mathcal{P})} \right] \\ &= \int_{\mathbb{C}^{n_t \times n_c}} p(\mathbf{X}) p(\mathbf{Y}|\mathbf{X},\mathbf{H})^{\frac{1}{1+\rho}} \mathrm{e}^{r(\|\mathbf{X}\|^2 - n_c \mathcal{P})} \, \mathrm{d}\mathbf{X} \,. \end{aligned}$$

By substituting in the above equation the expressions for $p(\mathbf{X})$ and $p(\mathbf{Y}|\mathbf{X},\mathbf{H})$, we obtain

$$\begin{aligned} t(\mathbf{Y},\mathbf{H}) &= c \int \mathrm{e}^{\mathsf{Tr}\left\{-\mathbf{L}\mathbf{X}\mathbf{X}^{\mathsf{H}} + \frac{1}{1+\rho}\left(\mathbf{H}^{\mathsf{H}}\mathbf{Y}\mathbf{X}^{\mathsf{H}} + \mathbf{X}\mathbf{Y}^{\mathsf{H}}\mathbf{H}\right)\right\}} \, \mathrm{d}\mathbf{X} \\ &\overset{(a)}{=} c \frac{\mathrm{e}^{\mathsf{Tr}\left\{\frac{1}{(1+\rho)^2}\mathbf{H}\mathbf{L}^{-1}\mathbf{H}^{\mathsf{H}}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}\right\}}}{\pi^{-n_c n_t} |\mathbf{L}|^{n_c}} \,, \end{aligned} \tag{21}$$

where $c = \mathrm{e}^{-r n_c \mathcal{P} - \frac{\|\mathbf{Y}\|^2}{1+\rho}} / (\gamma^{n_c n_t} \pi^{n_c n_t + \frac{n_r n_c}{1+\rho}})$ and

$$\mathbf{L} = \left(\frac{1}{\gamma} - r\right)\mathbf{I} + \frac{\mathbf{H}^{\mathsf{H}}\mathbf{H}}{1+\rho} \,, \tag{22}$$

and where the equality $(a)$ relies on the result in [8, Appendix B]. We now compute the integral w.r.t. $\mathbf{Y}$ appearing in (6). We have

$$\begin{aligned} s(\mathbf{H}) &= \int_{\mathbb{C}^{n_r \times n_c}} t(\mathbf{Y},\mathbf{H})^{1+\rho} \, \mathrm{d}\mathbf{Y} \\ &= \int_{\mathbb{C}^{n_r \times n_c}} \left[ c \frac{\mathrm{e}^{\mathsf{Tr}\left\{\frac{1}{(1+\rho)^2}\mathbf{H}\mathbf{L}^{-1}\mathbf{H}^{\mathsf{H}}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}\right\}}}{\pi^{-n_c n_t} |\mathbf{L}|^{n_c}} \right]^{(1+\rho)} \mathrm{d}\mathbf{Y} \\ &= \frac{\mathrm{e}^{-r n_c \mathcal{P}(1+\rho)}}{|\gamma \mathbf{L}|^{n_c(1+\rho)}} \int_{\mathbb{C}^{n_r \times n_c}} \frac{\mathrm{e}^{-\mathsf{Tr}\{(\mathbf{I} - \frac{1}{1+\rho}\mathbf{H}\mathbf{L}^{-1}\mathbf{H}^{\mathsf{H}})\mathbf{Y}\mathbf{Y}^{\mathsf{H}}\}}}{\pi^{n_r n_c}} \, \mathrm{d}\mathbf{Y} \,. \end{aligned}$$

The above integral can be solved by using the property

$$\int_{\mathbb{C}^{n_r \times n_c}} \frac{\mathrm{e}^{-\mathsf{Tr}\{\mathbf{A}^{-1}\mathbf{Y}\mathbf{Y}^{\mathsf{H}}\}}}{\pi^{n_r n_c} |\mathbf{A}|^{n_c}} \, \mathrm{d}\mathbf{Y} = 1$$

which holds for any invertible square matrix $\mathbf{A}$. Then we get

$$\begin{aligned} s(\mathbf{H}) &= \frac{\mathrm{e}^{-r n_c \mathcal{P}(1+\rho)}}{|\gamma \mathbf{L}|^{n_c(1+\rho)}} \left| \mathbf{I} - \frac{1}{1+\rho}\mathbf{H}\mathbf{L}^{-1}\mathbf{H}^{\mathsf{H}} \right|^{-n_c} \\ &= \frac{\mathrm{e}^{-r n_c \mathcal{P}(1+\rho)}}{|\gamma \mathbf{L}|^{n_c(1+\rho)} (\gamma^{-1} - r)^{n_c n_t}} |\mathbf{L}|^{n_c} \\ &= \frac{\mathrm{e}^{-r n_c \mathcal{P}(1+\rho)}}{|\gamma \mathbf{L}|^{n_c \rho} \alpha^{n_c n_t}} \end{aligned} \tag{23}$$

where $\alpha = 1 - \gamma r$. We immediately observe that $s(\mathbf{H})$ is a function of the non-zero eigenvalues, $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$, of $\mathbf{H}\mathbf{H}^{\mathsf{H}}$, through $|\mathbf{L}|$. Indeed, by using the definition of $\mathbf{L}$ given in (22), we have

$$|\gamma\mathbf{L}| = \alpha^{n_t - m}\prod_{i=1}^{m}\left(\alpha + \frac{\gamma\lambda_i}{1+\rho}\right)$$

were $m = \min\{n_r, n_t\}$. Thus we can write

$$s(\mathbf{H}) = \tilde{s}(\mathbf{\Lambda}) = \frac{\mathrm{e}^{-rn_c\mathcal{P}(1+\rho)}}{\alpha^{n_c(n_t+\rho(n_t-m))}}\prod_{i=1}^{m}\left(\alpha + \frac{\gamma\lambda_i}{1+\rho}\right)^{-n_c\rho}.$$

If, as in our assumptions, the density of $\mathbf{H}$ depends only on $\mathbf{H}^{\mathsf{H}}\mathbf{H}$, the outer integral in the expression of the error exponent (i.e., the integration over $\mathbf{H}$) can be computed as

$$\begin{aligned}\mathcal{E} &= \mathbb{E}_{\mathbf{H}}[s(\mathbf{H})] \\ &= \int_{\mathbb{C}^{n_r\times n_t}} p(\mathbf{H})s(\mathbf{H})\,\mathrm{d}\mathbf{H} \\ &= \int_{\mathbb{R}^{+m}} p(\mathbf{\Lambda})\tilde{s}(\mathbf{\Lambda})\,\mathrm{d}\mathbf{\Lambda} \end{aligned} \tag{24}$$

where in the last equality we first applied the change of integration variable $\mathbf{H}^{\mathsf{H}}\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathsf{H}}$ (with $\mathbf{U}$ being a unitary matrix) and then the result in [9, eq. (93)]. By substituting in (24) the expression for $\tilde{s}(\mathbf{\Lambda})$, we obtain (12).

Theorem 3.2 can be proved in a similar way. Specifically, the integral in (10) is evaluated by using the expression for $p(\mathbf{Y}|\mathbf{X},\mathbf{H})$ in (20) and by resorting to [8, Appendix B], i.e.,

$$\begin{aligned} w(\mathbf{X},\mathbf{X}',\mathbf{H}) &= \frac{\mathrm{e}^{-\|\mathbf{H}\mathbf{X}\|^2/2-\|\mathbf{H}\mathbf{X}'\|^2/2}}{\pi^{n_r n_c}}\cdot \\ &\quad \int_{\mathbb{C}^{n_r\times n_c}}\mathrm{e}^{-\mathsf{Tr}\{\mathbf{Y}\mathbf{Y}^{\mathsf{H}}-\frac{\mathbf{Y}(\mathbf{X}+\mathbf{X}')^{\mathsf{H}}\mathbf{H}^{\mathsf{H}}}{2}-\frac{\mathbf{H}(\mathbf{X}+\mathbf{X}')\mathbf{Y}^{\mathsf{H}}}{2}\}}\,\mathrm{d}\mathbf{Y} \\ &= \mathrm{e}^{-\|\mathbf{H}\mathbf{X}\|^2/2-\|\mathbf{H}\mathbf{X}'\|^2/2+\|\mathbf{H}(\mathbf{X}+\mathbf{X}')\|^2/4} \\ &= \mathrm{e}^{-\|\mathbf{H}(\mathbf{X}-\mathbf{X}')\|^2/4}. \end{aligned} \tag{25}$$

As for the computation of (9), we first average w.r.t. $\mathbf{X}'$. This average too admits closed-form expression by virtue of [8, Appendix B]. Indeed, we have

$$\begin{aligned} v(\mathbf{X},\mathbf{H}) &= \mathbb{E}_{\mathbf{X}'}\left[\mathrm{e}^{r\|\mathbf{X}'\|^2}w(\mathbf{X},\mathbf{X}',\mathbf{H})^{\frac{1}{\rho}}\right] \\ &= \int_{\mathbb{C}^{n_t\times n_c}} p(\mathbf{X}')\mathrm{e}^{r\|\mathbf{X}'\|^2}\mathrm{e}^{-\|\mathbf{H}(\mathbf{X}-\mathbf{X}')\|^2/4\rho}\,\mathrm{d}\mathbf{X}' \\ &= \frac{\mathrm{e}^{-\|\mathbf{H}\mathbf{X}\|^2/4\rho}}{(\pi\gamma)^{n_t n_c}}\cdot \\ &\quad \int_{\mathbb{C}^{n_t\times n_c}}\mathrm{e}^{-\mathsf{Tr}\left\{\mathbf{X}'\mathbf{X}'^{\mathsf{H}}\mathbf{L}_e-\frac{\mathbf{H}^{\mathsf{H}}\mathbf{H}\mathbf{X}\mathbf{X}'^{\mathsf{H}}}{4\rho}-\frac{\mathbf{X}^{\mathsf{H}}\mathbf{H}^{\mathsf{H}}\mathbf{H}\mathbf{X}'}{4\rho}\right\}}\,\mathrm{d}\mathbf{X}' \\ &= \frac{\mathrm{e}^{-\|\mathbf{H}\mathbf{X}\|^2/4\rho}}{(\gamma)^{n_t n_c}|\mathbf{L}_e|^{n_c}}\mathrm{e}^{\mathsf{Tr}\{\mathbf{W}\mathbf{X}\mathbf{X}^{\mathsf{H}}\}} \end{aligned} \tag{26}$$

where

$$\mathbf{L}_e = \left(\frac{1}{\gamma} - r\right)\mathbf{I} + \frac{1}{4\rho}\mathbf{H}^{\mathsf{H}}\mathbf{H}$$

and

$$\mathbf{W} = \frac{\mathbf{H}^{\mathsf{H}}\mathbf{H}\mathbf{L}_e^{-1}\mathbf{H}^{\mathsf{H}}\mathbf{H}}{16\rho^2}.$$

We now average w.r.t. $\mathbf{X}$ and we obtain

$$\begin{aligned} s_e(\mathbf{H}) &= \mathbb{E}_{\mathbf{X}}\left[\mathrm{e}^{r\|\mathbf{X}\|^2}v(\mathbf{X},\mathbf{H})\right] \\ &= \int_{\mathbb{C}^{n_t\times n_c}}\frac{\mathrm{e}^{-\mathsf{Tr}\left\{(\mathbf{L}_e-\mathbf{W})\mathbf{X}\mathbf{X}^{\mathsf{H}}\right\}}}{(\pi\gamma^2)^{n_t n_c}|\mathbf{L}_e|^{n_c}}\,\mathrm{d}\mathbf{X} \\ &= |\gamma^2\mathbf{L}_e(\mathbf{L}_e-\mathbf{W})|^{-n_c} \\ &= \left|\alpha^2\mathbf{I}+\frac{\alpha\gamma}{2\rho}\mathbf{H}^{\mathsf{H}}\mathbf{H}\right|^{-n_c} \\ &= \alpha^{-2n_c(n_t-m)}\prod_{i=1}^{m}\left(\alpha^2+\frac{\alpha\gamma\lambda_i}{2\rho}\right)^{-n_c} \end{aligned} \tag{27}$$

where we recall that $\alpha = 1-\gamma r$ and $m = \min\{n_r, n_t\}$. Note that also $s_e(\mathbf{H})$ depends on the eigenvalues $\mathbf{\Lambda}$ of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$. Under the assumption that the density of $\mathbf{H}$ depends only on $\mathbf{H}^{\mathsf{H}}\mathbf{H}$, the integral in (9) provides the result reported in (13).

## APPENDIX B
## PROOF OF PROPOSITION 4.1

Following Theorems 3.1 and 3.2, the only quantity that has to be provided to perform error exponents evaluation is the joint law of the non-zero ordered eigenvalues of $\mathbf{H}\mathbf{H}^{\mathsf{H}}$. If $\mathbf{H}$ is given by (14), where each factor is an i.i.d. Gaussian matrix, then [10]

$$p(\mathbf{\Lambda}) = \frac{\mathcal{V}(\mathbf{\Lambda})}{\mathcal{Z}_m}|\mathbf{G}(\mathbf{\Lambda})|. \tag{28}$$

In (28), $\mathcal{V}(\mathbf{\Lambda}) = \prod_{1\le\ell<k\le m}(\lambda_k-\lambda_\ell)$ is the Vandermonde determinant of $\mathbf{\Lambda}$, $\mathcal{Z}_m$ a normalizing constant such that

$$\mathcal{Z}_m = \int_{\mathbb{R}^{+m}}\mathcal{V}(\mathbf{\Lambda})|\mathbf{G}(\mathbf{\Lambda})|\,\mathrm{d}\mathbf{\Lambda}$$

and $\mathbf{G}$ is an $m\times m$ matrix such that

$$(\mathbf{G})_{i,j} = G_{0,M}^{M,0}\left(\begin{array}{c} - \\ n_M,\ldots,n_2,n_1+i-1 \end{array} \middle| \lambda_j\right),$$

for $i,j = 1,\ldots,m$ and $G(\cdot)$ is the the Meijer G function [11, Ch. 8].

Then, the matrix integral $\mathcal{E}$ boils down to

$$\mathcal{E} = \frac{k}{\mathcal{Z}_m}\int_{\mathbb{R}^{+m}}\mathcal{V}(\mathbf{\Lambda})|\mathbf{G}(\mathbf{\Lambda})|\prod_{i=1}^{m}\left(\alpha+\frac{\gamma\lambda_i}{1+\rho}\right)^{-n_c\rho}\mathrm{d}\mathbf{\Lambda}. \tag{29}$$

As far as the EEE evaluation is concerned,

$$\mathcal{E}_e = \frac{k_e}{\mathcal{Z}_m}\int_{\mathbb{R}^{+m}}\mathcal{V}(\mathbf{\Lambda})|\mathbf{G}(\mathbf{\Lambda})|\prod_{i=1}^{m}\left(\alpha^2+\frac{\alpha\gamma\lambda_i}{2\rho}\right)^{-n_c\rho}\mathrm{d}\mathbf{\Lambda} \tag{30}$$

is to be computed. Both (29) and (30) can be expressed in closed-form according to [12, Corollary I], so that

$$\mathcal{E} = \frac{m!k}{\mathcal{Z}_m}|\mathbf{Z}|, \tag{31}$$

with the entries of $\mathbf{Z}$ given by

$$\int_0^{+\infty}\lambda^{j-1}\frac{G_{0,M}^{M,0}\left(\begin{array}{c} - \\ n_M,\ldots,n_2,n_1+i-1 \end{array} \middle| \lambda\right)}{\left(\alpha+\frac{\gamma}{1+\rho}\lambda\right)^{n_c\rho}}\,\mathrm{d}\lambda.$$

The above integral can be solved by using the result in [11, 7.811.5] yielding (16). Similarly

$$\mathcal{E}_e = \frac{m! k_e}{\mathcal{Z}_m} |\mathbf{Z}_e| \tag{32}$$

where after applying [12, Corollary I] the entries of $\mathbf{Z}_e$ are given by

$$\int_0^{+\infty} \lambda^{j-1} \frac{G_{0,M}^{M,0}\left( \begin{array}{c} - \\ n_M, \ldots, n_2, n_1{+}i{-}1 \end{array} \middle| \lambda \right)}{\left(\alpha^2 + \frac{\alpha\gamma\lambda}{2\rho}\right)^{n_c \rho}} \, \mathrm{d}\lambda \,.$$

Again, this integral can be expressed in closed form by virtue of [11, 7.811.5] yielding (18).

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