POLITECNICO DI TORINO
Doctoral Degree in Aerospace Engineering

Thesis

# Development of refined models for multilayered composite and sandwich structures Analytical formulation, FEM implementation and experimental assessment 

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"e volta nostra poppa nel mattino,
de' remi facemmo ali al folle volo"
(Dante Alighieri, Inferno, Canto XXVI)

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Le metodologie appaiono multidisciplinari, presentando aspetti teorici, numerici e sperimentali. I risultati sperimentali risultano ben correlati con gli sviluppi metodologici.

I risultati presentati sono originali, interessanti e potenzialmente in grado di migliorare l'efficienza delle tecniche numeriche applicate nel contesto della ricerca.

Nel colloquio il candidato dimostra una ottima padronanza degli argomenti affrontati e piena consapevolezza delle potenzialità delle tematiche trattate.

La Commissione unanime giudica eccellente il lavoro svolto e propone che al dott. IURLARO Luigi venga conferito il titolo di Dottore di Ricerca.

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## Introduction

Over the last three decades, composite materials have been increasingly used in different engineering field (military and civilian aircraft, aerospace vehicles, naval and civil structures) due to their high stiffness-to-weight and strength-to-weight ratios. Nowadays, relatively thick laminated composite and sandwich materials with one hundred or more layers find their applications in primary load-bearing structural components of the modern aircraft. To ensure a reliable design and failure prediction, accurate evaluation of the strain/stress state is mandatory.

A high-fidelity analysis of multilayered composite and sandwich structures can be achieved by adopting detailed 3D finite element models that turn into a cumbersome modeling at high computational cost. Thus, most of the researchers efforts are devoted to the development of approximated models wherein assumptions on the distribution of displacements and/or stresses are made, thus reducing the 3 -dimensional analysis to a 2 dimensional problem. Generally speaking, three kinds of 2D approximated models are available: Equivalent Single Layer models, Layer-wise theories and the Zigzag models. Even if computationally efficient, the Equivalent Single Layer models are not accurate in terms of local response prediction; the Layer-wise theories, that on the contrary ensure great accuracy, are computationally very expensive. In the ' 80 s , thanks to the original works by Prof. Di Sciuva, a new modeling strategy of multilayered composite and sandwich structures arose: the so-called Zigzag theories, wherein accuracy comparable with that proper of the Layer-wise models is achieved but saving the computational cost. Inspired by early Prof. Di Sciuva's works, many researchers developed Zigzag models in
the course of these thirty years, contributing to progress in the multilayered composite and sandwich structures modeling field.

For accuracy, computational cost and efficient finite element implementation, the most remarkable Zigzag model, inspired by the Prof. Di Sciuva's work, is the Refined Zigzag Theory, recently developed by Dr. Tessler, Prof. Di Sciuva and Prof. Gherlone. From its first appearance, the Refined Zigzag Theory has experienced several developments in terms of beam and plate finite element implementations and has been extensively assessed on static problems. Other researchers, by using the Refined Zigzag Theory as basis, developed models with higher-order kinematics and including secondary effects as the normal deformability one, obtaining fair results.

In this context, the present research work is motivated, first of all, by the necessity to go further into the investigation about the Refined Zigzag Theory prediction capabilities, to highlight and solve possible flaws and to develop an original higher-order zigzag model able to produce accurate results in those situations wherein the Refined Zigzag Theory is not adequate. Taking in mind the motivation of the present research work, the Thesis is organized following a logical (and chronological) order.

Chapter 1 is devoted to a detailed discussion and review of models for the analysis of multilayered composite and sandwich structures, highlighting merits and deficiencies.

In Chapter 2, the Refined Zigzag Theory for plates is briefly recalled. For the first time, the non-linear governing equations are obtained and specialized to the linear bending, undamped free vibrations and linear buckling problems. Moreover, due to the growing interest towards the advanced functionally graded materials, the Refined Zigzag Theory is extended to the analysis of multilayered plates embedding functionally graded layers. Finally, a comparison of the Refined Zigzag function with the Murakami's one is set.

In Chapter 3, the first substantial attempt to enhance the Refined Zigzag Theory is made. Based on the Reissner Mixed Variational Theorem, the Mixed Refined Zigzag Theory is developed in order to improve the constitutive transverse shear stresses prediction and, as consequence, the transverse shear stiffness estimation.

The formulation of a novel higher-order zigzag model (called (3,2)-Mixed Refined Zigzag Theory), originated by the Refined Zigzag Theory, is introduced in Chapter 4, wherein the discussion is focused on plate problems. Moreover, an extension to the thermo-mechanical analysis, but limited to the beam problems for sake of conciseness, is presented. Based on the Reissner Mixed Variational Theorem, the novel higher-order zigzag model includes the transverse normal deformability effect, by extending the
transverse displacement approximation with respect to the Refined Zigzag Theory, and the transverse normal stress, neglected in the Refined Zigzag Theory. The kinematics, in terms of in-plane displacements assumption, is also enriched with respect to the Refined Zigzag Theory. The formulation of the novel higher-order zigzag model is motivated by the need of a computational model able to accurately analyze the response of thick multilayer composite beams/plates wherein secondary effects (the higher-order displacements patterns along the thickness, the transverse normal deformability and the transverse normal stress), neglected by the Refined Zigzag Theory, become significant.

Finite elements implementations are the subject of Chapter 5. Firstly, a novel Refined Zigzag Theory-based beam element, employing exact static shape functions, is presented. Secondly, a (3,2)-Mixed Refined Zigzag Theory-based beam element suitable for a thermo-mechanical analysis of thick multilayered composite and sandwich beams is formulated. Later, a (3,2)-Mixed Refined Zigzag Theory-based plate element is introduced.

Chapters 6 and 7 are devoted to the numerical results. In Chapter 6, only analytical solutions, that is exact or approximated ones by using the Rayleigh-Ritz's method, are presented. A in-deep investigation of the Refined Zigzag Theory prediction capabilities in problems concerning the linear bending, free vibrations and buckling problems of multilayered composite and sandwich plates, also including functionally graded layers, is carried out. To assess the improvements of the Mixed Refined Zigzag Theory with respect to the original displacement-based model formulation, the linear bending and free vibrations problems of laminated composite and sandwich plates are taken into consideration. Finally, the (3,2)-Mixed Refined Zigzag Theory performances are tested on the bending problem of a thick laminated composite plate. The finite element results, collected in Chapter 7, are devoted to the assessment of the RZT-based beam element employing exact static shape functions, both on static and free vibrations problems, and comparing the results with those obtained by means an already developed beam element based on the same underlying theory. The (3,2)-Mixed Refined Zigzag-based beam and plate elements, once their convergence is proved, are used to solve static and dynamic problems.

Finally, in Chapter 8, the results of an experimental campaign acted out are presented along with the experimental set-up used. The tests concern the four-point bending test of sandwich beams, for the static analysis, and hammer test on cantilever sandwich beams, for the free vibrations problems. The experimental results are compared with those obtained by using the Refined Zigzag Theory-based beam element as further theory assessment.

The present research activity supports the great accuracy of the Refined Zigzag Theory and for this reason deals with some overlooked aspects, as the application to the functionally graded materials (Chapter 2), the mixed formulation (Chapter 3), the implementation of a beam finite element employing exact static shape functions (Chapter 5) and the correlation with experimental results (Chapter 8). By enriching the Refined Zigzag Theory and using the Reissner Mixed Variational Theorem, a novel higher-order mixed zigzag model, called (3,2)-Mixed Refined Zigzag Theory is developed (Chapter 4). The higher-order zigzag model constitutes the underlying theory for a beam finite element, suitable for a thermo-mechanical analysis, and a plate element, formulated taking into account only mechanical loads. The results presented (Chapters 6-8), along with those already published in the open literature by other authors, still encourage the use of the Refined Zigzag Theory in the analysis of relatively thick multilayered composite and sandwich structures. Moreover, when the transverse normal stress and the transverse normal deformability effects are not negligible, the (3,2)-Mixed Refined Zigzag Theory appears proficient to solve these cases in virtue also of its efficient finite element implementations.

The author's auspice is that the models belonging to the Refined Zigzag Theory class becomes to attract attention of the companies involved in the design and analysis of multilayered composite and sandwich structures and of the finite element commercial codes that still implemented models not suitable for the analysis of composite and sandwich structures, as extensively demonstrated.

## Preliminaries

The objective of this section is to briefly review some basic equations, concerning the mechanics of orthotropic materials, that are abundantly used in the theoretical developments. Many details, above all about the mathematics, are omitted and the interested reader can refer to [Reddy, 2004] for a complete discussion.

## Strain-displacement relations

Consider a deformable body of known geometry, constitution, load and boundary conditions. Each material points of the body is referred to a Cartesian coordinate system ( $x_{1}, x_{2}, x_{3}$ ) and the Cartesian components of the displacement vector $\mathbf{u}$ are $U_{i}$ where the Latin index takes values 1, 2 and 3 .

According to the standard solid mechanics [Reddy, 2004], the strain is measured by using the Green-Lagrange strain tensor $\mathbf{E}$, that is defined in terms of displacement gradients as [Reddy, 2004]

$$
\begin{equation*}
E=\frac{1}{2}\left[\nabla \mathbf{u}+(\nabla \mathbf{u})^{T}+\nabla \mathbf{u} \cdot(\nabla \mathbf{u})^{T}\right] \tag{P.1}
\end{equation*}
$$

Performing the dot product, the orthogonal components of the Green-Lagrange tensor are given by

$$
\begin{equation*}
E_{j k}=\frac{1}{2}\left(\partial_{k} U_{j}+\partial_{j} U_{k}+\partial_{j} U_{i} \partial_{k} U_{i}\right) \tag{P.2}
\end{equation*}
$$

where the notation $\partial_{i}$ is used to denote the derivative with respect to the coordinate $x_{i}$. If the displacement gradient is small, that is $|\nabla \mathbf{u}| \ll 1$, the $\mathbf{E}$ tensor reduces to the infinitesimal strain tensor, $\varepsilon$, which components read as

$$
\begin{equation*}
\varepsilon_{i j}=\frac{1}{2}\left(\partial_{i} U_{j}+\partial_{j} U_{i}\right) \tag{P.3}
\end{equation*}
$$

## Constitutive material law

Herein, the constitutive equation of linear elasticity for the case of infinitesimal deformation of an orthotropic material, referred to as the Hooke's law, is discussed.

Denoted with $\boldsymbol{\sigma}$ the stress tensor, the material constitutive law reads as [Reddy, 2004]

$$
\begin{equation*}
\sigma_{i j}=\tilde{C}_{i j k l} \varepsilon_{k l} \tag{P.4}
\end{equation*}
$$

where $\tilde{C}_{i j k l}$ is the fourth-order tensor of material parameters, called stiffness tensor, and for an orthotropic material it depends on nine independent coefficients.

With reference to Eq. (P.4), a plane stress state with respect to the $x_{I}-x_{2}$ plane is characterized by the following condition

$$
\begin{equation*}
\sigma_{33}=0 \tag{P.5}
\end{equation*}
$$

The constitutive material law, according to the plane stress state, reads as

$$
\begin{align*}
& \sigma_{\alpha \beta}=C_{\alpha \beta \gamma \delta} \varepsilon_{\gamma \delta} \\
& \tau_{\alpha 3}=\tilde{C}_{\alpha 3 \beta 3} \gamma_{\beta 3} \tag{P.6}
\end{align*}
$$

where the Greek index takes values 1 and 2 and $\gamma_{\alpha 3}$ is the engineering transverse shear strain, defined as $\gamma_{\alpha 3}=2 \varepsilon_{\alpha 3}$. In Eq. (0.6), the reduced stiffness coefficients, $C_{\alpha \beta \gamma \delta}$, appear and read as

$$
\begin{equation*}
C_{\alpha \beta \gamma \delta}=\tilde{C}_{\alpha \beta \gamma \delta}-\frac{\tilde{C}_{\alpha \beta 33} \tilde{C}_{\gamma \delta 33}}{\tilde{C}_{3333}} \tag{P.7}
\end{equation*}
$$

whereas the stiffness coefficients related with the transverse shear stresses are not affected by the plane-stress assumptions. For these stiffness coefficients, since the subscript depends only on two indices, a different notation is adopted: $Q_{\alpha \beta}=\tilde{C}_{\alpha 3 \beta 3}$.

It is worth to note that, by making use of Eq. (P.6), it is possible to state in an equivalent way, called mixed form, the constitutive law, Eq. (P.4). By introducing the compliant coefficient, $S_{33}=\tilde{C}_{3333}^{-1}$, the constitutive material law in mixed form reads as

$$
\begin{gather*}
\sigma_{\alpha \beta}=C_{\alpha \beta \gamma \delta} \varepsilon_{\gamma \delta}+S_{33} R_{\alpha \beta} \sigma_{33} \\
\tau_{\alpha 3}=Q_{\alpha \beta} \gamma_{\beta 3}  \tag{P.8}\\
\varepsilon_{33}=S_{33} \sigma_{33}-S_{33} R_{\alpha \beta} \varepsilon_{\alpha \beta}
\end{gather*}
$$

wherein $R_{\alpha \beta}=\tilde{C}_{\alpha \beta 33}$.
When the thermal effect due to a temperature variation, $\Theta=T-T_{0}$, with respect to a reference one, $T_{0}$, has to be taken into consideration, the constitutive law is enriched by the thermal contribution and the mixed form of the constitutive law, Eq. (P.8), becomes [Gherlone et al., 2007]

$$
\begin{gather*}
\sigma_{\alpha \beta}=C_{\alpha \beta \gamma \delta} \varepsilon_{\gamma \delta}+S_{33} R_{\alpha \beta} \sigma_{33}-\omega_{\alpha \beta} \Theta \\
\tau_{\alpha 3}=Q_{\alpha \beta} \gamma_{\beta 3}  \tag{P.9}\\
\varepsilon_{33}=S_{33} \sigma_{33}-S_{33} R_{\alpha \beta} \varepsilon_{\alpha \beta}+S_{33} \lambda_{33} \Theta
\end{gather*}
$$

where $\omega_{\alpha \beta}=\lambda_{\alpha \beta}-S_{33} \lambda_{33} R_{\alpha \beta}$ and $\lambda_{i j}=\tilde{C}_{i j k} \alpha_{k l}$ with $\alpha_{k l}$ denoting the thermal expansion coefficients in the geometric axes.

## Chapter 1

## Theories for laminated plates

## 1. Introduction

In the last thirty years, composite materials have been increasingly used in many industrial applications due to their high specific mechanical properties, reduced weight, high corrosion and fatigue resistance. Along with these positive aspects, multilayered composite structures offer the possibility to tailor the mechanical properties according to the specific application by choosing carefully the fiber orientation and the stacking sequence. In aerospace field, the increasing use of composite and sandwich materials for primary load bearing structural components requires computational tools able to accurately predict the stress field in order to achieve reliable design and failure analysis. In particular near the geometric singularities, like as holes or free edges, where the stress field becomes three-dimensional and stress intensity factors arise, the computational model has to be very accurate, above all on the prediction of the transverse stresses, due to the key role played by them in damage mechanisms like as debonding and delamination.

It is trivial to note that the actual stress field in multilayered composite structures can be computed by exact Elasticity-based solutions. In the open literature, several exact solutions are available: Pagano [Pagano, 1969; Pagano, 1970] developed exact solutions for crossply and sandwich plates in the framework of linear Elasticity; Srinivas and Rao [Srinivas et
al., 1970; Srinivas et al., 1971] obtained exact solution for the static and dynamic analysis of thick laminates; Noor and Burton [Noor et al., 1990], Savoia and Reddy [Savoia et al., 1992] published solutions for cross-ply and antisymmetric angle-ply rectangular plates. It is easy to note that exact Elasticity solutions are available only for a limited set of geometries, loads and boundary conditions (in the case of plates, mainly bi-sinusoidal transverse pressure and simply support boundary condition). By using the Fourier series, the exact Elasticity solution can be easily extended to case of a plate with a general load. When an exact solution is not available, stress analysis can be performed by high-fidelity 3D finite elements models that turns out to be accurate but computationally expensive. From this point of view, an interesting solution is to develop approximate 2D models: in these models, the distribution along one of the coordinate axes of the reference frame is assumed for the primary variables, thus reducing a 3 D problem to a 2 D one. In the framework of the displacement-based plate/shell models, the through-the-thickness distribution of displacement components is assumed whereas the behavior of the elastic body in the plane generated by the remaining two coordinate axes is recovered by solving the governing equilibrium equations along with the variationally consistent boundary conditions.

In this chapter, the models formulated for the analysis of multilayered composite and sandwich plate/shell structures are examined following a common accepted classification [Reddy, 2004]. The models can be divided into (i) displacement-based models, wherein the primary variables are only the displacement components on which the assumptions are made; and (ii) mixed models, wherein displacements and stresses are assumed independently. Along with this classification, the plate/shell models can be divided into (a) Equivalent Single Layer (ESL) models and (b) Layer-wise (LW) ones, according to the type of kinematics assumed. In the former class of structural theories, the kinematics assumed is $C^{l}$-continuous, that is the displacement components and their derivatives are assumed to vary continuously along the plate/shell thickness. This means that the multilayer structure is substituted with a plate/shell made by an equivalent single layer. On the contrary, LW models postulates a $C^{0}$-continuous kinematics, that is a distribution of displacements (first of all, in-plane displacements) continuous along the thickness with first order derivatives showing a jump at layer interfaces (the reason of this discontinuity will be explained hereinafter). Moreover, according to [Ghugal et al., 2002] the LW models are further divided into (a) layer dependent, wherein the number of kinematic variables increases with the number of layers; and (b) layer independent, wherein the
number of variables remains constant and independent on the number of layers. A particular class of LW layer independent models are the so-called Zigzag (ZZ) models. In this context, the terminology "layer-wise models" is used to indicate only the layer dependent LW theories, whereas the ZZ models are treated separately.

## 2. Variational statement

According to the classification before mentioned, the plate/shell models can be divided into displacement-based and mixed models, depending on the type of primary variables. This difference reflects also the variational principle on which the model is formulated. In fact, displacement-based models are formulated via the Virtual Displacements Principle [Reddy, 2004] whereas mixed models are developed using the Reissner Mixed Variational Theorem [Reissner, 1950]. In this paragraph, the two variational statement are briefly recalled.

### 2.1. Virtual Displacements Principle

All the displacement-based models are formulated via the Virtual Displacement Principle. Omitting many details that the reader can find in [Reddy, 2004], it is important to remark that the Virtual Displacements Principle is the weak form of the 3D Elasticity equations that are solved by assuming as arbitrary test functions the virtual displacements [Zienkiewicz et al., 2000]. In formula, the principle reads as

$$
\begin{equation*}
\int_{\Omega} \delta \boldsymbol{\varepsilon}^{T} \boldsymbol{\sigma} d \Omega-\int_{\Omega} \delta \mathbf{u}^{T} \mathbf{b} d \Omega-\int_{\Gamma} \delta \mathbf{u}^{T} \mathbf{t} d \Gamma=0 \tag{1.1}
\end{equation*}
$$

where $\Omega, \Gamma$ are the body volume and the external surface, respectively; $\boldsymbol{\varepsilon}, \boldsymbol{\sigma}$ are the strain and stress vectors and $\mathbf{u}$ contains the displacement components. The body forces and the external surface applied loads are quoted as $\mathbf{b}, \mathbf{t}$, respectively. Recognizing the virtual variation of the strain energy, $\delta U$, the virtual variation of the work done by inertial forces, $\delta W_{i}$, and that done by external applied loads, $\delta W$, the Virtual Displacement Principle appears also as

$$
\begin{equation*}
\delta U-\delta W_{i}-\delta W=0 \tag{1.2}
\end{equation*}
$$

that represents the common way to present the Principle of Virtual Work (D'Alembert's Principle).

When a displacement-based model is developed, the only primary variables are the displacements thus the strain field and the stress one come from the strain-displacement relations and the constitutive material law, respectively.

### 2.2. Reissner Mixed Variational Theorem

In the framework of multilayered composite structures modeling, an important role is played by the Reissner Mixed Variational Theorem [Reissner, 1950], since it allows for an independent assumption on displacements and transverse shear and normal stresses. It is interesting to note that the Reissner Mixed Variational Theorem is a particular case of a more general two-field variational principle, the Hellinger-Reissner one [Hellinger, 1914] that allows independent assumption on displacements and all the six stress tensor components. Details about the two principles can be found in [Zienkiewicz et al., 2000]; in this context the Reissner Mixed Variational Theorem is stated and the quantities involved explained. The above cited theorem reads as

$$
\begin{align*}
& \int_{\Omega} \delta \boldsymbol{\varepsilon}^{T} \boldsymbol{\sigma} d \Omega+\int_{\Omega} \delta \boldsymbol{\tau}^{a T}\left(\boldsymbol{\gamma}-\boldsymbol{\gamma}^{a}\right) d \Omega+\int_{\Omega} \delta \sigma_{33}^{a}\left(\varepsilon_{33}-\varepsilon_{33}^{a}\right) d \Omega- \\
& \quad \int_{\Omega} \delta \mathbf{u}^{T} \mathbf{b} d \Omega-\int_{\Gamma} \delta \mathbf{u}^{T} \mathbf{t} d \Gamma=0 \tag{1.3}
\end{align*}
$$

where $\boldsymbol{\tau}^{a}, \sigma_{33}^{a}$ denote the assumed transverse shear and normal stresses, respectively, while $\gamma^{a}, \varepsilon_{33}^{a}$ denote the transverse shear and normal strains coming from the constitutive relations, using $\boldsymbol{\tau}^{a}, \sigma_{33}^{a}$. The counterpart, that is the strain coming from the straindisplacement relations, are denoted as $\gamma, \varepsilon_{33}$. It is important to remark that, in this case, the stress vector $\boldsymbol{\sigma}$ contains, for the transverse shear and normal stress part, the assumed stresses.

Due to the arbitrary virtual variation of the stress variables, the solution procedure of the Reissner Mixed Variational Theorem leads the compatibility terms between the strains coming from the assumed displacements and those coming from the constitutive equations to be solved in a form wherein the integration is limited to the plate thickness, henceforward called weak form, that is

$$
\begin{align*}
& \left\langle\delta \boldsymbol{\tau}^{a T}\left(\boldsymbol{\gamma}-\boldsymbol{\gamma}^{a}\right)\right\rangle=0 \\
& \left\langle\delta \sigma_{33}^{a}\left(\varepsilon_{33}-\varepsilon_{33}^{a}\right)\right\rangle=0 \tag{1.4}
\end{align*}
$$

wherein the $\langle\ldots\rangle$ notation stands for the integration over the entire thickness.
By solving the compatibilities, Eq. (1.4), the assumed stresses are expressed in terms of the kinematic variables, thus reducing the complexity of the model.

## 3. Equivalent Single Layer Models

The Equivalent Single Layer models, called also Smeared models, assume a $C^{l}$ continuous distribution along the thickness of displacements. In this way, the ESL models substitute the multilayered plate/shell with an equivalent single layer plate/shell. Due to this feature, the ESL models are generally an extension to multilayered structures of the models developed for homogeneous isotropic plates/shells.

The typical assumption of an ESL model for a generic displacement component, $u_{i}(\mathbf{x}, z, t)$, reads as

$$
\begin{equation*}
u_{i}(\mathbf{x}, z, t)=\sum_{j=0}^{N} f_{j}(z) u_{i}^{(j)}(\mathbf{x}, t) \tag{1.5}
\end{equation*}
$$

where $f_{j}(z)$ represents the base functions of the through-the-thickness assumption of the displacement component $u_{i}(\mathbf{x}, z, t)$ and $u_{i}^{(j)}(\mathbf{x}, t)$ are the model kinematic variables that are determined by solving the governing equations and the variationally consistent boundary conditions.

Generally speaking, as base functions, powers of the thickness coordinate, that is $f_{j}(z)=z^{j}$, are chosen and depending on the maximum order in the polynomial expansion, it is possible to distinguish between first-order ELS models and higher-order ESL ones. In the open literature are also available ESL models adopting the trigonometric functions as base for the through-the-thickness distribution of displacements. In the framework of the trigonometric ESL models, are noticeable the works done by Touratier [Touratier, 1991] that, to the best author knowledge, was the first to adopt trigonometric functions in smeared models.

It is not intent of this paragraph to carry out a detailed review of ESL models, thus the author addresses the interested reader to review works [Ghugal et al., 2002; Wanji et al., 2008; Khandan et al., 2012].

### 3.1. Classical Laminated Plate Theory

The Classical Laminated Plate Theory (CLPT) represents the extension to multilayered plates of the Classical Plate Theory (CLT) developed by Kirchhoff for isotropic plates. It is the simplest ESL model and the assumed kinematics reads as

$$
\begin{align*}
& U_{\alpha}(\mathbf{x}, z, t)=u_{\alpha}(\mathbf{x}, t)-z \partial_{\alpha} w(\mathbf{x}, t) \\
& U_{z}(\mathbf{x}, z, t)=w(\mathbf{x}, t) \tag{1.6}
\end{align*}
$$

where the subscript $\alpha=1,2$. Consistent with the kinematic assumptions of the CLPT, $u_{\alpha}, w_{\alpha}$ represent the in-plane and transverse displacements of a point located on the reference plane (i.e., $z=0$ ) of the plate. The kinematics in Eq.(1.6) implies that the crosssection remains plane after deformation and transverse shear and normal deformations are neglected, thus accounting only for bending and in-plane stretching.

It is well-known the role played by the transverse shear stresses on the plate/shell structural response. Neglecting the transverse shear effect leads to relevant errors: for the isotropic plate, the CLPT can be applied when the span-to-thickness ratio of the plate is $a / 2 h>30$ (that is, for thin plate); for a multilayered laminate, the minimum value of the span-to-thickness ratio that justifies the application of the CLPT increases up to 50. Along with the span-to-thickness ratio, the application of the CLPT depends also on the stiffness ratio between the adjacent layers: the use of the CLPT for highly heterogeneous stacking sequences leads to relevant errors, even if the span-to-thickness ratio is greater that 50 .

### 3.2. First-Order Shear Deformation Theory

The First-Order Shear Deformation Theory (FSDT) [Whitney et al., 1970] represents the extension to the multilayered laminate of the Reissner-Mindlin plate theory [Mindlin, 1951; Reissner, 1945]. With respect to the CLPT, the FSDT relaxes the hypothesis suggested by Kirchhoff allowing a transverse shear deformation of the plate. The kinematics of the FSDT reads as

$$
\begin{align*}
& U_{\alpha}(\mathbf{x}, z, t)=u_{\alpha}(\mathbf{x}, t)+z \theta_{\alpha}(\mathbf{x}, t) \\
& U_{z}(\mathbf{x}, z, t)=w(\mathbf{x}, t) \tag{1.7}
\end{align*}
$$

where $\theta_{\alpha}$ represents the rotation around the $x_{\beta}$ coordinate axis.
Consistent with the displacement field in Eq. (1.7), the transverse shear strain is assumed constant along the plate thickness, thus yielding to a piece-wise constant
distribution of transverse shear stress that violates the Elasticity conditions at layer interfaces. Moreover, since the distribution is constant along the thickness, the zerotransverse shear stresses condition at the top and bottom plate surface is not satisfied. The gap between the transverse shear stresses distribution provided by the FSDT and that obtained by the Elasticity requires the use of a shear correction factor [Whitney, 1973; Vlachoutsis, 1992; Hutchinson, 2001] in the FSDT that acts reducing the shear stiffness of the plate. The shear correction factor estimation is not easy since it depends on the stacking sequence, the geometry and the boundary and loading conditions.

By the inclusion of the transverse shear effect, the FSDT could be able to provide moderately accurate global responses (maximum deflection, first natural frequency and buckling load) if an adequate shear correction factor is adopted and the plate features a span-to-thickness ratio $a / 2 h>20$. Although the global responses are moderately accurate, the through-the-thickness distribution of displacements and stresses for a multilayered plate deviates from that provided by the exact solution due to the FSDT through-thethickness $C^{l}$-continuity kinematic assumptions.

### 3.3. High-Order Shear Deformation Theory

As remarked before, the inclusion of the transverse shear effect in the FSDT guarantees an increase in accuracy, on condition that the plate remains thin. By increasing the plate thickness, the through-the-thickness linear distribution of the displacements provided by the FSDT is no more correct as suggested by the Pagano exact Elasticity solution [Pagano, 1969; Pagano, 1970]. In fact, the thickness effect acts making the displacements distribution non-linear along the thickness and requires a description with a higher-order polynomial.

The term High-Order Shear Deformation Theory (HSDT) refers to a set of ESL models adopting a polynomial assumption for the displacements with a order greater than one. The reader can refer to [Ghugal et al., 2002; Wanji et al., 2008; Khandan et al., 2012] for a detailed review of the HSDTs.

A fundamental contribution in the framework of HSDTs was given by the third-order shear deformation theory developed by Reddy [Reddy, 1984; Reddy, 1990] and assuming the following kinematics

$$
\begin{align*}
& U_{\alpha}(\mathbf{x}, z, t)=u_{\alpha}(\mathbf{x}, t)+z \theta_{\alpha}(\mathbf{x}, t)+z^{3}\left(-\frac{4}{3 h^{2}}\right)\left(\theta_{\alpha}+\partial_{\alpha} w\right)  \tag{1.8}\\
& U_{z}(\mathbf{x}, z, t)=w(\mathbf{x}, t)
\end{align*}
$$

The transverse shear stresses distribution provided by this model is parabolic along the thickness direction and able to satisfy the zero-transverse shear stresses condition on the top and bottom plate surface. For this reason, the model does not require any shear correction factors. It is worth to note that, due to smeared-type approximation of displacements, the transverse shear stresses are discontinuous at the interface between two layers with different mechanical properties.

## 4. Layer-wise Models

The multilayered composite structures are characterized by a severe transverse anisotropy since they are made by the superposition of considerably different layers. The transverse anisotropy is responsible for the zigzag effect, that is, the distribution along the thickness of in-plane displacements is not $C^{l}$-continuous but $C^{0}$-continuous, showing a change in the slope at layer interfaces. The jump of the first-order derivative with respect to the thickness coordinate of the in-plane displacement at layer interface derives from equilibrium consideration: according to the Cauchy's theorem, the transverse shear stresses have to be equal at layer interfaces. The unique way to ensure a continuity condition on these stresses having layers with different mechanical properties is to ensure a jump in the transverse shear strains, obtainable with a discontinuous first-order derivative of in-plane displacements with respect to the thickness coordinate at layer interfaces. It is easy to understand that the zigzag effect is the main responsible for the inaccuracy of the ESL models in predicting the local (through-the-thickness distribution of displacements and stresses) and global (maximum deflection, natural frequencies and buckling load) response of multilayered structures.

The problem of reproducing a $C^{0}$-continuous displacement field along the thickness can be addressed by adopting LW models, wherein an ESL-like assumption is made for every single layer. According to Reddy [Reddy, 2004], the LW models are divided into partial theories and full theories: the former assume a $C^{0}$-continuous distribution only for the inplane displacements; on the contrary, the latter ones include the zigzag effect also in the transverse displacement assumption. The purpose of this paragraph is only to explain the
basic idea of the LW theories, without giving a detailed review. Readers interested may refer to [Ghugal et al., 2002; Wanji et al., 2008; Khandan et al., 2012].

For the sake of simplicity, partial LW models are taken into consideration. A typical displacements assumption of a partial LW model reads as

$$
\begin{align*}
& U_{\alpha}^{(k)}(\mathbf{x}, z, t)=u_{\alpha}^{(k)}(\mathbf{x}, t)+z \theta_{\alpha}^{(k)}(\mathbf{x}, t)+\ldots .+z^{n} \omega_{\alpha}^{(k)}(\mathbf{x}, t) \\
& U_{z}^{(k)}(\mathbf{x}, z, t)=w(\mathbf{x}, t) \tag{1.9}
\end{align*}
$$

where the superscript ( $k$ ) means that the quantities are referred to the $k$ th layer. As Eq. (1.9) shows, the LW models adopt for each layers an ESL-like assumption with a polynomial degree up to $n$. Along with the assumption of Eq. (1.9), the continuity conditions on displacements and transverse stresses have to be satisfied at layer interfaces. These contact conditions only reduce the total number of kinematic variables that still remains dependent on the number of layers.

With LW models, high accuracy is obtained at the expense of a substantial computational cost that increases with the number of layers. This makes the LW models inadequate for the analysis of multilayered structures made by one hundred or more layers, as the structures for real applications are.

## 5. Zigzag Models

The basic idea of the Zigzag models, pioneered by Di Sciuva [Di Sciuva, 1983; Di Sciuva, 1984a,b; Di Sciuva, 1986] is to model the actual cross-section distortion typical of multilayered structures by using a limited and fixed number of kinematic variables, in order to preserve the computational cost. The kinematics of a Zigzag model can be presented in a multi-scale view: the assumed displacement field is given by the superposition of a coarse kinematics and a fine one. The former represents the behavior on the total laminate thickness scale whereas the fine kinematics describes the behavior on the layer thickness scale. An ESL model is adequate to constitute the coarse kinematics due to the $C^{l}$-continuous assumption. The layer refinement, that is the fine kinematics, is given by the product of a priori known piecewise continuous function of the thickness coordinate, called zigzag function, and a kinematic variable function of the in-plane coordinate axes, called zigzag amplitude, that rules the magnitude of the zigzag effect. In this way, a Zigzag model retains the same kinematic variables of the ESL model adopted for the coarse kinematics in addition to the zigzag amplitudes, one in each directions, thus resulting in a constant number of the kinematic-variables model.

The typical Zigzag kinematics (for the sake of simplicity, constant transverse displacement distribution and ESL coarse kinematics with polynomial basis functions are considered) reads as

$$
\begin{align*}
& U_{\alpha}^{(k)}(\mathbf{x}, z, t)=u_{\alpha}(\mathbf{x}, t)+z \theta_{\alpha}(\mathbf{x}, t)+\ldots .+z^{n} \omega_{\alpha}(\mathbf{x}, t)+\Pi_{\alpha}^{(k)}(z) \psi_{\alpha}(\mathbf{x}, t) \\
& U_{z}^{(k)}(\mathbf{x}, z, t)=w(\mathbf{x}, t) \tag{1.10}
\end{align*}
$$

where the Greek subscript takes values $\alpha=1,2$ and $u_{\alpha}(\mathbf{x}, t), \theta_{\alpha}(\mathbf{x}, t), \ldots ., \omega_{\alpha}(\mathbf{x}, t)$ represent the kinematic variables of the coarse kinematics, $\Pi_{\alpha}^{(k)}(z)$ the zigzag function and $\psi_{\alpha}(\mathbf{x}, t)$ the zigzag amplitude. On the whole, the Zigzag model in Eq. (1.10) has $2(n+1)+1$ kinematic variables coming from the coarse kinematics plus two zigzag amplitudes, giving a constant $2(n+1)+3$ number of variables irrespective of the number of layers.

### 5.1. The choice of the zigzag function

Once the ESL model describing the coarse kinematics has been chosen, the fundamental step in the development of a Zigzag model is the choice of the zigzag function: the predictive capabilities of models assuming the same ESL coarse kinematics depend on the zigzag function. In the open literature, two kind of zigzag functions are available: the former is the Di Sciuva's zigzag function [Di Sciuva, 1983; Di Sciuva, 1984a,b; Di Sciuva, 1986] originally developed in the framework of a displacement-based zigzag model; the second one, attributed to Murakami [Murakami, 1986], was introduced, for the first time, in the framework of a Reissner Mixed Variational Theorem-based zigzag model.

The Di Sciuva's zigzag function is physically-based since it is formulated in order to satisfy, a priori, the continuity conditions of transverse shear stresses at layer interfaces. The fulfillment of continuity of transverse shear stresses leads the zigzag function to be defined on the basis of the mechanical proprieties of the layers and changes accordingly the stacking sequence.

On the contrary, Murakami derived his zigzag function examining the distribution along the thickness of the in-plane displacements of a periodic laminate, that is a laminate featuring a stacking sequence made by the succession of the same two materials. As the exact Elasticity solution suggests, in this case the through-the-thickness distribution of inplane displacements shows a periodicity also in the slope change, that is the same in magnitude but opposite in sign from an interface to the subsequent one. The Murakami's
zigzag function is defined in order to reproduce this periodic behavior and for this reason is no mechanical properties-dependent.

For a long time, the Di Sciuva and Murakami zigzag functions have been involved in the development of zigzag models, both displacement-based and mixed ones, without an in-deep investigation and comparison of the predictive capabilities of the zigzag functions being performed. Recently, Gherlone [Gherlone, 2013] has covered this lack in the open literature, carried out a comparison of the two zigzag functions in the framework of a firstorder zigzag model, that is a model assuming the FSDT as coarse kinematics, both displacement-based and mixed one, focusing on the elasto-static response of multilayered composite and sandwich beams. The conclusion of the investigation performed by Gherlone states the superior predictive capabilities of the Di Sciuva's zigzag function over the Murakami's one. The same conclusions made by Gherlone [Gherlone, 2013] have been lately confirmed by Iurlaro et al. [Iurlaro et al.,2014a] also for the static, free vibrations and buckling load problems of sandwich plates.

For a detailed description of the zigzag functions and their use in the development of a First-Order Zigzag model, see Chapter 2.

### 5.2. Review zigzag models

The research activity of this Thesis focuses on Zigzag models and for this reason the author considers worthwhile a review of the main works done in the last thirty years, distinguishing the Zigzag models on the basis of the zigzag function employed.

As pioneer of Zigzag theories, of remarkable importance are the works done by Di Sciuva. In addition to the already cited early work [Di Sciuva, 1983; Di Sciuva, 1984a,b], it is important to mention the paper published in the framework of a linear zigzag model [Di Sciuva, 1984a,b; Di Sciuva, 1986; Di Sciuva, 1987] and in the framework of the cubic zigzag model [Di Sciuva, 1990; Di Sciuva, 1992; Di Sciuva, 1994; Di Sciuva, 1997] by the same author. While the Di Sciuva's linear zigzag model is obtained as refinement of the FSDT, the cubic one is formulated starting from the Reddy's third-order shear deformation theory and is able to satisfy a priori the continuity of transverse shear stresses at layer interfaces and the zero-conditions at the top and bottom plate surfaces. The Di Sciuva's cubic zigzag model is further developed in order to represent the interlaminar slip that occurs at damage interface [Di Sciuva, 1997]. Inspired by the Di Sciuva's works, many zigzag model arose. In the framework of cubic models, a contribution was given by Cho and Parmenter [Cho et al., 1993] and recently by Nemeth [Nemeth, 2012]. By using the
sublaminates approach, a 3D finite element based on a linear zigzag model accounting for a linear distribution across the sublaminate thickness of a transverse displacement was formulated by Cho [Cho et al., 2000]. An important contribution in the development of zigzag models was given by Icardi: in [Icardi, 2001a,b; Icardi, 2005] a higher-order zigzag model accounting for a piece-wise cubic distribution of in-plane displacements and piecewise four-order distribution of transverse displacement is developed. The transverse shear and normal stresses continuity conditions are not sufficient and the continuity condition on the gradient of transverse normal stress is enforced at layer interfaces. In [Icardi, 2011] an adaptive approach is proposed: the assumed kinematics can be refined with higher-order terms in order to improve locally the solution, where necessary. The coupled thermo-electro-mechanical analysis by using Zigzag models are performed in [Kapuria et al., 2003; Topdar et al., 2004; Kapuria et al., 2004]. In [Oh et al., 2005] a cubic zigzag model able to solve the elasto-dynamic equations of multilayered composite plates with multiple delaminations is presented. Examples of trigonometric ESL model in conjunction with Di Sciuva's zigzag function can be found in [Vidal et al., 2006; Arya et al., 2002]. A cubic zigzag model, very similar to the Di Sciuva's one, is developed in [Xiaohui et al., 2011; Xiaohui et al., 2012] and a finite element not requiring the continuity of the first order derivative of the transverse displacement is formulated. Interesting is the predictor-corrector approach introduced in [Lee et al., 1996] where the Di Sciuva's linear zigzag theory is used in the predictor step to obtain the distribution of the transverse shear stresses coming from integration of the Elasticity equilibrium equations. In the corrector step, that distribution is used in conjunction with a higher-order model in order to enhance the prediction of displacements and stresses. Recently, based on the Di Sciuva's original researches, a linear zigzag model, called Refined Zigzag Theory, for beams/plates/shells has been developed by Tessler, Di Sciuva and Gherlone [Tessler et al., 2007; Tessler et al., 2009a,b; Di Sciuva et al., 2010; Tessler et al., 2010a,b; Tessler et al., 2011; Gherlone et al., 2011; Versino, 2012; Versino et al., 2013; Versino et al., 2014]. The Refined Zigzag Theory has received remarkable attention by several researchers, among these are worth considering Narita and co-workers [Honda et al., 2013], Oñate [Oñate et al., 2012; Eijo et al., 2013a,b,c] and Flores [Flores, 2014]. Appealing developments in the framework of Refined Zigzag Theory are found in [Barut et al., 2012; Barut et al., 2013] wherein the Refined Zigzag Theory kinematics has been enriched with a smeared quadratic term in the in-plane assumptions and transverse displacement.

The Murakami's zigzag function has been extensively used by Carrera and co-workers [Carrera, 2000; Carrera, 2004; Brischetto et al., 2009a,b; Carrera et al., 2009]. Other works employing the Murakami's zigzag function are those of Bhaskar [Ali et al., 1999; Umasree et al., 2006], Ganapathi [Ganapathi et al., 2001; Ganapathi et al., 2002], D'Ottavio [D'Ottavio et al., 2006a,b] and Vidal and Polit [Vidal et al., 2011].

## 6. Multiple models methods

In this chapter, a brief mention of the Multiple models methods is done, addressing the interesting readers to Reddy [Reddy, 2004] for a detailed description.

The key idea of the Multiple models methods is to use different models in the analysis of a large and complex structure, in order to adopt more accurate but also computationally expensive models only where it is necessary, for example near an hole, and less accurate models in the other structural regions. In this way, this method allows an optimal use of the computational resources.

Two kind of multiple models methods exist: (i) sequential methods; the problem is solved by using a low accurate model with the purpose of obtaining the boundary conditions for a localized analysis of sensible zones that is performed in a second step adopting high-fidelity models; (ii) simultaneous methods; the problem is solved by adopting different models or different level of discretization.

The interested reader is addressed to the Reddy's book [Reddy, 2004] for further details.

## Chapter 2

## Refined Zigzag Theory

## 1. Motivation

As previously stated in Chapter 1, the Di Sciuva's early works inspired several researchers working in the field of structural analysis, motivating them to adopt the zigzag concept in the modeling of multilayered composite and sandwich structures. With respect to the existing shear deformable theories for beams and plates, the improvement in the response prediction of the elasto-static and elasto-dynamic behavior of multilayered structures made by Di Sciuva was remarkable. Above all, the inclusion of the zigzag contribution into the Timoshenko beam and the Reissner-Mindlin plate kinematics, leads the Di Sciuva's model to predict accurately the through-the-thickness distribution of inplane displacements and stresses, if compared with the exact Elasticity solution, of thick and highly heterogeneous beams/plates and to satisfy the interlaminar transverse shear stresses continuity. Nevertheless, as highlighted in Tessler et al. [Tessler et al., 2009a], the early linear zigzag model proposed by Di Sciuva [Di Sciuva, 1983; Di Sciuva, 1984] was affected by serious drawbacks. Briefly, the shortcomings arisen in the original Di Sciuva's model were (i) the shear-force inconsistency, that is, the transverse shear forces coming from the integrated transverse shear stresses do not match with those obtained by the constitutive transverse shear stresses, and (ii) the $C^{l}$-continuity requirement for a finite
element implementation, thus resulting in a less efficient formulation with respect to $C^{0}$ continuous finite elements. Recently, Wanji and co-workers solve the problem of the $C^{l}$ continuity condition, by developing a $C^{0}$-type element [Xiaohui et al., 2012].

In order to preserve the accurate global and local prediction capabilities provided by the former Di Sciuva's linear zigzag model and, simultaneously, to overcome the drawbacks above discussed, recently the Refined Zigzag Theory (RZT) has been proposed by Tessler, Di Sciuva and Gherlone [Tessler et al., 2007; Tessler et al., 2009a,b; Di Sciuva et al., 2010; Tessler et al., 2010a,b; Tessler et al., 2011; Gherlone et al., 2011; Versino, 2012; Versino et al., 2013; Versino et al., 2014]. The RZT belongs to the class of displacementbased zigzag models accounting for a piecewise linear distribution of in-plane displacements and a constant transverse one.

In this Chapter, the theoretical bases of the RZT are recalled starting from the assumed kinematics. Successively, by using the D'Alembert Principle and adopting the Von Kàrmàn strain-displacement relations, the non-linear governing equations for plates are obtained and specialized to the linear bending problem and linear eigenvalues problem of free vibrations and buckling loads. Later, a brief presentation of the advanced functionally graded materials is carried out and the extension of the RZT to functionally graded plates presented. Finally, in a general notation, a first-order zigzag model is presented, letting open the possibility to chose the zigzag function. A section is reserved to describe the Murakami's zigzag function and to highlight the main differences with respect to the Refined Zigzag function, that belongs to the Di Sciuva's type zigzag function.

The content of this Chapter have been subject of publications on International Journals, in particular, the non-linear governing RZT equations are derived in [Iurlaro et al., 2013a], wherein they are used to solve the linear bending, undamped free vibrations and linear buckling problems of sandwich plates. The extension of the RZT to the advanced functionally graded materials is presented in [Iurlaro et al., 2014b], wherein the bending and free vibrations problems of sandwich panels embedding functionally graded layers, both as face-sheets and core, are solved.

## 2. Refined Zigzag Theory for plates: displacements, strains and stresses

Consider a laminated plate of uniform thickness $2 h$ with $N$ perfectly bonded orthotropic layers, of thickness $2 h^{(k)}$, as shown in Figure 2.1. The orthogonal Cartesian coordinate system $\left(x_{l}, x_{2}, z\right)$ is taken as reference where the thickness coordinate $z$ ranges from $-h$ to $+h$.

The middle reference plane (or midplane) of the plate, $S_{m}$, is placed on the ( $x_{1}, x_{2}$ )-plane. The plate is bounded by a cylindrical edge surface, $S$, constituted by two distinct surfaces, $S_{u}$ and $S_{s}$, on which the geometrical and mechanical boundary conditions are enforced, respectively. Moreover, the intersection of the surface $S$ and of the ( $x_{1}, x_{2}$ )-plane is the curve $C$ which represents the perimeter of the midplane, $S_{m}$. As for the edge surface, the curve $C$ is composed by two distinct curves, $C_{u}$ and $C_{s}$, originated by the intersection of $S_{u}$ and $S_{s}$ with the $\left(x_{1}, x_{2}\right)$-plane, respectively. Finally, $S_{t}$ and $S_{b}$ represent the top and bottom external surfaces of the plate (at $z=h$ and $z=-h$ ), respectively. The plate represented in Figure 2.1 is subjected to a transverse pressure loads, applied on the midplane $S_{m}$, to surface tractions, acting on the top, $S_{t}$, and on the bottom, $S_{b}$, surfaces and to traction stresses prescribed on $S_{s}$.

### 2.1. Kinematic assumptions

According to the Refined Zigzag Theory kinematic assumptions [Tessler et al., 2010a,b], the orthogonal components of the displacement vector read as

$$
\begin{align*}
& U_{\alpha}^{(k)}(\mathbf{x}, z, t)=u_{\alpha}(\mathbf{x}, t)+z \theta_{\alpha}(\mathbf{x}, t)+\phi_{\alpha}^{(k)}(z) \psi_{\alpha}(\mathbf{x}, t)  \tag{2.1}\\
& U_{z}(\mathbf{x}, z, t)=w(\mathbf{x}, t)
\end{align*}
$$

where the superscript $(k)$ is used to denote quantities corresponding to the $k$ th lamina and $t$ represent the time variable. The subscript $\alpha=1,2$ denotes the component of the displacement vector along the $x_{\alpha}$-coordinate axis while the notation $\mathbf{x} \equiv\left(x_{1}, x_{2}\right)$ has been used.

The RZT displacement field, Eq. (2.1), as any zigzag theories, results from the superposition of a coarse kinematics and a fine one. As coarse kinematics the FSDT has been assumed, whereas, the behavior on the layer thickness scale is reproduced by the layer-wise contribution given by the product $\phi_{\alpha}^{(k)}(z) \psi_{\alpha}(\mathbf{x}, t)$. Since the FSDT has been assumed, $u_{\alpha}, \theta_{\alpha}, w$ are the classical kinematic variables representing the in-plane uniform displacement, the bending rotation and the transverse deflection, respectively. The RZT adds to the FSDT kinematics a piecewise linear contribution given by the product of the zigzag amplitude, $\psi_{\alpha}(\mathbf{x}, t)$, and the zigzag function, $\phi_{\alpha}^{(k)}(z)$. The zigzag function is an $a$ priori known function of the thickness coordinate, thus the RZT model for plate results in a fixed (seven) number of kinematic variables: the five FSDT variables plus the two zigzag amplitudes.


Figure 2.1 General plate notation.

### 2.2. Refined zigzag function

The key aspect in each zigzag theory is the formulation of the zigzag function adopted. As stated in Chapter 1, the Di Sciuva's zigzag function comes from the fulfillment of the interlaminar transverse shear stresses continuity conditions that leads, according to what has been explained previously, to a constant through-the-thickness distribution of transverse shear stresses in the framework of a linear zigzag model.

In order to overcome the shortcomings that affected the Di Sciuva's model, in RZT the interlaminar transverse shear stresses continuity condition has been only partially satisfied, allowing a discontinuity of these stresses at layer interfaces. The results, if compared with the exact Elasticity solution, demonstrate remarkable accuracy in terms of in-plane displacements and stresses distribution along the thickness, along with a through-thethickness distribution of transverse shear stresses that approximate in an average way the exact distribution in each layer. In other words, the fulfillment of the transverse shear stresses continuity condition leads a reduced kinematic-variables model (like the original Di Sciuva's one) to be over-constrained; the RZT, relaxing the continuity conditions, adds a variable to the kinematics that allows for a piecewise constant distribution of transverse shear stresses, accurate in an average sense, and that avoids the shear forces inconsistency.

In what follow, the RZT zigzag function estimation procedure is recalled. For further details, reader can refer to [Tessler et al., 2007; Tessler et al., 2009a,b; Di Sciuva et al., 2010; Tessler et al., 2010a,b; Tessler et al., 2011; Gherlone et al., 2011; Versino, 2012; Versino et al., 2013; Versino et al., 2014].

Consistent with the RZT kinematics and by using the linear strain-displacement relations, the transverse shear strains read as

$$
\begin{align*}
\gamma_{\alpha z}^{(k)} & =\gamma_{\alpha}+\partial_{z} \phi_{\alpha}^{(k)}(z) \psi_{\alpha}(\mathbf{x}, t)  \tag{2.2}\\
\gamma_{\alpha} & \equiv \theta_{\alpha}(\mathbf{x}, t)+\partial_{\alpha} w(\mathbf{x}, t)
\end{align*}
$$

The transverse shear stresses come straightforwardly from the Hooke's law, thus

$$
\begin{equation*}
\tau_{\alpha z}^{(k)}=Q_{\alpha \beta}^{(k)} \gamma_{\beta z}^{(k)} ; \quad(\alpha, \beta=1,2) \tag{2.3}
\end{equation*}
$$

An alternative way of measuring the shear strain is by introducing the strain measure [Tessler et al., 2010a,b]

$$
\begin{equation*}
\eta_{\alpha} \equiv \gamma_{\alpha}-\psi \tag{2.4}
\end{equation*}
$$

Thus, combining Eq. (2.3) and Eq. (2.4), an alternative expression for the transverse shear stress is obtained

$$
\begin{equation*}
\tau_{\alpha z}^{(k)}=Q_{\alpha \beta}^{(k)} \eta_{\beta}+Q_{\alpha \beta}^{(k)}\left(1+\partial_{z} \phi_{\beta}^{(k)}(z)\right) \psi_{\beta}=Q_{\alpha \beta}^{(k)} \eta_{\beta}+\tilde{\tau}_{\alpha z}^{(k)} ; \quad(\alpha, \beta=1,2) \tag{2.5}
\end{equation*}
$$

The transverse shear stress is thus composed by two contributions: the continuity condition is enforced only on the zigzag-dependent contribution, $\tilde{\tau}_{\alpha z}^{(k)}$, that is, on the transverse shear stress obtained by vanishing the shear measure $\eta_{\beta}$ and considering the diagonal contribution of the transverse shear stress vector, that is $\alpha=\beta$. The constraint reads as

$$
\begin{equation*}
\tilde{\tau}_{\alpha z}^{(k)}=\tilde{\tau}_{\alpha z}^{(k+1)} \Rightarrow Q_{\alpha \alpha}^{(k)}\left(1+\partial_{z} \phi_{\alpha}^{(k)}(z)\right) \psi_{\alpha}=Q_{\alpha \alpha}^{(k+1)}\left(1+\partial_{z} \phi_{\alpha}^{(k+1)}(z)\right) \psi_{\alpha} \tag{2.6}
\end{equation*}
$$

It is easy to realize that constraint in Eq. (2.6) actually involves a combination of the shear modulus of the layer and the zigzag function, that is, Eq. (2.6) resolves into the following condition

$$
\begin{equation*}
Q_{\alpha \alpha}^{(k)}\left(1+\partial_{z} \phi_{\alpha}^{(k)}(z)\right)=G_{\alpha} \tag{2.7}
\end{equation*}
$$

where $G_{\alpha}$ is the weighted-average transverse shear stiffness coefficient of the lamina level coefficient $Q_{\alpha \alpha}^{(k)}$.

From Eq. (2.7), the spatial derivative of the zigzag function can be recovered and reads as

$$
\begin{equation*}
\partial_{z} \phi_{\alpha}^{(k)}(z)=\frac{G_{\alpha}}{Q_{\alpha \alpha}^{(k)}}-1 \tag{2.8}
\end{equation*}
$$

The problem now moves on the definition of the weighted-average transverse shear stiffness coefficient $G_{\alpha}$ : the RZT kinematic assumptions adopt the FSDT as coarse kinematics and, as a consequence, the $\theta_{\alpha}$ variable represents the average bending rotation of the cross-section. In order to make $\theta_{\alpha}$ the average bending rotation, the following relation has to be satisfied

$$
\begin{equation*}
\frac{1}{2 h}\left\langle\partial_{z} \phi_{\alpha}^{(k)}(z)\right\rangle=0 \tag{2.9}
\end{equation*}
$$

By using the expression for the spatial derivative of the zigzag function, Eq. (2.8), into Eq. (2.9), the weighted-average transverse shear stiffness coefficient is obtained

$$
\begin{equation*}
G_{\alpha}=\left(\frac{1}{2 h}\left\langle\frac{1}{Q_{\alpha \alpha}^{(k)}}\right\rangle\right)^{-1}=\left(\frac{1}{h} \sum_{k=1}^{N} \frac{h^{(k)}}{Q_{\alpha \alpha}^{(k)}}\right)^{-1} \tag{2.10}
\end{equation*}
$$

Once the stacking sequence is defined, the transverse shear moduli of each layer are known and the weighted-average transverse shear stiffness coefficient can be computed by using Eq. (2.10). Successively, the first order derivative of the zigzag function in each layer derives from Eq. (2.8). Since the zigzag function is a piecewise linear function of the thickness coordinate, its spatial derivative is piecewise constant through-the-thickness. In order to completely define the zigzag function, two additional conditions have to be enforced. In the former Di Sciuva's model, this problem was addressed by choosing the fixed layer; in the RZT the missing conditions derive from Eq. (2.9) [Di Sciuva et al., 2010]

$$
\begin{equation*}
\phi_{\alpha}^{(1)}(z=-h)=\phi_{\alpha}^{(N)}(z=h)=0 \tag{2.11}
\end{equation*}
$$

By using Eq. (2.11) , Eq. (2.8) and Eq. (2.10), the zigzag function is completely defined and, as it is easy to realize, is dependent only on the shear moduli and the thickness of each layer. This means that once the stacking sequence is defined, the zigzag function is
obtained. In Figure 2.2 an example of the through-the-thickness plot of the zigzag functions in each direction for a three-layer plate is given.

(a) Zigzag function in the $\left(x_{1}, z\right)$-plane

(b) Zigzag function in the $\left(x_{2}, z\right)$-plane

Figure 2.2 Through-the-thickness distribution of the zigzag functions.
In the case of homogeneous plates, the zigzag functions vanish identically and the displacement field, Eq. (2.1), reduces to that of FSDT. Recently, Tessler et. al. [Tessler et al., 2010a,b; Tessler et al., 2011] showed that within RZT, the homogeneous plates should be modeled as laminated plates with infinitesimally small differences in the transverse shear moduli of the material layers (homogeneous limit methodology), thus producing highly accurate response predictions. Moreover, Gherlone [Gherlone, 2013] showed that when the external layers of a laminate are weaker than the adjacent layers, in terms of transverse shear stiffness, the RZT zigzag functions can be adapted naturally to the effective shear properties of the stacking sequence and lead to accurate results.

### 2.3. Non-linear strains and stresses

In order to develop a plate theory which accounts for moderately large deflection and small strains, the von Kàrmàn's non-linear strain-displacement relations are used [ChuengYuan, 1980]. Consistent with the displacement field of Eq. (2.1), the in-plane and transverse shear strains are [Iurlaro et al., 2013a]

$$
\begin{align*}
& 2 \varepsilon_{\alpha \beta}^{(k)}=\partial_{\beta} u_{\alpha}+\partial_{\alpha} u_{\beta}+z\left(\partial_{\beta} \theta_{\alpha}+\partial_{\alpha} \theta_{\beta}\right)+\phi_{\alpha}^{(k)} \partial_{\beta} \psi_{\alpha}+\phi_{\beta}^{(k)} \partial_{\alpha} \psi_{\beta}+\partial_{\alpha} w \partial_{\beta} w  \tag{2.12}\\
& \gamma_{\alpha z}^{(k)}=\gamma_{\alpha}+\beta_{\alpha}^{(k)} \psi_{\alpha}
\end{align*}
$$

where $\gamma_{\alpha} \equiv \partial_{\alpha} w+\theta_{\alpha}$ and $\beta_{\alpha}^{(k)} \equiv \partial_{z} \phi_{\alpha}^{(k)}$. Hooke's constitutive relations are then invoked to compute the stresses

$$
\begin{equation*}
\sigma_{\alpha \beta}^{(k)}=C_{\alpha \beta \gamma \delta}^{(k)} \varepsilon_{\gamma \delta}^{(k)} ; \quad \tau_{\alpha z}^{(k)}=Q_{\alpha \beta}^{(k)} \gamma_{\beta z}^{(k)} \tag{2.13}
\end{equation*}
$$

where $C_{\alpha \beta \gamma \delta}^{(k)}$ and $Q_{\alpha \beta}^{(k)}$ are the transformed elastic stiffness coefficients referred to the ( $\mathbf{x}, z$ ) coordinate system and relative to the plane-stress condition that assumes that transverse normal stress is negligibly small with respect to the in-plane stresses.

## 3. Non-linear equations of motion

The non-linear plate governing equations and the variationally consistent boundary conditions are derived by using the D'Alembert's principle that reads as

$$
\begin{equation*}
\delta U-\delta W_{i}-\delta W_{e}=0 \tag{2.14}
\end{equation*}
$$

where $\delta$ is the variational operator, $U, W_{i}$ and $W_{e}$ are the strain energy, the work done by the inertial forces and that done by the applied external loads, respectively. The variation of the strain energy is given by

$$
\begin{equation*}
\delta U \equiv \int_{S_{m}}\left\langle\sigma_{11}^{(k)} \delta \varepsilon_{11}^{(k)}+\sigma_{22}^{(k)} \delta \delta_{22}^{(k)}+\tau_{12}^{(k)} \delta \gamma_{12}^{(k)}+\tau_{1 z}^{(k)} \delta \gamma_{1 z}^{(k)}+\tau_{2 z}^{(k)} \delta \gamma_{2 z}^{(k)}\right\rangle d S \tag{2.15}
\end{equation*}
$$

By using the displacement field, Eq. (2.1), coupled with the non-linear straindisplacement relations, Eq. (2.12), and the constitutive material law, Eq. (2.13), the virtual variation of the strain energy, in tense notation, results as

$$
\begin{align*}
& \delta U=-\int_{S_{m}}\left\{\partial_{\beta} N_{\alpha \beta} \delta u_{\alpha}+\left(\partial_{\beta} M_{\alpha \beta}-Q_{\alpha}\right) \delta \theta_{\alpha}+\left(\partial_{\beta} M_{\alpha \beta}^{\phi}-Q_{\alpha}^{\phi}\right) \delta \psi_{\alpha}+\right. \\
&\left.\partial_{\alpha}\left[\left(N_{\alpha \beta} \partial_{\beta} w\right)+Q_{\alpha}\right] \delta w\right\} d S+  \tag{2.16}\\
& \int_{C_{\sigma}}\left\{N_{\alpha \beta} \delta u_{\alpha}+M_{\alpha \beta} \delta \theta_{\alpha}+M_{\alpha \beta}^{\phi} \delta \psi_{\alpha}+\left(N_{\alpha \beta} \partial_{\alpha} w+Q_{\beta}\right) \delta w\right\} n_{\beta} d \Gamma
\end{align*}
$$

where $n_{\alpha}$ denotes the direction cosine of $\mathbf{n}$, the unit outward vector normal to C , with respect to the in-plane coordinate $x_{\alpha}$. Moreover, the following membrane, bending and transverse shear stress resultants are introduced

$$
\begin{align*}
& \left(N_{\alpha \beta}, M_{\alpha \beta}, M_{\alpha \beta}^{\phi}\right) \equiv\left\langle\left(1, z, \phi_{\alpha}^{(k)}(z)\right) \sigma_{\alpha \beta}^{(k)}\right\rangle \\
& \left(Q_{\alpha}, Q_{\alpha}^{\phi}\right) \equiv\left\langle\left(1, \partial_{z} \phi_{\alpha}^{(k)}(z)\right) \tau_{\alpha z}^{(k)}\right\rangle \tag{2.17}
\end{align*}
$$

The variation of the work done by external applied loads reads as

$$
\begin{align*}
\delta W_{e} \equiv & \int_{S_{m}} \bar{q}(\mathbf{x}, t) \delta U_{z} d S+\int_{S_{\sigma}}\left(\bar{T}_{\alpha} \delta U_{\alpha}^{(k)}+\bar{T}_{z} \delta U_{z}\right) d S+  \tag{2.18}\\
& \int_{S_{m}}\left(\bar{p}_{\alpha}^{t}(\mathbf{x}, t) \delta U_{\alpha}^{(N)}(z=+h)+\bar{p}_{\alpha}^{b}(\mathbf{x}, t) \delta U_{\alpha}^{(1)}(z=-h)\right) d S
\end{align*}
$$

where the equivalence among the surfaces, $S_{l}=S_{b}=S_{m}$, is set. Introducing the displacement components definition into Eq.(2.18), yields

$$
\begin{align*}
\delta W_{e}= & \int_{S_{m}} \bar{q}(\mathbf{x}, t) \delta w d S+ \\
& \int_{C_{\sigma}}\left\langle\bar{T}_{\alpha}\left(\delta u_{\alpha}+z \delta \theta_{\alpha}+\phi_{\alpha}{ }^{(k)} \delta \psi_{\alpha}\right)+\bar{T}_{z} \delta w\right) d \Gamma+ \\
& \int_{S_{m}}\left[\bar{p}_{\alpha}^{t}(\mathbf{x}, t)\left(\delta u_{\alpha}+h \delta \theta_{\alpha}+\phi_{\alpha}{ }^{(N)}(z=+h) \delta \psi_{\alpha}\right)\right] d S+  \tag{2.19}\\
& \int_{S_{m}}\left[\bar{p}_{\alpha}^{b}(\mathbf{x}, t)\left(\delta u_{\alpha}-h \delta \theta_{\alpha}+\phi_{\alpha}{ }^{(1)}(z=-h) \delta \psi_{\alpha}\right)\right] d S
\end{align*}
$$

Introducing the following definitions

$$
\begin{align*}
& \bar{p}_{\alpha} \equiv \bar{p}_{\alpha}^{t}+\bar{p}_{\alpha}^{b} \\
& \bar{m}_{\alpha} \equiv h\left(\bar{p}_{\alpha}^{t}-\bar{p}_{\alpha}^{b}\right) \tag{2.20}
\end{align*}
$$

the variation of the work done by external loads reads

$$
\begin{align*}
\delta W_{e}= & \int_{S_{m}}\left(\bar{p}_{\alpha}(\mathbf{x}, t) \delta u_{\alpha}+\bar{m}_{\alpha}(\mathbf{x}, t) \delta \theta_{\alpha}+\bar{q}(\mathbf{x}, t) \delta w\right) d S+ \\
& \int_{C_{\sigma}}\left[\bar{N}_{\alpha n} \delta u_{\alpha}+\bar{M}_{\alpha n} \delta \theta_{\alpha}+\bar{M}_{\alpha n}^{\phi} \delta \psi_{\alpha}+\bar{V}_{z n} \delta w\right] d \Gamma \tag{2.21}
\end{align*}
$$

with the force and moment resultants of the prescribed tractions follows

$$
\begin{equation*}
\left(\bar{N}_{\alpha n}, \bar{M}_{\alpha n}, \bar{M}_{\alpha n}^{\phi}, \bar{V}_{z n}\right) \equiv\left\langle\left(\bar{T}_{\alpha}, z \bar{T}_{\alpha}, \phi_{\alpha}^{(k)} \bar{T}_{\alpha}, \bar{T}_{z}\right)\right\rangle \tag{2.22}
\end{equation*}
$$

The virtual work of the inertial forces is

$$
\begin{equation*}
\delta W_{i} \equiv-\int_{S_{m}}\left\langle\rho^{(k)}\left(\ddot{U}_{\alpha}^{(k)} \delta U_{\alpha}^{(k)}+\ddot{U}_{z} \delta U_{z}\right)\right) d S \tag{2.23}
\end{equation*}
$$

where $\rho^{(k)}$ is the material mass density of the $k$ th layer. Moreover, the dot indicates differentiation with respect to the time variable. Substituting the displacement field and
performing integration through the thickness, gives rise to the 2-D form of the virtual work of inertial forces

$$
\begin{gather*}
\delta W_{i}=-\int_{S_{m}}\left[\left(I_{0} \ddot{u}_{\alpha}+I_{1} \ddot{\theta}_{\alpha}+I_{0}^{\phi_{\alpha}} \ddot{\psi}_{\alpha}\right) \delta u_{\alpha}+\left(I_{1} \ddot{u}_{\alpha}+I_{2} \ddot{\theta}_{\alpha}+I_{1}^{\phi_{\alpha}} \ddot{\psi}_{\alpha}\right) \delta \theta_{\alpha}+\right.  \tag{2.24}\\
\left.\left(I_{0}^{\phi_{\alpha}} \ddot{u}_{\alpha}+I_{1}^{\phi_{\alpha}} \ddot{\theta}_{\alpha}+I_{2}^{\phi_{\alpha}} \ddot{\psi}_{\alpha}\right) \delta \psi_{\alpha}+I_{0} \ddot{w} \delta w\right] d S
\end{gather*}
$$

the mass moments of inertia are defined as

$$
\begin{align*}
& \left(I_{0}, I_{1}, I_{2}\right) \equiv\left\langle\rho^{(k)}\left(1, z, z^{2}\right)\right\rangle \\
& \left(I_{0}^{\phi_{\alpha}}, I_{1}^{\phi_{\alpha}}, I_{2}^{\phi_{\alpha}}\right) \equiv\left\langle\rho^{(k)}\left(\phi_{\alpha}^{(k)}, z \phi_{\alpha}^{(k)},\left(\phi_{\alpha}^{(k)}\right)^{2}\right)\right\rangle \tag{2.25}
\end{align*}
$$

By introducing Eqs. (2.16), (2.21) and (2.24) into the D'Alembert principle, the nonlinear differential equations of motion are obtained [Iurlaro et al., 2013a]

$$
\begin{array}{ll}
\delta u_{\alpha}: & \partial_{\beta} N_{\alpha \beta}+\bar{p}_{\alpha}=I_{0} \ddot{u}_{\alpha}+I_{1} \ddot{\theta}_{\alpha}+I_{0}^{\phi_{\alpha}} \ddot{\psi}_{\alpha} \\
\delta w: & \partial_{\alpha} Q_{\alpha}+\partial_{\alpha}\left(N_{\alpha \beta} \partial_{\beta} w\right)+\bar{q}=I_{0} \ddot{w} \\
\delta \theta_{\alpha}: & \partial_{\beta} M_{\alpha \beta}-Q_{\alpha}+\bar{m}_{\alpha}=I_{1} \ddot{u}_{\alpha}+I_{2} \ddot{\theta}_{\alpha}+I_{1}^{\phi_{\alpha}} \ddot{\psi}_{\alpha}  \tag{2.26}\\
\delta \psi_{\alpha}: & \partial_{\beta} M_{\alpha \beta}^{\phi}-Q_{\alpha}^{\phi}=I_{0}^{\phi_{\alpha}} \ddot{u}_{\alpha}+I_{1}^{\phi_{\alpha}} \ddot{\theta}_{\alpha}+I_{2}^{\phi_{\alpha}} \ddot{\psi}_{\alpha}
\end{array}
$$

along with the variationally consistent boundary conditions

$$
\begin{array}{cccccl}
u_{\alpha}=\bar{u}_{\alpha} & \text { on } & C_{u} & \text { or } & N_{\alpha \beta} n_{\beta}=\bar{N}_{\alpha n} & \text { on } C_{s} \\
w=\bar{w}_{n} & \text { on } & C_{u} & \text { or } & \mathrm{Q}_{\alpha} n_{\alpha}+N_{\alpha \beta} \partial_{\alpha} w n_{\beta}=\bar{V}_{z n} & \text { on } C_{s} \\
\theta_{\alpha}=\bar{\theta}_{\alpha} & \text { on } & C_{u} & \text { or } & M_{\alpha \beta} n_{\beta}=\bar{M}_{\alpha n} & \text { on } C_{s}  \tag{2.27}\\
\psi_{\alpha}=\bar{\psi}_{\alpha} & \text { on } & C_{u} & \text { or } & M_{\alpha \beta}^{\phi} n_{\beta}=\bar{M}_{\alpha n}^{\phi} & \text { on } C_{s}
\end{array}
$$

It is worthwhile to note that Eqs. (2.26) represent a generalization of the FSDT governing equations. In fact, the RZT displacement field is given by the superposition of the FSDT displacement field and of a through-the-thickness piecewise linear contribution related to the zigzag kinematic variables, $\psi_{\alpha}$. Thus, the FSDT equations of motion can be recovered by the first three equations in Eqs. (2.26) neglecting all the mass moments of inertia multiplying the second order time derivative of $\psi_{\alpha}$. Furthermore, since the von Kàrmàn strain-displacement relations are used, the non-linear contribution appears only in the third equation: in order to recover the linear equations of motion, the contribution given by the in-plane stress resultants to this equation has to be neglected.

The plate constitutive equations are derived by introducing the strain-displacement relations and the constitutive material law in Eqs. (2.17) and then integrating over the laminate thickness. To be consistent with [Tessler et al., 2010a], an extended notation is adopted according to which the force and moment stress resultants are collected in the following vector

$$
\begin{align*}
\mathbf{N}_{m}^{T} \equiv & \left(N_{1}, N_{2}, N_{12}\right)=\left\langle\left(\sigma_{11}^{(k)}, \sigma_{22}^{(k)}, \tau_{12}^{(k)}\right)\right\rangle \\
\mathbf{M}_{b}^{T} \equiv & \equiv\left(M_{1}, M_{1}^{\phi}, M_{2}, M_{2}^{\phi}, M_{12}, M_{12}^{\phi}, M_{21}^{\phi}\right)= \\
& \left\langle\left(z \sigma_{11}^{(k)}, \phi_{1}^{(k)} \sigma_{11}^{(k)}, z \sigma_{22}^{(k)}, \phi_{2}^{(k)} \sigma_{22}^{(k)}, z \tau_{12}^{(k)}, \phi_{1}^{(k)} \tau_{12}^{(k)}, \phi_{2}^{(k)} \tau_{12}^{(k)}\right)\right\rangle  \tag{2.28}\\
\mathbf{Q}_{s}^{T} \equiv & \left(Q_{2}, Q_{2}^{\phi}, Q_{1}, Q_{1}^{\phi}\right)=\left\langle\left(\tau_{2 z}^{(k)}, \beta_{2}^{(k)} \tau_{2 z}^{(k)}, \tau_{1 z}^{(k)}, \beta_{1}^{(k)} \tau_{1 z}^{(k)}\right)\right\rangle
\end{align*}
$$

Consequently, the constitutive equations appear as

$$
\left\{\begin{array}{c}
\mathbf{N}_{m}  \tag{2.29}\\
\mathbf{M}_{b} \\
\mathbf{Q}_{s}
\end{array}\right\}=\left[\begin{array}{ccc}
\mathbf{A} & \mathbf{B} & \mathbf{0} \\
\mathbf{B}^{T} & \mathbf{D} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{G}
\end{array}\right]\left\{\begin{array}{c}
\mathbf{e}_{m}^{n l} \\
\mathbf{e}_{b} \\
\mathbf{e}_{s}
\end{array}\right\}
$$

with the non-linear membrane, linear bending and transverse shear strain measures defined, respectively, as

$$
\begin{align*}
& \mathbf{e}_{m}^{n l} \equiv\left[\partial_{1} u_{1}+1 / 2\left(\partial_{1} w\right)^{2}, \partial_{2} u_{2}+1 / 2\left(\partial_{2} w\right)^{2}, \partial_{2} u_{1}+\partial_{1} u_{2}+\partial_{1} w \partial_{2} w\right]^{T} \\
& \mathbf{e}_{b} \equiv\left[\partial_{1} \theta_{1}, \partial_{1} \psi_{1}, \partial_{2} \theta_{2}, \partial_{2} \psi_{2}, \partial_{2} \theta_{1}+\partial_{1} \theta_{2}, \partial_{2} \psi_{1}, \partial_{1} \psi_{2}\right]^{T}  \tag{2.30}\\
& \mathbf{e}_{s} \equiv\left[\partial_{2} w+\theta_{2}, \psi_{2}, \partial_{1} w+\theta_{1}, \psi_{1}\right]^{T}
\end{align*}
$$

and the stiffness matrices

$$
\left.\begin{array}{l}
\mathbf{A} \equiv\langle\mathbf{C}\rangle ; \quad \mathbf{B} \equiv\left\langle\mathbf{C B}_{\phi}\right\rangle ; \quad \mathbf{D} \equiv\left\langle\mathbf{B}_{\phi}{ }^{T} \mathbf{C} \mathbf{B}_{\phi}\right\rangle ; \quad \mathbf{G} \equiv\left\langle\mathbf{B}_{\beta}{ }^{T} \mathbf{Q} \mathbf{B}_{\beta}\right\rangle \\
\mathbf{B}_{\phi} \equiv\left[\begin{array}{cccccc}
z & \phi_{1}^{(k)} & 0 & 0 & 0 & 0 \\
0 & 0 & z & \phi_{2}^{(k)} & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & z & \phi_{1}^{(k)}
\end{array} \phi_{2}^{(k)}\right.
\end{array}\right] ; \quad \mathbf{B}_{\beta} \equiv\left[\begin{array}{cccc}
1 & \beta_{2}^{(k)} & 0 & 0  \tag{2.31}\\
0 & 0 & 1 & \beta_{1}^{(k)}
\end{array}\right] .
$$

where $\mathbf{C}, \mathbf{Q}$ are the reduced elastic stiffness coefficient in the usual matrix notation [Tessler et al., 2010a].

By introducing Eqs. (2.29) into Eqs. (2.26), the equations of motion in terms of the kinematic variables are obtained.

### 3.1. Linear bending

In order to derive the equilibrium equations for the static linear response of the plate subjected only to transverse pressure and surface tractions applied on the top and bottom plate surfaces, the membrane strains are linearized with respect to the displacement components, that is $\mathbf{e}_{m} \equiv\left\{u_{1,1}, u_{2,2}, u_{1,2}+u_{2,1}\right\}^{T}$, and the inertial terms are neglected. Thus, the static linear equilibrium equations read as [Tessler et al., 2010a]

$$
\begin{array}{cc}
\partial_{\beta} N_{\alpha \beta}+\bar{p}_{\alpha}=0 ; & \partial_{\alpha} Q_{\alpha}+\bar{q}=0 \\
\partial_{\beta} M_{\alpha \beta}-Q_{\alpha}+\bar{m}_{\alpha}=0 ; & \partial_{\beta} M_{\alpha \beta}^{\phi}-Q_{\alpha}^{\phi}=0 \tag{2.32}
\end{array}
$$

### 3.2. Free vibrations

The governing equations for linear free vibrations of the plate may be obtained from Eqs. (2.26) by neglecting the non-linear terms of the membrane strain measures and by discarding the external loads [Iurlaro et al., 2013a]

$$
\begin{align*}
& \partial_{\beta} N_{\alpha \beta}=I_{0} \ddot{u}_{\alpha}+I_{1} \ddot{\theta}_{\alpha}+I_{0}^{\phi_{\alpha}} \ddot{\psi}_{\alpha} \\
& \partial_{\alpha} Q_{\alpha}=I_{0} \ddot{w} \\
& \partial_{\beta} M_{\alpha \beta}-Q_{\alpha}=I_{1} \ddot{u}_{\alpha}+I_{2} \ddot{\theta}_{\alpha}+I_{1}^{\phi_{\alpha}} \ddot{\psi}_{\alpha}  \tag{2.33}\\
& \partial_{\beta} M_{\alpha \beta}^{\phi}-Q_{\alpha}^{\phi}=I_{0}^{\phi_{\alpha}} \ddot{u}_{\alpha}+I_{1}^{\phi_{\alpha}} \ddot{\theta}_{\alpha}+I_{2}^{\phi_{\alpha}} \ddot{\psi}_{\alpha}
\end{align*}
$$

### 3.3. Linear buckling

The governing equations of the linearized problem of buckling for symmetrically laminated plates subjected to uniformly distributed in-plane stress resultants, $\bar{N}_{\alpha n}$, can be formulated by using the Euler's method of the adjacent equilibrium configurations. It is assumed that the plate remains flat during the pre-buckling equilibrium state and that the external in-plane stress resultants vary neither in magnitude nor in direction during the buckling [Brush et al., 1975]. Under these assumptions, the linearized stability equations read [Iurlaro et al., 2013a]

$$
\begin{align*}
& \partial_{\beta} N_{\alpha \beta}^{*}=0 \\
& \partial_{\alpha} Q_{\alpha}^{*}+N_{\alpha \beta,{ }_{e q}} \partial_{\alpha} \partial_{\beta} w^{*}=0 \\
& \partial_{\beta} M_{\alpha \beta}^{*}-Q_{\alpha}^{*}=0  \tag{2.34}\\
& \partial_{\beta} M_{\alpha \beta}^{\phi *}-Q_{\alpha}^{\phi *}=0
\end{align*}
$$

with the appropriate homogeneous boundary conditions. The force and moment stress resultants appearing in Eq. (2.34), and denoted with ( ${ }^{*}$ ), are increment with respect to the pre-buckling state, $N_{\alpha \beta, e q}$ are the in-plane stress resultants at the equilibrium state.

## 4. Extension of RZT to functionally graded structures

The Refined Zigzag Theory was originally developed to model the typical cross-section distortion of multilayered structures made by orthotropic materials, that is structures having a piecewise constant distribution of the mechanical properties along the laminate thickness. The challenging aim of modeling the in-plane displacements distribution slope change was addressed by enriching the FSDT kinematics with a zigzag contribution guided by the zigzag function.

Starting from the early 1980s, an advanced class of composite materials arose, the functionally graded materials (FGMs), wherein two or more phases are combined together in order to obtain a synergic combination of their mechanical properties. The main difference with respect to the traditional fiber-reinforced composite materials is the continuous and smooth through-the-thickness variation of the mechanical and thermal properties. The continuous changing in material properties along the layer thickness gives the possibility to realize multilayered structures wherein the mismatch of proprieties at layer interfaces is removed, thus preventing delamination or spalling failure caused by mismatch of the thermal expansion coefficient between layers. An in-depth investigation covering all the issues concerning the FGMs can be found in [Birman et al., 2007].

Generally speaking, FGMs are particulate composites wherein the fraction volume of the phases varies along the grading direction, that is the thickness one, thus tailoring the distribution of mechanical and thermal proprieties. According to the microstructures, examples of FGMs showing an isotropic and orthotropic behavior are available [Birman et al., 2007]. The main problem in the FGMs modeling is to find the equivalent mechanical properties, that is, the mechanical properties of the FGM considered as a whole. For this purpose, several homogenization methods are available in literature [Jha et al., 2013; Zuiker, 1995; Yin et al., 2004]. More appealing results the possibility to obtain an analytical law governing the distribution of the Young's modulus, the Poisson ratio and the thermal expansion coefficient. The interest in finding an analytical law for the mechanical properties distribution comes from the interest in obtaining a 3D Elasticity-based closed form solutions for some basic problems, in a similar manner to what has been done for
orthotropic multilayered structures. In the open literature, two properties gradation laws are very common: the polynomial and the exponential one [Jha et al., 2013].

The importance of a 3D Elasticity-based solution is to highlight the main effects that a 2D approximated theory has to be able to reproduce in order to accurately model a functionally graded structure. The continuous and smooth variation of the mechanical properties makes the Elasticity governing equations a set of partial differential equations with variable coefficients, thus the Pagano solution procedure [Pagano, 1969] is no longer applicable. A simple solution is to adopt a "sub-layers strategy", that is, a functionally graded layer is divided into a certain number of mathematical layers wherein the mechanical properties are constant and equal to an average value. This procedure, known as modified Pagano solution [Wu et al., 2010], converges by increasing the number of sublayers.

The 3D investigations demonstrate that the smooth and gradual variation of the mechanical properties along the thickness direction makes the behavior of the functionally graded structures quite different from that of traditional multilayered composite ones. In particular, the through-the-thickness distribution of in-plane displacements can exhibit a higher-order pattern different from the piecewise linear one that may be observed in the traditional multilayered orthotropic structures (even if thin or moderately thick). Thus, the 3D solution guides the researchers in the formulation of 2D approximated theories. In the open literature, many efforts have been made [Zenkour, 2004; Zenkour, 2005a, b; Reddy, 2000; Natarajan et al., 2012; Das et al., 2006; Xiang et al., 2013; Abrate, 2006; Abrate, 2008; Wu et al., 2010].

Traditionally, the Zigzag theories, and consequently, the RZT, have been developed in order to model structures wherein a mismatch in the mechanical properties takes place. When functionally graded structures are considered, the discontinuity in the material properties is removed, therefore smeared models could be accurate in predicting the through-the-thickness distribution of displacements and stresses. Moreover, in order to reproduce the higher-order distribution of quantities that are typical of functionally graded structures, the models could be more complex than the FSDT; for example, the third-order shear deformation theory by Reddy [Reddy, 2000] should be adopted.

An alternative way of reproducing the higher-order behavior of functionally graded structures is by enriching the FSDT with a non-linear contribution. This strategy can be
employed by extending the RZT to FGMs-made multilayered structures [Iurlaro et al., 2014b].

### 4.1. Refined zigzag function for functionally graded materials

The displacement field of the RZT in case of functionally graded structures remains that given in Eq. (2.1). Along with the displacement field, all the others conditions, like as the loading ones, remains the same with respect to what has been previously stated.

The difference between the previous case and the present one relies on the constitutive material law: in the former case, the elastic stiffness coefficients were constant along the layer thickness whereas in the latter one they obey to an analytical law, thus becoming variable along the layer thickness. The generalized Hooke's law for functionally graded materials, in compact notation, read as

$$
\begin{equation*}
\sigma_{\alpha \beta}^{(k)}=C_{\alpha \beta \gamma \delta}^{(k)}(z) \varepsilon_{\gamma \delta}^{(k)} ; \quad \tau_{\alpha z}^{(k)}=Q_{\alpha \beta}^{(k)}(z) \gamma_{\beta z}^{(k)} \tag{2.35}
\end{equation*}
$$

where $C_{\alpha \beta \gamma \delta}^{(k)}(z)$ and $Q_{\alpha \beta}^{(k)}(z)$ are the transformed elastic stiffness coefficients computed following the analogous relations for the traditional materials [Reddy, 2004]. The problem is thus to generalize the derivation of the zigzag function to the functionally graded materials.

According to the original RZT formulation, the zigzag functions are derived using two conditions that are here generalized to the functionally graded materials case [Iurlaro et al., 2014b]:
(i) $\quad \theta_{\alpha}$ is assumed to be the average rotations of the transverse normal. This implied that the values of the zigzag functions $\phi_{\alpha}^{(k)}(z)$ at the top and bottom plate surfaces have to be equal. For simplicity, this value is set to zero.
(ii) A partial through-the-thickness continuity condition on the transverse shear stresses is used to determine a relation for the zigzag function slope.

Recalling the same procedure developed previously and taking in mind Eq. (2.35), the definition of the weighted-average transverse shear stiffness coefficient, that appears in the expression of the zigzag function slope, reads as

$$
\begin{equation*}
G_{\alpha}=\left(\frac{1}{2 h} \sum_{k=1}^{N} \int_{z_{(k-1)}}^{z_{(k)}} \frac{d z}{Q_{\alpha \alpha}^{(k)}(z)}\right)^{-1} \tag{2.36}
\end{equation*}
$$

where $z_{(k-l)}$ and $z_{(k)}$ denote the bottom and top $k$ th layer interfaces, respectively; consequently, the spatial derivative of the zigzag function follow

$$
\begin{equation*}
\beta_{\alpha}^{(k)}(z)=\frac{G_{\alpha}}{Q_{\alpha \alpha}^{(k)}(z)}-1 \tag{2.37}
\end{equation*}
$$

The inherent variability of the mechanical properties along the thickness direction of the functionally graded layers affects the zigzag function that ceases to be a piecewise-linear function, as for the traditional multilayered composite and sandwich structures. For multilayered structures with functionally graded layers, $\phi_{\alpha}^{(k)}(z)$ is a piecewise-non-linear function whose shape depends on the grading law of the transverse shear stiffness.

### 4.2. Governing equations and constitutive relations

Consider a laminated plate of uniform thickness $2 h$ with $N$ perfectly bonded functionally graded layers as shown in Figure 2.1.

The governing equations are derived by using the D'Alembert's principle, Eq. (2.14), in the analogous manner of Sect. 2 and the same results are obtained both in terms of governing equations and variationally consistent boundary conditions.

The main differences with respect to Sect. 2 relies on the definition of the mass moments of inertia since the functionally graded materials exhibit a through-the-thickness distribution of mass density

$$
\begin{align*}
& \left(I_{0}, I_{1}, I_{2}\right) \equiv\left\langle\rho^{(k)}(z)\left(1, z, z^{2}\right)\right\rangle \\
& \left(I_{0}^{\phi_{\alpha}}, I_{1}^{\phi_{\alpha}}, I_{2}^{\phi_{\alpha}}\right) \equiv\left\langle\rho^{(k)}(z)\left(\phi_{\alpha}^{(k)}(\mathrm{z}), z \phi_{\alpha}^{(k)}(\mathrm{z}),\left(\phi_{\alpha}^{(k)}(z)\right)^{2}\right)\right\rangle \tag{2.38}
\end{align*}
$$

and on the stiffness matrices, due to the variation of the elastic stiffness coefficients

$$
\begin{align*}
& \mathbf{A} \equiv\langle\mathbf{C}(z)\rangle ; \quad \mathbf{B} \equiv\left\langle\mathbf{C}(z) \mathbf{B}_{\phi}(z)\right\rangle \\
& \mathbf{D} \equiv\left\langle\mathbf{B}_{\phi}(z)^{T} \mathbf{C}(z) \mathbf{B}_{\phi}(z)\right\rangle ; \quad \mathbf{G} \equiv\left\langle\mathbf{B}_{\beta}(z)^{T} \mathbf{Q}(z) \mathbf{B}_{\beta}(z)\right\rangle \\
& \mathbf{B}_{\phi}(z) \equiv\left[\begin{array}{ccccccc}
z & \phi_{1}^{(k)}(z) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z & \phi_{2}^{(k)}(z) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z & \phi_{1}^{(k)}(z) & \phi_{2}^{(k)}(z)
\end{array}\right] ;  \tag{2.39}\\
& \mathbf{B}_{\beta}(z) \equiv\left[\begin{array}{cccc}
1 & \beta_{2}^{(k)}(z) & 0 & 0 \\
0 & 0 & 1 & \beta_{1}^{(k)}(z)
\end{array}\right]
\end{align*}
$$

## 5. First-order zigzag models

Although no commonly used in the open literature, the term first-order zigzag model can be used to group all the displacement-based models enriching the FSDT kinematics by adding a piecewise linear contribution given by the product of a priori known zigzag function, $\Pi_{\alpha}^{(k)}(z)$, and an additional kinematic variable, the zigzag amplitude $\psi(\mathbf{x}, t)$, function of the in-plane coordinates. According to these assumptions, the displacement field read as

$$
\begin{align*}
& U_{\alpha}^{(k)}(\mathbf{x}, z, t)=u_{\alpha}(\mathbf{x}, t)+z \theta_{\alpha}(\mathbf{x}, t)+\Pi_{\alpha}^{(k)}(z) \psi(\mathbf{x}, t)  \tag{2.40}\\
& U_{z}(\mathbf{x}, z, t)=w(\mathbf{x}, t)
\end{align*}
$$

where, for the sake of simplicity, a model assuming a constant distribution of transverse displacement is considered.

According to the kinematics in Eq. (2.40), the first-order zigzag models can be distinguished based on the zigzag function adopted. From this point of view, the Refined Zigzag Theory can be recovered by Eq. (2.40) assuming $\Pi_{\alpha}^{(k)}(z)=\phi_{\alpha}^{(k)}(z)$.

In the open literature, two kind of zigzag functions are available: (i) the Di Sciuva's type and (ii) the Murakami's zigzag function. The (i) refers to a methodology according to which define a zigzag function, that is the fulfillment of the interlaminar transverse shear stresses continuity conditions, rather than to a precise zigzag function. On the contrary, the (ii) defines a precise zigzag function, formulated on the basis of the mechanical behavior of periodic laminates. Thus, since the first-order zigzag models are distinguished on the basis of the zigzag function adopted, two models can be formulated: the Di Sciuva's type, for example the RZT, and the Murakami's type, that is a model like that in Eq. (2.40) wherein the Murakami's zigzag function, quoted herein as $S_{\alpha}^{(k)}(z)$, is adopted. It is worth to note that, even if the RZT postulates a partial continuity condition transverse shear stresses, it belongs to the Di Sciuva's type zigzag functions, since it derives by handling the transverse shear stresses and resulting in a mechanical properties-dependent zigzag function. The RZT has been already discussed (see Sect. 2.2); in what follows, a brief presentation of the Murakami's zigzag function is given. Readers can refer to the original Murakami's work [Murakami, 1986] for more details.

### 5.1. Murakami's zigzag function

Before defining the Murakami's zigzag function, it appears useful to introduce notations: the $z$-coordinate of an interface is denoted with $z_{(i)}$, with $i=1,2, . . N$ and $z_{(0)}=-h$. The distance of the layer mid-plane from the reference frame is denoted with $z_{m}^{(k)}$ while the layer thickness is $2 h^{(k)}$ (see Figure 2.3). According to this notation, the local nondimensional thickness coordinate, $\zeta^{(k)} \in[-1 ; 1]$, is defined as

$$
\begin{equation*}
\zeta^{(k)} \equiv \frac{z-z_{m}^{(k)}}{h^{(k)}} \tag{2.41}
\end{equation*}
$$



Figure 2.3 Thickness notation.
Consistent with the thickness notation introduced, the Murakami's zigzag functions is defined as

$$
\begin{equation*}
S_{\alpha}^{(k)}(z) \equiv(-1)^{k} \zeta^{(k)} \tag{2.42}
\end{equation*}
$$

wherein $\zeta^{(k)} \equiv\left(z-z_{m}^{(k)}\right) / h^{(k)}$.
The Murakami's zigzag function has the following properties:
(i) It ranges between the values -1 and +1 ;
(ii) It is independent of the mechanical properties of the layers and is equal in both the directions, that is $S_{1}^{(k)}(z)=S_{2}^{(k)}(z)$;
(iii) The only parameter involved in the definition of $S_{\alpha}^{(k)}(z)$ is the layer index $k$.

In Figure 2.4 the Murakami's zigzag function is compared with the RZT one for a symmetric three-layer plate and the resulting kinematics of the two first-order zigzag


Figure 2.4 Comparison of zigzag functions.


Figure 2.5 Comparison of first-order zigzag models kinematics.
models adopting the Murakami's zigzag function and the RZT ones is highlighted in Figure 2.5 .

Since the Murakami's zigzag function takes values ranging between -1 and +1 , the fine kinematics participates to the resulting one by adding a value that, at layer interfaces, is equal to the zigzag amplitudes, $\psi_{\alpha}$. On the contrary, the Refined Zigzag Theory adds a contribution that is always given by the product of the zigzag function value at the considered $z$-coordinate value and the zigzag amplitude, with the exception of the top and bottom plate surface, where the fine kinematics does not participate to the resulting kinematics as consequence of the vanishing zigzag function (see Figure 2.5).

### 5.2. Governing equations and constitutive relations

The RZT represents a special case of the first-order zigzag model given in Eq.(2.40), wherein the RZT zigzag function is adopted. In order to obtain the governing equations and variationally consistent boundary conditions of the general case, the same procedure, previously explained, has to be followed. For sake of brevity, the same equations are not reported and the reader is addressed to Sect. 2 for details about the derivation of the equilibrium equations.

## Chapter 3

## Mixed Refined Zigzag Theory

## 1. Introduction

The most challenging purpose of any theory for multilayered composite and sandwich structures is to accurately model the arising transverse shear stress field. Above all, the importance of an accurate evaluation of transverse shear stresses is remarked by the role played by these stresses in the failure processes, like as delamination and debonding. This motivated several researchers to develop analytical and numerical models wherein attention has been focused on the transverse shear stresses [Kant et al., 2000].

In order to accurately model the transverse shear stresses, a post processing technique in conjunction with an ESL or a Zigzag model is strongly required. In the open literature, several methods are suggested: a well established one is based on the integration of the local 3D equilibrium equations once the in-plane stresses have been estimated. This technique is not suited for a finite element implementation due to the presence of the derivatives of the in-plane stresses which require high-order shape functions to be computed. The order of the shape functions required by a post processing technique that integrates the equilibrium equations could be reduced by following the approach suggested by Rolfes et al. [Rolfes et al., 1998]: here, by neglecting the effect of the in-plane stresses resultants and by assuming a cylindrical bending in each plane, the transverse shear
stresses are expressed in terms of their force resultants, so reducing the order of the derivate.

Beside this technique, several predictor-corrector approaches have been proposed. Based on the FSDT, Noor and Burton [Noor et al., 1989] employ the predictor-corrector approach to estimate the shear correction factors: in the predictor phase, the transverse shear stresses distribution coming from the constitutive equations and those resulting from integration of the equilibrium equations are used to compute the shear correction factors; in the corrector phase, an enhanced prediction of the stresses is reached due to the shear correction factors estimated in the previous step. In the framework of Zigzag theories, Lee and Cao [Lee et al., 1996] resort to a predictor-corrector approach: in the predictor step, the Di Sciuva's linear zigzag model is used to obtain the transverse shear stresses by integration of the equilibrium equations; in the corrector phase, the transverse shear stresses computed serve to reconstruct an enhanced displacements and stresses field by using a higher-order zigzag model. The idea of using model of different order is also present in Cho and Kim [Cho et al., 1996]: by assuming an equivalence between the shear strain energy computed with a lower order model and with another more accurate, relations between the kinematic variables of the two models are established. In this way, the predictor phase solve the problem with the lower order model, while in the corrector phase, more accurate solution is computed by means of the higher-order model making use of the relations set before.

A novel post processing technique is suggested by Bhar and Satsangi [Bhar et al., 2011]: this approach is based on a least square error method. Once the solution has been computed by means of an HSDT, it is possible to compute the error in the equilibrium equations assuming a piecewise parabolic approximation for the transverse shear stresses. The error is minimized with respect to the transverse shear stresses at the top and bottom surface of the layer and with respect to the shear stresses resultants. In this way, a set of two algebraic equations for each layer is achieved. By using the continuity conditions of the transverse shear stresses at the layer interfaces, the layer-based systems are assembled and solved in terms of shear stresses at interfaces.

From the brief literature review above, it is clear that a post-process of the in-plane results or complex predictor-corrector strategies are necessary to provide accurate estimation of the transverse shear stresses. The desirable objective is to obtain accurate constitutive transverse shear stresses. A way to overcome the need of a post-processing
technique is to develop mixed approach based on the Reissner Mixed Variational Theorem [Reissner, 1950], wherein a stress field may be assumed independently from the displacement one. Thus, the stresses become, together with the displacements, primary variables in the analysis. Murakami [Murakami, 1986; Toledano et al., 1987], by assuming a quadratic approximation for the transverse shear stresses in each layer, was the first to develop a mixed model using a zigzag kinematics. In the framework of FSDT, Auricchio and Sacco [Auricchio et al., 2003] presented a model based on a more general mixed variational formulation. In the same work, Auricchio and Sacco compared two different strategies of modeling the transverse shear stresses: in the first one, the shear stresses are approximated by means of polynomials of the second order in the thickness coordinate; in the second one, the transverse shear stresses pattern is derived from the integration of the local equilibrium equations. In the framework of the Reissner Mixed Variational Theorembased models, remarkable is the work of Kim and Cho [Kim et al., 2007] wherein the FSDT is adopted as kinematics while the assumed transverse shear stress field derive from a higher-order zigzag model. By establishing a relation between the kinematic variables of the two models, the solution obtained by the mixed model is used in order to obtain a better description of displacements and stresses by means of the higher-order zigzag model.

The objective of this Chapter is to develop a mixed Refined Zigzag Theory, $\mathrm{RZT}^{(\mathrm{m})}$, via the Reissner Mixed Variational Theorem. In particular, the $\mathrm{RZT}^{(\mathrm{m})}$ model assumes the RZT kinematics along with an independent transverse shear stress field, derived by using the strategy proposed by Tessler [Tessler, 2014], according to which the profile of the stresses derives from integration of the local equilibrium equations. To the best author's knowledge, the idea of assuming in a mixed approach a transverse shear stresses field coming from the integration of the local equilibrium equations appears for the first time in [Auricchio et al., 2003]. From this point of view, the novelty in the Tessler' strategy consists in a generalization of the procedure proposed in [Auricchio et al., 2003]. In fact, while in [Auricchio et al. 2003] the external surface loads are neglected, in [Tessler, 2014] the traction conditions at the top and bottom plate faces are satisfied also in the presence of external loads. The work done by Tessler [Tessler, 2014] is focused on the beam problems; recently, the $\mathrm{RZT}^{(\mathrm{m})}$ has been extended to the plate problems in [Iurlaro et al., 2013b], wherein slight modification of the original procedure is required to overcome some problems that may arise by adopting the Tessler's strategy (see Sect. 3.2). In the framework of mixed approach, the most common assumption on transverse shear stresses
is the layer-wise polynomial one, introduced by Murakami [Murakami, 1986]. To be thorough, the Murakami's assumption [Murakami, 1986] is briefly recalled. Moreover, by using a general formalism for the assumed transverse shear stresses, the derived $\mathrm{RZT}^{(\mathrm{m})}$ equations hold true also by adopting the layer-wise polynomial assumption.

## 2. Reissner Mixed Variational Theorem

When the term mixed is used to denote a variational principle in Elasticity, it means that the principle assumes secondary variables as independent along with the primary variables. In other words, mixed means that the independent variables belong to different fields, for example displacements and stresses may represent the independent variables of a variational principle in Elasticity [Reddy, 2002].

The Reissner Mixed Variational Theorem, allowing independent assumption on displacements and transverse shear and normal stresses, is a special case of a more general principle for an elastic body, the Hellinger-Reissner one [Reddy, 2002]. The Reissner Mixed Variational Theorem is derived from the Hellinger-Reissner principle by assuming that strains are related with stresses by constitutive relations. The Reissner Mixed Variational Theorem is here briefly recalled.

The total potential energy functional [Reddy, 2002] is given by

$$
\begin{equation*}
\Pi\left(u_{i}, \varepsilon_{i j}\right) \equiv \int_{V}\left(W\left(\varepsilon_{i j}\right)-f_{i} u_{i}\right) d V-\int_{S_{\sigma}} \hat{t}_{i} u_{i} d S \tag{3.1}
\end{equation*}
$$

where the Latin index take values $1,2,3$ and $W\left(\varepsilon_{i j}\right)$ is the strain energy density function, $f_{i}$ the body force vector component, $u_{i}$ the displacement vector component and $\hat{t}_{i}$ the specified traction vector on boundary $S_{\sigma}$ of the volume $V$ occupied by the body.

Now, the linear strain-displacement relations

$$
\begin{equation*}
\varepsilon_{i j}=\frac{1}{2}\left(\partial_{j} u_{i}+\partial_{i} u_{j}\right) \tag{3.2}
\end{equation*}
$$

and the displacement boundary conditions on $S_{u}$

$$
\begin{equation*}
u_{i}=\hat{u}_{i} \quad \text { on } S_{u} \tag{3.3}
\end{equation*}
$$

are introduced in the functional of Eq. (3.1) by means of the Lagrange's multipliers method (in this case, for physical consideration, the Lagrange's multiplier related with constraint in

Eq. (3.2) is the stress component, while that related with constraint in Eq. (3.3) is a traction vector component). The modified functional is defined

$$
\begin{align*}
\bar{\Pi}\left(u_{i}, \varepsilon_{i j}, \sigma_{i j}\right) & \equiv \int_{V}\left(W\left(\varepsilon_{i j}\right)-f_{i} u_{i}\right) d V+\int_{V} \sigma_{i j}\left[\varepsilon_{i j}-\frac{1}{2}\left(\partial_{j} u_{i}+\partial_{i} u_{j}\right)\right] d V  \tag{3.4}\\
& -\int_{S_{\sigma}} \hat{t}_{i} u_{i} d S-\int_{S_{u}} t_{i}\left(u_{i}-\hat{u}_{i}\right) d S
\end{align*}
$$

According to Reissner's hypothesis, the strains are related with stresses by a constitutive law

$$
\begin{equation*}
\varepsilon_{i j}=\partial_{\sigma_{i j}} W_{c}\left(\sigma_{i j}\right) \tag{3.5}
\end{equation*}
$$

where $W_{c}\left(\sigma_{i j}\right)$ is the complementary strain energy density function; in this way the strain can be eliminated by Eq. (3.4) using Eq. (3.5). Moreover, the existence of a constitutive law leads to a relation between the complementary and the strain energy density function that reads as

$$
\begin{equation*}
W_{c}\left(\sigma_{i j}\right)=\sigma_{i j} \varepsilon_{i j}-W\left(\varepsilon_{i j}\right) \tag{3.6}
\end{equation*}
$$

Introducing Eq. (3.6) into Eq. (3.4), the Reissner's functional is obtained

$$
\begin{align*}
\Pi_{R}\left(u_{i}, \sigma_{i j}\right) & \equiv \int_{V}\left(\frac{1}{2}\left(\partial_{j} u_{i}+\partial_{i} u_{j}\right) \sigma_{i j}-W_{c}\left(\sigma_{i j}\right)-f_{i} u_{i}\right) d V \\
& -\int_{S_{\sigma}} \hat{t}_{i} u_{i} d S-\int_{S_{u}} t_{i}\left(u_{i}-\hat{u}_{i}\right) d S \tag{3.7}
\end{align*}
$$

As consequence of Eq. (3.5), the strain components are given by [Kim et al., 2007]

$$
\begin{align*}
& \varepsilon_{\alpha \beta}=\partial_{\sigma_{\alpha \beta}} W_{c}\left(\sigma_{i j}\right)=\frac{1}{2}\left(\partial_{\beta} u_{\alpha}+\partial_{\alpha} u_{\beta}\right) \\
& \varepsilon_{\alpha z}=\partial_{\tau_{\alpha z}} W_{c}\left(\sigma_{i j}\right)=\frac{1}{2} \tilde{C}_{\alpha 3 \beta 3}^{-1} \tau_{\beta z}^{a} \equiv \varepsilon_{\alpha z}^{a}=\frac{1}{2} \gamma_{\alpha z}^{a}  \tag{3.8}\\
& \varepsilon_{z z}=\partial_{\sigma_{z z}} W_{c}\left(\sigma_{i j}\right)=\tilde{C}_{3333}^{-1}\left(\sigma_{z z}^{a}-\tilde{C}_{33 \alpha \beta} \varepsilon_{\alpha \beta}\right) \equiv \varepsilon_{z z}^{a}
\end{align*}
$$

where $\tilde{C}_{\alpha \beta \gamma \delta}$ are the elastic material coefficients. Moreover, the superscript $a$ denotes the transverse strains derived from the complementary strain energy density function, that is from the assumed transverse shear and normal stresses, $\tau_{\alpha z}^{a}$ and $\sigma_{z z}^{a}$, in order to distinguish them from the same strains coming from displacements.

Since the Reissner Mixed Variational Theorem assumes as independent variables the displacements and transverse stresses, the variation become

$$
\begin{align*}
\delta \Pi_{R} & =\int_{V}\left[\left(\sigma_{i j} \delta \varepsilon_{i j}+\gamma_{\alpha z} \delta \tau_{\alpha z}^{a}+\varepsilon_{z z} \delta \sigma_{z z}^{a}\right)-\left(\gamma_{\alpha z}^{a} \delta \tau_{\alpha z}^{a}+\varepsilon_{z z}^{a} \delta \sigma_{z z}^{a}\right)-f_{i} \delta u_{i}\right] d V \\
& -\int_{S_{\sigma}} \hat{t}_{i} \delta u_{i} d S-\int_{S_{u}} \delta t_{i}\left(u_{i}-\hat{u}_{i}\right) d S \tag{3.9}
\end{align*}
$$

Rearranging Eq. (3.9), the variation of the Reissner functional appears as

$$
\begin{align*}
\delta \Pi_{R} & =\int_{V}\left[\left(\sigma_{\alpha \beta} \delta \varepsilon_{\alpha \beta}+\tau_{\alpha z}^{a} \delta \gamma_{\alpha z}+\sigma_{z z}^{a} \delta \varepsilon_{z z}\right)+\delta \tau_{\alpha z}^{a}\left(\gamma_{\alpha z}-\gamma_{\alpha z}^{a}\right)+\delta \sigma_{z z}^{a}\left(\varepsilon_{z z}-\varepsilon_{z z}^{a}\right)\right] d V \\
& -\int_{V} f_{i} \delta u_{i} d V-\int_{S_{\sigma}} \hat{t}_{i} \delta u_{i} d S-\int_{S_{u}} \delta t_{i}\left(u_{i}-\hat{u}_{i}\right) d S \tag{3.10}
\end{align*}
$$

The stationary condition of the functional, $\delta \Pi_{R}=0$, gives the Euler-Lagrange equations, along with the boundary conditions, of the Reissner Mixed Variational Theorem. Among the Euler-Lagrange equations, two constraints of compatibility between the transverse shear and normal strains coming from the constitutive law and those derived by using the strain-displacement relations, appear. In formula

$$
\begin{align*}
& \gamma_{\alpha z}-\gamma_{\alpha z}^{a}=0 \\
& \varepsilon_{z z}-\varepsilon_{z z}^{a}=0 \tag{3.11}
\end{align*}
$$

## 3. RZT $^{(m)}$ plate model: kinematics and transverse shear stresses

A mixed RZT model, here quoted as $\mathrm{RZT}^{(\mathrm{m})}$, is developed via the Reissner Mixed Variational Theorem. In agreement with RZT model, the $\mathrm{RZT}^{(\mathrm{m})}$ neglects the transverse normal stress too, thus the variational statement on which the $\mathrm{RZT}^{(\mathrm{m})}$ model is based becomes

$$
\begin{equation*}
\delta \Pi_{R}=\int_{V}\left[\left(\sigma_{\alpha \beta} \delta \varepsilon_{\alpha \beta}+\tau_{\alpha z}^{a} \delta \gamma_{\alpha z}\right)+\delta \tau_{\alpha z}^{a}\left(\gamma_{\alpha z}-\gamma_{\alpha z}^{a}\right)\right] d V-\delta W_{e}=0 \tag{3.12}
\end{equation*}
$$

where the inertial forces are neglected and $\delta W_{e}$ represents the virtual variation of the work done by external loads. With respect to Eq. (3.10), in Eq. (3.12) the boundary terms on $S_{u}$ and $S_{\sigma}$ disappear since, analogously to the Virtual Work Principle, it is supposed that the body experiences a virtual displacement from the equilibrium configuration that is able to satisfy the geometrical and mechanical boundary conditions.

The Reissner Mixed Variational Theorem requires independent assumption on displacements and transverse shear stresses. The RZT ${ }^{(\mathrm{m})}$ model assumes as kinematics that of the RZT model while the assumed transverse shear stresses are derived by a novel
procedure proposed by Tessler [Tessler, 2014] in the framework of mixed RZT for beams and here extended to the plate problem. The procedure proposed by Tessler [Tessler, 2014] is not the unique possibility: in the open literature, in fact, is common to adopt the transverse shear stress assumption introduced by Murakami [Murakami, 1986]. For completeness, both strategies are presented in the following.

Regardless the type of assumption, formally, the assumed transverse shear stresses can be expressed as

$$
\begin{equation*}
\boldsymbol{\tau}^{a}=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{f}_{\mathbf{v}}(\mathbf{x})+\mathbf{Z}_{\mathbf{n}}(z) \mathbf{n}_{\mathbf{v}}(\mathbf{x}) \tag{3.13}
\end{equation*}
$$

where $\boldsymbol{\tau}^{a}=\left\{\begin{array}{ll}\tau_{1 z}^{a} & \tau_{2 z}^{a}\end{array}\right\}^{T}$ collects the assumed transverse shear stresses, the thickness coordinate dependent matrices, $\mathbf{Z}_{\mathbf{f}}(z)$ and $\mathbf{Z}_{\mathbf{n}}(z)$, rule the stresses profile along the laminate thickness, $\mathbf{f}_{\mathrm{v}}(\mathbf{x})$ contains stresses function of the in-plane coordinates, and the external surface traction are collected in $\mathbf{n}_{\mathbf{v}}(\mathbf{x})=\left\{\begin{array}{cccc}\bar{p}_{1}^{b} & \bar{p}_{1}^{t} & \bar{p}_{2}^{b} & \bar{p}_{2}^{t}\end{array}\right\}^{T}$.

### 3.1. Polynomial assumption

The use of polynomial approximation for the transverse shear stresses in a mixed-field formulation appears for the first time in [Murakami, 1986], and since then has been adopted by many investigators. For each material layer, a polynomial thickness distribution is assumed and is expressed as

$$
\begin{equation*}
\tau_{\alpha z}^{a^{(k)}}=\tau_{\alpha z}^{b(k)} F_{b}^{(k)}(z)+\tau_{\alpha z}^{t(k)} F_{t}^{(k)}(z)+T_{\alpha}^{(k)} F_{m}^{(k)}(z) \tag{3.14}
\end{equation*}
$$

where $\tau_{\alpha z}^{b(k)}$ and $\tau_{\alpha z}^{t(k)}$ are the values of the transverse shear stresses at the bottom and top interface of the $k$ th layer, respectively; $T_{\alpha}^{(k)}$ stands for the average shear stress in the $k$ th layer of thickness $2 h^{(k)}$

$$
\begin{equation*}
T_{\alpha}^{(k)} \equiv \frac{1}{2 h^{(k)}} \int_{z_{(k-1)}}^{z_{(k)}} \tau_{\alpha z}^{q^{(k)}} d z \tag{3.15}
\end{equation*}
$$

Moreover, the base functions used for the approximation in the thickness direction of the stresses in Eq. (3.14) are defined as

$$
\begin{align*}
& F_{b}^{(k)}=\frac{3}{4} \zeta^{(k)^{2}}-\frac{1}{2} \zeta^{(k)}-\frac{1}{4} ; \\
& F_{t}^{(k)}=\frac{3}{4} \zeta^{(k)^{2}}+\frac{1}{2} \zeta^{(k)}-\frac{1}{4} ;  \tag{3.16}\\
& F_{m}^{(k)}=\frac{3}{2}\left(1-\zeta^{(k)^{2}}\right)
\end{align*}
$$

where $\zeta^{(k)} \equiv\left(z-z_{m}^{(k)}\right) / h^{(k)} \in[-1 ;+1], z_{m}^{(k)}$ representing the coordinate of the $k$ th midplane.
The formalism in Eq. (3.13) can be easily recovered by enforcing the traction conditions in Eq. (3.14) and the continuity conditions of transverse shear stresses at layer interfaces.

Although the approximation in Eq. (3.14) is widespread in the open literature, Auricchio and Sacco [Auricchio et al., 2003] highlighted a serious drawback of this kind of assumption. In fact, when the number of stress variables increases, that is by increasing the number of layers, the compatibility term in the Reissner's functional is enforced stronger and stronger leading the assumed stresses to fit with those coming from the kinematic assumptions yielding to no improvements or to highly non-smooth through-the-thickness distributions of the assumed transverse shear stresses.

### 3.2. Equilibrium-based assumption

Three-dimensional equilibrium equations of Elasticity are commonly used in an attempt to derive improved, layer interface-continuous transverse shear stresses. Auricchio and Sacco [Auricchio et al., 2003], motivating by the unsuccessful results provided by the polynomial assumption of transverse shear stresses, use an equilibrium-integration approach to derive transverse shear stresses for the FSDT-based plate analysis. Moreover, an $a d$ hoc function has to be added to the integrated transverse shear stresses in order to satisfy the traction-free boundary conditions on the top bounding surface, whereas the bottom zero-traction condition was enforced a priori.

Recently, Tessler [Tessler, 2014] presented a mixed-field formulation for RZT beams, which derives the transverse shear stress from the two-dimensional Elasticity equilibrium equations. A key step in the formulation is that the transverse shear stress is made to satisfy exactly the first (axial) equilibrium equation, and hence it satisfies a priori the top and bottom traction conditions of arbitrary distributions, including the special cases of zerotraction conditions. The derived stress is also fully continuous along layer interfaces. The problem is reduced to replacing two second-order derivatives of the kinematic variables with two unknown stress functions that are determined using Reissner's mixed-field
theorem.
Herein, the Tessler' methodology [Tessler, 2014] is used to derive the transverse shear stresses for the plate case. By neglecting the body forces, the first two equilibrium equations of Elasticity read as

$$
\begin{equation*}
\partial_{z} \tau_{\alpha z}=-\left(\partial_{\alpha} \sigma_{\alpha \alpha}+\partial_{\beta} \tau_{\alpha \beta}\right), \quad \beta \neq \alpha \tag{3.17}
\end{equation*}
$$

Integrating with respect to the $z$-coordinate and enforcing the traction conditions at the bottom plate surface $(z=-h)$ yields

$$
\begin{equation*}
\tau_{\alpha z}=-\bar{p}_{\alpha}^{b}-\int_{-h}^{z}\left(\partial_{\alpha} \sigma_{\alpha \alpha}+\partial_{\beta} \tau_{\alpha \beta}\right) d z, \quad \beta \neq \alpha \tag{3.18}
\end{equation*}
$$

In order to derive an expression of the transverse shear stresses in terms of RZT kinematic variables, the Hooke's law and the linear strain-displacements relations are introduced in Eq. (3.18). In this way, after some straightforward manipulations, the transverse shear stresses involving eighteen second-order partial derivatives of the RZT kinematic variables, $u_{\alpha}, \theta_{\alpha}$ and $\psi_{\alpha}$, are obtained. The high number of stress variables, that is of kinematic variables derivatives, can cause the same problems of inaccuracy experienced in [Auricchio et al., 2003] with the polynomial approximation of transverse stresses. To circumvent the over fitting deficiency, the simple strategy pursued herein is to simplify the expression of transverse shear stresses in terms of kinematic variables by adopting the cylindrical bending hypothesis. In this manner, each transverse shear stress is related only to the second order partial derivatives, with respect to the $x_{\alpha}$ coordinate, of the kinematic variables $u_{\alpha}, \theta_{\alpha}$ and $\psi_{\alpha}$. Thus, the integrated shear stresses become (see Appendix 1)

$$
\begin{equation*}
\tau_{\alpha z}=-\bar{p}_{\alpha}^{b}-\left(\int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z\right) \partial_{\alpha \alpha} u_{\alpha}-\left(\int_{-h}^{z} z C_{\alpha \alpha}^{(k)} d z\right) \partial_{\alpha \alpha} \theta_{\alpha}-\left(\int_{-h}^{z} \phi_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)} d z\right) \partial_{\alpha \alpha} \psi_{\alpha} \tag{3.19}
\end{equation*}
$$

where a contracted notation (the Voigt-Kelvin notation) for the elastic coefficients, $C_{\alpha \alpha}^{(k)}$, is adopted.

In order to include in Eq. (3.19) the traction load applied on the top plate surface, $\bar{p}_{\alpha}^{t}$, the integration is extended to the entire laminate thickness, obtaining

$$
\begin{equation*}
\bar{p}_{\alpha}^{t}=-\bar{p}_{\alpha}^{b}-\left\langle C_{\alpha \alpha}^{(k)}\right\rangle \partial_{\alpha \alpha} u_{\alpha}-\left\langle z C_{\alpha \alpha}^{(k)}\right\rangle \partial_{\alpha \alpha} \theta_{\alpha}-\left\langle\phi_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)}\right\rangle \partial_{\alpha \alpha} \psi_{\alpha} \tag{3.20}
\end{equation*}
$$

The Eq. (3.20) can be used to obtain the second order derivative

$$
\begin{equation*}
\partial_{\alpha \alpha} u_{\alpha}=-\left\langle C_{\alpha \alpha}^{(k)}\right\rangle^{-1}\left[\left(\bar{p}_{\alpha}^{b}+\bar{p}_{\alpha}^{t}\right)+\left\langle z C_{\alpha \alpha}^{(k)}\right\rangle \partial_{\alpha \alpha} \theta_{\alpha}+\left\langle\phi_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)}\right\rangle \partial_{\alpha \alpha} \psi_{\alpha}\right] \tag{3.21}
\end{equation*}
$$

By introducing Eq. (3.21) into Eq. (3.19), the transverse shear stresses read as

$$
\begin{align*}
\tau_{\alpha z}= & \left(\left\langle C_{\alpha \alpha}^{(k)}\right\rangle^{-1} \int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z-1\right) \bar{p}_{\alpha}^{b}+\left(\left\langle C_{\alpha \alpha}^{(k)}\right\rangle^{-1} \int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z\right) \bar{p}_{\alpha}^{t}+ \\
& \left(\left\langle C_{\alpha \alpha}^{(k)}\right\rangle^{-1}\left\langle z C_{\alpha \alpha}^{(k)}\right\rangle \int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z-\int_{-h}^{z} z C_{\alpha \alpha}^{(k)} d z\right) \partial_{\alpha \alpha} \theta_{\alpha}+  \tag{3.22}\\
& \left(\left\langle C_{\alpha \alpha}^{(k)}\right\rangle^{-1}\left\langle\phi_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)}\right\rangle \int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z-\int_{-h}^{z} \phi_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)} d z\right) \partial_{\alpha \alpha} \psi_{\alpha}
\end{align*}
$$

The Eq. (3.22) represents the transverse shear stresses obtained by integration of the three-dimensional equilibrium equations and adopting the RZT kinematics. The procedure followed allows to identify the $z$-coordinate shape function that rule the profile of the equilibrium-based transverse shear stresses. It is worth to note that these shape functions depend only on the mechanical properties of the layers, thus they are known a priori once the lamination is set.

For convenience, the shape functions are quoted as follows

$$
\begin{align*}
& Z_{\alpha}^{p b}(z)=\left\langle C_{\alpha \alpha}^{(k)}\right\rangle^{-1} \int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z-1 ; \quad Z_{\alpha}^{p t}(z)=\left\langle C_{\alpha \alpha}^{(k)}\right\rangle^{-1} \int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z \\
& Z_{\alpha}^{\theta}(z)=\left\langle C_{\alpha \alpha}^{(k)}\right\rangle^{-1}\left\langle z C_{\alpha \alpha}^{(k)}\right\rangle \int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z-\int_{-h}^{z} z C_{\alpha \alpha}^{(k)} d z  \tag{3.23}\\
& Z_{\alpha}^{\psi}(z)=\left\langle C_{\alpha \alpha}^{(k)}\right\rangle^{-1}\left\langle\phi_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)}\right\rangle \int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z-\int_{-h}^{z} \phi_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)} d z
\end{align*}
$$

and it is easy to note that the shape functions satisfy the following relations

$$
\begin{array}{ll}
Z_{\alpha}^{p b}(z=-h)=1 ; & Z_{\alpha}^{p b}(z=h)=0 \\
Z_{\alpha}^{p t}(z=-h)=0 ; & Z_{\alpha}^{p t}(z=h)=1 ; \\
Z_{\alpha}^{\theta}(z=-h)=0 ; & Z_{\alpha}^{\theta}(z=h)=0  \tag{3.24}\\
Z_{\alpha}^{\psi}(z=-h)=0 ; & Z_{\alpha}^{\psi}(z=h)=0
\end{array}
$$

thus ensuring the fulfillment of the traction conditions at the bottom and top plate surfaces.
The idea proposed by Tessler [Tessler, 2014] is to adopt these functions as base for the approximation of the assumed transverse shear stresses, that is

$$
\begin{equation*}
\tau_{\alpha z}^{a}=Z_{\alpha}^{p b}(z) \bar{p}_{\alpha}^{b}+Z_{\alpha}^{p t}(z) \bar{p}_{\alpha}^{t}+Z_{\alpha}^{\theta}(z) f_{\alpha}^{\theta}+Z_{\alpha}^{\psi}(z) f_{\alpha}^{\psi} \tag{3.25}
\end{equation*}
$$

where the second order derivatives that appear in Eq. (3.22) are substituted with stress variables, $f_{\alpha}^{\theta}$ and $f_{\alpha}^{\psi}$, functions of the in-plane coordinates. It is worth to note that Eq. (3.25) exhibits the same formalism of Eq. (3.13), thus no manipulations are required to fit with Eq. (3.13).

## 4. RZT $^{(\mathbf{m})}$ governing equations and constitutive relations

Herein, the problem of a laminated plate subjected to surface traction and transverse distributed loads is considered. Refer to Chapter 2 for the problem statement and the notation.

The governing equations and the variationally consistent boundary conditions are derived by searching for the stationary condition of the functional (Eq. (3.12)) wherein the RZT kinematics is adopted and the assumed transverse shear stresses are those given in Eq. (3.25). Thus, the governing equations and the boundary conditions derive by setting the first variation of the Reissner' functional $\Pi_{R}$ equal to zero (Eq. (3.12)). The transverse shear stresses assumption does not involve kinematic variables, thus the vanishing condition of the Reissner' functional can be decomposed as

$$
\left\{\begin{array}{l}
\delta_{u} \Pi_{R}=\int_{V}\left(\sigma_{\alpha \beta} \delta \varepsilon_{\alpha \beta}+\tau_{\alpha z}^{a} \delta \gamma_{\alpha z}\right) d V-\delta W_{e}=0  \tag{3.26}\\
\delta_{\tau^{a}} \Pi_{R}=\int_{V} \delta \tau_{\alpha z}^{a}\left(\gamma_{\alpha z}-\gamma_{\alpha z}^{a}\right) d V=0
\end{array}\right.
$$

where $\delta_{u}$ and $\delta_{\tau^{a}}$ denote the virtual variation of the functional with respect to the displacement components and assumed transverse shear stresses, respectively. Thus, Eq. (3.26) means that the solution of the stationary condition of the Reissner' functional can be performed by substitution, that is, once the second line of Eq. (3.26) is solved, the result can be used to satisfy the first line of Eq. (3.26). Henceforth, second line in Eq. (3.26) is called the weak form of the compatibility constraint.

Here, the solution of the weak form of the compatibility constraint follows. In order to make easier the derivation, the tensor notation is substituted with the matrix one.

Consistent with the $\mathrm{RZT}^{(\mathrm{m})}$ kinematics, the transverse shear strains are given by

$$
\gamma=\left\{\begin{array}{l}
\gamma_{1 z}  \tag{3.27}\\
\gamma_{2 z}
\end{array}\right\} \equiv\left\{\begin{array}{l}
\partial_{z} U_{1}^{(k)}+\partial_{1} U_{z} \\
\partial_{z} U_{2}^{(k)}+\partial_{2} U_{z}
\end{array}\right\}=\left\{\begin{array}{l}
\theta_{1}+\partial_{1} w \\
\theta_{2}+\partial_{2} w
\end{array}\right\}+\left[\begin{array}{cc}
\beta_{1}^{(k)} & 0 \\
0 & \beta_{2}^{(k)}
\end{array}\right]\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}
$$

The transverse shear strains coming from the assumed transverse shear stresses are computed by reverting the Hooke's law, that is

$$
\boldsymbol{\gamma}^{a}=\left\{\begin{array}{l}
\gamma_{1 z}  \tag{3.28}\\
\gamma_{2 z}
\end{array}\right\}^{a}=\mathbf{D}_{\mathbf{T}} \boldsymbol{\tau}^{a}
$$

where $\mathbf{D}_{\mathbf{T}}=\mathbf{Q}^{-1}$ represents the shear deformability matrix.
The assumed transverse shear stress in Eq. (3.25) is arranged in the vector form given in Eq. (3.13). By using Eq. (3.25) in vector form, and Eq. (3.27) and (3.28), the weak form of the compatibility constraint becomes

$$
\left\langle\delta \mathbf{f}_{\mathbf{v}} \mathbf{Z}_{\mathbf{f}}^{T}\left[\left\{\begin{array}{l}
\theta_{1}+\partial_{1} w  \tag{3.29}\\
\theta_{2}+\partial_{2} w
\end{array}\right\}+\left[\begin{array}{cc}
\beta_{1}^{(k)} & 0 \\
0 & \beta_{2}^{(k)}
\end{array}\right]\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}-\mathbf{D}_{\mathbf{T}}\left(\mathbf{Z}_{\mathbf{f}} \mathbf{f}_{\mathbf{v}}+\mathbf{Z}_{\mathbf{n}} \mathbf{n}_{\mathbf{v}}\right)\right]\right\rangle=0
$$

where, for the arbitrary of the virtual variation, the integration over the body volume is substituted with the integration over the laminate thickness. Solving Eq. (3.29), the stress variables vector is obtained in terms of kinematic variables and surface traction applied loads

$$
\begin{align*}
\mathbf{f}_{\mathrm{v}}= & \left\langle\mathbf{Z}_{\mathrm{f}}{ }^{T} \mathbf{D}_{\mathrm{T}} \mathbf{Z}_{\mathrm{f}}\right\rangle^{-1}\left[\left\langle\mathbf{Z}_{\mathrm{f}}{ }^{T}\right\rangle\left\{\begin{array}{l}
\theta_{1}+\partial_{1} w \\
\theta_{2}+\partial_{2} w
\end{array}\right\}+\left\langle\mathbf{Z}_{\mathrm{f}}{ }^{T}\left[\begin{array}{cc}
\beta_{1}^{(k)} & 0 \\
0 & \beta_{2}^{(k)}
\end{array}\right]\right\rangle\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}\right]-  \tag{3.30}\\
& \left\langle\mathbf{Z}_{\mathrm{f}}{ }^{T} \mathbf{D}_{\mathrm{T}} \mathbf{Z}_{\mathrm{f}}\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathrm{f}}{ }^{T} \mathbf{D}_{\mathrm{T}} \mathbf{Z}_{\mathrm{n}}\right\rangle \mathbf{n}_{\mathrm{v}}
\end{align*}
$$

that can be written in a compact form

$$
\mathbf{f}_{\mathrm{v}}=\mathbf{A}_{\gamma}\left\{\begin{array}{l}
\theta_{1}+\partial_{1} w  \tag{3.31}\\
\theta_{2}+\partial_{2} w
\end{array}\right\}+\mathbf{A}_{\psi}\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}+\mathbf{A}_{n} \mathbf{n}_{\mathrm{v}}
$$

Introducing Eq. (3.31) into Eq. (3.13), the expression of the assumed transverse shear stresses able to satisfy the weak form of the compatibility constraint is derived

$$
\boldsymbol{\tau}^{a}=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{A}_{\gamma}\left\{\begin{array}{c}
\theta_{1}+\partial_{1} w  \tag{3.32}\\
\theta_{2}+\partial_{2} w
\end{array}\right\}+\mathbf{Z}_{\mathbf{f}}(z) \mathbf{A}_{\psi}\left\{\begin{array}{c}
\psi_{1} \\
\psi_{2}
\end{array}\right\}+\left(\mathbf{Z}_{\mathbf{n}}(z)+\mathbf{Z}_{\mathbf{f}}(z) \mathbf{A}_{n}\right) \mathbf{n}_{\mathbf{v}}(\mathbf{x})
$$

It is convenient to further simplify Eq. (3.32) introducing the following definition

$$
\begin{equation*}
\mathbf{T}_{\gamma}(z)=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{A}_{\gamma} ; \quad \mathbf{T}_{\psi}(z)=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{A}_{\psi} ; \quad \mathbf{T}_{p}(z)=\left(\mathbf{Z}_{\mathbf{n}}(z)+\mathbf{Z}_{\mathbf{f}}(z) \mathbf{A}_{n}\right) \tag{3.33}
\end{equation*}
$$

thus yielding

$$
\boldsymbol{\tau}^{a}=\mathbf{T}_{\gamma}(z)\left\{\begin{array}{l}
\theta_{1}+\partial_{1} w  \tag{3.34}\\
\theta_{2}+\partial_{2} w
\end{array}\right\}+\mathbf{T}_{\psi}(z)\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}+\mathbf{T}_{p}(z) \mathbf{n}_{\mathbf{v}}(\mathbf{x})
$$

Even if the $\mathrm{RZT}^{(\mathrm{m})}$ model assumes transverse shear stresses coming from integration of the three-dimensional equilibrium equations, Eq. (3.25), it is worth to note that the obtained equations hold true also if the polynomial layerwise assumption (Sect. 3.1) is kept, thanks to the general formalism, Eq. (3.13), adopted.

Once the weak form of the compatibility constraint is solved, the governing equations and the variationally consistent boundary conditions derive from the first line of Eq. (3.26), that is

$$
\begin{equation*}
\int_{V}\left(\sigma_{\alpha \beta} \delta \varepsilon_{\alpha \beta}+\tau_{\alpha z}^{a} \delta \gamma_{\alpha z}\right) d V-\delta W_{e}=0 \tag{3.35}
\end{equation*}
$$

By using the assumed displacement field, the constitutive material law, the straindisplacement relations and the assumed transverse shear stresses, Eq. (3.35) can be expressed in terms of the kinematic variables, after integration by parts

$$
\begin{align*}
& -\int_{S_{m}}\left\{\partial_{\beta} N_{\alpha \beta} \delta u_{\alpha}+\left(\partial_{\beta} M_{\alpha \beta}-Q_{\alpha}\right) \delta \theta_{\alpha}+\left(\partial_{\beta} M_{\alpha \beta}^{\phi}-Q_{\alpha}^{\phi}\right) \delta \psi_{\alpha}+\partial_{\alpha} Q_{\alpha} \delta w\right\} d S+ \\
& \int_{C_{\sigma}}\left\{N_{\alpha \beta} \delta u_{\alpha}+M_{\alpha \beta} \delta \theta_{\alpha}+M_{\alpha \beta}^{\phi} \delta \psi_{\alpha}+\left(N_{\alpha \beta} \partial_{\alpha} w+Q_{\beta}\right) \delta w\right\} n_{\beta} d \Gamma-\delta W_{e}=0 \tag{3.36}
\end{align*}
$$

where $n_{\alpha}$ denotes the direction cosine of the unit outward vector normal to $C$, with respect to the in-plane coordinate $x_{\alpha}$. Moreover, the following membrane, bending and transverse shear stress resultants are introduced

$$
\begin{align*}
& \left(N_{\alpha \beta}, M_{\alpha \beta}, M_{\alpha \beta}^{\phi}\right) \equiv\left\langle\left(1, z, \phi_{\alpha}^{(k)}(z)\right) \sigma_{\alpha \beta}^{(k)}\right\rangle \\
& \left(Q_{\alpha}, Q_{\alpha}^{\phi}\right) \equiv\left\langle\left(1, \partial_{z} \phi_{\alpha}^{(k)}(z)\right) \tau_{\alpha z}^{a}\right\rangle \tag{3.37}
\end{align*}
$$

The virtual variation of the work done by external loads read as

$$
\begin{align*}
\delta W_{e} \equiv & \int_{S_{m}} \bar{q}(\mathbf{x}, t) \delta U_{z} d S+\int_{S_{\sigma}}\left(\bar{T}_{\alpha} \delta U_{\alpha}^{(k)}+\bar{T}_{z} \delta U_{z}\right) d S+  \tag{3.38}\\
& \int_{S_{m}}\left(\bar{p}_{\alpha}^{t}(\mathbf{x}, t) \delta U_{\alpha}^{(N)}(z=+h)+\bar{p}_{\alpha}^{b}(\mathbf{x}, t) \delta U_{\alpha}^{(1)}(z=-h)\right) d S
\end{align*}
$$

and introducing the displacement components definition, it can be expressed as

$$
\begin{array}{r}
\delta W_{e}=\int_{S_{m}}\left(\bar{p}_{\alpha}(\mathbf{x}, t) \delta u_{\alpha}+\bar{m}_{\alpha}(\mathbf{x}, t) \delta \theta_{\alpha}+\bar{q}(\mathbf{x}, t) \delta w\right) d S+ \\
\int_{C_{\sigma}}\left[\bar{N}_{\alpha n} \delta u_{\alpha}+\bar{M}_{\alpha n} \delta \theta_{\alpha}+\bar{M}_{\alpha n}^{\phi} \delta \psi_{\alpha}+\bar{V}_{z n} \delta w\right] d \Gamma \tag{3.39}
\end{array}
$$

where the force and moment resultants of the prescribed tractions

$$
\begin{equation*}
\left(\bar{N}_{\alpha n}, \bar{M}_{\alpha n}, \bar{M}_{\alpha n}^{\phi}, \bar{V}_{z n}\right) \equiv\left\langle\left(\bar{T}_{\alpha}, z \bar{T}_{\alpha}, \phi_{\alpha}^{(k)} \bar{T}_{\alpha}, \bar{T}_{z}\right)\right\rangle \tag{3.40}
\end{equation*}
$$

and the resultants of the applied surface tractions

$$
\begin{align*}
& \bar{p}_{\alpha} \equiv \bar{p}_{\alpha}^{t}+\bar{p}_{\alpha}^{b} \\
& \bar{m}_{\alpha} \equiv h\left(\bar{p}_{\alpha}^{t}-\bar{p}_{\alpha}^{b}\right) \tag{3.41}
\end{align*}
$$

are introduced. Finally, the linear governing equations of $\mathrm{RZT}^{(\mathrm{m})}$ plate model are obtained

$$
\begin{array}{ll}
\delta u_{\alpha}: & \partial_{\beta} N_{\alpha \beta}+\bar{p}_{\alpha}=0 \\
\delta w: & \partial_{\alpha} Q_{\alpha}+\bar{q}=0 \\
\delta \theta_{\alpha}: & \partial_{\beta} M_{\alpha \beta}-Q_{\alpha}+\bar{m}_{\alpha}=0  \tag{3.42}\\
\delta \psi_{\alpha}: & \partial_{\beta} M_{\alpha \beta}^{\phi}-Q_{\alpha}^{\phi}=0
\end{array}
$$

along with the variationally consistent boundary conditions

$$
\begin{array}{rlllll}
u_{\alpha}=\bar{u}_{\alpha} & \text { on } & C_{u} & \text { or } & N_{\alpha \beta} n_{\beta}=\bar{N}_{\alpha n} & \text { on } C_{s} \\
w=\bar{w}_{2} & \text { on } & C_{u} & \text { or } & \mathrm{Q}_{\alpha} n_{\alpha}=\bar{V}_{z n} & \text { on } C_{s} \\
\theta_{\alpha}=\bar{\theta}_{\alpha} & \text { on } & C_{u} & \text { or } & M_{\alpha \beta} n_{\beta}=\bar{M}_{\alpha n} & \text { on } C_{s}  \tag{3.43}\\
\psi_{\alpha}=\bar{\psi}_{\alpha} & \text { on } & C_{u} & \text { or } & M_{\alpha \beta}^{\phi} n_{\beta}=\bar{M}_{\alpha n}^{\phi} & \text { on } C_{s}
\end{array}
$$

It is worth to note that the governing equations of $\mathrm{RZT}^{(\mathrm{m})}$ model are equal to those of the linear RZT model [Tessler et al., 2010]. The membrane, bending and shear force and moment stress resultants are defined as

$$
\begin{align*}
\mathbf{N}_{m}^{T} \equiv & \left(N_{1}, N_{2}, N_{12}\right)=\left\langle\left(\sigma_{11}^{(k)}, \sigma_{22}^{(k)}, \tau_{12}^{(k)}\right)\right\rangle \\
\mathbf{M}_{b}^{T} \equiv & \equiv\left(M_{1}, M_{1}^{\phi}, M_{2}, M_{2}^{\phi}, M_{12}, M_{12}^{\phi}, M_{21}^{\phi}\right)= \\
& \left\langle\left(z \sigma_{11}^{(k)}, \phi_{1}^{(k)} \sigma_{11}^{(k)}, z \sigma_{22}^{(k)}, \phi_{2}^{(k)} \sigma_{22}^{(k)}, z \tau_{12}^{(k)}, \phi_{1}^{(k)} \tau_{12}^{(k)}, \phi_{2}^{(k)} \tau_{12}^{(k)}\right)\right\rangle  \tag{3.44}\\
\mathbf{Q}_{s}^{T} \equiv & \left(Q_{2}, Q_{2}^{\phi}, Q_{1}, Q_{1}^{\phi}\right)=\left\langle\left(\tau_{2 z}^{a}, \beta_{2}^{(k)} \tau_{2 z}^{a}, \tau_{12}^{a}, \beta_{1}^{(k)} \tau_{1 z}^{a}\right)\right\rangle
\end{align*}
$$

Since the transverse shear stress assumption does not affect the in-plane behavior, the membrane and bending force and moment stress resultants are equal to the RZT ones, that is

$$
\left\{\begin{array}{l}
\mathbf{N}_{m}  \tag{3.45}\\
\mathbf{M}_{b}
\end{array}\right\}=\left[\begin{array}{cc}
\mathbf{A} & \mathbf{B} \\
\mathbf{B}^{T} & \mathbf{D}
\end{array}\right]\left\{\begin{array}{l}
\mathbf{e}_{m} \\
\mathbf{e}_{b}
\end{array}\right\}
$$

where the stiffness matrices are, analogously to RZT, defined as

$$
\begin{align*}
\mathbf{A} & \equiv\langle\mathbf{C}\rangle ; \quad \mathbf{B} \equiv\left\langle\mathbf{C B}_{\phi}\right\rangle ; \quad \mathbf{D} \equiv\left\langle\mathbf{B}_{\phi}{ }^{T} \mathbf{C} \mathbf{B}_{\phi}\right\rangle ; \\
\mathbf{B}_{\phi} & \equiv\left[\begin{array}{ccccccc}
z & \phi_{1}^{(k)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z & \phi_{2}^{(k)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z & \phi_{1}^{(k)} & \phi_{2}^{(k)}
\end{array}\right] ; \tag{3.46}
\end{align*}
$$

while the linear membrane and bending strain measures defined, respectively, as

$$
\begin{align*}
& \mathbf{e}_{m} \equiv\left[\partial_{1} u_{1}, \partial_{2} u_{2}, \partial_{2} u_{1}+\partial_{1} u_{2}\right]^{T} \\
& \mathbf{e}_{b} \equiv\left[\partial_{1} \theta_{1}, \partial_{1} \psi_{1}, \partial_{2} \theta_{2}, \partial_{2} \psi_{2}, \partial_{2} \theta_{1}+\partial_{1} \theta_{2}, \partial_{2} \psi_{1}, \partial_{1} \psi_{2}\right]^{T} \tag{3.47}
\end{align*}
$$

The shear force resultants are, in this model, obtained by integration of the assumed transverse shear stresses, that can be arranged in the following form

$$
\begin{equation*}
\boldsymbol{\tau}^{a}=\mathbf{T}_{\tau}(z) \mathbf{e}_{s}+\mathbf{T}_{p}(z) \mathbf{n}_{\mathbf{v}}(\mathbf{x}) \tag{3.48}
\end{equation*}
$$

where the shear measure and the shape functions matrix are

$$
\begin{align*}
& \boldsymbol{\tau}^{a}=\mathbf{T}_{\tau}(z) \mathbf{e}_{s}+\mathbf{T}_{p}(z) \mathbf{n}_{\mathbf{v}}(\mathbf{x}) \\
& \mathbf{T}_{\tau} \equiv\left[\begin{array}{ll}
\mathbf{T}_{\gamma} & \mathbf{T}_{\psi}
\end{array}\right]  \tag{3.49}\\
& \mathbf{e}_{s} \equiv\left[\theta_{1}+\partial_{1} w, \theta_{2}+\partial_{2} w, \psi_{1}, \psi_{2}\right]
\end{align*}
$$

Thus, by integrating the assumed transverse shear stresses, the shear force resultants appear

$$
\begin{align*}
\mathbf{Q}_{s} & \equiv\left\langle\mathbf{B}_{\beta}^{T} \boldsymbol{\tau}^{a}\right\rangle=\left\langle\mathbf{B}_{\beta}^{T} \mathbf{T}_{\tau}(z)\right\rangle \mathbf{e}_{s}+\left\langle\mathbf{B}_{\beta}^{T} \mathbf{T}_{p}(z)\right\rangle \mathbf{n}_{\mathrm{v}}=\mathbf{G e}_{s}+\mathbf{G}_{\mathrm{n}} \mathbf{n}_{\mathrm{v}} \\
\mathbf{B}_{\beta} & =\left[\begin{array}{cccc}
0 & 0 & 1 & \beta_{2}^{(k)} \\
1 & \beta_{1}^{(k)} & 0 & 0
\end{array}\right] \tag{3.50}
\end{align*}
$$

Finally, the constitutive relations can be arranged in the following form

$$
\left\{\begin{array}{l}
\mathbf{N}_{m}  \tag{3.51}\\
\mathbf{M}_{b} \\
\mathbf{Q}_{s}
\end{array}\right\}=\left[\begin{array}{ccc}
\mathbf{A} & \mathbf{B} & \mathbf{0} \\
\mathbf{B}^{T} & \mathbf{D} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{G}
\end{array}\right]\left\{\begin{array}{l}
\mathbf{e}_{m} \\
\mathbf{e}_{b} \\
\mathbf{e}_{s}
\end{array}\right\}+\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{0} \\
\mathbf{G}_{\mathbf{n}}
\end{array}\right] \mathbf{n}_{\mathbf{v}}
$$

It is worth to note that, with respect to the RZT model, the $\mathrm{RZT}^{(\mathrm{m})}$ introduces an additional shear stiffness matrix $\mathbf{G}_{\mathbf{n}}$, related with the surface traction loads $\mathbf{n}_{\mathrm{v}}$, while the shear stiffness matrices related with the kinematic variables, $\mathbf{G}$, are computed not using the material constitutive law, that is by using the elastic material coefficient, but using Eq. (3.34) that could be considered as a modified constitutive law consistent with the transverse shear stresses assumption.

## Appendix 1. RZT integrated transverse shear stresses

Herein, the way to obtain Eq. (3.19) starting from Eq. (3.18) is elucidated.
The constitutive material law, given in Eq. (2.13) is here recalled

$$
\begin{equation*}
\sigma_{\alpha \beta}^{(k)}=C_{\alpha \beta \gamma \delta}^{(k)} \varepsilon_{\gamma \delta}^{(k)} \tag{A1.1}
\end{equation*}
$$

where $C_{\alpha \beta \gamma \delta}^{(k)}$ are the transformed elastic stiffness coefficients referred to the $(\mathbf{x}, z)$ coordinate system and relative to the plane-stress condition, and $\varepsilon_{\gamma \delta}^{(k)}$ the strain components. According to Eq. (A1.1), the in-plane normal and shear stresses read as

$$
\begin{align*}
& \sigma_{11}^{(k)}=C_{11}^{(k)} \varepsilon_{11}^{(k)}+C_{22}^{(k)} \varepsilon_{22}^{(k)}+C_{16}^{(k)} \gamma_{12}^{(k)} ; \\
& \sigma_{22}^{(k)}=C_{21}^{(k)} \varepsilon_{11}^{(k)}+C_{22}^{(k)} \varepsilon_{22}^{(k)}+C_{26}^{(k)} \gamma_{12}^{(k)} ;  \tag{A1.2}\\
& \tau_{12}^{(k)}=C_{16}^{(k)} \varepsilon_{11}^{(k)}+C_{26}^{(k)} \varepsilon_{22}^{(k)}+C_{66}^{(k)} \gamma_{12}^{(k)}
\end{align*}
$$

wherein the full-index notation is adopted. According to the cylindrical bending assumption in the ( $x_{1}, z$ )-plane, the stresses in Eq. (A1.2) become

$$
\begin{align*}
& \sigma_{11}^{(k)}=C_{11}^{(k)} \varepsilon_{11}^{(k)}=C_{11}^{(k)}\left(\partial_{1} u_{1}+z \partial_{1} \theta_{1}+\phi_{1}^{(k)} \partial_{1} \psi_{1}\right)  \tag{A1.3}\\
& \sigma_{22}^{(k)}=0 ; \tau_{12}^{(k)}=0
\end{align*}
$$

where the strain is expressed in terms of the kinematic variables by using the linear straindisplacement relations. Similarly, the cylindrical bending in the $\left(x_{2}, z\right)$-plane leads to

$$
\begin{align*}
& \sigma_{22}^{(k)}=C_{22}^{(k)} \varepsilon_{22}^{(k)}=C_{22}^{(k)}\left(\partial_{2} u_{2}+z \partial_{2} \theta_{2}+\phi_{2}^{(k)} \partial_{2} \psi_{2}\right)  \tag{A1.4}\\
& \sigma_{11}^{(k)}=0 ; \quad \tau_{12}^{(k)}=0
\end{align*}
$$

Thus, using a contract notation, Eq. (A1.3) and Eq. (A1.4) are formally condensed in

$$
\begin{align*}
& \sigma_{\alpha \alpha}^{(k)}=C_{\alpha \alpha}^{(k)}\left(\partial_{\alpha} u_{\alpha}+z \partial_{\alpha} \theta_{\alpha}+\phi_{\alpha}^{(k)} \partial_{\alpha} \psi_{\alpha}\right) \\
& \sigma_{\beta \beta}^{(k)}=0 ; \quad \tau_{\alpha \beta}^{(k)}=0 \tag{A1.5}
\end{align*}
$$

and Eq. (A1.5) can be used to treat the cylindrical bending assumption in the $\left(x_{\alpha}, z\right)$ plane.
By substituting Eq. (A1.5) in Eq. (3.18), the transverse shear stress, obtained by integration of the local 3D equilibrium equations and under the cylindrical bending assumption, reads as

$$
\begin{equation*}
\tau_{\alpha z}=-\bar{p}_{\alpha}^{b}-\int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)} \partial_{\alpha \alpha} u_{\alpha}+z C_{\alpha \alpha}^{(k)} \partial_{\alpha \alpha} \theta_{\alpha}+C_{\alpha \alpha}^{(k)} \phi_{\alpha}^{(k)} \partial_{\alpha \alpha} \psi_{\alpha}\right) d z, \quad \beta \neq \alpha \tag{A1.6}
\end{equation*}
$$

and, after some manipulations, the transverse shear stress becomes

$$
\begin{equation*}
\tau_{\alpha z}=-\bar{p}_{\alpha}^{b}-\left(\int_{-h}^{z} C_{\alpha \alpha}^{(k)} d z\right) \partial_{\alpha \alpha} u_{\alpha}-\left(\int_{-h}^{z} z C_{\alpha \alpha}^{(k)} d z\right) \partial_{\alpha \alpha} \theta_{\alpha}-\left(\int_{-h}^{z} C_{\alpha \alpha}^{(k)} \phi_{\alpha}^{(k)} d z\right) \partial_{\alpha \alpha} \psi_{\alpha} \tag{A1.7}
\end{equation*}
$$

## Chapter 4

## (3,2)-Mixed Refined Zigzag Theory

## 1. Introduction

The increasing use of composite and sandwich materials for primary load-bearing components in the form of thick multilayered beam/plate/shell-like structures, requires accurate stress prediction in order to achieve reliable design. The plane stress assumption, on which the majority of plate models are based, ceases to be valid when the laminate thickness increases thus yielding to a fully three-dimensional stress field. Moreover, the increasing thickness affects the distribution along the $z$-coordinate of the in-plane displacements that stop to be piecewise linear along the laminate thickness assuming a higher-order pattern even within each single layer.

From these observations, it appears clear that a model suitable for the analysis of thick laminates has to be able to account for a non-linear distribution of in-plane displacements along the thickness of each layer and the transverse normal deformability effect.

In the past, extensive efforts were devoted to the development of higher-order plate theories including the transverse normal deformability effect. In the framework of ESL models, the $\{m, n\}$-order models proposed by Tessler and co-workers [Tessler, 1993; Cook et al., 1998; Barut et al., 2001] deserve special mention. The $\{m, n\}$ notation indicates the order of the polynomial assumption for the in-plane displacements $(m)$ and transverse one
(n). Moreover, in [Tessler, 1993; Cook et al., 1998; Barut et al., 2001] the assumptions involve also the transverse shear strains and the transverse normal stress, adopting a smeared parabolic distribution and a cubic one, respectively. The strains and stress assumptions are subjected to the Elasticity-based equilibrium conditions of zero transverse shear stresses and zero-value transverse normal stress gradient, at the top and bottom plate surface. Moreover, in order to further reduce the number of variables, least-squares compatibility constraint between the strains coming from the displacement field and those coming from the assumed strains or stress is enforced. However, due to the smeared approximation of in-plane displacements, the $\{m, n\}$-order plate models [Tessler, 1993; Cook et al., 1998; Barut et al., 2001] suffers from a not accurate description of displacements distribution along the thickness of a multilayered structures.

The early enhancement of the zigzag model kinematics is attributed to Di Sciuva [Di Sciuva, 1992] which developed a cubic zigzag model able to satisfy the interlaminar continuity conditions on transverse shear stresses. The cubic Di Sciuva's model [Di Sciuva, 1992] postulates a constant transverse displacement and is based on the plane stress hypothesis, thus neglecting the transverse normal deformability. A remarkable contribution to the early developments of cubic zigzag model is addressed to Cho and Parmenter [Cho et al., 1993]. In the context of higher-order zigzag models, it is worth to mention the work done by Icardi [Icardi, 2001b], wherein a piece-wise cubic and a piece-wise fourth-order assumption for in-plane and transverse displacements, respectively, is done. Interlaminar continuity conditions on transverse shear stresses are enforced along with continuity requirements on transverse normal stress and its gradient, whereas traction conditions are enforced at the top and bottom plate surfaces along with zero-value transverse normal stress gradient. Recently, Tessler and co-workers [Barut et al., 2012; Barut et al., 2013] enrich the RZT kinematics adding a piece-wise parabolic contribution to the in-plane displacements description and assuming a smeared parabolic distribution for the transverse displacement. The transverse normal stress is independently assumed through the thickness in the form of a smeared cubic polynomial involving two stress variables. Subsequently, the stress variables are expressed in terms of the kinematic ones by satisfying a leastsquare statement. Finally, the governing equations and the variationally consistent boundary conditions are derived by the Virtual Work Principle.

In this Chapter, a mixed higher-order zigzag model, quoted as $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$, is developed via the Reissner Mixed Variational Theorem. The assumed kinematics postulates a piece-wise
cubic distribution along the laminate thickness of in-plane displacements and a smeared parabolic pattern for the transverse one. The assumed transverse shear stresses are derived with the aid of the local three-dimensional equilibrium equations whereas the assumed transverse normal stress is smeared cubic along the thickness as suggested by the Elasticity solution.

## 2. Reissner Mixed Variational Theorem

The variational statement by means the $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$ model is formulated is the Reissner Mixed Variational Theorem in the complete form, that is taking into account also the transverse normal stress

$$
\begin{gather*}
\int_{V}\left[\left(\sigma_{\alpha \beta} \delta \varepsilon_{\alpha \beta}+\tau_{\alpha z}^{a} \delta \gamma_{\alpha z}+\sigma_{z z}^{a} \delta \varepsilon_{z z}\right)+\delta \tau_{\alpha z}^{a}\left(\gamma_{\alpha z}-\gamma_{\alpha z}^{a}\right)+\right.  \tag{4.1}\\
\left.\delta \sigma_{z z}^{a}\left(\varepsilon_{z z}-\varepsilon_{z z}^{a}\right)\right] d V-\delta W_{e}=0
\end{gather*}
$$

where $W_{e}$ represents the work done by the applied load.
Similarly to the $\mathrm{RZT}^{(\mathrm{m})}$ model, the solution procedure of the variational statement can be performed by splitting the contribution as follows

$$
\left\{\begin{array}{l}
\int_{V}\left(\sigma_{\alpha \beta} \delta \varepsilon_{\alpha \beta}+\tau_{\alpha z}^{a} \delta \gamma_{\alpha z}+\sigma_{z z}^{a} \delta \varepsilon_{z z}\right) d V-\delta W_{e}=0  \tag{4.2}\\
\int_{V} \delta \tau_{\alpha z}^{a}\left(\gamma_{\alpha z}-\gamma_{\alpha z}^{a}\right) d V=0 \\
\int_{V} \delta \sigma_{z z}^{a}\left(\varepsilon_{z z}-\varepsilon_{z z}^{a}\right) d V=0
\end{array}\right.
$$

The weak form of the compatibility constraint on the assumed transverse shear and normal stresses, second and third line of Eq. (4.2), can be solved separately due to the independence of the stress assumption from the kinematic variables. Thus, once the assumed stress profile is stated, the weak form of the compatibility constraint can be solved. Successively, the first line of Eq. (4.2) is fulfilled by using the solution of the second and third line of Eq. (4.2).

In what follows, the assumed kinematics is presented and the procedure for the definition of a novel zigzag function explained. Moreover, the assumed transverse normal stress is stated and the weak form of the compatibility constraint solved. This result is firstly used in the derivation of the assumed transverse shear stresses, due to the coupling produced by the mixed form of the Hooke's law between in-plane stresses and transverse
normal one, and finally involved, along with the solution of the compatibility constraint on transverse shear stresses, in the fulfillment of the first line of Eq. (4.2).

## 3. Higher-order zigzag kinematics

Consider a laminated plate of uniform thickness $2 h$ with $N$ perfectly bonded orthotropic layers, of thickness $2 h^{(k)}$, as shown in Figure 4.1. The orthogonal Cartesian coordinate system $\left(x_{l}, x_{2}, z\right)$ is taken as reference where the thickness coordinate $z$ ranges from $-h$ to $+h$. The middle reference plane (or midplane) of the plate, $S_{m}$, is placed on the ( $x_{1}, x_{2}$ )-plane. The plate is bounded by a cylindrical edge surface, $S$, constituted by two distinct surfaces, $S_{u}$ and $S_{s}$ on which the geometrical and mechanical boundary conditions are enforced, respectively. Moreover, the intersection of the surface $S$ and of the ( $x_{1}, x_{2}$ )-plane is the curve $C$ which represents the perimeter of the midplane, $S_{m}$. As for the edge surface, the curve $C$ is composed by two distinct curves, $C_{u}$ and $C_{s}$, originated by the intersection of $S_{u}$ and $S_{s}$ with the ( $x_{1}, x_{2}$ )-plane, respectively. Finally, $S_{t}$ and $S_{b}$ represent the top and bottom external surfaces of the plate (at $z=h$ and $z=-h$ ), respectively. The plate represented in Figure 4.1 is subjected to a transverse pressure loads, applied on $S_{b}$ and $S_{t}$, to surface tractions, acting on the top, $S_{t}$, and on the bottom, $S_{b}$, surface and to traction stresses, prescribed on $S_{s}$.


Figure 4.1 General plate notation.

Even if not already published, the higher-order kinematic assumptions for the in-plane displacements have been formulated by Gherlone and Tessler. In this Thesis, those assumptions are enriched with a through-the-thickness parabolic distribution of the transverse displacement. According to that kinematics, the orthogonal components of the displacement vector read as

$$
\begin{align*}
& U_{\alpha}^{(k)}(\mathbf{x}, z, t)=u_{\alpha}(\mathbf{x}, t)+z \theta_{\alpha}(\mathbf{x}, t)+z^{2} \chi_{\alpha}(\mathbf{x}, t)+z^{3} \omega_{\alpha}(\mathbf{x}, t)+\phi_{\alpha}^{(k)}(z) \psi_{\alpha}(\mathbf{x}, t)  \tag{4.3}\\
& U_{z}(\mathbf{x}, z, t)=H_{b}^{w}(z) w_{b}(\mathbf{x}, t)+H_{t}^{w}(z) w_{t}(\mathbf{x}, t)+H_{a}^{w}(z) \bar{w}(\mathbf{x}, t)
\end{align*}
$$

where the superscript $(k)$ is used to denote quantities corresponding to the $k$ th lamina and $t$ represent the time variable. The subscript $\alpha=1,2$ denotes the component of the displacement vector along the $x_{\alpha}$-coordinate axis while the notation $\mathbf{x} \equiv\left(x_{1}, x_{2}\right)$ has been used.

The kinematic assumption in Eq. (4.3) is an enrichment of a RZT-like displacement field: the in-plane displacements, $U_{\alpha}^{(k)}$, are given by the superposition of the RZT-like inplane displacements and a smeared quadratic and a cubic contribution. Thus, the $u_{\alpha}, \theta_{\alpha}, \psi_{\alpha}$ and $\phi_{\alpha}^{(k)}$ represent, respectively, the uniform in-plane displacement, the rotation along the $\beta$-axis, the zigzag amplitude and the zigzag function, that is different from that of the RZT and will be later defined (Sect. 3.1). The $\chi_{\alpha}$ and $\omega_{\alpha}$ are regarded as additional kinematic variables accounting for the actual distortion of the normal in a thick plate. Instead, the transverse displacement, $U_{z}$, is assumed to vary in an ESL-view quadratically along the thickness direction. Thus, $w_{b}, w_{t}$ are the bottom and top transverse displacements, respectively, whereas the average transverse displacement and the base functions used in the approximation are defined as

$$
\begin{array}{cc}
\bar{w}=\frac{1}{2 h}\left\langle U_{z}(z)\right\rangle ; & H_{b}^{w}(z)=-\frac{1}{4}-\frac{1}{2 h} z+\frac{3}{4 h^{2}} z^{2}  \tag{4.4}\\
H_{t}^{w}(z)=-\frac{1}{4}+\frac{1}{2 h} z+\frac{3}{4 h^{2}} z^{2} & H_{a}^{w}(z)=\frac{3}{2}-\frac{3}{2 h^{2}} z^{2}
\end{array}
$$

The (3,2)-order zigzag kinematics in Eq. (4.3) involves 13 unknown variables, independent on the number of layers. In order to reduce the computational cost also in the view of a finite element implementation, the kinematics can be condensed introducing a novel zigzag function, piecewise cubic, thus yielding to the following reduced kinematics

$$
\begin{align*}
& U_{\alpha}^{(k)}(\mathbf{x}, z, t)=u_{\alpha}(\mathbf{x}, t)+z \theta_{\alpha}(\mathbf{x}, t)+\mu_{\alpha}^{(k)}(z) \psi_{\alpha}(\mathbf{x}, t)  \tag{4.5}\\
& U_{z}(\mathbf{x}, z, t)=H_{b}^{w}(z) w_{b}(\mathbf{x}, t)+H_{t}^{w}(z) w_{t}(\mathbf{x}, t)+H_{a}^{w}(z) \bar{w}(\mathbf{x}, t)
\end{align*}
$$

where $\mu_{\alpha}^{(k)}(z)$ represents the novel third-order zigzag function.

### 3.1. Derivation of the zigzag function

Consistent with the assumed kinematics (Eq. (4.3)) and by using the strain-displacement relations and the constitutive material law, the transverse shear stress read as

$$
\begin{equation*}
\tau_{\alpha z}^{(k)}=Q_{\alpha \beta}^{(k)}\left(\partial_{z} U_{\beta}^{(k)}+\partial_{\beta} U_{z}\right) \tag{4.6}
\end{equation*}
$$

By introducing the strain measure $\eta_{\alpha} \equiv \theta_{\alpha}-\psi_{\alpha}+\partial_{\alpha} U_{z}$, the transverse shear stress in Eq. (4.6) can be rearranged in the following form

$$
\begin{equation*}
\tau_{\alpha z}^{(k)}=Q_{\alpha \beta}^{(k)} \eta_{\beta}+Q_{\alpha \beta}^{(k)}\left[\left(1+\partial_{z} \phi_{\beta}^{(k)}\right) \psi_{\beta}+2 \chi_{\beta} z+3 \omega_{\beta} z^{2}\right]=Q_{\alpha \beta}^{(k)} \eta_{\beta}+\tilde{\tau}_{\beta z}^{(k)} \tag{4.7}
\end{equation*}
$$

Similar to the RZT, the conditions by means of which is possible to define the zigzag function are enforced on the zigzag-dependent contribution of the transverse shear stress, $\tilde{\tau}_{\beta z}^{(k)}$, and considering only the diagonal contribution, that is for $\alpha=\beta$. The conditions require $(i)$ the zero-value of $\tilde{\tau}_{\alpha z}^{(k)}$ at the top and bottom plate surface, that is

$$
\begin{equation*}
\left.Q_{\alpha \alpha}^{(k)}\left[\left(1+\partial_{z} \phi_{\alpha}^{(k)}\right) \psi_{\alpha}+2 \chi_{\alpha} z+3 \omega_{\alpha} z^{2}\right]\right|_{z= \pm h}=0 \tag{4.8}
\end{equation*}
$$

By solving Eq. (4.8), the additional kinematic variables, $\chi_{\alpha}$ and $\omega_{\alpha}$, are expressed in terms of the RZT degrees of freedom

$$
\begin{align*}
& \chi_{\alpha}=-\frac{\partial_{z} \phi_{\alpha}^{(N)}(h)-\partial_{z} \phi_{\alpha}^{(1)}(-h)}{4 h} \psi=-\chi_{0} \psi_{\alpha} \\
& \omega_{\alpha}=-\frac{2+\partial_{z} \phi_{\alpha}^{(N)}(h)+\partial_{z} \phi_{\alpha}^{(1)}(-h)}{6 h^{2}} \psi=-\omega_{0} \psi_{\alpha} \tag{4.9}
\end{align*}
$$

Substituting Eq. (4.9) into Eq. (4.7), it is then enforced the condition (ii): the continuity condition at layer interfaces of the zigzag dependent transverse shear stress, $\tilde{\tau}_{\alpha z}^{(k)}$, that is

$$
\begin{align*}
& \left.Q_{\alpha \alpha}^{(k)}\left[\left(1+\partial_{z} \phi_{\alpha}^{(k)}\right)-2 \chi_{0} z-3 \omega_{0} z^{2}\right]\right|_{z=z_{(k-1)}}= \\
& \left.Q_{\alpha \alpha}^{(k+1)}\left[\left(1+\partial_{z} \phi_{\alpha}^{(k+1)}\right)-2 \chi_{0} z-3 \omega_{0} z^{2}\right]\right|_{z=z_{(k)}} \tag{4.10}
\end{align*}
$$

It is worth to note that Eq. (4.10) supplies $N-1$ conditions, equal to the layer interfaces, whereas, in order to completely define the zigzag contribution, $N+1$ conditions are required. Thus, two conditions remain to enforce.

Finally, by introducing Eq. (4.9) into Eq. (4.3), the displacements field read as

$$
\begin{align*}
& U_{\alpha}^{(k)}(\mathbf{x}, z, t)=u_{\alpha}(\mathbf{x}, t)+z \theta_{\alpha}(\mathbf{x}, t)+\left(-z^{2} \chi_{0}-z^{3} \omega_{0}+\phi_{\alpha}^{(k)}(z)\right) \psi_{\alpha}(\mathbf{x}, t)  \tag{4.11}\\
& U_{z}(\mathbf{x}, z, t)=H_{b}^{w}(z) w_{b}(\mathbf{x}, t)+H_{t}^{w}(z) w_{t}(\mathbf{x}, t)+H_{a}^{w}(z) \bar{w}(\mathbf{x}, t)
\end{align*}
$$

and the novel zigzag function can easily be identified

$$
\begin{equation*}
\mu_{\alpha}^{(k)}(z)=\left(-z^{2} \chi_{0}-z^{3} \omega_{0}+\phi_{\alpha}^{(k)}(z)\right) \tag{4.12}
\end{equation*}
$$

In order to completely define the zigzag function, conditions (iii) are fulfilled: the vanishing condition of the zigzag function at the top and bottom plate surface, that is

$$
\begin{equation*}
\mu_{\alpha}^{(1)}(-h)=\mu_{\alpha}^{(N)}(h)=0 \tag{4.13}
\end{equation*}
$$

## 4. Assumed transverse stresses

The assumed transverse stresses are continuous along the thickness and able to satisfy the traction conditions at the top and bottom plate surface. Two different approaches are employed in the approximation of the assumed transverse shear and normal stress. In the following, the two strategies are explained.

### 4.1. Transverse normal stress

The already cited models [Tessler, 1993; Cook et al., 1998; Barut et al., 2001], wherein the transverse normal stress is introduced, assumed a smeared cubic profile of transverse normal stress being inspired by the Elasticity solution. In agreement with this observation, the transverse normal stress is postulated smeared cubic, thus represented as a power series up to third order

$$
\begin{equation*}
\sigma_{z z}^{a}=\sum_{k=0}^{3} a_{k} z^{k} \tag{4.14}
\end{equation*}
$$

In order to get a transverse normal stress able to satisfy the traction conditions at the top and bottom plate surface, the equilibrium conditions are enforced, that is

$$
\begin{align*}
& \sigma_{z z}^{a}(z=-h)=\sum_{k=0}^{3} a_{k}(-h)^{k}=-\bar{q}^{b} \\
& \sigma_{z z}^{a}(z=h)=\sum_{k=0}^{3} a_{k}(h)^{k}=\bar{q}^{t} \tag{4.15}
\end{align*}
$$

By solving conditions in Eq. (4.15), a two-parameter assumption on the transverse normal stress is obtained

$$
\begin{align*}
& \sigma_{z z}^{a}=\mathbf{P}(z) \mathbf{q}_{\mathbf{v}}+\mathbf{L}(z) \mathbf{q}_{\mathbf{z}} \\
& \mathbf{P}(z)=\left[\begin{array}{ll}
z^{2}-h^{2} & z^{3}-z h^{2}
\end{array}\right]  \tag{4.16}\\
& \mathbf{L}(z)=\left[\begin{array}{ll}
\frac{1}{2}(z / h-1) & \frac{1}{2}(z / h+1)
\end{array}\right]
\end{align*}
$$

where $\mathbf{q}_{\mathrm{v}}=\left\{\begin{array}{ll}q_{v 1} & q_{v 2}\end{array}\right\}^{T}$ is the unknown stress vector and $\mathbf{q}_{\mathbf{z}}=\left\{\begin{array}{cc}\bar{q}_{z}^{b} & \bar{q}_{z}^{t}\end{array}\right\}^{T}$ collects the transverse pressure loads applied at the bottom and top plate surface.

Once the assumed transverse normal stress is defined, the weak form of the compatibility constraint is solved, that is

$$
\begin{equation*}
\left\langle\delta \sigma_{z z}^{a}\left(\varepsilon_{z z}-\varepsilon_{z z}^{a}\right)\right\rangle=0 \tag{4.17}
\end{equation*}
$$

where the integral over the body volume is replaced by the integral over the laminate thickness in virtue of the arbitrary virtual variation of the stress variables.

Consistent with the assumed kinematics, Eq. (4.5), the transverse normal strain reads as

$$
\begin{equation*}
\varepsilon_{z z} \equiv \partial_{z} U_{z}=\partial_{z} H_{b}^{w}(z) w_{b}+\partial_{z} H_{t}^{w}(z) w_{t}+\partial_{z} H_{a}^{w}(z) \bar{w} \tag{4.18}
\end{equation*}
$$

whereas the transverse normal strain coming from the assumed transverse normal stress is computed reverting the Hooke's law, that is

$$
\begin{equation*}
\varepsilon_{z z}^{a} \equiv S_{33}^{(k)} \sigma_{z z}^{a}-S_{33}^{(k)} R_{\alpha \beta}^{(k)} \varepsilon_{\alpha \beta} \tag{4.19}
\end{equation*}
$$

By introducing Eq. (4.18) and Eq. (4.19) into Eq. (4.17), the weak form of compatibility constraint reads as

$$
\begin{gather*}
\left\langle\delta \mathbf { q } _ { \mathbf { v } } { } ^ { T } \mathbf { P } ( z ) ^ { T } \left[\partial_{z} H_{b}^{w}(z) w_{b}+\partial_{z} H_{t}^{w}(z) w_{t}+\partial_{z} H_{a}^{w}(z) \bar{w}+\right.\right. \\
\left.\left.\quad-S_{33}^{(k)} \mathbf{P}(z) \mathbf{q}_{\mathbf{v}}-S_{33}^{(k)} \mathbf{L}(z) \mathbf{q}_{\mathbf{z}}+S_{33}^{(k)} R_{\alpha \beta}^{(k)} \varepsilon_{\alpha \beta}\right]\right\rangle=0 \tag{4.20}
\end{gather*}
$$

with solution

$$
\begin{align*}
\mathbf{q}_{\mathbf{v}} & =\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left[\left\langle\mathbf{P}^{T} \partial_{z} H_{b}^{w}\right\rangle w_{b}+\left\langle\mathbf{P}^{T} \partial_{z} H_{t}^{w}\right\rangle w_{t}+\left\langle\mathbf{P}^{T} \partial_{z} H_{a}^{w}\right\rangle \bar{w}\right. \\
& \left.-\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{L}\right\rangle \mathbf{q}_{\mathbf{z}}+\frac{1}{2}\left\langle\mathbf{P}^{T} S_{33}^{(k)} R_{\alpha \beta}^{(k)}\right\rangle\left(\partial_{\beta} U_{\alpha}^{(k)}+\partial_{\alpha} U_{\beta}^{(k)}\right)\right] \tag{4.21}
\end{align*}
$$

where the $z$-coordinate dependence is omitted for the sake of brevity.
The assumed transverse normal stress able to fulfill the weak form of the compatibility constraint is obtained by introducing Eq. (4.21) into Eq. (4.16)

$$
\begin{align*}
\sigma_{z z}^{a}= & \frac{1}{2} \mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}^{T} S_{33}^{(k)} R_{\alpha \beta}^{(k)}\right\rangle\left(\partial_{\beta} u_{\alpha}+\partial_{\alpha} u_{\beta}\right)+ \\
& \frac{1}{2} \mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}^{T} z S_{33}^{(k)} R_{\alpha \beta}^{(k)}\right\rangle\left(\partial_{\beta} \theta_{\alpha}+\partial_{\alpha} \theta_{\beta}\right)+ \\
& \frac{1}{2} \mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}^{T} \mu_{\alpha}^{(k)} S_{33}^{(k)} R_{\alpha \beta}^{(k)}\right\rangle \partial_{\beta} \psi_{\alpha}+  \tag{4.22}\\
& \frac{1}{2} \mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}^{T} \mu_{\beta}^{(k)} S_{33}^{(k)} R_{\alpha \beta}^{(k)}\right\rangle \partial_{\alpha} \psi_{\beta}+ \\
& \mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}^{T} \partial_{z} H_{b}^{w}\right\rangle w_{b}+\mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}^{T} \partial_{z} H_{t}^{w}\right\rangle w_{t}+ \\
& \mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}^{T} \partial_{z} H_{a}^{w}\right\rangle \bar{w}+\left(\mathbf{L}-\mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{L}\right\rangle\right) \mathbf{q}_{z}
\end{align*}
$$

In order to simplify the notation, the assumed transverse normal stress can be arranged in the following form

$$
\begin{align*}
\sigma_{z z}^{a}= & A_{\alpha \beta}^{u}(z)\left(\partial_{\beta} u_{\alpha}+\partial_{\alpha} u_{\beta}\right)+A_{\alpha \beta}^{\theta}(z)\left(\partial_{\beta} \theta_{\alpha}+\partial_{\alpha} \theta_{\beta}\right)+ \\
& A_{\alpha \beta}^{\psi}(z) \partial_{\beta} \psi_{\alpha}+A_{\beta \alpha}^{\psi}(z) \partial_{\alpha} \psi_{\beta}+  \tag{4.23}\\
& A_{b}^{w}(z) w_{b}+A_{t}^{w}(z) w_{t}+A_{a}^{w}(z) \bar{w}+\mathbf{A}^{\mathbf{q z}}(z) \mathbf{q}_{z}
\end{align*}
$$

where the definition of the shape functions can be inferred by comparing Eq. (4.22) with Eq. (4.23).

### 4.2. Transverse shear stresses

The assumption on the transverse shear stresses follows the same procedure employed in the development of the $\mathrm{RZT}^{(\mathrm{m})}$ model, that is by integration of the three-dimensional equilibrium equations. In this way, a continuous across the thickness distribution of transverse shear stresses able to satisfy the traction condition at the bottom plate surface is obtained. As in the $\mathrm{RZT}^{(\mathrm{m})}$ model, by substituting the second order derivatives of a kinematic variable, the tangential load applied at the top is introduced yielding to assumed continuous transverse shear stresses able to satisfy the traction condition also at the top plate surface. Moreover, in order to circumvent over fitting problems, the cylindrical
bending assumption is adopted here.
By neglecting the body forces, the first two Elasticity equilibrium equations, under the cylindrical bending assumption, read as

$$
\begin{equation*}
\partial_{z} \tau_{\alpha z}=-\partial_{\alpha} \sigma_{\alpha \alpha} \tag{4.24}
\end{equation*}
$$

Integrating with respect to the $z$-coordinate and enforcing the traction conditions at the bottom plate surface yields

$$
\begin{equation*}
\tau_{\alpha z}=-\bar{p}_{\alpha}^{b}-\int_{-h}^{z} \partial_{\alpha} \sigma_{\alpha \alpha} d z \tag{4.25}
\end{equation*}
$$

According to the mixed form of Hooke's law, the in-plane stress, under the cylindrical bending assumption, is given by

$$
\begin{equation*}
\sigma_{\alpha \alpha}=C_{\alpha \alpha}^{(k)} \varepsilon_{\alpha \alpha}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} \sigma_{z z}^{a} \tag{4.26}
\end{equation*}
$$

Similarly to the RZT $^{(\mathrm{m})}$ model (see Chapter 3), by introducing Eq. (4.26) into Eq. (4.25) and by using the strain-displacement relations and Eq. (4.23), the transverse shear stresses read as

$$
\begin{align*}
& \tau_{\alpha z}=-\bar{p}_{\alpha}^{b}-\partial_{\alpha \alpha} u_{\alpha} \int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right) d z-\partial_{\alpha \alpha} \theta_{\alpha} \int_{-h}^{z}\left(z C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{\theta}\right) d z- \\
& \partial_{\alpha \alpha} \psi_{\alpha} \int_{-h}^{z}\left(\mu_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{\psi}\right) d z-\partial_{\alpha} w_{b} \int_{-h}^{z} S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{b}^{w} d z-  \tag{4.27}\\
& \partial_{\alpha} w_{t} \int_{-h}^{z} S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{t}^{w} d z-\partial_{\alpha} \bar{w} \int_{-h}^{z} S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{a}^{w} d z-\partial_{\alpha} \mathbf{q}_{\mathbf{z}} \int_{-h}^{z} S_{33}^{(k)} R_{\alpha \alpha}^{(k)} \mathbf{A}^{\mathrm{q}} d z
\end{align*}
$$

Integration is extended up to the top plate surface, in order to derive a relation of the second order derivative of the uniform in-plane displacement taking into account also the surface loads applied at the top

$$
\begin{align*}
\partial_{\alpha \alpha} u_{\alpha}=- & \left\langle C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right\rangle^{-1}\left[\left(\bar{p}_{\alpha}^{t}+\bar{p}_{\alpha}^{b}\right)-\right. \\
& \left\langle z C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{\theta}\right\rangle \partial_{\alpha \alpha} \theta_{\alpha}- \\
& \left\langle\mu_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{\psi}\right\rangle \partial_{\alpha \alpha} \psi_{\alpha}-  \tag{4.28}\\
& \left\langle S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{b}^{w}\right\rangle \partial_{\alpha} w_{b}-\left\langle S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{t}^{w}\right\rangle \partial_{\alpha} w_{t}- \\
& \left.\left\langle S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{a}^{w}\right\rangle \partial_{\alpha} \bar{w}-\left\langle S_{33}^{(k)} R_{\alpha \alpha}^{(k)} \mathbf{A}^{\text {qz }}\right\rangle \partial_{\alpha} \mathbf{q}_{\mathbf{z}}\right]
\end{align*}
$$

Subsequently, Eq. (4.28) is introduced in Eq. (4.27) yielding to a transverse shear stress expression that, in a tense notation, reads as

$$
\begin{align*}
\tau_{\alpha z}= & Z_{\alpha}^{p b}(z) \bar{p}_{\alpha}^{b}+Z_{\alpha}^{p t}(z) \bar{p}_{\alpha}^{t}+Z_{\alpha}^{\theta}(z) \partial_{\alpha \alpha} \theta_{\alpha}+Z_{\alpha}^{\psi}(z) \partial_{\alpha \alpha} \psi_{\alpha}+  \tag{4.29}\\
& Z_{\alpha}^{w b}(z) \partial_{\alpha} w_{b}+Z_{\alpha}^{w t}(z) \partial_{\alpha} w_{t}+Z_{\alpha}^{w a}(z) \partial_{\alpha} w_{a}+\mathbf{Z}_{\alpha}^{q z}(z) \partial_{\alpha} \mathbf{q}_{z}
\end{align*}
$$

where the $z$-coordinate shape function, defined in Eq. (4.30), fulfill the vanishing condition at the top and bottom plate surface with except for $Z_{\alpha}^{p b}(z)$ and $Z_{\alpha}^{p t}(z)$ that assume zero and unit value in order to satisfy the traction condition at the external laminate surfaces.

$$
\begin{align*}
Z_{\alpha}^{p b}(z)= & \left\langle C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right\rangle^{-1} \int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right) d z-1 ; \\
Z_{\alpha}^{p t}(z)= & \left\langle C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right\rangle^{-1} \int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right) d z ; \\
Z_{\alpha}^{\theta}(z)= & \left\langle C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right\rangle^{-1}\left\langle z C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{\theta}\right\rangle \int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right) d z- \\
& \int_{-h}^{z}\left(z C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{\theta}\right) d z ; \\
Z_{\alpha}^{\psi}(z)= & \left\langle C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right\rangle^{-1}\left\langle\mu_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{\mu}\right\rangle \int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right) d z \\
& -\int_{-h}^{2}\left(\mu_{\alpha}^{(k)} C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{\mu}\right) d z ;  \tag{4.30}\\
Z_{\alpha}^{w b}(z)= & \left\langle C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right\rangle^{-1}\left\langle S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{b}^{w}\right\rangle \int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right) d z- \\
& \int_{-h}^{z} S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{b}^{w} d z ; \\
Z_{\alpha}^{w t}(z)= & \left\langle C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right\rangle^{-1}\left\langle S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{t}^{w}\right\rangle \int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right) d z- \\
& \int_{-h}^{z} S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{t}^{w} d z ; \\
Z_{\alpha}^{w a}(z)= & \left\langle C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right\rangle^{-1}\left\langle S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{a}^{w}\right\rangle \int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{k)} A_{\alpha \alpha}^{u}\right) d z- \\
& \int_{-h}^{z} S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{a}^{w} d z ;
\end{align*}
$$

$$
\begin{aligned}
\mathbf{Z}_{\alpha}^{q z}(z)= & \left\langle C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right\rangle^{-1}\left\langle S_{33}^{(k)} R_{\alpha \alpha}^{(k)} \mathbf{A}^{\mathrm{qz}}\right\rangle \int_{-h}^{z}\left(C_{\alpha \alpha}^{(k)}+S_{33}^{(k)} R_{\alpha \alpha}^{(k)} A_{\alpha \alpha}^{u}\right) d z- \\
& \int_{-h}^{z} S_{33}^{(k)} R_{\alpha \alpha}^{(k)} \mathbf{A}^{\mathrm{qz}} d z ;
\end{aligned}
$$

Integration of the local equilibrium equations allows to identify $z$-coordinate shape functions that could be involved in the approximation of the assumed transverse shear stresses. The second order derivatives in Eq. (4.29) are substituted with unknown stresses function of the in-plane coordinates, thus the assumed transverse shear stresses read as

$$
\begin{align*}
\tau_{\alpha z}^{a}= & Z_{\alpha}^{p b}(z) \bar{p}_{\alpha}^{b}+Z_{\alpha}^{p t}(z) \bar{p}_{\alpha}^{t}+Z_{\alpha}^{\theta}(z) f_{\alpha}^{\theta}+Z_{\alpha}^{\psi}(z) f_{\alpha}^{\psi}+  \tag{4.31}\\
& Z_{\alpha}^{w b}(z) f_{\alpha}^{w b}+Z_{\alpha}^{w t}(z) f_{\alpha}^{w t}+Z_{\alpha}^{w a}(z) f_{\alpha}^{w a}+\mathbf{Z}_{\alpha}^{q z}(z) \partial_{\alpha} \mathbf{q}_{\mathbf{z}}
\end{align*}
$$

Once the assumed transverse shear stresses are defined, the weak form of the compatibility constraint can be solved and the result used, coupled with Eq. (4.23), in the solution of the first line of Eq. (4.2).

Firstly, the weak form of the compatibility constraint is solved. Here, to make easier the derivation, the tensor notation is substituted with matrix one

$$
\begin{equation*}
\left\langle\delta \boldsymbol{\tau}^{a^{T}}\left(\boldsymbol{\gamma}-\boldsymbol{\gamma}^{a}\right)\right\rangle=0 \tag{4.32}
\end{equation*}
$$

where the integration over the body volume is substituted with integration over the laminate thickness in virtue of the arbitrary virtual variation. By using the straindisplacement relations and consistent with the assumed kinematics, Eq. (4.5), the transverse shear strains read as

$$
\begin{align*}
\boldsymbol{\gamma}=\left\{\begin{array}{l}
\gamma_{1 z} \\
\gamma_{2 z}
\end{array}\right\}= & \left\{\begin{array}{l}
\theta_{1} \\
\theta_{2}
\end{array}\right\}+\left[\begin{array}{cc}
\partial_{z} \mu_{1}^{(k)} & 0 \\
0 & \partial_{z} \mu_{2}^{(k)}
\end{array}\right]\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}+ \\
& H_{b}^{w}\left\{\begin{array}{l}
\partial_{1} w_{b} \\
\partial_{2} w_{b}
\end{array}\right\}+H_{t}^{w}\left\{\begin{array}{l}
\partial_{1} w_{t} \\
\partial_{2} w_{t}
\end{array}\right\}+H_{a}^{w}\left\{\begin{array}{l}
\partial_{1} \bar{w} \\
\partial_{2} \bar{w}
\end{array}\right\} \tag{4.33}
\end{align*}
$$

while the transverse shear strains coming from the assumed stresses, derives by reverting the Hooke's law

$$
\boldsymbol{\gamma}^{a}=\left\{\begin{array}{l}
\gamma_{1 z}  \tag{4.34}\\
\gamma_{2 z}
\end{array}\right\}^{a}=\mathbf{D}_{\mathbf{T}} \boldsymbol{\tau}^{a}
$$

The assumed transverse shear stresses in Eq. (4.31) can be arranged in a matrix form as

$$
\begin{align*}
& \boldsymbol{\tau}^{a}=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{f}_{\mathbf{v}}+\mathbf{Z}_{\mathbf{n}}(z) \mathbf{n}_{\mathbf{v}}+\mathbf{Z}_{\mathbf{q}}(z) \mathbf{q} \\
& \mathbf{f}_{\mathbf{v}}^{T}=\left\{f_{1}^{\theta}, f_{2}^{\theta}, f_{1}^{\psi}, f_{2}^{\psi}, f_{1}^{w b}, f_{2}^{w b}, f_{1}^{w t}, f_{2}^{w t}, f_{1}^{w a}, f_{2}^{w a}\right\}  \tag{4.35}\\
& \mathbf{q}^{T}=\left\{\begin{array}{llll}
\partial_{1} \bar{q}^{b} & \partial_{2} \bar{q}^{b} & \partial_{1} \bar{q}^{t} & \partial_{2} \bar{q}^{t}
\end{array}\right\}
\end{align*}
$$

where the matrix shape function can be easily identified by comparing Eq. (4.31) with Eq. (4.35), and $\mathbf{n}_{\mathrm{v}}$ is the vector containing the tangential loads, as defined in Chapter 3.

By introducing Eq. (4.33), (4.34) and (4.35) into Eq. (4.32), performing the variation

$$
\begin{align*}
& \left\langle\delta \mathbf { f } _ { \mathrm { v } } { } ^ { T } \mathbf { Z } _ { \mathrm { f } } ^ { T } \left[\left\{\begin{array}{l}
\theta_{1} \\
\theta_{2}
\end{array}\right\}+\left[\begin{array}{cc}
\partial_{z} \mu_{1}^{(k)} & 0 \\
0 & \partial_{z} \mu_{2}^{(k)}
\end{array}\right]\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}+H_{b}^{w}\left\{\begin{array}{l}
\partial_{1} w_{b} \\
\partial_{2} w_{b}
\end{array}\right\}+\right.\right. \\
& \left.\left.H_{t}^{w}\left\{\begin{array}{l}
\partial_{1} w_{t} \\
\partial_{2} w_{t}
\end{array}\right\}+H_{a}^{w}\left\{\begin{array}{l}
\partial_{1} \bar{w} \\
\partial_{2} \bar{w}
\end{array}\right\}-\mathbf{D}_{\mathbf{T}}\left(\mathbf{Z}_{\mathbf{f}} \mathbf{f}_{\mathrm{v}}+\mathbf{Z}_{\mathbf{n}} \mathbf{n}_{\mathrm{v}}+\mathbf{Z}_{\mathbf{q}} \mathbf{q}\right)\right]\right\rangle=0 \tag{4.36}
\end{align*}
$$

and solving with respect to the stress unknowns vector

$$
\begin{align*}
& \mathbf{f}_{\mathrm{v}}=\left\langle\mathbf{Z}_{\mathbf{f}}{ }^{T} \mathbf{D}_{\mathbf{T}} \mathbf{Z}_{\mathbf{f}}\right\rangle^{-1}\left[\left\langle\mathbf{Z}_{\mathbf{f}}{ }^{T}\right\rangle\left\{\begin{array}{l}
\theta_{1} \\
\theta_{2}
\end{array}\right\}+\left\langle\mathbf{Z}_{\mathbf{f}}{ }^{T}\left[\begin{array}{cc}
\partial_{z} \mu_{1}^{(k)} & 0 \\
0 & \partial_{z} \mu_{2}^{(k)}
\end{array}\right]\right\rangle\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}+\right. \\
& \left.\left\langle\mathbf{Z}_{\mathbf{f}}{ }^{T} H_{b}^{w}\right\rangle\left\{\begin{array}{l}
\partial_{1} w_{b} \\
\partial_{2} w_{b}
\end{array}\right\}+\left\langle\mathbf{Z}_{\mathbf{f}}{ }^{T} H_{t}^{w}\right\rangle\left\{\begin{array}{l}
\partial_{1} w_{t} \\
\partial_{2} w_{t}
\end{array}\right\}+\left\langle\mathbf{Z}_{\mathbf{f}}{ }^{T} H_{a}^{w}\right\rangle\left\{\begin{array}{l}
\partial_{1} \bar{w} \\
\partial_{2} \bar{w}
\end{array}\right\}\right]-  \tag{4.37}\\
& \left\langle\mathbf{Z}_{\mathrm{f}}{ }^{T} \mathbf{D}_{\mathrm{T}} \mathbf{Z}_{\mathbf{f}}\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathrm{f}}{ }^{T} \mathbf{D}_{\mathrm{T}} \mathbf{Z}_{\mathrm{n}}\right\rangle \mathbf{n}_{\mathrm{v}}-\left\langle\mathbf{Z}_{\mathrm{f}}{ }^{T} \mathbf{D}_{\mathrm{T}} \mathbf{Z}_{\mathbf{f}}\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathrm{f}}{ }^{T} \mathbf{D}_{\mathrm{T}} \mathbf{Z}_{\mathbf{q}}\right\rangle \mathbf{q}
\end{align*}
$$

In a more convenient compact form, Eq. (4.36) reads as

$$
\begin{align*}
\mathbf{f}_{\mathrm{v}}=\mathbf{B}_{\theta}\left\{\begin{array}{l}
\theta_{1} \\
\theta_{2}
\end{array}\right\}+\mathbf{B}_{\psi}\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}+\mathbf{B}_{w b}\left\{\begin{array}{l}
\partial_{1} w_{b} \\
\partial_{2} w_{b}
\end{array}\right\}+\mathbf{B}_{w t}\left\{\begin{array}{l}
\partial_{1} w_{t} \\
\partial_{2} w_{t}
\end{array}\right\}+ \\
\mathbf{B}_{w a}\left\{\begin{array}{l}
\partial_{1} \bar{w} \\
\partial_{2} \bar{w}
\end{array}\right\}-\mathbf{B}_{n} \mathbf{n}_{\mathrm{v}}-\mathbf{B}_{q} \mathbf{q} \tag{4.38}
\end{align*}
$$

Finally, by using the solution of the weak form of the compatibility constraint, Eq. (4.38), into Eq. (4.35), the assumed transverse shear stresses are arranged in the form

$$
\begin{gather*}
\boldsymbol{\tau}^{a}=\mathbf{T}^{\theta}(z)\left\{\begin{array}{l}
\theta_{1} \\
\theta_{2}
\end{array}\right\}+\mathbf{T}^{\psi}(z)\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}+\mathbf{T}_{b}^{w}(z)\left\{\begin{array}{l}
\partial_{1} w_{b} \\
\partial_{2} w_{b}
\end{array}\right\}+\mathbf{T}_{t}^{w}(z)\left\{\begin{array}{l}
\partial_{1} w_{t} \\
\partial_{2} w_{t}
\end{array}\right\}+ \\
\mathbf{T}_{a}^{w}(z)\left\{\begin{array}{l}
\partial_{1} \bar{w} \\
\partial_{2} \bar{w}
\end{array}\right\}+\mathbf{T}_{p}(z) \mathbf{n}_{\mathrm{v}}+\mathbf{T}_{q}(z) \mathbf{q} \tag{4.39}
\end{gather*}
$$

where are used the following definitions

$$
\begin{align*}
& \mathbf{T}^{\theta}(z)=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{B}_{\theta} ; \quad \mathbf{T}^{\psi}(z)=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{B}_{\psi} ; \\
& \mathbf{T}_{b}^{w}(z)=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{B}_{w b} ; \quad \mathbf{T}_{t}^{w}(z)=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{B}_{w t} ; \quad \mathbf{T}_{a}^{w}(z)=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{B}_{w a}  \tag{4.40}\\
& \mathbf{T}_{p}(z)=\mathbf{Z}_{\mathbf{n}}(z)-\mathbf{Z}_{\mathbf{f}}(z) \mathbf{B}_{n} ; \quad \mathbf{T}_{q}(z)=\mathbf{Z}_{\mathbf{q}}(z)-\mathbf{Z}_{\mathbf{f}}(z) \mathbf{B}_{q}
\end{align*}
$$

## 5. Governing equations and constitutive relations

According to the Reissner Mixed Variational Theorem, the equations governing the elastostatic behavior and the variationally consistent boundary conditions are derived by solving

$$
\begin{equation*}
\int_{V}\left(\sigma_{\alpha \beta} \delta \varepsilon_{\alpha \beta}+\tau_{\alpha z}^{a} \delta \gamma_{\alpha z}+\sigma_{z z}^{a} \delta \varepsilon_{z z}\right) d V-\delta W_{e}=0 \tag{4.41}
\end{equation*}
$$

By using the strain-displacement relations and performing the integration by parts, the first contribution at the left-hand side of Eq. (4.41) reads as

$$
\begin{align*}
& -\int_{S_{m}}\left\{\partial_{\beta} N_{\alpha \beta} \delta u_{\alpha}+\left(\partial_{\beta} M_{\alpha \beta}-Q_{\alpha}\right) \delta \theta_{\alpha}+\left(\partial_{\beta} M_{\alpha \beta}^{\phi}-Q_{\alpha}^{\phi}\right) \delta \psi_{\alpha}+\right. \\
& \left.\quad\left(\partial_{\alpha} Q_{\alpha z}^{b}-N_{z z}^{b}\right) \delta w_{b}+\left(\partial_{\alpha} Q_{\alpha z}^{t}-N_{z z}^{t}\right) \delta w_{t}+\left(\partial_{\alpha} Q_{\alpha z}^{a}-N_{z z}^{a}\right) \delta \bar{w}\right\} d S+  \tag{4.42}\\
& \int_{C_{\sigma}}\left\{N_{\alpha \beta} \delta u_{\alpha}+M_{\alpha \beta} \delta \theta_{\alpha}+M_{\alpha \beta}^{\phi} \delta \psi_{\alpha}+Q_{\beta z}^{b} \delta w_{b}+Q_{\beta z}^{t} \delta w_{t}+Q_{\beta z}^{a} \delta \bar{w}\right\} n_{\beta} d \Gamma
\end{align*}
$$

where $n_{\alpha}$ denotes the direction cosine the unit outward vector normal to $C$, with respect to the in-plane coordinate $x_{\alpha}$. Moreover, the following membrane, bending and transverse shear stress resultants are introduced

$$
\begin{align*}
& \left(N_{\alpha \beta}, M_{\alpha \beta}, M_{\alpha \beta}^{\mu}\right) \equiv\left\langle\left(1, z, \mu_{\alpha}^{(k)}(z)\right) \sigma_{\alpha \beta}^{(k)}\right\rangle \\
& \left(Q_{\alpha}, Q_{\alpha}^{\mu}\right) \equiv\left\langle\left(1, \partial_{z} \mu_{\alpha}^{(k)}(z)\right) \tau_{\alpha z}^{a}\right\rangle \\
& \left(Q_{\alpha}^{b}, Q_{\alpha}^{t}, Q_{\alpha}^{a}\right) \equiv\left\langle\left(H_{b}^{w}(z), H_{t}^{w}(z), H_{a}^{w}(z)\right) \tau_{\alpha z}^{a}\right\rangle  \tag{4.43}\\
& \left(N_{z z}^{b}, N_{z z}^{t}, N_{z z}^{a}\right) \equiv\left\langle\left(\partial_{z} H_{b}^{w}(z), \partial_{z} H_{t}^{w}(z), \partial_{z} H_{a}^{w}(z)\right) \sigma_{z z}^{a}\right\rangle
\end{align*}
$$

The variation of the work done by the external loads reads as

$$
\begin{align*}
\delta W_{e} \equiv & \int_{S_{b}} \bar{q}^{b} \delta U_{z}(z=-h) d S+\int_{S_{t}} \bar{q}^{t} \delta U_{z}(z=h) d S+ \\
& \int_{S_{\sigma}}\left(\bar{T}_{\alpha} \delta U_{\alpha}^{(k)}+\bar{T}_{z} \delta U_{z}\right) d S+\int_{S_{m}}\left(\bar{p}_{\alpha}^{t} \delta U_{\alpha}^{(N)}(z=h)+\bar{p}_{\alpha}^{b} \delta U_{\alpha}^{(1)}(z=-h)\right) d S \tag{4.44}
\end{align*}
$$

By introducing the equivalence among the surfaces $S_{t}=S_{b}=S_{m}$ and the assumed kinematics, the variation of the external work becomes

$$
\begin{gather*}
\delta W_{e} \equiv \int_{S_{m}} \bar{q}^{b} \delta w_{b} d S+\int_{S_{m}} \bar{q}^{t} \delta w_{t} d S+\int_{S_{m}}\left(\left(\bar{p}_{\alpha}^{t}+\bar{p}_{\alpha}^{b}\right) \delta u_{\alpha}+h\left(\bar{p}_{\alpha}^{t}-\bar{p}_{\alpha}^{b}\right) \delta \theta_{\alpha}\right) d S \\
\int_{C_{\sigma}}\left\langle\bar{T}_{\alpha} \delta\left(u_{\alpha}+z \theta_{\alpha}+\mu_{\alpha}^{(k)} \psi_{\alpha}\right)+\bar{T}_{z} \delta\left(H_{b}^{w} w_{b}+H_{t}^{w} w_{t}+H_{a}^{w} \bar{w}\right) d \Gamma\right. \tag{4.45}
\end{gather*}
$$

Using the definition

$$
\begin{align*}
& \bar{p}_{\alpha} \equiv \bar{p}_{\alpha}^{t}+\bar{p}_{\alpha}^{b} \\
& \bar{m}_{\alpha} \equiv h\left(\bar{p}_{\alpha}^{t}-\bar{p}_{\alpha}^{b}\right) \tag{4.46}
\end{align*}
$$

and introducing the force and moment resultants of the prescribed tractions

$$
\begin{equation*}
\left(\bar{N}_{\alpha n}, \bar{M}_{\alpha n}, \bar{M}_{\alpha n}^{\mu}, \bar{V}_{z n}^{b}, \bar{V}_{z n}^{t}, \bar{V}_{z n}^{a}\right) \equiv\left\langle\left(\bar{T}_{\alpha}, z \bar{T}_{\alpha}, \mu_{\alpha}^{(k)} \bar{T}_{\alpha}, \bar{T}_{z} H_{b}^{w}, \bar{T}_{z} H_{t}^{w}, \bar{T}_{z} H_{a}^{w}\right)\right\rangle \tag{4.47}
\end{equation*}
$$

the virtual variation of the external work appears

$$
\begin{align*}
\delta W_{e}= & \int_{S_{m}}\left(\bar{p}_{\alpha}(\mathbf{x}, t) \delta u_{\alpha}+\bar{m}_{\alpha}(\mathbf{x}, t) \delta \theta_{\alpha}+\bar{q}^{t}(\mathbf{x}, t) \delta w_{t}+\bar{q}^{b}(\mathbf{x}, t) \delta w_{b}\right) d S+ \\
& \int_{C_{\sigma}}\left[\bar{N}_{\alpha n} \delta u_{\alpha}+\bar{M}_{\alpha n} \delta \theta_{\alpha}+\bar{M}_{\alpha n}^{\phi} \delta \psi_{\alpha}+\bar{V}_{z n}^{b} \delta w_{b}+\bar{V}_{z n}^{t} \delta w_{t}+\bar{V}_{z n}^{a} \delta \bar{w}\right] d \Gamma \tag{4.48}
\end{align*}
$$

By equating Eq. (4.42) and (4.48), the linear governing equations are derived

$$
\begin{array}{ll}
\delta u_{\alpha}: & \partial_{\beta} N_{\alpha \beta}+\bar{p}_{\alpha}=0 \\
\delta \theta_{\alpha}: & \partial_{\beta} M_{\alpha \beta}-Q_{\alpha z}+\bar{m}_{\alpha}=0 \\
\delta \psi_{\alpha}: & \partial_{\beta} M_{\alpha \beta}^{\mu}-Q_{\alpha z}^{\mu}=0  \tag{4.49}\\
\delta w_{b}: & \partial_{\alpha} Q_{\alpha z}^{b}-N_{z z}^{b}+\bar{q}^{b}=0 \\
\delta w_{t}: & \partial_{\alpha} Q_{\alpha z}^{t}-N_{z z}^{t}+\bar{q}^{t}=0 \\
\delta \bar{w}: & \partial_{\alpha} Q_{\alpha z}^{a}-N_{z z}^{a}=0
\end{array}
$$

along with the variationally consistent boundary conditions

$$
\begin{array}{rllllll}
u_{\alpha} & =\bar{u}_{\alpha} & \text { on } & C_{u} & \text { or } & N_{\alpha \beta} n_{\beta}=\bar{N}_{\alpha n} & \text { on } C_{s} \\
\theta_{\alpha} & =\bar{\theta}_{\alpha} & \text { on } & C_{u} & \text { or } & M_{\alpha \beta} n_{\beta}=\bar{M}_{\alpha n} & \text { on } C_{s} \\
\psi_{\alpha} & =\bar{\psi}_{\alpha} & \text { on } & C_{u} & \text { or } & M_{\alpha \beta}^{\mu} n_{\beta}=\bar{M}_{\alpha n}^{\mu} &  \tag{4.50}\\
w_{b} & \text { on } C_{s} \\
w_{b} & \text { on } & C_{u} & \text { or } & Q_{\alpha z}^{b} n_{\alpha}=\bar{V}_{z n}^{b} & \text { on } C_{s} \\
w_{t} & =\bar{w}_{t} & \text { on } & C_{u} & \text { or } & Q_{\alpha z}^{t} n_{\alpha}=\bar{V}_{z n}^{t} & \text { on } C_{s} \\
\bar{w} & =\overline{\bar{w}} & \text { on } & C_{u} & \text { or } & Q_{\alpha z}^{a} n_{\alpha}=\bar{V}_{z n}^{a} & \text { on } C_{s}
\end{array}
$$

The model constitutive relations come by expressing the stresses in terms of the kinematic variables in Eq. (4.43) and by using the definition of the assumed transverse shear and normal stresses as derived after fulfillment of the weak form of the compatibility constraint, that is Eq. (4.23) and (4.39). To ease the derivation, firstly the assumed transverse normal stress is rewritten in a more convenient form, that is

$$
\begin{equation*}
\sigma_{z z}^{a}=\mathbf{A}_{m}(z) \mathbf{e}_{m}+\mathbf{A}_{b}(z) \mathbf{e}_{b}+\mathbf{A}_{w}(z) \mathbf{w}+\mathbf{A}^{\mathbf{q z}}(z) \mathbf{q}_{\mathbf{z}} \tag{4.51}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathbf{e}_{m} \equiv\left[\partial_{1} u_{1}, \partial_{2} u_{2}, \partial_{2} u_{1}+\partial_{1} u_{2}\right]^{T} \\
& \mathbf{e}_{b} \equiv\left[\partial_{1} \theta_{1}, \partial_{1} \psi_{1}, \partial_{2} \theta_{2}, \partial_{2} \psi_{2}, \partial_{2} \theta_{1}+\partial_{1} \theta_{2}, \partial_{2} \psi_{1}, \partial_{1} \psi_{2}\right]^{T}  \tag{4.52}\\
& \mathbf{w} \equiv\left[w_{b}, w_{t}, \bar{w}\right]^{T}
\end{align*}
$$

while the shape function matrices $\mathbf{A}_{m}(z), \mathbf{A}_{b}(z), \mathbf{A}_{w}(z)$ derive by comparing Eq. (4.51) with Eq. (4.23). The membrane, bending and normal stress resultants are collected as

$$
\begin{align*}
\mathbf{N}_{m}^{T} \equiv & \equiv\left(N_{1}, N_{2}, N_{12}\right)=\left\langle\left(\sigma_{11}^{(k)}, \sigma_{22}^{(k)}, \tau_{12}^{(k)}\right)\right\rangle \\
\mathbf{M}_{b}^{T} \equiv & \equiv\left(M_{1}, M_{1}^{\mu}, M_{2}, M_{2}^{\mu}, M_{12}, M_{12}^{\mu}, M_{21}^{\mu}\right)= \\
& \left\langle\left(z \sigma_{11}^{(k)}, \mu_{1}^{(k)} \sigma_{11}^{(k)}, z \sigma_{22}^{(k)}, \mu_{2}^{(k)} \sigma_{22}^{(k)}, z \tau_{12}^{(k)}, \mu_{1}^{(k)} \tau_{12}^{(k)}, \mu_{2}^{(k)} \tau_{12}^{(k)}\right)\right\rangle  \tag{4.53}\\
\mathbf{N}_{z}^{T} \equiv & \equiv\left(N_{z z}^{b}, N_{z z}^{t}, N_{z z}^{a}\right)=\left\langle\left(H_{b}^{w}, H_{t}^{w}, H_{a}^{w}\right) \sigma_{z z}^{a}\right\rangle
\end{align*}
$$

while the shear stress resultants are

$$
\begin{align*}
& \mathbf{Q}^{T} \equiv\left\{Q_{1}, Q_{1}^{\phi}, Q_{2}, Q_{2}^{\phi}, Q_{1}^{b}, Q_{1}^{t}, Q_{1}^{a}, Q_{2}^{b}, Q_{2}^{t}, Q_{2}^{a}\right\}= \\
& \left\langle\tau_{1 z}^{a}, \partial_{z} \mu_{1}^{(k)} \tau_{1 z}^{a}, \tau_{2 z}^{a}, \partial_{z} \mu_{2}^{(k)} \tau_{2 z}^{a}, H_{b}^{w} \tau_{1 z}^{a}, H_{t}^{w} \tau_{1 z}^{a}, H_{a}^{w} \tau_{1 z}^{a}, H_{b}^{w} \tau_{2 z}^{a}, H_{t}^{w} \tau_{2 z}^{a}, H_{a}^{w} \tau_{2 z}^{a}\right\rangle \tag{4.54}
\end{align*}
$$

Consequently, the constitutive equations appear as

$$
\left\{\begin{array}{l}
\mathbf{N}_{m}  \tag{4.55}\\
\mathbf{M}_{b} \\
\mathbf{N}_{z} \\
\mathbf{Q}_{s}
\end{array}\right\}=\left[\begin{array}{cccc}
\mathbf{A} & \mathbf{B} & \mathbf{L} & \mathbf{0} \\
\mathbf{P} & \mathbf{S} & \mathbf{T} & \mathbf{0} \\
\mathbf{E} & \mathbf{F} & \mathbf{H} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}
\end{array}\right]\left\{\begin{array}{l}
\mathbf{e}_{m} \\
\mathbf{e}_{b} \\
\mathbf{w} \\
\mathbf{e}_{s}
\end{array}\right\}+\left[\begin{array}{ccc}
\mathbf{A}_{\mathbf{q}} & \mathbf{0} & \mathbf{0} \\
\mathbf{P}_{\mathbf{q}} & \mathbf{0} & \mathbf{0} \\
\mathbf{E}_{\mathbf{q}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{G}_{\mathbf{n}} & \mathbf{G}_{\mathbf{q}}
\end{array}\right]\left\{\begin{array}{c}
\mathbf{q}_{\mathbf{z}} \\
\mathbf{n}_{\mathbf{v}} \\
\mathbf{q}
\end{array}\right\}
$$

where the transverse shear strain measure vector is

$$
\begin{equation*}
\mathbf{e}_{s} \equiv\left[\theta_{1}, \theta_{2}, \psi_{1}, \psi_{2}, \partial_{1} w_{b}, \partial_{2} w_{b}, \partial_{1} w_{t}, \partial_{2} w_{t}, \partial_{1} \bar{w}, \partial_{2} \bar{w}\right]^{T} \tag{4.56}
\end{equation*}
$$

with the stiffness matrices defined as

$$
\begin{align*}
& \mathbf{A} \equiv\left\langle\mathbf{C}+S_{33}^{(k)} \mathbf{R} \mathbf{A}_{m}\right\rangle ; \mathbf{B} \equiv\left\langle\mathbf{C B}_{\mu}+S_{33}^{(k)} \mathbf{R} \mathbf{A}_{b}\right\rangle ; \mathbf{L} \equiv\left\langle S_{33}^{(k)} \mathbf{R} \mathbf{A}_{w}\right\rangle \\
& \mathbf{P} \equiv\left\langle\mathbf{B}_{\mu}^{T} \mathbf{C}+S_{33}^{(k)} \mathbf{B}_{\mu}^{T} \mathbf{R} \mathbf{A}_{m}\right\rangle ; \mathbf{S} \equiv\left\langle\mathbf{B}_{\mu}^{T} \mathbf{C B}_{\mu}+S_{33}^{(k)} \mathbf{B}_{\mu}^{T} \mathbf{R} \mathbf{A}_{b}\right\rangle ; \mathbf{T} \equiv\left\langle S_{33}^{(k)} \mathbf{B}_{\mu}^{T} \mathbf{R} \mathbf{A}_{w}\right\rangle \\
& \mathbf{E} \equiv\left\langle S_{33}^{(k)} \mathbf{H}_{w} \mathbf{R} \mathbf{A}_{m}\right\rangle ; \mathbf{F} \equiv\left\langle S_{33}^{(k)} \mathbf{H}_{w} \mathbf{R} \mathbf{A}_{b}\right\rangle ; \mathbf{H} \equiv\left\langle S_{33}^{(k)} \mathbf{H}_{w} \mathbf{R} \mathbf{A}_{w}\right\rangle  \tag{4.57}\\
& \mathbf{G} \equiv\left\langle\mathbf{B}_{\lambda}^{T} \mathbf{T}_{\tau}\right\rangle
\end{align*}
$$

$$
\begin{aligned}
& \mathbf{A}_{\mathbf{q}} \equiv\left\langle S_{33}^{(k)} \mathbf{R} \mathbf{A}^{\mathbf{q z}}\right\rangle ; \mathbf{P}_{\mathbf{q}} \equiv\left\langle S_{33}^{(k)} \mathbf{B}_{\mu}^{T} \mathbf{R} \mathbf{A}^{\mathrm{qz}}\right\rangle ; \mathbf{E}_{\mathbf{q}} \equiv\left\langle S_{33}^{(k)} \mathbf{H}_{w} \mathbf{R} \mathbf{A}^{\mathbf{q} \mathrm{z}}\right\rangle \\
& \mathbf{G}_{\mathbf{n}} \equiv\left\langle\mathbf{B}_{\lambda}^{T} \mathbf{T}_{p}\right\rangle ; \mathbf{G}_{\mathbf{q}} \equiv\left\langle\mathbf{B}_{\lambda}^{T} \mathbf{T}_{q}\right\rangle
\end{aligned}
$$

and

$$
\left.\begin{array}{l}
\mathbf{R} \equiv\left[\begin{array}{lll}
R_{11}^{(k)} & R_{22}^{(k)} & R_{12}^{(k)}
\end{array}\right]^{T} ; \mathbf{H}_{w} \equiv\left[\begin{array}{llll}
H_{b}^{w} & H_{t}^{w} & H_{a}^{w}
\end{array}\right] \\
\mathbf{B}_{\mu} \equiv\left[\begin{array}{ccccccc}
z & \mu_{1}^{(k)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z & \mu_{2}^{(k)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z & \mu_{1}^{(k)} & \mu_{2}^{(k)}
\end{array}\right]  \tag{4.58}\\
\mathbf{B}_{\lambda} \equiv\left[\begin{array}{ccccccccc}
1 & \partial_{z} \mu_{1}^{(k)} & 0 & 0 & H_{b}^{w} & H_{t}^{w} & H_{a}^{w} & 0 & 0 \\
0 & 0 & 1 & \partial_{z} \mu_{2}^{(k)} & 0 & 0 & 0 & H_{b}^{w} & H_{t}^{w}
\end{array} H_{a}^{w}\right.
\end{array}\right],
$$

## 6. Thermo-mechanical beam model

In this paragraph, the $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$ model for the beam problem is introduced. The purpose is double: firstly, the relations that are derived in this paragraph are required for the finite element implementation that will be introduced in Chapter 5; secondly, the coupling between mechanical and thermal loads on the governing equations is assessed on a simpler case. Since the previous paragraphs are devoted to a detailed description of the $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$ model, the discussion of this section will be briefly and some details will be omitted.

Consider a beam of length $L$, and cross-section with area $A=2 h x b$ made of $N$ orthotropic layers perfectly bonded together. The beam is located in a Cartesian coordinate reference frame $\left(x_{1}, x_{2}, z\right)$, where $x_{1}$ denotes the beam longitudinal axis, and $z$ is referred to the thickness coordinate (see Figure 4.2).

Moreover, the beam is subjected to forces for unit length applied at the top and bottom surface (see Figure 4.3). Furthermore, the beam is subjected to prescribed axial and shear tractions acting on the two end cross-sections.


Figure 4.2 Beam geometry and reference frame.


Figure 4.3 General beam notation.
Consistent with the $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$ model, the assumed kinematics for the axial and the transverse displacement reads as

$$
\begin{align*}
& U_{1}^{(k)}\left(x_{1}, z, t\right)=u_{1}\left(x_{1}, t\right)+z \theta_{1}\left(x_{1}, t\right)+\mu_{1}^{(k)}(z) \psi_{1}\left(x_{1}, t\right)  \tag{4.59}\\
& U_{z}\left(x_{1}, z, t\right)=H_{b}^{w}(z) w_{b}\left(x_{1}, t\right)+H_{t}^{w}(z) w_{t}\left(x_{1}, t\right)+H_{a}^{w}(z) \bar{w}\left(x_{1}, t\right)
\end{align*}
$$

The assumed transverse normal stress is continuous along the beam thickness and able to satisfy the traction conditions at the top and bottom beam surface

$$
\begin{gather*}
\sigma_{z z}^{a}=\mathbf{P}(z) \mathbf{q}_{\mathbf{v}}+\mathbf{L}(z) \mathbf{q}_{\mathbf{z}} \\
\mathbf{q}_{\mathbf{v}}=\left\{\begin{array}{ll}
q_{v 1} & q_{v 2}
\end{array}\right\}^{T} ; \mathbf{q}_{\mathbf{z}}=\left\{\begin{array}{ll}
\bar{q}^{b} / b & \bar{q}^{t} / b
\end{array}\right\}^{T} \tag{4.60}
\end{gather*}
$$

where $\mathbf{q}_{v}$ is the unknown stress variables and $\mathbf{q}_{\mathbf{z}}$ collects the forces for unit length applied at the bottom and top surface.

Similar to the plate model, the weak form of the compatibility constraint

$$
\begin{equation*}
b\left\langle\delta \sigma_{z z}^{a}\left(\varepsilon_{z z}-\varepsilon_{z z}^{a}\right)\right\rangle=0 \tag{4.61}
\end{equation*}
$$

has to be satisfied. By introducing the assumed transverse normal stress, Eq. (4.61), and using the strain-displacement relations accounting for the thermal field (see Preliminaries) along with the assumed kinematics, Eq. (4.61) becomes

$$
\begin{align*}
& b\left\langle\delta \mathbf { q } _ { \mathrm { v } } { } ^ { T } \mathbf { P } ( z ) ^ { T } \left[\partial_{z} H_{b}^{w} w_{b}+\partial_{z} H_{t}^{w} w_{t}+\partial_{z} H_{a}^{w} \bar{w}-S_{33}^{(k)}\left(\mathbf{P}(z) \mathbf{q}_{\mathrm{v}}+\mathbf{L}(z) \mathbf{q}_{\mathrm{z}}\right)+\right.\right. \\
& \left.\left.S_{33}^{(k)} R_{11}^{(k)}\left(\partial_{1} u_{1}+z \partial_{1} \theta_{1}+\mu_{1}^{(k)} \partial_{1} \psi_{1}\right)-S_{33}^{(k)} \lambda_{33}^{(k)} \Delta T\left(x_{1}, z\right)\right]\right\rangle=0 \tag{4.62}
\end{align*}
$$

where $\Delta T\left(x_{1}, z\right)$ denotes the temperature distribution. The equation is solved in terms of the unknown stress variables, that is

$$
\begin{align*}
\mathbf{q}_{\mathrm{v}}= & \left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left[\left\langle\mathbf{P}^{T} \partial_{z} H_{b}^{w}\right\rangle w_{b}+\left\langle\mathbf{P}^{T} \partial_{z} H_{t}^{w}\right\rangle w_{t}+\left\langle\mathbf{P}^{T} \partial_{z} H_{a}^{w}\right\rangle \bar{w}+\right. \\
& \left\langle\mathbf{P}^{T} S_{33}^{(k)} R_{11}^{(k)}\right\rangle \partial_{1} u_{1}+\left\langle\mathbf{P}^{T} z S_{33}^{(k)} R_{11}^{(k)}\right\rangle \partial_{1} \theta_{1}+\left\langle\mathbf{P}^{T} \mu_{1}^{(k)} S_{33}^{(k)} R_{11}^{(k)}\right\rangle \partial_{1} \psi_{1}-  \tag{4.63}\\
& \left.\left\langle\mathbf{P}(z)^{T} S_{33}^{(k)} \mathbf{L}(z)\right\rangle \mathbf{q}_{z}-\left\langle\mathbf{P}(z)^{T} S_{33}^{(k)} \lambda_{33}^{(k)} \Delta T(z)\right\rangle \Delta T\left(x_{1}\right)\right]
\end{align*}
$$

where the temperature field is assumed to be written as $\Delta T\left(x_{1}, z\right)=\Delta T\left(x_{1}\right) \Delta T(z)$. Finally, the unknown stress variables vector, Eq. (4.63), is introduced in the assumed transverse normal stress, Eq. (4.61)

$$
\begin{gather*}
\sigma_{z z}^{a}=A^{u}(z) \partial_{1} u_{1}+A^{\theta}(z) \partial_{1} \theta_{1}+A^{\psi}(z) \partial_{1} \psi_{1}+A_{b}^{w}(z) w_{b}+A_{t}^{w}(z) w_{t}+A_{a}^{w}(z) \bar{w}+ \\
\mathbf{A}^{g z}(z) \mathbf{q}_{z}-A^{\Delta T}(z) \Delta T\left(x_{1}\right)  \tag{4.64}\\
A^{\Delta T}(z)=\mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}(z)^{T} S_{33}^{(k)} \lambda_{33}^{(k)} \Delta T(z)\right\rangle
\end{gather*}
$$

where the $z$-coordinate shape functions are introduced, similar to plate model.

The assumed transverse shear stress is approximated to vary along the thickness by adopting the $z$-coordinate shape functions that derive by integration of the local equilibrium equations, that is

$$
\begin{equation*}
\tau_{1 z}\left(x_{1}, z\right)=-\bar{p}^{b}\left(x_{1}\right) / b-\int_{-h}^{z} \partial_{1} \sigma_{11}^{(k)} d z \tag{4.65}
\end{equation*}
$$

By using the constitutive material law, Eq. (4.65) becomes

$$
\begin{align*}
& \tau_{1 z}\left(x_{1}, z\right)=-\bar{p}^{b}\left(x_{1}\right) / b-\partial_{11} u_{1} \int_{-h}^{z}\left(C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{u}(z)\right) d z \\
& \quad-\partial_{11} \theta_{1} \int_{-h}^{z}\left(z C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{\theta}(z)\right) d z-\partial_{11} \psi_{1} \int_{-h}^{z}\left(\mu_{1}^{(k)} C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{\psi}(z)\right) d z  \tag{4.66}\\
& -\partial_{1} w_{b} \int_{-h}^{z} S_{33}^{(k)} R_{11}^{(k)} A_{b}^{w}(z) d z-\partial_{1} w_{t} \int_{-h}^{z} S_{33}^{(k)} R_{11}^{(k)} A_{t}^{w}(z) d z-\partial_{1} \bar{w} \int_{-h}^{z} S_{33}^{(k)} R_{11}^{(k)} A_{a}^{w}(z) d z \\
& -\partial_{1} \mathbf{q}_{z}^{T} \int_{-h}^{z} S_{33}^{(k)} R_{11}^{(k)} \mathbf{A}^{\mathrm{qz}}(z)^{T} d z+\partial_{1} \Delta T\left(x_{1}\right) \int_{-h}^{z}\left(\omega_{11}^{(k)} \Delta T(z)+S_{33}^{(k)} R_{11}^{(k)} A^{\Delta T}(z)\right) d z
\end{align*}
$$

Again, integration is extended up to the top surface and the traction condition satisfied

$$
\begin{align*}
& \bar{p}^{t}\left(x_{1}\right) / b=-\bar{p}^{b}\left(x_{1}\right) / b-\partial_{11} u_{1}\left\langle C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{u}(z)\right\rangle \\
& \quad-\partial_{11} \theta_{1}\left\langle\left(z C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{\theta}(z)\right)\right\rangle-\partial_{11} \psi_{1}\left\langle\mu_{1}^{(k)} C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{\mu}(z)\right\rangle \\
& \quad-\partial_{1} w_{b}\left\langle S_{33}^{(k)} R_{11}^{(k)} A_{b}^{w}(z)\right\rangle-\partial_{1} w_{t}\left\langle S_{33}^{(k)} R_{11}^{(k)} A_{t}^{w}(z)\right\rangle-\partial_{1} \bar{w}\left\langle S_{33}^{(k)} R_{11}^{(k)} A_{a}^{w}(z)\right\rangle  \tag{4.67}\\
& \quad-\partial_{1} \mathbf{q}_{\mathbf{z}}{ }^{T}\left\langle S_{33}^{(k)} R_{11}^{(k)} \mathbf{A}^{q z}(z)^{T}\right\rangle+\partial_{1} \Delta T\left(x_{1}\right)\left\langle\omega_{11}^{(k)} \Delta T(z)+S_{33}^{(k)} R_{11}^{(k)} A^{\Delta T}(z)\right\rangle
\end{align*}
$$

Eq. (4.67) is used to obtain the second-order derivative of the uniform axial displacement in terms of the other kinematic variables derivatives and the external applied loads

$$
\begin{align*}
\partial_{11} u_{1}= & -\left\langle C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{u}(z)\right\rangle^{-1}\left[\left(\bar{p}^{b}\left(x_{1}\right)+\bar{p}^{t}\left(x_{1}\right)\right) / b\right. \\
& -\left\langle z C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{\theta}(z)\right\rangle \partial_{11} \theta_{1} \\
& -\left\langle\mu_{1}^{(k)} C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{\psi}(z)\right\rangle \partial_{11} \psi_{1} \\
& -\left\langle S_{33}^{(k)} R_{11}^{(k)} A_{b}^{w}(z)\right\rangle \partial_{1} w_{b}-\left\langle S_{33}^{(k)} R_{11}^{(k)} A_{t}^{w}(z)\right\rangle \partial_{1} w_{t}  \tag{4.68}\\
& -\left\langle S_{33}^{(k)} R_{11}^{(k)} A_{a}^{w}(z)\right\rangle \partial_{1} \bar{w}-\left\langle S_{33}^{(k)} R_{11}^{(k)} \mathbf{A}^{q z}(z)^{T}\right\rangle^{T} \partial_{1} \mathbf{q}_{z} \\
& \left.+\left\langle S_{33}^{(k)} \omega_{11}^{(k)} \Delta T(z)+S_{33}^{(k)} R_{11}^{(k)} A^{\Delta T}(z)\right\rangle \partial_{1} \Delta T\left(x_{1}\right)\right]
\end{align*}
$$

By introducing Eq. (4.68) into Eq. (4.66), the $z$-coordinate shape functions that rule the pattern along the thickness of the transverse shear stress appear. In tense notation, the shear stress coming from integration of the three-dimensional equilibrium equations reads as

$$
\begin{align*}
\tau_{1 z}= & Z^{p b}(z) \bar{p}^{b}+Z^{p t}(z) \bar{p}^{t}+Z^{\theta}(z) \partial_{11} \theta_{1}+Z^{\psi}(z) \partial_{11} \psi_{1}+ \\
& Z^{w b}(z) \partial_{1} w_{b}+Z^{w t}(z) \partial_{1} w_{t}+Z^{w a}(z) \partial_{1} w_{a}+\mathbf{Z}^{q z}(z) \partial_{1} \mathbf{q}_{z}- \\
& \partial_{1} \Delta T\left(x_{1}\right)\left\langle f^{\Delta T}(z)\right\rangle\left\langle C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{u}\right\rangle^{-1} \int_{-h}^{z}\left(C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{u}\right) d z+  \tag{4.69}\\
& \partial_{1} \Delta T\left(x_{1}\right) \int_{-h}^{z} f^{\Delta T}(z) d z
\end{align*}
$$

where $f^{\Delta T}(z)=S_{33}^{(k)} R_{11}^{(k)} A^{\Delta T}(z)+S_{33}^{(k)} \omega_{11}^{(k)} \Delta T(z)$.
As for the plate model, the shape functions of the integrated transverse shear stress are used as base for the transverse shear stress assumption, that is

$$
\begin{align*}
\tau_{1 z}^{a}= & Z^{p b}(z) \bar{p}^{b}+Z^{p t}(z) \bar{p}^{t}+Z^{\theta}(z) f^{\theta}+Z^{\psi}(z) f^{\psi}+ \\
& Z^{w b}(z) f_{b}^{w}+Z^{w t}(z) f_{t}^{w}+Z^{w a}(z) f_{a}^{w}+Z^{q z}(z) \partial_{1} \mathbf{q}_{z}+Z^{\Delta T}(z) \partial_{1} \Delta T\left(x_{1}\right) \tag{4.70}
\end{align*}
$$

where $f^{\theta}, f^{\psi}, f_{b}^{w}, f_{t}^{w}, f_{a}^{w}$ are unknown stress variables and the meaning of $Z^{\Delta T}(z)$ is easily derivable by comparing Eq. (4.70) with Eq. (4.69). In order to make easier the derivation, the Eq. (4.70) is arranged in matrix notation

$$
\begin{gather*}
\tau_{1 z}^{a}=\mathbf{Z}_{\mathbf{f}}(z) \mathbf{f}_{\mathbf{v}}+\mathbf{Z}_{\mathbf{n}}(z) \mathbf{n}_{\mathbf{v}}+\mathbf{Z}^{q z}(z) \partial_{1} \mathbf{q}_{\mathbf{Z}}+Z^{\Delta T}(z) \partial_{1} \Delta T\left(x_{1}\right) \\
\mathbf{n}_{\mathrm{v}}^{T}=\left\{\begin{array}{ll}
\bar{p}^{b} & \bar{p}^{t}
\end{array}\right\} \tag{4.71}
\end{gather*}
$$

where the unknown stress variables are collected in $\mathbf{f}_{\mathrm{v}}$.
Once the assumed transverse shear stress is advanced, the weak form of the compatibility constraint has to be fulfilled

$$
\begin{equation*}
b\left\langle\delta \tau_{1 z}^{a}\left(\gamma_{1 z}-\gamma_{1 z}^{a}\right)\right\rangle=0 \tag{4.72}
\end{equation*}
$$

By introducing Eq. (4.70), and by expressing the shear strains coming from the assumed kinematics and that from the assumed transverse shear stress, Eq. (4.72) becomes

$$
\begin{align*}
&\left\langle\delta \mathbf { f } _ { \mathrm { v } } ^ { T } \mathbf { Z } _ { \mathbf { f } } ( z ) ^ { T } \left[ H_{b}^{w} \partial_{1} w_{b}+H_{t}^{w} \partial_{1} w_{t}+H_{a}^{w} \partial_{1} \bar{w}+\theta_{1}+\partial_{z} \mu_{1}^{(k)} \psi_{1}\right.\right.  \tag{4.73}\\
&\left.\left.\quad-D_{T}\left(\mathbf{Z}_{\mathbf{f}}(z) \mathbf{f}_{\mathrm{v}}+\mathbf{Z}_{\mathbf{n}}(z) \mathbf{n}_{\mathrm{v}}+\mathbf{Z}^{q z}(z) \partial_{1} \mathbf{q}_{\mathbf{z}}+Z^{\Delta T}(z) \partial_{1} \Delta T\left(x_{1}\right)\right)\right]\right\rangle=0
\end{align*}
$$

Solving for the unknown stress variables vector, it is obtained

$$
\begin{align*}
\mathbf{f}_{\mathrm{v}}= & \left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{f}}(z)\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} H_{b}^{w}\right\rangle \partial_{1} w_{b}+ \\
& \left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{f}}(z)\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} H_{t}^{w}\right\rangle \partial_{1} w_{t}+ \\
& \left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{f}}(z)\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} H_{a}^{w}\right\rangle \partial_{1} \bar{w}+ \\
& \left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{f}}(z)\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T}\right\rangle \theta_{1}+  \tag{4.74}\\
& \left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{f}}(z)\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} \partial_{z} \mu_{1}^{(k)}\right\rangle \psi_{1}- \\
& \left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{f}}(z)\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{n}}(z)\right\rangle \mathbf{n}_{\mathbf{v}}- \\
& \left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{f}}(z)\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}^{G z}(z)\right\rangle \partial_{1} \mathbf{q}_{\mathbf{z}}- \\
& \left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{f}}(z)\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} Z^{\Delta T}(z)\right\rangle \partial_{1} \Delta T\left(x_{1}\right)
\end{align*}
$$

Stress variables vector is now introduced in Eq. (4.71) and the assumed transverse shear stress able to satisfy the weak form of the compatibility constraint reads as

$$
\begin{align*}
\tau_{1 z}^{a}= & T^{\theta}(z) \theta_{1}+T^{\psi}(z) \psi_{1}+T_{b}^{w}(z) \partial_{1} w_{b}+T_{t}^{w}(z) \partial_{1} w_{t}+T_{a}^{w}(z) \partial_{1} \bar{w}+ \\
& \mathbf{T}_{p}(z) \mathbf{n}_{\mathbf{v}}+\mathbf{T}_{q}(z) \partial_{1} \mathbf{q}_{\mathbf{z}}+T_{\Delta T}(z) \partial_{1} \Delta T\left(x_{1}\right) \tag{4.75}
\end{align*}
$$

The governing equations and the variationally consistent boundary conditions related with the beam problem (Figure 4.3) are derived by solving the remaining part of the Reissner Mixed Variational Theorem, that is

$$
\begin{equation*}
\int_{V}\left(\sigma_{11} \delta \varepsilon_{11}+\tau_{1 z}^{a} \delta \gamma_{1 z}+\sigma_{z z}^{a} \delta \varepsilon_{z z}\right) d V-\delta W_{e}=0 \tag{4.76}
\end{equation*}
$$

By using the strain-displacement relations coupling with the constitutive material law and consistent with the assumed kinematics, integration by parts leads to the governing equations

$$
\begin{array}{ll}
\delta u_{1}: & \partial_{1} N_{11}+\bar{p}_{1}=0 \\
\delta \theta_{1}: & \partial_{1} M_{11}-Q_{1}+\bar{m}_{1}=0 \\
\delta \psi_{1}: & \partial_{1} M_{11}^{\mu}-Q_{1}^{\mu}=0 \\
\delta w_{b}: & \partial_{1} Q_{1}^{b}-N_{z z}^{b}+\bar{q}^{b}=0  \tag{4.77}\\
\delta w_{t}: & \partial_{1} Q_{1}^{t}-N_{z z}^{t}+\bar{q}^{t}=0 \\
\delta \bar{w}: & \partial_{1} Q_{1}^{a}-N_{z z}^{a}=0
\end{array}
$$

and the variationally consistent boundary conditions

$$
\begin{array}{llllll}
u_{1}=\bar{u}_{1} & \text { on } & x_{1}=0, L & \text { or } & N_{11}=\left.\bar{N}_{11}\right|_{x_{1}} & \text { on } \\
x_{1}=0, L \\
\theta_{1}=\bar{\theta}_{1} & \text { on } & x_{1}=0, L & \text { or } & M_{11}=\left.\bar{M}_{11}\right|_{x_{1}} & \text { on }  \tag{4.78}\\
x_{1}=0, L \\
\psi_{1}=\bar{\psi}_{1} & \text { on } & x_{1}=0, L & \text { or } & M_{11}^{\mu}=\left.\bar{M}_{11}^{\mu}\right|_{x_{1}} & \text { on } \\
x_{1}=0, L \\
w_{b}=\bar{w}_{b} & \text { on } & x_{1}=0, L & \text { or } & Q_{1}^{b}=\left.\bar{V}_{1}^{b}\right|_{x_{1}} & \text { on } \\
x_{1}=0, L \\
w_{t}=\bar{w}_{t} & \text { on } & x_{1}=0, L & \text { or } & Q_{1}^{t}=\left.\bar{V}_{1}^{t}\right|_{x_{1}} & \text { on } \\
x_{1}=0, L \\
\bar{w}=\overline{\bar{w}} & \text { on } & x_{1}=0, L & \text { or } & Q_{1}^{a}=\left.\bar{V}_{1}^{a}\right|_{x_{1}} & \text { on } \\
x_{1}=0, L
\end{array}
$$

where the resultants of the prescribed traction applied in $x_{1}=0, L$ read as

$$
\begin{align*}
& \quad\left(\left.\bar{N}_{11}\right|_{x_{1}},\left.\bar{M}_{11}\right|_{x_{1}},\left.\bar{M}_{11}^{\mu}\right|_{x_{1}},\left.\bar{V}_{1}^{b}\right|_{x_{1}},\left.\bar{V}_{1}^{t}\right|_{x_{1}},\left.\bar{V}_{1}^{a}\right|_{x_{1}}\right) \equiv  \tag{4.79}\\
& \quad b\left\langle\left(T_{1}^{x_{1}}, z T_{1}^{x_{1}}, \mu_{1}^{(k)} T_{1}^{x_{1}}, T_{z}^{x_{1}} H_{b}^{w}, T_{z}^{x_{1}} H_{t}^{w}, T_{z}^{x_{1}} H_{a}^{w}\right)\right\rangle
\end{align*}
$$

It is worth to note that Eq. (4.77) hides, under the stress resultants, the dependence on the temperature variation. In order to revel that, the constitutive relations and the stress resultants are shown. Consistent with notation in Eq. (4.52) and Eq. (4.56), the strain measures are defined as

$$
\begin{array}{cc}
\mathbf{e}_{m} \equiv \partial_{1} u_{1} & \mathbf{e}_{b} \equiv\left[\partial_{1} \theta_{1}, \partial_{1} \psi_{1}\right]^{T} \\
\mathbf{w} \equiv\left[w_{b}, w_{t}, \bar{w}\right]^{T} & \mathbf{e}_{s} \equiv\left[\theta_{1}, \psi_{1}, \partial_{1} w_{b}, \partial_{1} w_{t}, \partial_{1} \bar{w}\right]^{T} \tag{4.80}
\end{array}
$$

The membrane, bending, normal and shear stress resultants are

$$
\begin{gather*}
\mathbf{N}^{T} \equiv\left(N_{1}, M_{1}, M_{1}^{\mu}\right) \equiv b\left\langle\sigma_{11}^{(k)}\left(1, z, \mu_{1}^{(k)}\right)\right\rangle \\
\mathbf{N}_{z}^{T} \equiv\left(N_{z z}^{b}, N_{z z}^{t}, N_{z z}^{a}\right)=b\left\langle\left(H_{b}^{w}, H_{t}^{w}, H_{a}^{w}\right) \sigma_{z z}^{a}\right\rangle  \tag{4.81}\\
\mathbf{Q}_{s}^{T} \equiv\left\{Q_{1}, Q_{1}^{\phi}, Q_{1}^{b}, Q_{1}^{t}, Q_{1}^{a}\right\}=b\left\langle\tau_{1 z}^{a}, \partial_{z} \mu_{1}^{(k)} \tau_{1 z}^{a}, H_{b}^{w} \tau_{1 z}^{a}, H_{t}^{w} \tau_{1 z}^{a}, H_{a}^{w} \tau_{1 z}^{a}\right\rangle
\end{gather*}
$$

According to Eq. (4.80) and Eq. (4.79), integrating the stresses over the cross-section area, the constitutive relations appear

$$
\begin{align*}
\left\{\begin{array}{l}
\mathbf{N} \\
\mathbf{N}_{z} \\
\mathbf{Q}_{s}
\end{array}\right\}=\left[\begin{array}{llll}
\mathbf{A} & \mathbf{B} & \mathbf{L} & \mathbf{0} \\
\mathbf{P} & \mathbf{S} & \mathbf{T} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}
\end{array}\right]\left\{\begin{array}{l}
\mathbf{e}_{m} \\
\mathbf{e}_{b} \\
\mathbf{w} \\
\mathbf{e}_{s}
\end{array}\right\}+\left[\begin{array}{ccc}
\mathbf{A}_{\mathbf{q}} & \mathbf{0} & \mathbf{0} \\
\mathbf{P}_{\mathbf{q}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{G}_{\mathbf{n}} & \mathbf{G}_{\mathbf{q}}
\end{array}\right]\left\{\begin{array}{c}
\mathbf{q}_{\mathbf{z}} \\
\mathbf{n}_{\mathbf{v}} \\
\mathbf{q}
\end{array}\right\}-  \tag{4.82}\\
{\left[\begin{array}{cc}
\mathbf{A}_{\Delta T} & \mathbf{0} \\
\mathbf{P}_{\Delta T} & \mathbf{0} \\
\mathbf{0} & -\mathbf{G}_{\Delta T}
\end{array}\right]\left\{\begin{array}{c}
\Delta T\left(x_{1}\right) \\
\partial_{1} \Delta T\left(x_{1}\right)
\end{array}\right\} }
\end{align*}
$$

where the stiffness matrices are defined as

$$
\begin{align*}
& \mathbf{A} \equiv b\left\langle\mathbf{B}_{z}^{T} Q_{11}^{(k)}\right\rangle+b\left\langle\mathbf{B}_{z}^{T} S_{33}^{(k)} R_{11}^{(k)} A^{u}(z)\right\rangle ; \\
& \mathbf{B} \equiv b\left\langle\mathbf{B}_{z}^{T}\left[\begin{array}{ll}
z Q_{11}^{(k)} & \mu_{1}^{(k)} Q_{11}^{(k)}
\end{array}\right]\right\rangle+b\left\langle\mathbf{B}_{z}^{T} S_{33}^{(k)} R_{11}^{(k)}\left[\begin{array}{ll}
A^{\theta}(z) & A^{\mu}(z)
\end{array}\right]\right\rangle ; \\
& \mathbf{L} \equiv b\left\langle\mathbf{B}_{z}^{T} S_{33}^{(k)} R_{11}^{(k)}\left[\begin{array}{lll}
A_{b}^{w}(z) & A_{t}^{w}(z) & A_{a}^{w}(z)
\end{array}\right]\right\rangle ; \\
& \mathbf{A}_{\mathbf{q}} \equiv b\left\langle\mathbf{B}_{z}^{T} \mathbf{A}^{q z}(z)\right\rangle ; \mathbf{A}_{\Delta T} \equiv b\left\langle\mathbf{B}_{z}^{T} \omega_{11}^{(k)} \Delta T(z)\right\rangle+b\left\langle\mathbf{B}_{z}^{T} S_{33}^{(k)} R_{11}^{(k)} A^{\Delta T}(z)\right\rangle ; \\
& \mathbf{P} \equiv b\left\langle\mathbf{H}_{w}^{T} A^{u}(z)\right\rangle ; \mathbf{S} \equiv b\left\langle\mathbf{H}_{w}^{T}\left[\begin{array}{ll}
A^{\theta}(z) & A^{\mu}(z)
\end{array}\right]\right\rangle ;  \tag{4.83}\\
& \mathbf{T} \equiv b\left\langle\mathbf{H}_{w}^{T}\left[\begin{array}{lll}
A_{b}^{w}(z) & A_{t}^{w}(z) & A_{a}^{w}(z)
\end{array}\right]\right\rangle ; \\
& \mathbf{P}_{\mathbf{q}} \equiv b\left\langle\mathbf{H}_{w}^{T} \mathbf{A}^{q z}(z)\right\rangle ; \mathbf{P}_{\Delta T} \equiv b\left\langle\mathbf{B}_{z}^{T} A^{\Delta T}(z)\right\rangle ; \\
& \mathbf{G} \equiv b\left\langle\mathbf{B}_{\lambda}^{T}\left[\begin{array}{llll}
T^{\theta}(z) & T^{\psi}(z) & T_{b}^{w}(z) & T_{t}^{w}(z)
\end{array} T_{a}^{w}(z)\right]\right\rangle ; \\
& \mathbf{G}_{\mathbf{n}} \equiv b\left\langle\mathbf{B}_{\lambda}^{T} \mathbf{T}_{q}(z)\right\rangle ; \mathbf{G}_{\mathbf{q}} \equiv b\left\langle\mathbf{B}_{\lambda}^{T} \mathbf{T}_{q}(z)\right\rangle ; \mathbf{G}_{\Delta T} \equiv b T_{\Delta T}(z)
\end{align*}
$$

and the following definitions are introduced

$$
\begin{align*}
& \mathbf{H}_{w} \equiv\left[\begin{array}{lll}
H_{b}^{w}(z) & H_{t}^{w}(z) & H_{a}^{w}(z)
\end{array}\right] ; \\
& \mathbf{B}_{\mu} \equiv\left[\begin{array}{llll}
1 & z & \mu_{1}^{(k)}(z)
\end{array}\right] ; \mathbf{B}_{\lambda} \equiv\left[\begin{array}{lll}
1 & \partial_{1} \mu_{1}^{(k)}(z) & \mathbf{H}_{w}
\end{array}\right] \tag{4.84}
\end{align*}
$$

## Chapter 5

## Finite Elements Formulation

## 1. Introduction

The formulations of beam/plate/shell finite elements based on zigzag theories abound in the open literature. In the framework of the Refined Zigzag Theory, several researchers have recently developed finite elements achieving interesting results. Two RZT-based beam elements were developed, almost simultaneously, by Gherlone et al. [Gherlone et al., 2011] and Oñate et al. [Oñate et al., 2010]. As demonstrated by Gherlone [Gherlone et al., 2011], the RZT-based beam element suffers for the shear locking phenomenon, a deficiency that affects the isoparametric elements and consisting in an erroneous overestimation of the transverse shear stiffness. In open literature, several solutions have been proposed to address the shear locking deficiency [Reddy, 1997]. Although the underlying theory is the same, the two RZT beam elements [Oñate et al., 2010; Gherlone et al., 2011] have different peculiarities as result of two different approaches adopted to circumvent the shear locking. Gherlone et al. [Gherlone et al., 2011] adopted the anisoparametric interpolation scheme, as proposed by Tessler et al. [Tessler et al., 1981] for the shear locking suppression, yielding to a 3 -nodes/9-dof's beam element. The anisoparametric interpolation requires that the bending degree of freedom, typically the bending rotation, is approximated with the linear Lagrange polynomials whereas the
deflection degree of freedom (the transverse displacement) is approximated with a quadratic Lagrange polynomials. The anisoparametric interpolation scheme leads to the existence of additional nodes with only extra deflection degrees of freedom. In order to recover an isoparametric-like nodal pattern but saving the anisoparametric interpolation, a constrained element can be formulated. Gherlone et al. [Gherlone et al., 2011] enforced three different constraints yielding to as much constrained 2 -nodes/ 8 -dof's elements: a discussion on the best choice of the constraint condition is presented by the authors. In Oñate et al. [Oñate et al., 2010], a simple 2-nodes/8-dof's linear Langrangian beam element was developed and the shear locking avoided adopting the reduced integration technique. The comparison on the two modeling strategies is not still performed in the open literature. Following the same idea of Gherlone et al. [Gherlone et al., 2011], recently Versino et al. [Versino et al., 2013] formulated an anisoparametric triangular plate element, both unconstrained ( 6 -nodes/24-dof's) and constrained ( 3 -nodes/21-dof's), and the element was implemented in the $A B A Q U S^{\circledR}$ finite element commercial code via an user-element subroutine. By using the RZT model, Oñate and co-workers [Eijo et al., 2013] developed a 4-nodes/28-dof's bilinear quadrilateral element wherein a linear shear strain field is assumed to avoid shear locking.

Generalization of the RZT model for shell structures and relative finite element implementations was performed by Versino et al. [Versino et al., 2014], in the framework of small displacements and rotation, and Flores [Flores, 2014], for large displacements and rotations analysis.

The formulation of higher-order Refined Zigzag Theory models by Barut et al. [Barut et al., 2012] leads to the implementation of a triangular plate element [Barut et al., 2013] based on the extended theory.

In this Chapter, several finite element formulations are introduced: firstly, a novel 2-nodes/8-dof's RZT-based beam element employing an exact static stiffness matrix is developed; secondly, beam and plate elements based on the mixed (3,2)-Refined Zigzag Theory (aka $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$ ) are developed.

## 2. Exact static stiffness RZT beam element

As stated in Sect.1, the RZT-based beam element has already been formulated. Moreover, two different formulations are available: one proposed by Gherlone et al. [Gherlone et al., 2011] and the second given by Oñate et al. [Oñate et al., 2010]. The main
difference relies on the shape functions: in [Gherlone et al., 2011] a Lagrangian anisoparametric (linear for the in-plane degrees of freedom and parabolic for the transverse displacement) element is formulated whereas in [Oñate et al., 2010] a linear Lagrangian element is proposed. The shear locking is circumvented in [Gherlone et al., 2011] in virtue of the anisoparametric interpolation strategy, whereas in [Oñate et al., 2010] the reduced integration of the shear contribution to the stiffness matrix is required. Although these basic differences, both the elements follow the usual procedure according to which the degree of freedom variation along the axial beam coordinate is independent on the remaining ones and approximated by using polynomial shape functions that relate it with its nodal values. Only in [Gherlone et al., 2011], by developing a constrained element, an interdependent interpolation scheme is achieved.

A more interesting way to find an interpolation for the field variables, that is to find shape functions, is to solve the homogeneous part of the static equilibrium equations. In the open literature, several works have been published wherein the shape functions are derived from the exact static solution. Eisenberger [Eisenberger, 1994] developed a 2-nodes/4dof's Timoshenko beam element wherein the shape functions are derived by solving the homogeneous differential equations of the model adopting the power series solution method. Later, Pilkey et al. [Pilkey, 1996] proposed an element adopting the exact static stiffness matrix suitable for the analysis of general cross-section beams, thus accounting for the coupling between the displacement into the two perpendicular planes. Noteworthy is the work done by Reddy in [Reddy, 1997], wherein several finite elements, based on Timoshenko and Reddy's [Reddy, 2004] theory, are developed and compared. In particular, the comparison is focused on the performances of the isoparametric finite elements, those adopting the reduced integration and those employing exact shape functions. The higher-order shear deformation theory proposed by Heyliger and Reddy [Heyliger et al., 1988] has been used in [Murthy et al., 2005] to develop an exact beam element. Extension to the 3D Timoshenko beam element adopting exact shape functions has been performed in [Luo, 2008].

The use of exact static (henceforward denoted as consistent) shape functions results in several benefits: the stiffness matrix is exact, that is no errors of discretization are introduced in the stiffness. Due to the exact stiffness matrix, the element shows superconvergent behavior if compared with the isoparametric version of the same element. The consistent shape functions ensure that the finite element achieves exact solution, in the
nodes and inside the element, for concentrated loads applied at nodes, whereas, in case of distributed loads, the finite element solution fits with the exact one only at the nodes, while inside the element the convergence is reached by using a number of elements greater than the minimum one (that depends on the boundary conditions) but lower to that required by the isoparametric element [Reddy, 1997]. Even though the inertial forces are neglected, the consistent mass matrix along with the exact stiffness one produces faster convergence if compared with isoparametric element, also in dynamic problems. Moreover, the consistent shape functions introduce an interdependence between the field variables that allows to avoid numerical drawbacks, as well as the shear locking [Luo, 2008].

In this Chapter, the RZT-based 2-nodes/8-dof's beam element adopting consistent shape functions is formulated. The consistent mass matrix and the exact stiffness one are computed along with the force vector.

In the subsequent section, the basic RZT equations for the beam problem are briefly recalled.

### 2.1. Refined Zigzag Theory for beams

Consider a beam of length $L$ and cross-section with area $A=2 h x b$, made by $N$ orthotropic layers perfectly bonded together. The beam is located in a Cartesian coordinate system ( $x_{1}, x_{2}, z$ ), where $x_{1}$ denotes the beam longitudinal axis, and $z$ is referred to the thickness coordinate (see Figure 5.1).


Figure 5.1 Beam geometry and reference frame.
Moreover, the beam is subjected to transverse pressure applied at the top and the bottom surface along with surface traction acting on the top and bottom surface (see Figure 5.2).

Furthermore, the beam is subjected to prescribed axial and shear tractions acting on the end cross-sections.


Figure 5.2 Load applied system.
According to the Refined Zigzag Theory, the orthogonal components of the displacement vector are defined as [Tessler et al., 2007]

$$
\begin{align*}
& U_{1}^{(k)}\left(x_{1}, z, t\right)=u_{1}\left(x_{1}, t\right)+z \theta_{1}\left(x_{1}, t\right)+\phi_{1}^{(k)}(z) \psi_{1}\left(x_{1}, t\right)  \tag{5.1}\\
& U_{z}\left(x_{1}, z, t\right)=w\left(x_{1}, t\right)
\end{align*}
$$

where $u_{1}\left(x_{1}\right), \theta_{1}\left(x_{1}\right), \psi_{1}\left(x_{1}\right)$ and $\phi_{1}^{(k)}(z)$ are the uniform axial displacement, the bending rotation, the zigzag amplitude and the zigzag function, respectively.

By using the Virtual Work Principle, the static governing equilibrium equations and variationally consistent boundary conditions read as [Gherlone et al., 2011]

$$
\begin{align*}
& \partial_{1} N_{1}+\bar{p}^{b}+\bar{p}^{t}=0 \\
& \partial_{1} M_{1}-V_{1}+h\left(\bar{p}^{t}-\bar{p}^{b}\right)=0 \\
& \partial_{1} V_{1}+\bar{q}^{t}+\bar{q}^{b}=0 \\
& \partial_{1} M_{\phi}-V_{\phi}=0 \tag{5.2}
\end{align*}
$$

$$
\begin{array}{lllll}
\text { either } & u\left(x_{\alpha}\right)=\bar{u}_{\alpha} & \text { or } & N_{1}\left(x_{\alpha}\right)=\bar{N}_{1} \\
\text { either } & \theta\left(x_{\alpha}\right)=\bar{\theta}_{\alpha} & \text { or } & M_{1}\left(x_{\alpha}\right)=\bar{M}_{1} \\
\text { either } & w\left(x_{\alpha}\right)=\bar{w}_{\alpha} & \text { or } & V_{1}\left(x_{\alpha}\right)=\bar{V}_{1} \\
\text { either } & \psi\left(x_{\alpha}\right)=\bar{\psi}_{\alpha} & \text { or } & M_{\phi}\left(x_{\alpha}\right)=\bar{M}_{\phi}
\end{array}
$$

where the bar-superscripted symbols denote the prescribed displacements and stress resultants; the applied loads are depicted in Figure 5.2 and the following definitions for the stress resultants are introduced [Tessler et al., 2007]

$$
\begin{equation*}
\left(N_{1}, M_{1}, M_{\phi}, V_{1}, V_{\phi}\right) \equiv b\left\langle\left(\sigma_{1}^{(k)}, z \sigma_{1}^{(k)}, \phi_{1}^{(k)} \sigma_{1}^{(k)}, \tau_{1 z}^{(k)}, \beta_{1}^{(k)} \tau_{1 z}^{(k)}\right)\right\rangle \tag{5.3}
\end{equation*}
$$

Performing the integration over the beam cross-section, the constitutive equations of the beam model are obtained [Tessler et al., 2007]

$$
\left.\begin{array}{l}
\left\{\begin{array}{l}
N_{1} \\
M_{1} \\
M_{\phi}
\end{array}\right\}=\left[\begin{array}{lll}
A_{11} & B_{12} & B_{13} \\
B_{12} & D_{11} & D_{12} \\
B_{13} & D_{12} & D_{22}
\end{array}\right]\left\{\begin{array}{c}
\partial_{1} u_{1} \\
\partial_{1} \theta_{1} \\
\partial_{1} \psi_{1}
\end{array}\right\}
\end{array}\right\} \begin{aligned}
& \left\{\begin{array}{l}
V_{1} \\
V_{\phi}
\end{array}\right\}=\left[\begin{array}{ll}
Q_{11}^{\lambda} & Q_{12}^{\lambda} \\
Q_{12}^{\lambda} & Q_{22}^{\lambda}
\end{array}\right]\left\{\begin{array}{c}
\theta_{1}+\partial_{1} w \\
\psi_{1}
\end{array}\right\} \tag{5.4}
\end{aligned}
$$

where the stiffness coefficients are defined in [Tessler et al., 2007].
By using the constitutive equations, Eq. (5.4), the equilibrium equations in terms of the kinematic variables are obtained

$$
\begin{align*}
& A_{11} \partial_{11} u_{1}+B_{12} \partial_{11} \theta_{1}+B_{13} \partial_{11} \psi_{1}+\bar{p}^{b}+\bar{p}^{t}=0 \\
& Q_{11}^{\lambda}\left(\partial_{11} w+\partial_{1} \theta_{1}\right)+Q_{12}^{\lambda} \partial_{1} \psi_{1}+\bar{q}^{t}+\bar{q}^{b}=0 \\
& B_{12} \partial_{11} u_{1}+D_{11} \partial_{11} \theta_{1}+D_{12} \partial_{11} \psi_{1}-Q_{11}^{\lambda}\left(\partial_{1} w+\theta_{1}\right)-Q_{12}^{\lambda} \psi_{1}+2 h\left(\bar{p}^{t}-\bar{p}^{b}\right)=0  \tag{5.5}\\
& B_{13} \partial_{11} u_{1}+D_{12} \partial_{11} \theta_{1}+D_{22} \partial_{11} \psi_{1}-Q_{12}^{\lambda}\left(\partial_{1} w+\theta_{1}\right)-Q_{22}^{\lambda} \psi_{1}=0
\end{align*}
$$

The solution of the homogeneous part of the equilibrium equations, Eq. (5.5), appears in [Tessler et al., 2007] and reads as

$$
\begin{align*}
u_{1}\left(x_{1}\right)= & \left(-C_{8}+C_{3} C_{7}-\frac{C_{2} C_{7}}{R^{2} D_{11}^{*}}\right)\left(a_{1} \cosh \left(R x_{1}\right)+a_{2} \sinh \left(R x_{1}\right)\right) \\
& -\frac{C_{2} C_{7} a_{3}}{2 D_{11}^{*}} x_{1}^{2}+a_{6} x_{1}+a_{7} \\
\theta_{1}\left(x_{1}\right)= & \left(-C_{3}+\frac{C_{2}}{R^{2} D_{11}^{*}}\right)\left(a_{1} \cosh \left(R x_{1}\right)+a_{2} \sinh \left(R x_{1}\right)\right) \\
+ & \frac{C_{2} a_{3}}{2 D_{11}^{*} x_{1}^{2}+a_{4} x_{1}+a_{7}}  \tag{5.6}\\
\psi_{1}\left(x_{1}\right)= & a_{1} \cosh \left(R x_{1}\right)+a_{2} \sinh \left(R x_{1}\right)+a_{3} \\
w\left(x_{1}\right)= & {\left[\frac{C_{3}}{R}-\frac{C_{2}}{R^{3} D_{11}^{*}}+\frac{C_{2} C_{5}}{R D_{11}^{* *}}-\frac{C_{4}}{R}+R\left(C_{6}+C_{3} C_{5}\right)\right]\left(\psi_{1}\left(x_{1}\right)-a_{3}\right) } \\
& -\frac{C_{2} a_{3}}{6 D_{11}^{*}} x_{1}^{3}-\frac{a_{4}}{2} x_{1}^{2}+\left[\left(\frac{C_{2} C_{5}}{D_{11}^{*}}-C_{4}\right) a_{3}-a_{5}\right] x+a_{8}
\end{align*}
$$

where $a_{i}(i=1, . .8)$ are unknown constants that are determined by enforcing the appropriate boundary conditions. Moreover, $C_{i}(i=1, . .8), D_{11}^{*}, D_{12}^{*}, D_{22}^{*}$ and $R$ are combination of stiffness coefficients defined in [Tessler et al., 2007].

### 2.2. Consistent shape functions

The topology of the beam element consists in a simple 2-nodes/4-dof's (see Figure 5.3).

$$
\left\{\begin{array}{l}
u_{0} \\
\theta_{0} \\
\psi_{0} \\
w_{0}
\end{array}\right\} \bigcirc\left\{\begin{array}{c}
u_{L} \\
\theta_{L} \\
\psi_{L} \\
w_{L}
\end{array}\right\}
$$

Figure 5.3 Beam element topology.
The solution of the homogeneous part of Euler-Lagrange equations, Eq. (5.6), can be used to derive consistent shape functions for the element in Figure 5.3 by enforcing the equivalence with the nodal degrees of freedom at the location $x_{I}=0$ and $x_{I}=L$
$\left[\begin{array}{cccccccc}\Gamma & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \Pi & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \Omega & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \Gamma \cosh (R L) & \Gamma \sinh (R L) & -\frac{C_{2} C_{7}}{2 D_{11}^{*}} L^{2} & 0 & 0 & L & 1 & 0 \\ \Pi \cosh (R L) & \Pi \sinh (R L) & \frac{C_{2} L^{2}}{2 D_{11}^{*}} & L & 0 & 0 & 1 & 0 \\ \cosh (R L) & \sinh (R L) & 1 & 0 & 0 & 0 & 0 & 0 \\ \Omega \cosh (R L) & \Omega \sinh (R L) & \frac{C_{2} C_{5} L}{D_{11}^{*}}-\frac{C_{2} L^{3}}{6 D_{11}^{*}} & -\frac{L^{2}}{2} & -L & 0 & 0 & 1\end{array}\right] \mathbf{a}=\left\{\begin{array}{c}u_{0} \\ \theta_{0} \\ \psi_{0} \\ w_{0} \\ u_{L} \\ \theta_{L} \\ \psi_{L} \\ w_{L}\end{array}\right\}$
with
$\Omega=\frac{C_{3}-C_{4}}{R}-\frac{C_{2}\left(1+R^{2} C_{5}\right)}{R^{3} D_{11}^{*}}+R\left(C_{6}+C_{3} C_{5}\right) ; \Gamma=-C_{8}+C_{3} C_{7}-\frac{C_{2} C_{7}}{R^{2} D_{11}^{*}}$
$\Pi=-C_{3}+\frac{C_{2}}{R^{2} D_{11}^{*}}$
and $\mathbf{a}$ is the vector of the eight unknowns $a_{i}(i=1, . .8)$. By solving in terms of a and substituting into Eq. (5.6), after collecting the nodal values degrees of freedom, the shape functions appear (see Appendix 1 for the expressions).

### 2.3. Mass matrix, exact stiffness matrix and consistent load vector

According to the usual finite element notation, the shape functions matrix is denoted by $\mathbf{N}$ and the degrees of freedom are interpolated by using their nodal values, organized in $\mathbf{q}_{\mathrm{e}}$, that is

$$
\begin{gather*}
\left\{\begin{array}{c}
u_{1}\left(x_{1}\right) \\
\theta_{1}\left(x_{1}\right) \\
\psi_{1}\left(x_{1}\right) \\
w\left(x_{1}\right)
\end{array}\right\}=\mathbf{N} \mathbf{q}^{(e)}  \tag{5.8}\\
\mathbf{q}^{(e)}=\left\{u_{0}, \theta_{0}, \psi_{0}, w_{0}, u_{L}, \theta_{L}, \psi_{L}, w_{L}\right\}^{T}
\end{gather*}
$$

The consistent shape functions given in Appendix 1 give rise to an interdependent interpolation scheme, that is each kinematic variable is approximated by using not only its nodal values but involving also the nodal values of the remaining variables. The full interdependence is held true for the axial displacement, whereas the bending rotation, the zigzag amplitude and the deflection degrees of freedom keep out the axial displacement nodal values from their approximation.

The mass and stiffness matrix, and the load vector derive by using Eq. (5.8) in the dynamic version of the Virtual Work Principle

$$
\begin{align*}
& \int_{V} \rho \delta \mathbf{u}^{T} \ddot{\mathbf{u}} d V+ \int_{V} \delta \boldsymbol{\varepsilon}^{T} \boldsymbol{\sigma} d V \\
&=\int_{0}^{L} q \delta w d V+  \tag{5.9}\\
& \int_{0}^{L}\left[\bar{p}^{b} \delta u(z=-h)+\bar{p}^{t} \delta u(z=h)\right] d x_{1}
\end{align*}
$$

Consistent with Eq. (5.8), the strain vector read as

$$
\begin{gather*}
\boldsymbol{\varepsilon}=\mathbf{B}^{(e)} \mathbf{q}^{(e)} \\
\mathbf{B}^{(e)}=\left[\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 1 & 0 & z & \phi_{1}^{(k)} \\
0 & 1 & \beta_{1}^{(k)} & 0 & 0 & 0 & 0 & 1
\end{array}\right]\left[\begin{array}{c}
\mathbf{N} \\
\partial_{1} \mathbf{N}
\end{array}\right] \tag{5.10}
\end{gather*}
$$

and the stress vector is related with strain by means of the constitutive material matrix

$$
\boldsymbol{\sigma}=\mathbf{D} \boldsymbol{\varepsilon}=\left[\begin{array}{cc}
E_{1}^{(k)} & 0  \tag{5.11}\\
0 & G_{1 z}^{(k)}
\end{array}\right] \boldsymbol{\varepsilon}
$$

By using Eq. (5.8), (5.10) and (5.11) into Eq. (5.9), the mass matrix, the stiffness matrix and the load vector appear

$$
\begin{gather*}
\mathbf{M}^{(e)}=\int_{V} \rho \mathbf{N}^{T} \mathbf{H}^{T}(z) \mathbf{H}(z) \mathbf{N} d V \\
\mathbf{K}^{(e)}=\int_{V} \mathbf{B}^{(e) T} \mathbf{D} \mathbf{B}^{(e)} d V \\
\mathbf{F}^{(e)}=\int_{0}^{L}\left[\begin{array}{lll}
\bar{p}^{b} & \bar{p}^{t} & q
\end{array}\right] \stackrel{\mathbf{N}}{ } d x_{1}  \tag{5.12}\\
\mathbf{H}(z)=\left[\begin{array}{cccc}
1 & z & \phi_{1}^{(k)} & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
\end{gather*}
$$

where $\widehat{\mathbf{N}}$ is a combination of shape functions and the element-level equilibrium equation is

$$
\begin{equation*}
\mathbf{M}^{(e)} \ddot{\boldsymbol{q}}^{(e)}+\mathbf{K}^{(e)} \mathbf{q}^{(e)}=\mathbf{F}^{(e)} \tag{5.13}
\end{equation*}
$$

The stiffness matrix calculated by using the shape functions coming from the solution of the Euler-Lagrange equations is exact, that is no discretization errors are introduced for the stiffness evaluation, whereas the mass matrix is not and errors in the total mass evaluation are still present. Due to the exact stiffness evaluation, the beam element formulated is expected to show super-convergent behavior in static analysis and faster convergence with respect to the isoparametric independent element in dynamic analysis.

## 3. (3,2)- Mixed Refined Zigzag Theory beam element

In this paragraph, the formulation of the (3,2)-Mixed Refined Zigzag Theory-based beam element suitable for a coupled thermo-mechanical analysis of laminated beams is presented by using the relations obtained in Chapter 4.

As already highlighted in Sect. 1, the RZT-based elements suffer for the shear locking deficiency; in order to avoid shear locking, the (3,2)-Mixed Refined Zigzag Theory -based beam element is developed employing the anisoparametric interpolation strategy. Successively, two constrained beam elements are formulated.

### 3.1. Kinematic restatement

In the framework of finite elements, it could be useful to develop a rotation-free element, that is an element without rotational degrees of freedom, above all when problems concerning small displacements and high rotations have to be solved. The matter is to perform a change in variables, i.e. to express the bending rotation degree of freedom in terms of in-plane displacements at the top and bottom beam surface. This implies that the kinematics of the model should be more conveniently expressed in terms of the displacement values at the top and bottom beam surface. The change in variables is straightforward by considering the meaning of the degrees of freedom and reads as

$$
\begin{equation*}
u_{1}=\frac{u_{t}+u_{b}}{2} ; \quad \theta_{1}=\frac{u_{t}-u_{b}}{2 h} \tag{5.14}
\end{equation*}
$$

where $u_{t}, u_{b}$ are the in-plane displacements at the upper and lower beam surfaces. Consistent with Eq.(4.60), the kinematics becomes

$$
\begin{align*}
& U_{1}^{(k)}\left(x_{1}, z\right)=L_{1}(z) u_{b}\left(x_{1}\right)+L_{2}(z) u_{t}\left(x_{1}\right)+\mu^{(k)}(z) \psi_{1}\left(x_{1}\right)  \tag{5.15}\\
& U_{z}\left(x_{1}, z\right)=H_{b}^{w}(z) w_{b}\left(x_{1}\right)+H_{t}^{w}(z) w_{t}\left(x_{1}\right)+H_{m}^{w}(z) \bar{w}\left(x_{1}\right)
\end{align*}
$$

where $L_{1}(z)$ and $L_{2}(z)$ are the linear Lagrange's polynomials. By enforcing the change in variables also to Eq. (4.64) and Eq. (4.75), the assumed transverse normal stress reads as

$$
\begin{align*}
\sigma_{z z}^{a}= & A_{b}^{u}(z) \partial_{1} u_{b}+A_{t}^{u}(z) \partial_{1} u_{t}+A^{\psi}(z) \partial_{1} \psi_{1}+ \\
& A_{b}^{w}(z) w_{b}+A_{t}^{w}(z) w_{t}+A_{a}^{w}(z) \bar{w}+\mathbf{A}^{q z}(z) \mathbf{q}_{z}-A^{\Delta T}(z) \Delta T\left(x_{1}\right) \tag{5.16}
\end{align*}
$$

and the assumed transverse shear stress becomes

$$
\begin{align*}
\tau_{1 z}^{a}= & T_{b}^{u}(z) u_{b}+T_{t}^{u}(z) u_{t}+T^{\psi}(z) \psi_{1}+T_{b}^{w}(z) \partial_{1} w_{b}+T_{t}^{w}(z) \partial_{1} w_{t}+T_{a}^{w}(z) \partial_{1} \bar{w}+ \\
& \mathbf{T}_{p}(z) \mathbf{n}_{\mathbf{v}}+\mathbf{T}_{q}(z) \partial_{1} \mathbf{q}_{\mathbf{z}}+T_{\Delta T}(z) \partial_{1} \Delta T\left(x_{1}\right) \tag{5.17}
\end{align*}
$$

where the novel $z$-coordinate functions $A_{b}^{u}(z), A_{t}^{u}(z), T_{b}^{u}(z)$ and $T_{t}^{u}(z)$ derive by simple manipulation of the formulas.

### 3.2. Nine-node, fifteen-dof's anisoparametric element

Following the anisoparametric interpolation strategy, the lowest order beam element is characterized by a linear interpolation of the in-plane degrees of freedom and by a quadratic polynomial interpolation for the transverse displacement, that is

$$
\begin{align*}
\mathbf{u} \equiv\left\{\begin{array}{l}
\left\{\begin{array}{l}
u_{b} \\
u_{t} \\
\psi
\end{array}\right\} \\
\left\{\begin{array}{l}
w_{b} \\
w_{t} \\
\bar{w}
\end{array}\right\}
\end{array}\right\}=\left[\begin{array}{cccc}
N_{1}^{L} \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & N_{2}^{L} \mathbf{I}_{3 \times 3} \\
\mathbf{0}_{3 \times 3} & N_{1}^{Q} \mathbf{I}_{3 \times 3} & N_{3}^{Q} \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\
N_{2}^{Q} \mathbf{I}_{3 \times 3}
\end{array}\right]\left[\begin{array}{c}
\mathbf{q}_{\mathbf{e},}^{T} \\
w_{b}^{*} \\
w_{t}^{*} \\
\bar{w}^{*} \\
\mathbf{q}_{\mathbf{e} 2}^{T}
\end{array}\right\}  \tag{5.18}\\
\mathbf{q}_{\mathbf{e} i}=\left\{u_{\left.b_{i}, u_{t i}, \psi_{i}, w_{b_{i}}, w_{t i}, \bar{w}_{i}\right\} \quad(i=1,2)}\right.
\end{align*}
$$

where $\mathbf{I}_{3 \times 3}$ and $\mathbf{0}_{3 x 3}$ are the $3 \times 3$ identity and zero matrix, respectively, and

$$
\begin{gather*}
{\left[N_{1}^{L} ; N_{2}^{L}\right]=\left[\frac{1}{2}(1-\zeta) ; \frac{1}{2}(1+\zeta)\right]} \\
{\left[N_{1}^{Q} ; N_{2}^{Q} ; N_{3}^{Q}\right]=\left[\frac{1}{2} \zeta(1-\zeta) ; \frac{1}{2} \zeta(1+\zeta) ;\left(1-\zeta^{2}\right)\right]} \tag{5.19}
\end{gather*}
$$

are the linear and the quadratic Lagrange polynomials, where $\zeta=2 x_{1} / L^{(e)}-1$ is the nondimensional axial coordinate and $L^{(e)}$ is the beam element length. The topology of this element is reported in Figure 5.4.

It is worth to note that, contrary to the in-plane and transverse displacements of the top and bottom beam surfaces, the zigzag rotation and the average transverse displacement degrees of freedom are only formally attributed to the nodes placed on the half-thickness coordinate, since they cannot be calculated at a precise thickness coordinate.


Figure 5.4 Anisoparametric virgin beam element topology $\left(\Omega_{0}\right)$.

### 3.3. Six-node, twelve-dof's constrained anisoparametric element

The shape functions matrix in Eq. (5.18) shows that each degree of freedom is approximated by using its nodal values, as in the standard finite element approximation. A
way to eliminate the additional $w$-dof's is to introduce an interdependency of these dof's from the other ones. This is achieved by satisfying a constraint that basically tends to reduce of one order the polynomial degree that expresses the transverse shear strain. In the $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$ model two ways to measure the shear strain are possible: the first one is based on the transverse shear strain coming from the linear strain-displacement relations, $\gamma_{1 z}=\theta_{1}+\partial_{1} U_{z}$; the second one is using the definition of the transverse shear strain measure, $\eta=\theta_{1}+\partial_{1} U_{z}-\psi_{1}$. Thus, two constrained beam elements are developed, namely $\Omega_{\gamma}$ and $\Omega_{\eta}$, according to the constraint enforced (the subscript denotes the constrained quantity). Since the anisoparametric element of Eq. (5.18), hence called also virgin and denoted by $\Omega_{0}$, introduces three additional $w$-dof's, the constraint has to be enforced to three $z$-locations, that is

$$
\left\{\begin{array}{l}
\partial_{1} \gamma_{1 z}(z=-h)=0  \tag{5.20}\\
\partial_{1} \gamma_{1 z}(z=0)=0 \\
\partial_{1} \gamma_{1 z}(z=h)=0
\end{array}\right.
$$

in the case of $\Omega_{\gamma}$ element, whereas the $\Omega_{\eta}$ requires that

$$
\left\{\begin{array}{l}
\partial_{1} \eta(z=-h)=0  \tag{5.21}\\
\partial_{1} \eta(z=0)=0 \\
\partial_{1} \eta(z=h)=0
\end{array}\right.
$$

Solving Eqs. (5.20) and (5.21) in terms of $w_{b}^{*}, w_{t}^{*}$ and $\bar{w}^{*}$ and substituting the expression in Eq. (5.18), the shape functions matrix of the constrained elements is obtained and read as

$$
\begin{gather*}
\mathbf{u} \equiv\left\{\begin{array}{c}
\left\{\begin{array}{l}
\left\{\begin{array}{l}
u_{b} \\
u_{t} \\
\psi
\end{array}\right\} \\
\left\{\begin{array}{c}
w_{b} \\
w_{t} \\
\bar{w}
\end{array}\right\}
\end{array}\right\}=\left[\begin{array}{ccc}
N_{1}^{L} \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} & N_{2}^{L} \mathbf{I}_{3 \times 3} \\
\mathbf{N}_{3 \times 3}^{w u} & N_{1}^{L} \mathbf{I}_{3 \times 3} & -\mathbf{N}_{3 \times 3}^{w u 3}
\end{array} N_{2}^{L} \mathbf{I}_{3 \times 3}\right.
\end{array}\right]\left\{\begin{array}{c}
\mathbf{q}_{\mathbf{e} 1}^{T} \\
\mathbf{q}_{\mathbf{e} 2}^{T}
\end{array}\right\}=\mathbf{N} \mathbf{q}^{(e)}  \tag{5.22}\\
\mathbf{q}_{\mathbf{e} i}=\left\{u_{\left.b_{i}, u_{t_{i}}, \psi_{i}, w_{b_{i}}, w_{t_{i}}, \bar{w}_{i}\right\} \quad(i=1,2)}\right.
\end{gather*}
$$

where

$$
\mathbf{N}_{3 \times 3}^{w u}=\frac{L^{(e)} N_{3}^{Q}}{8} \mathbf{1}_{3 \times 1}\left[\begin{array}{lll}
\frac{1}{2 h} & -\frac{1}{2 h} & \alpha \tag{5.23}
\end{array}\right]
$$

The vector $\mathbf{1}_{3 x 1}$ is a $3 \times 1$ vector made by ones and the parameter $\alpha$ distinguishes between $\Omega_{\gamma}(\alpha=0)$ and $\Omega_{\eta}(\alpha=1)$. Since in the $\Omega_{\gamma}$ element the constraint is applied on $\gamma_{1 z}$, the zigzag rotation is not involved in the interpolation of the deflection degrees of freedom, contrary to the $\Omega_{\eta}$ element that is developed enforcing a constraint on the transverse shear measure. For this reason, the parameter $\alpha$ takes values equal to zero or one respectively for $\Omega_{\gamma}$ and $\Omega_{\eta}$ element.

The resulting beam element topology for $\Omega_{\gamma}$ and $\Omega_{\eta}$ is reported in Figure 5.5.


Figure 5.5 Anisoparametric constraint beam element topology ( $\Omega_{\gamma}, \Omega_{\eta}$ ).

### 3.4. Temperature field representation

The temperature field representation uses the topology defined for the kinematic variables and can assume different distribution along the axial direction depending on the element considered. In particular, the use of the virgin element allows a parabolic distribution of the temperature along the axial direction, due to the additional nodes located in the middle of the beam element (see Figure 5.4). By adopting the constrained elements, the topology (see Figure 5.5) keeps out nodes placed in the middle of the element thus allowing only for a linear distribution of the temperature. On the contrary, the through-thethickness distribution of the temperature is assumed to be quadratic, regardless the type of element.

For the virgin element, the temperature distribution is given by

$$
\begin{equation*}
\Delta T\left(x_{1}, z\right)=\Delta T(z) \Delta T\left(x_{1}\right)=\mathbf{H}_{\Delta T}(z) \mathbf{N}_{\Delta T}\left(x_{1}\right) \boldsymbol{\Theta} \tag{5.24}
\end{equation*}
$$

with

$$
\begin{gather*}
\mathbf{H}_{\Delta T}(z)=\left[H_{b}^{w}(z)\right. \\
H_{t}^{w}(z)  \tag{5.25}\\
\mathbf{N}_{\Delta T}\left(x_{1}^{w}\right)=\left[\begin{array}{lll}
\mathbf{N}^{Q} & \mathbf{0}_{1 \times 3} & \mathbf{0}_{1 \times 3} \\
\mathbf{0}_{1 \times 3} & \mathbf{N}^{Q} & \mathbf{0}_{1 \times 3} \\
\mathbf{0}_{1 \times 3} & \mathbf{0}_{1 \times 3} & \mathbf{N}^{Q}
\end{array}\right] \\
\mathbf{N}^{Q}=\left[\begin{array}{lll}
N_{1}^{Q}\left(x_{1}\right) & N_{1}^{Q}\left(x_{1}\right) & N_{1}^{Q}\left(x_{1}\right)
\end{array}\right]
\end{gather*}
$$

where the polynomials that appear in $\mathbf{H}_{\Delta T}(z)$ are the same used for the transverse displacement assumption (see Chapter 4) and where the nodal temperature values (see Figure 5.6) are collected in $\boldsymbol{\Theta}$ as follows

$$
\begin{equation*}
\mathbf{\Theta}^{T}=\left\{T_{b_{1}}, T_{b_{2}}, T_{b}^{*}, T_{t_{1}}, T_{t_{2}}, T_{t}^{*}, T_{m_{1}}, T_{m_{2}}, T_{m}^{*}\right\} \tag{5.26}
\end{equation*}
$$



Nodal values

- $T_{t_{i}} \triangle T_{m_{i}} \square T_{b_{i}} ○ T_{t}^{*} \Delta T_{m}{ }^{*} \square T_{b}^{*}$

Figure 5.6 Nodal values of temperature.
For the constrained elements, the distribution of temperature along the axial direction is linear combined with a parabolic through-the-thickness one. In this case, the temperature field is given by

$$
\begin{equation*}
\Delta T\left(x_{1}, z\right)=\Delta T(z) \Delta T\left(x_{1}\right)=\mathbf{H}_{\Delta T}(z) \mathbf{N}_{\Delta T}\left(x_{1}\right) \boldsymbol{\Theta} \tag{5.27}
\end{equation*}
$$

where the following definitions are introduced

$$
\begin{gather*}
\mathbf{H}_{\Delta T}(z)=\left[\begin{array}{lll}
H_{b}^{w}(z) & H_{t}^{w}(z) & H_{a}^{w}(z)
\end{array}\right] \\
\mathbf{N}_{\Delta T}\left(x_{1}\right)=\left[\begin{array}{lll}
\mathbf{N}^{L} & \mathbf{0}_{1 \times 2} & \mathbf{0}_{1 \times 2} \\
\mathbf{0}_{1 \times 2} & \mathbf{N}^{L} & \mathbf{0}_{1 \times 2} \\
\mathbf{0}_{1 \times 2} & \mathbf{0}_{1 \times 2} & \mathbf{N}^{L}
\end{array}\right] ; \mathbf{N}^{L}=\left[\begin{array}{ll}
N_{1}^{L}\left(x_{1}\right) & N_{1}^{L}\left(x_{1}\right)
\end{array}\right] \tag{5.28}
\end{gather*}
$$

and the nodal values of temperature, consistent with the nodal pattern in Figure 5.7, are collected in $\boldsymbol{\Theta}$ as follows

$$
\begin{equation*}
\boldsymbol{\Theta}^{T}=\left\{T_{b_{1}}, T_{b_{2}}, T_{t_{1}}, T_{t_{2}}, T_{m_{1}}, T_{m_{2}}\right\} \tag{5.29}
\end{equation*}
$$



Figure 5.7 Nodal values of temperature.

### 3.5. Consistent mass matrix, stiffness matrix and consistent load vector

The consistent mass matrix, the stiffness matrix and the consistent load vector are obtained by introducing the variables interpolation, Eq. (5.18) for the virgin element or Eq. (5.22) for the constrained elements, into the dynamic version of the Reissner Mixed Variational Theorem. Since the assumed transverse normal and shear stresses, Eq. (5.16) and (5.17), are already able to fulfill the weak form of the compatibility constraint, the remaining part of the Reissner Mixed Variational Theorem concerns the equilibrium between the work done by stresses and strains, that of inertial forces and external loads is considered, that is

$$
\begin{equation*}
\int_{V} \rho \delta \mathbf{u}^{T} \ddot{\mathbf{u}} d V+\int_{V} \delta \boldsymbol{\varepsilon}^{T} \boldsymbol{\sigma} d V=\delta W_{e} \tag{5.30}
\end{equation*}
$$

Consistent with the finite element interpolation of the kinematic variables, the strain vector is given by

$$
\begin{gather*}
\boldsymbol{\varepsilon}=\mathbf{B}^{(e)} \mathbf{q}^{(e)} \\
\boldsymbol{\varepsilon}^{T}=\left\{\begin{array}{lll}
\varepsilon_{11}^{(k)} & \gamma_{1 z}^{(k)} & \varepsilon_{z z}
\end{array}\right\} \tag{5.31}
\end{gather*}
$$

whereas the stress vector read as

$$
\begin{align*}
\boldsymbol{\sigma} & =\mathbf{S}^{(e)} \mathbf{q}^{(e)}+\mathbf{S}_{f}^{(e)} \mathbf{f}_{e}-\mathbf{S}_{\Delta T}^{(e)} \boldsymbol{\Theta} \\
\boldsymbol{\sigma}^{T} & =\left\{\begin{array}{lll}
\sigma_{11}^{(k)} & \tau_{1 z}^{a} & \sigma_{z z}^{a}
\end{array}\right\} \tag{5.32}
\end{align*}
$$

where the strain and stress matrices are defined in Appendix 2.
By introducing Eq. (5.31) and Eq. (5.32) in the work done by inertial forces and that done by stresses and strains, the element mass and stiffness matrices appear

$$
\begin{gather*}
\mathbf{M}^{(e)}=\int_{V} \rho \mathbf{N}^{T} \mathbf{H}^{T}(z) \mathbf{H}(z) \mathbf{N} d V \\
\mathbf{K}^{(e)}=\int_{V} \mathbf{B}^{(e) T} \mathbf{S}^{(e)} d V  \tag{5.33}\\
\mathbf{F}_{\sigma}^{(e)}=\int_{V} \mathbf{B}^{(e) T} \mathbf{S}_{f}^{(e)} \mathbf{f}_{e} d V ; \mathbf{F}_{\Delta T}^{(e)}=\int_{V} \mathbf{B}^{(e) T} \mathbf{S}_{\Delta T}^{(e)} d V
\end{gather*}
$$

along with two vectors, $\mathbf{F}_{\sigma}^{(e)}$ and $\mathbf{F}_{\Delta T}^{(e)}$, that contribute to the right-hand-side of the elementlevel equilibrium equation. The $\mathbf{H}(z)$ matrix is defined in Appendix 2.

The load vector follows by the definition of the work done by external loads wherein the finite element approximation, Eq. (5.18) or Eq. (5.22) depending on the developed element, is introduced and reads as

$$
\mathbf{F}^{(e)}=\int_{0}^{L}\left\{\begin{array}{llll}
\bar{p}^{b} & \bar{p}^{t} & \bar{q}^{b} & \bar{q}^{t}
\end{array}\right\} \widehat{\mathbf{N}} d x_{1}+\sum_{i=1}^{n_{c}}\left\{\begin{array}{llll}
P_{i}^{b} & P_{i}^{t} & T_{i}^{b} & T_{i}^{t} \tag{5.34}
\end{array}\right\} \widehat{\mathbf{N}}\left(x_{i}\right)
$$

where $\widehat{\mathbf{N}}$ is a matrix made by the $1^{\text {st }}, 2^{\text {nd }}, 4^{\text {th }}$ and $5^{\text {th }}$ shape functions, $n_{c}$ is the number of concentrated loads, $P_{i}^{b}, P_{i}^{t}, T_{i}^{b}$ and $T_{i}^{t}$, acting along the axial and thickness directions and at the bottom and top surface, respectively, and applied in $x_{i}$.

Finally, the element-level equilibrium equation appears

$$
\begin{equation*}
\mathbf{M}^{(e)} \ddot{\boldsymbol{q}}^{(e)}+\mathbf{K}^{(e)} \mathbf{q}^{(e)}=\mathbf{F}^{(e)}-\mathbf{F}_{\sigma}^{(e)}+\mathbf{F}_{\Delta T}^{(e)} \boldsymbol{\Theta} \tag{5.35}
\end{equation*}
$$

## 4. Mixed (3,2)-Refined Zigzag Theory plate element

In this section, a $C^{0}$-continuous (3,2)-Mixed Refined Zigzag Theory bending rotationsfree plate element is developed.

Similar to the beam element, in order to avoid the shear locking deficiency that affects the isoparametric element, an anisoparametric interpolation scheme is adopted leading to the development of an element, called virgin, with extra nodes with only transverse displacements degrees of freedom. In order to reduce the number of nodal unknowns and to recovery an isoparametric-like nodal configuration, a constrained element is formulated by adopting the linked interpolation strategy developed by Tessler [Tessler et al., 1985] and employed by Gherlone [Gherlone et al., 2011] and Versino [Versino et al., 2013] in the formulation of beam and plate elements, respectively.

### 4.1. Kinematic restatement

In order to develop a bending rotations-free element, the $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$ kinematics is restated by adopting the change in variables of Eq. (5.14) in both directions. In this way, the restated kinematics read as

$$
\begin{align*}
& U_{\alpha}^{(k)}(\mathbf{x}, z, t)=L_{1}(z) u_{\alpha}^{b}(\mathbf{x}, t)+L_{2}(z) u_{\alpha}^{t}(\mathbf{x}, t)+\mu_{\alpha}^{(k)}(z) \psi_{\alpha}(\mathbf{x}, t) \\
& U_{z}(\mathbf{x}, z, t)=H_{b}^{w}(z) w_{b}(\mathbf{x}, t)+H_{t}^{w}(z) w_{t}(\mathbf{x}, t)+H_{a}^{w}(z) \bar{w}(\mathbf{x}, t) \tag{5.36}
\end{align*}
$$

where $L_{1}(z), L_{2}(z)$ are the linear Lagrange's polynomials and $u_{\alpha}^{b}, u_{\alpha}^{t}$ are the bottom and top in-plane displacements, respectively.

Consistent with the restated kinematics, the assumed transverse normal stress is

$$
\begin{align*}
\sigma_{z z}^{a}= & A_{\alpha \beta}^{t}(z) \partial_{\beta} u_{\alpha}^{t}+A_{\alpha \beta}^{b}(z) \partial_{\beta} u_{\alpha}^{b}+A_{\alpha \beta}^{\psi}(z) \partial_{\beta} \psi_{\alpha}+A_{\beta \alpha}^{\psi}(z) \partial_{\alpha} \psi_{\beta}+  \tag{5.37}\\
& A_{b}^{w}(z) w_{b}+A_{t}^{w}(z) w_{t}+A_{a}^{w}(z) \bar{w}+\mathbf{A}^{\mathrm{qz}}(z) \mathbf{q}_{\mathbf{z}}
\end{align*}
$$

where the novel $z$-coordinate shape functions, $A_{\alpha \beta}^{t}(z), A_{\alpha \beta}^{b}(z)$ are easily derivable by simple manipulation of Eq. (4.24). Along with the transverse normal stress, the assumed transverse shear stresses become

$$
\begin{gather*}
\boldsymbol{\tau}^{a}=\mathbf{T}_{b}^{u}(z)\left\{\begin{array}{l}
u_{1}^{b} \\
u_{2}^{b}
\end{array}\right\}+\mathbf{T}_{t}^{u}(z)\left\{\begin{array}{l}
u_{1}^{t} \\
u_{2}^{t}
\end{array}\right\}+\mathbf{T}^{y}(z)\left\{\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right\}+\mathbf{T}_{b}^{w}(z)\left\{\begin{array}{l}
\partial_{1} w_{b} \\
\partial_{2} w_{b}
\end{array}\right\}+ \\
\mathbf{T}_{t}^{w}(z)\left\{\begin{array}{l}
\partial_{1} w_{t} \\
\partial_{2} w_{t}
\end{array}\right\}+\mathbf{T}_{a}^{w}(z)\left\{\begin{array}{l}
\partial_{1} \bar{w} \\
\partial_{2} \bar{w}
\end{array}\right\}+\mathbf{T}_{p}(z) \mathbf{n}_{\mathrm{v}}+\mathbf{T}_{q}(z) \mathbf{q} \tag{5.38}
\end{gather*}
$$

wherein the $z$-coordinate matrices of shape functions $\mathbf{T}_{b}^{u}(z), \mathbf{T}_{t}^{u}(z)$ derive by enforcing the change in variables to Eq. (4.40).

### 4.2. Eighteen-node, thirty-six dof's anisoparametric element

By adopting the anisoparametric interpolation strategy to circumvent the shear locking, the lowest order element requires a linear interpolation of the in-plane displacements and zigzag amplitudes along with a quadratic approximation for the transverse displacements, that is

$$
\begin{array}{ccc}
u_{\alpha b}=\sum_{i=1}^{3} \xi_{i} u_{\alpha b}^{(i)} ; & u_{\alpha t}=\sum_{i=1}^{3} \xi_{i} u_{\alpha t}^{(i)} ; & \psi_{\alpha}=\sum_{i=1}^{3} \xi_{i} \psi_{\alpha}^{(i)} ; \\
w_{b}=\sum_{i=1}^{3} \xi_{i i} w_{b}^{(i)}+\xi_{i j} s_{b}^{(i)} ; & w_{t}=\sum_{i=1}^{3} \xi_{i i} w_{t}^{(i)}+\xi_{i j} w_{t}^{(i j)} ; & \bar{w}=\sum_{i=1}^{3} \xi_{i i} \bar{w}^{(i)}+\xi_{i j} \bar{w}^{(i j)} \tag{5.39}
\end{array}
$$

where $\xi_{i}$ are the linear element parametric coordinates [Zienkiewicz et al., 2000], $\xi_{i i}$ and $\xi_{i j}(i, j=1,2,3 ; i \neq j)$ are the quadratic shape functions given by [Zienkiewicz et al., 2000]

$$
\begin{equation*}
\xi_{i i}=\xi_{i}\left(2 \xi_{i}-1\right) ; \xi_{i j}=4 \xi_{i} \xi_{j} \tag{5.40}
\end{equation*}
$$

Equivalently, Eq. (5.39) can be arranged in the common adopted compact matrix notation, that is

$$
\begin{gather*}
\mathbf{u} \equiv \mathbf{N} \hat{\mathbf{q}}^{(e)} \\
\mathbf{u}^{T}=\left\{u_{1 b}, u_{2 b}, u_{1 t}, u_{2 t}, \psi_{1}, \psi_{2}, w_{b}, w_{t}, \bar{w}\right\}  \tag{5.41}\\
\hat{\mathbf{q}}^{(e) T}=\left\{\mathbf{u}_{1}^{T}, \mathbf{u}_{2}^{T}, \mathbf{u}_{3}^{T}, w_{b}^{(12)}, w_{t}^{(12)}, \bar{w}^{(12)}, w_{b}^{(23)}, w_{t}^{(23)}, \bar{w}^{(23)}, w_{b}^{(31)}, w_{t}^{(31)}, \bar{w}^{(31)}\right\} \\
\mathbf{u}_{i}^{T} \equiv\left\{u_{1 b}^{(i)}, u_{2 b}^{(i)}, u_{1 t}^{(i)}, u_{2 t}^{(i)}, \psi_{1}^{(i)}, \psi_{2}^{(i)}, w_{b}^{(i)}, w_{t}^{(i)}, \bar{w}^{(i)}\right\}
\end{gather*}
$$

The interpolation scheme of Eq. (5.39) is consistent with the element topology shown in Figure 5.8.

### 4.3. Nine-node, twenty-seven dof's constrained anisoparametric element

In order to reduce the number of nodal unknowns and to recover an isoparametric-like nodal configuration, a constrained element is developed employing the strategy proposed by Tessler [Tessler et al., 1985], adopted by Gherlone [Gherlone et al., 2011] and Versino [Versino et al., 2013] in the framework of RZT-based beam and plate element and previously involved in the formulation of the constrained $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$-based beam element.

A way to eliminate the extra- $w$ dof's leading to an isoparametric-like nodal pattern is to use the continuous edge constrains. One possibility is to adopt the constraint typically
adopted in the FSDT anisoparametric element [Tessler et. al., 1985] wherein the transverse shear strain along each element edge

$$
\begin{equation*}
\gamma_{n z}(z) \equiv \partial_{s} U_{z}+\theta_{n}=\partial_{s} U_{z}+\frac{1}{2 h}\left(u_{n t}-u_{n b}\right) \tag{5.42}
\end{equation*}
$$

is set to be constant along the local edge coordinate $s$ (see Figure 5.9), that is

$$
\begin{equation*}
\partial_{s} \gamma_{n z}(z)=0 \tag{5.43}
\end{equation*}
$$

In the framework of the RZT-like models, the transverse shear strain can be evaluated also taking into account the transverse shear strain measure wherein the zigzag amplitude effect is included. Thus, a second edge constraint that can be enforced reads as

$$
\begin{equation*}
\partial_{s} \eta_{n z}(z)=0 \tag{5.44}
\end{equation*}
$$

where the strain measure is defined as

$$
\begin{equation*}
\eta_{n z}(z)=\gamma_{n z}(z)-\psi_{n} \tag{5.45}
\end{equation*}
$$

Thus, depending on the constraint used, Eq. (5.43) or Eq. (5.44), two constrained elements can be formulated. In order to distinguish them, that based on Eq. (5.43) is denoted as $\Omega_{\gamma}$ whereas that adopting Eq. (5.44) is quoted as $\Omega_{\eta}$.

Consistent with the kinematic definitions (Figure 5.8), the top and bottom in-plane displacements and the zigzag amplitude oriented along the normal to the edge element are expressed in terms of the kinematic variables as

$$
\begin{align*}
& u_{n b}=-u_{1 b} \sin \alpha_{i j}+u_{2 b} \cos \alpha_{i j} \\
& u_{n t}=-u_{1 t} \sin \alpha_{i j}+u_{2 t} \cos \alpha_{i j}  \tag{5.46}\\
& \psi_{n}=\psi_{1} \cos \alpha_{i j}+\psi_{2} \sin \alpha_{i j}
\end{align*}
$$

where $\alpha_{i j}$ is the angle between the $i j$-edge and the $x_{I}$ axis.
It is worth to note that constraint in Eq. (5.43) and Eq. (5.44) are $z$-coordinate dependent due to the quadratic distribution of the transverse displacement, thus, as for the beam element formulation, the constraints are enforced at the three $z$-location, that is

$$
\left\{\begin{array}{l}
\partial_{s} \gamma_{n z}(z=-h)=0  \tag{5.47}\\
\partial_{s} \gamma_{n z}(z=0)=0 \\
\partial_{s} \gamma_{n z}(z=h)=0
\end{array}\right.
$$



Figure 5.8 Anisoparametric plate element topology $\left(\Omega_{0}\right)$.


Figure 5.9 Triangular element edge definitions of $s, n$ and $\alpha_{12}$.
in case of $\Omega_{\gamma}$ element, whereas in case of $\Omega_{\eta}$ element

$$
\left\{\begin{array}{l}
\partial_{s} \eta_{n z}(z=-h)=0  \tag{5.48}\\
\partial_{s} \eta_{n z}(z=0)=0 \\
\partial_{s} \eta_{n z}(z=h)=0
\end{array}\right.
$$

Enforcing constraints Eq. (5.47) or Eq. (5.48), the additional w-dof's, that is those defined at the center of the element edges, are expressed in terms of the corner nodal dof's

$$
\begin{align*}
\begin{aligned}
& w_{b}^{(i j)}= \frac{w_{b}^{(i)}+w_{b}^{(j)}}{2}+ \\
& \frac{x_{2}^{(j)}-x_{2}^{(i)}}{2 h}\left[\left(u_{1 t}^{(j)}-u_{1 b}^{(j)}+2 h \alpha \psi_{1}^{(j)}\right)-\left(u_{1 t}^{(i)}-u_{1 b}^{(i)}+2 h \alpha \psi_{1}^{(i)}\right)\right]+ \\
& \frac{x_{1}^{(i)}-x_{1}^{(j)}}{2 h}\left[\left(u_{2 t}^{(j)}-u_{2 b}^{(j)}+2 h \alpha \psi_{2}^{(j)}\right)-\left(u_{2 t}^{(i)}-u_{2 b}^{(i)}+2 h \alpha \psi_{2}^{(i)}\right)\right] \\
& w_{t}^{(i j)}= \frac{w_{t}^{(i)}+w_{t}^{(j)}}{2}+ \\
& \frac{x_{2}^{(j)}-x_{2}^{(i)}}{2 h}\left[\left(u_{1 t}^{(j)}-u_{1 b}^{(j)}+2 h \alpha \psi_{1}^{(j)}\right)-\left(u_{1 t}^{(i)}-u_{1 b}^{(i)}+2 h \alpha \psi_{1}^{(i)}\right)\right]+ \\
& \frac{x_{1}^{(i)}-x_{1}^{(j)}}{2 h}\left[\left(u_{2 t}^{(j)}-u_{2 b}^{(j)}+2 h \alpha \psi_{2}^{(j)}\right)-\left(u_{2 t}^{(i)}-u_{2 b}^{(i)}+2 h \alpha \psi_{2}^{(i)}\right)\right] \\
& \bar{w}^{(i j)}=\frac{\bar{w}^{(i)}+\bar{w}^{(j)}}{2}+ \frac{x_{2}^{(j)}-x_{2}^{(i)}}{2 h}\left[\left(u_{1 t}^{(j)}-u_{1 b}^{(j)}+2 h \alpha \psi_{1}^{(j)}\right)-\left(u_{1 t}^{(i)}-u_{1 b}^{(i)}+2 h \alpha \psi_{1}^{(i)}\right)\right]+ \\
& \frac{x_{1}^{(i)}-x_{1}^{(j)}}{2 h}\left[\left(u_{2 t}^{(j)}-u_{2 b}^{(j)}+2 h \alpha \psi_{2}^{(j)}\right)-\left(u_{2 t}^{(i)}-u_{2 b}^{(i)}+2 h \alpha \psi_{2}^{(i)}\right)\right]
\end{aligned} \tag{5.49a}
\end{align*}
$$

where $x_{1}^{(i, j)}, x_{2}^{(i, j)}$ are the in-plane coordinates of nodes $i$ and $j$ and the parameter $\alpha$ distinguishes between element $\Omega_{\gamma}(\alpha=0)$ and $\Omega_{\eta}(\alpha=1)$. By introducing Eqs. (5.49) in Eqs. (5.39), the interpolation of kinematic variables for the constrained element, consistent with the nodal pattern shown in Figure 5.10, is reached

$$
\begin{align*}
& u_{\alpha b}=\sum_{i=1}^{3} \xi_{i} u_{\alpha b}^{(i)} ; \quad u_{\alpha t}=\sum_{i=1}^{3} \xi_{i} u_{\alpha t}^{(i)} ; \quad \psi_{\alpha}=\sum_{i=1}^{3} \xi_{i} \psi_{\alpha}^{(i)} ; \\
& w_{b}=\sum_{i=1}^{3} \xi_{i} w_{b}^{(i)}+\sum_{i=1}^{3}\left[\left(u_{1 t}^{(i)}-u_{1 b}^{(i)}+2 h \alpha \psi_{1}^{(i)}\right) \varphi_{1 i}+\left(u_{2 t}^{(i)}-u_{2 b}^{(i)}+2 h \alpha \psi_{2}^{(i)}\right) \varphi_{2 i}\right] ; \\
& w_{t}=\sum_{i=1}^{3} \xi_{i} w_{t}^{(i)}+\sum_{i=1}^{3}\left[\left(u_{1 t}^{(i)}-u_{1 b}^{(i)}+2 h \alpha \psi_{1}^{(i)}\right) \varphi_{1 i}+\left(u_{2 t}^{(i)}-u_{2 b}^{(i)}+2 h \alpha \psi_{2}^{(i)}\right) \varphi_{2 i}\right] ;  \tag{5.50}\\
& \bar{w}=\sum_{i=1}^{3} \xi_{i} \bar{w}^{(i)}+\sum_{i=1}^{3}\left[\left(u_{1 t}^{(i)}-u_{1 b}^{(i)}+2 h \alpha \psi_{1}^{(i)}\right) \varphi_{1 i}+\left(u_{2 t}^{(i)}-u_{2 b}^{(i)}+2 h \alpha \psi_{2}^{(i)}\right) \varphi_{2 i}\right]
\end{align*}
$$

wherein $\varphi_{1 i}, \varphi_{2 i}$ are combination of the linear element parametric coordinates and are defined as

$$
\begin{equation*}
\varphi_{1 i}=\frac{1}{2 h} \frac{\xi_{i}}{2}\left(a_{j} \xi_{k}-a_{k} \xi_{j}\right) ; \quad \varphi_{2 i}=\frac{1}{2 h} \frac{\xi_{i}}{2}\left(b_{k} \xi_{j}-b_{j} \xi_{k}\right) \tag{5.51}
\end{equation*}
$$

where the subscripts are given by the cyclic permutation $i=1,2,3, j=2,3,1$ and $k=3,1,2$ and $a_{i} \equiv x_{1}^{(k)}-x_{1}^{(j)}, b_{i} \equiv x_{2}^{(j)}-x_{2}^{(k)}$.

By following the common used compact matrix notation, the kinematic variables are given by

$$
\begin{gather*}
\mathbf{u} \equiv \mathbf{N} \mathbf{q}^{(e)} \\
\mathbf{q}^{(e) T}=\left\{\mathbf{u}_{1}^{T}, \mathbf{u}_{2}^{T}, \mathbf{u}_{3}^{T}\right\}  \tag{5.52}\\
\mathbf{u}_{i}^{T} \equiv\left\{u_{1 b}^{(i)}, u_{2 b}^{(i)}, u_{1 t}^{(i)}, u_{2 t}^{(i)}, \psi_{1}^{(i)}, \psi_{2}^{(i)}, w_{b}^{(i)}, w_{t}^{(i)}, \bar{w}^{(i)}\right\}
\end{gather*}
$$

where the shape functions matrix can be easily defined by comparing Eq. (5.52) with Eq. (5.50).


Figure 5.10 Anisoparametric constrained plate element topology ( $\Omega_{\gamma}, \Omega_{\eta}$ ).

### 4.4. Consistent mass matrix, stiffness matrix and consistent load vector

The element consistent mass matrix, the stiffness matrix and the consistent load vector derive by introducing the finite element approximation into the Reissner Mixed Variational

Theorem part dealing with the balance between the work done by stresses and strains and those done by the external loads and the inertial forces, that is

$$
\begin{equation*}
\int_{V} \rho \delta \mathbf{u}^{T} \ddot{\mathbf{u}} d V+\int_{V} \delta \boldsymbol{\varepsilon}^{T} \boldsymbol{\sigma} d V=\delta W_{e} \tag{5.53}
\end{equation*}
$$

The strain vector is related with the nodal unknowns by means of the following relation

$$
\begin{gather*}
\boldsymbol{\varepsilon}=\mathbf{B}^{(e)} \mathbf{q}^{(e)} \\
\boldsymbol{\varepsilon}^{T}=\left\{\begin{array}{llllll}
\varepsilon_{11}^{(k)} & \varepsilon_{22}^{(k)} & \gamma_{12}^{(k)} & \gamma_{12}^{(k)} & \gamma_{12}^{(k)} & \varepsilon_{z z}
\end{array}\right\} \tag{5.54}
\end{gather*}
$$

where the definition of the $\mathbf{B}^{(e)}$ matrix is given in Appendix 3. The stress vector read as

$$
\begin{gather*}
\boldsymbol{\sigma}=\mathbf{S}^{(e)} \mathbf{q}^{(e)}+\mathbf{S}_{f}^{(e)} \mathbf{f}_{e} \\
\boldsymbol{\sigma}^{T}=\left\{\begin{array}{llllll}
\sigma_{11}^{(k)} & \sigma_{22}^{(k)} & \tau_{12}^{(k)} & \tau_{1 z}^{a} & \tau_{2 z}^{a} & \sigma_{z z}^{a}
\end{array}\right\} \tag{5.55}
\end{gather*}
$$

where $\mathbf{S}^{(e)}, \mathbf{S}_{f}^{(e)}, \mathbf{f}_{e}$ are declared in Appendix 3.
By using Eq. (5.54) and Eq. (5.55) into Eq. (5.53), the mass and stiffness matrices and the load vectors appear

$$
\begin{gather*}
\mathbf{M}^{(e)}=\int_{V} \rho \mathbf{N}^{T} \mathbf{H}^{T}(z) \mathbf{H}(z) \mathbf{N} d V \\
\mathbf{K}^{(e)}=\int_{V} \mathbf{B}^{(e) T} \mathbf{S}^{(e)} d V \\
\mathbf{F}_{\sigma}^{(e)}=\int_{V} \mathbf{B}^{(e) T} \mathbf{S}_{f}^{(e)} \mathbf{f}_{e} d V  \tag{5.56}\\
\mathbf{F}^{(e)}=\int_{S}\left\{\mathbf{n}_{\mathbf{v}}^{T} \quad \mathbf{q}_{\mathbf{z}}^{T}\right\} \widehat{\mathbf{N}} d S+\sum_{i=1}^{n_{c}} \mathbf{C N}\left(x_{i}\right)
\end{gather*}
$$

where the expression of $\mathbf{H}(z)$ is given in Appendix 3, whereas $\widehat{\mathbf{N}}$ is made by the $1^{\text {st }}, 2^{\text {nd }}, 3^{\text {rd }}$, $4^{\text {th }}, 6^{\text {th }}$ and $7^{\text {th }}$ rows of the shape functions matrix and vector $\mathbf{C}$ collects all the concentrated loads in the order consistent with that assumed for $\overline{\mathbf{N}}$.

Finally, the element-level equilibrium equation reads as

$$
\begin{equation*}
\mathbf{M}^{(e)} \ddot{\mathbf{q}}^{(e)}+\mathbf{K}^{(e)} \mathbf{q}^{(e)}=\mathbf{F}^{(e)}-\mathbf{F}_{\sigma}^{(e)} \tag{5.57}
\end{equation*}
$$

## Appendix 1. RZT-based beam finite element matrices

Here, the shape functions derived by the solution of the homogeneous part of the EulerLagrange's equations is given.

It is worth to note that the solution of Eq. (5.7) is expressed in terms of hyperbolic functions, that produces numerical drawbacks like round-off errors. In order to circumvent this problem, the solution is expressed in terms of the exponential function.

According to the order defined in Eq. (5.8), the shape functions matrix read as

$$
\mathbf{N}=\left[\begin{array}{l}
\mathbf{n}_{1} \\
\mathbf{n}_{2} \\
\mathbf{n}_{3} \\
\mathbf{n}_{4}
\end{array}\right]
$$

where (see next page)

$$
\mathbf{n}_{1}=\frac{1}{\Delta}\{X\}\left[\begin{array}{cccccccc}
0 & \Gamma(m-n) & -\Gamma \frac{L(m-n)}{2} & \Gamma \frac{\alpha_{1} \lambda_{n}+2 \alpha_{3}-\alpha_{6}}{2 \lambda_{n}} & 0 & -\Gamma(m-n) & -\Gamma \frac{L(m-n)}{2} & \Gamma \frac{\alpha_{2} \lambda_{n}-2 \alpha_{3}-2 \alpha_{5}}{2 \lambda_{n}} \\
0 & \Gamma(m+n) & -\Gamma \frac{L(m+n)}{2} & \Gamma \frac{\alpha_{1} \lambda_{n}-2 \alpha_{3}+\alpha_{6}}{2 \lambda_{n}} & 0 & -\Gamma(m+n) & -\Gamma \frac{L(m+n)}{2} & \Gamma \frac{\alpha_{2} \lambda_{n}+2 \alpha_{3}+2 \alpha_{5}}{2 \lambda_{n}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{C_{2} C_{7} m}{D_{11}^{*}} & \frac{C_{2} C_{7} L m}{2 D_{11}^{*}} & \frac{C_{2} C_{7} \alpha_{2}}{2 D_{11}^{*}} & 0 & -\frac{C_{2} C_{7} m}{D_{11}^{*}} & -\frac{C_{2} C_{7} L m}{2 D_{11}^{*}} & \frac{C_{2} C_{7} \alpha_{2}}{2 D_{11}^{*}} \\
-\frac{\Delta}{L} & -2 C_{2} C_{7} L \lambda_{p} & -C_{2} C_{7} L^{2} \lambda_{p} & -\frac{\alpha_{4}+2 \alpha_{2} \Gamma-\alpha_{8} \Gamma}{L} & \frac{\Delta}{L} & 2 C_{2} C_{7} L \lambda_{p} & C_{2} C_{7} L^{2} \lambda_{p} & -\frac{\alpha_{4}-2 \alpha_{2} \Gamma-\alpha_{8} \Gamma}{L} \\
\Delta & -2 \Gamma m & -\Gamma m & -\alpha_{1} \Gamma & 0 & 2 \Gamma m & \Gamma L m & -\alpha_{2} \Gamma
\end{array}\right]
$$

$$
\mathbf{n}_{3}=\frac{1}{\Delta}\{X\}\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
\Pi(m-n) & \Pi(m+n) & 0 & -\frac{C_{2} m}{D_{11}^{*}} & 2 C_{2} L \lambda_{p} \\
-\Pi \frac{L(m-n)}{2} & -\Pi \frac{L(m+n)}{2} & 0 & \frac{C_{2} L m}{2 D_{11}^{*}} & -\frac{\lambda_{p} C_{2} L^{3}-2 \alpha_{2}+\alpha_{8}}{L} \\
\Pi \frac{\alpha_{1} \lambda_{n}+2 \alpha_{3}-\alpha_{6}}{2 \lambda_{n}} & \Pi \frac{\alpha_{1} \lambda_{n}-2 \alpha_{3}+\alpha_{6}}{2 \lambda_{n}} & 0 & -\frac{\alpha_{2} C_{2}}{2 D_{11}^{*}} & 2\left(\alpha_{7} C_{2} L^{2}-\alpha_{2} \Pi+2 \Pi Z D_{11}^{*} \lambda_{p}\right) \\
0 & 0 & 0 & 0 & \alpha_{8}-2 D_{11}^{*}\left(4 \Omega \lambda_{n}+\Pi L \lambda_{p}\right) \\
-\Pi(m-n) & -\Pi(m+n) & 0 & \frac{C_{2} m}{D_{11}^{*}} & -\Pi \alpha_{1} \\
-\Pi \frac{L(m-n)}{2} & -\Pi \frac{L(m+n)}{2} & 0 & \frac{C_{2} L m}{2 D_{11}^{*}} & 2 \frac{\alpha_{2}-Z m}{L} \\
\Pi \frac{\alpha_{2} \lambda_{n}-2\left(\alpha_{3}+\alpha_{5}\right)}{2 \lambda_{n}} & \Pi \frac{\alpha_{2} \lambda_{n}+2\left(\alpha_{3}+\alpha_{5}\right)}{2 \lambda_{n}} & 0 & -\frac{\alpha_{2} C_{2}}{2 D_{11}^{*}} & 2 \frac{C_{2} \Omega \lambda_{n} L^{2}+\alpha_{2} \Pi-\Pi Z m}{L}
\end{array}\right.
$$

$$
\mathbf{n}_{4}=\frac{1}{\Delta}\{X\}\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{(m-n)}{} & (m+n) & 0 & 0 & 0 & -2 m \\
-\frac{L(m-n)}{2} & -\frac{L(m+n)}{2} & 0 & 0 & 0 & m L \\
\frac{\alpha_{1} \lambda_{n}+2 \alpha_{3}-\alpha_{6}}{2 \lambda_{n}} & \frac{\alpha_{1} \lambda_{n}-2 \alpha_{3}+\alpha_{6}}{2 \lambda_{n}} & 0 & 0 & 0 & -\alpha_{2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
-(m-n) & -(m+n) & 0 & 0 & 0 & 2 m \\
-\frac{L(m-n)}{2} & -\frac{L(m+n)}{2} & 0 & 0 & 0 & L m \\
\frac{\alpha_{2} \lambda_{n}-2\left(\alpha_{3}+\alpha_{5}\right)}{2 \lambda_{n}} & \frac{\alpha_{2} \lambda_{n}+2\left(\alpha_{3}+\alpha_{5}\right)}{2 \lambda_{n}} & 0 & 0 & 0 & -\alpha_{2}
\end{array}\right]^{T}
$$

and

$$
\begin{aligned}
& \{X\}=\left\{e^{R x_{1}}, e^{-R x_{1}}, x_{1}^{3}, x_{1}^{2}, x_{1}, 1\right\} ; \lambda_{p}=e^{R L}+1 ; \lambda_{n}=e^{R L}-1 ; \lambda_{p}^{2}=e^{2 R L}+1 ; \lambda_{n}^{2}=e^{2 R L}-1 ; \\
& m=2 D_{11}^{*} \lambda_{p} ; n=2 D_{11}^{*} \lambda_{n} ; \mathrm{Z}=L\left(C_{2} C_{5} / D_{11}^{*}-C_{4}\right)-C_{2} L^{3} /\left(6 D_{11}^{*}\right) ; \\
& \alpha_{2}=2 \Omega n+\Pi L n ; \alpha_{8}=2 m \mathrm{Z}+C_{2} L^{3} \lambda_{p} ; \\
& \alpha_{1}=\alpha_{8}-\alpha_{2} ; \alpha_{3}=D_{11}^{*}\left(2 \Omega \lambda_{n}^{2}+\Pi L \lambda_{p}^{2}\right) ; \\
& \alpha_{4}=C_{2} C_{7} L^{2}\left(2 \Omega \lambda_{n}+\Pi L \lambda_{p}\right) ; \alpha_{5}=-4 D_{11}^{*} Z^{R L}-C_{2} L^{3} e^{R L}+2 D_{11}^{*} \Pi L e^{R L} ; \\
& \alpha_{6}=4 D_{11}^{*} \mathrm{Z} \lambda_{p}^{2}+C_{2} L^{3} \lambda_{p}^{2}-4 D_{11}^{*} \Pi L e^{R L} ; \alpha_{7}=\Omega \lambda_{n}+\Pi L \lambda_{p} ; \Delta=\alpha_{8}-2 \alpha_{2}
\end{aligned}
$$

## Appendix 2. RZT $_{(3,2)}^{(\mathbf{m})}$ - based beam finite element matrices

The strain matrix is given by

$$
\begin{gathered}
\mathbf{B}^{(e) T}=\left[\begin{array}{c}
\mathbf{N} \\
\partial_{1} \mathbf{N}
\end{array}\right]^{T}\left[\begin{array}{ccc}
\mathbf{0}_{3 \times 1} & \partial_{z} \mathbf{p}_{1} & \mathbf{0}_{3 \times 1} \\
\mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \partial_{z} \mathbf{H}_{w} \\
\mathbf{p}_{1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} \\
\mathbf{0}_{3 \times 1} & \mathbf{H}_{w} & \mathbf{0}_{3 \times 1}
\end{array}\right] \\
\mathbf{p}_{1}{ }^{T} \equiv\left\{L_{1}(z), L_{2}(z), \mu_{1}^{(k)}(z)\right\} ; \mathbf{H}_{w}{ }^{T} \equiv\left\{H_{b}^{w}(z), H_{t}^{w}(z), H_{a}^{w}(z)\right\}
\end{gathered}
$$

Here, the stress matrices for the $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$ beam element are defined

$$
\begin{aligned}
& \mathbf{S}^{(e) T}=\left[\begin{array}{c}
\mathbf{N} \\
\partial_{1} \mathbf{N}
\end{array}\right]^{T}\left[\begin{array}{ccc}
\mathbf{0}_{3 \times 1} & \mathbf{T}_{u} & \mathbf{0}_{3 \times 1} \\
S_{33}^{(k)} R_{11}^{(k)} \mathbf{A}_{w} & \mathbf{0}_{3 \times 1} & \mathbf{A}_{w} \\
C_{11}^{(k)} \mathbf{p}_{1}+S_{33}^{(k)} R_{11}^{(k)} \mathbf{A}_{u} & \mathbf{0}_{3 \times 1} & \mathbf{A}_{u} \\
\mathbf{0}_{3 \times 1} & \mathbf{T}_{w} & \mathbf{0}_{3 \times 1}
\end{array}\right] \\
& \mathbf{A}_{u}{ }^{T} \equiv\left\{A_{b}^{u}(z), A_{t}^{u}(z), A^{\psi}(z)\right\} ; \mathbf{A}_{w}{ }^{T} \equiv\left\{A_{b}^{w}(z), A_{t}^{w}(z), A_{a}^{w}(z)\right\} \\
& \mathbf{T}_{u}{ }^{T} \equiv\left\{T_{b}^{u}(z), T_{t}^{u}(z), T^{\psi}(z)\right\} ; \mathbf{T}_{w}{ }^{T} \equiv\left\{T_{b}^{w}(z), T_{t}^{w}(z), T_{a}^{w}(z)\right\} \\
& \mathbf{S}_{f}^{(e)}=\left[\begin{array}{ccc}
\mathbf{0}_{1 \times 2} & S_{33}^{(k)} R_{11}^{(k)} \mathbf{A}^{q z}(z) & \mathbf{0}_{1 \times 2} \\
\mathbf{T}_{p}(z) & \mathbf{0}_{1 \times 2} & \mathbf{T}_{q}(z) \\
\mathbf{0}_{1 \times 2} & \mathbf{A}^{q z}(z) & \mathbf{0}_{1 \times 2}
\end{array}\right] ; \quad \mathbf{f}_{e}{ }^{T}=\left\{\begin{array}{lll}
\mathbf{n}_{\mathbf{v}}{ }^{T} & \mathbf{q}_{\mathbf{z}}{ }^{T} & \partial_{1} \mathbf{q}_{\mathbf{z}}{ }^{T}
\end{array}\right\} \\
& \mathbf{S}_{\Delta T}^{(e)}=\left[\begin{array}{cc}
S_{33}^{(k)} R_{11}^{(k)} \mathbf{A}_{\Delta T}(z)-S_{33}^{(k)} \omega_{11}^{(k)} \mathbf{H}_{\Delta T}(z) & 0 \\
0 & -\mathbf{T}_{\Delta T}(z) \\
\mathbf{A}_{\Delta T}(z) & 0
\end{array}\right]\left[\begin{array}{c}
\mathbf{N}_{\Delta T}\left(x_{1}\right) \\
\partial_{1} \mathbf{N}_{\Delta T}\left(x_{1}\right)
\end{array}\right]
\end{aligned}
$$

and the following definitions are used

$$
\begin{aligned}
& \mathbf{A}_{\Delta T}(z)=\mathbf{P}\left\langle\mathbf{P}^{T} S_{33}^{(k)} \mathbf{P}\right\rangle^{-1}\left\langle\mathbf{P}(z)^{T} S_{33}^{(k)} \lambda_{33}^{(k)} \mathbf{H}_{\Delta T}(z)\right\rangle \\
& \mathbf{T}_{\Delta T}(z)=\mathbf{Z}_{\Delta T}(z)-\mathbf{Z}_{\mathbf{f}}(z)\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\mathbf{f}}(z)\right\rangle^{-1}\left\langle\mathbf{Z}_{\mathbf{f}}(z)^{T} D_{T} \mathbf{Z}_{\Delta T}(z)\right\rangle \\
& \mathbf{Z}_{\Delta T}(z)=\int_{-h}^{z}\left(S_{33}^{(k)} R_{11}^{(k)} \mathbf{A}_{\Delta T}(\tilde{z})+S_{33}^{(k)} \omega_{11}^{(k)} \mathbf{H}_{\Delta T}(z)\right) d \tilde{z}- \\
& \left\langle C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{u}\right\rangle^{-1}\left(S_{33}^{(k)} R_{11}^{(k)} \mathbf{A}_{\Delta T}+S_{33}^{(k)} \omega_{11}^{(k)} \mathbf{H}_{\Delta T}(z)\right) \int_{-h}^{\tilde{z}}\left(C_{11}^{(k)}+S_{33}^{(k)} R_{11}^{(k)} A^{u}\right) d \tilde{z}
\end{aligned}
$$

The matrix involved in the definition of the mass matrix reads as

$$
\mathbf{H}(z)=\left[\begin{array}{cc}
\mathbf{p}_{1} & \mathbf{0}_{1 \times 3} \\
\mathbf{0}_{1 \times 3} & \mathbf{H}_{w}
\end{array}\right]
$$

## Appendix 3. $\mathrm{RZT}_{(3,2)}^{(\mathbf{m})}$ - based plate finite element matrices

Here, the matrices and vectors involved in the definition of the mass and stiffness matrices are introduced.

$$
\begin{gathered}
\mathbf{B}^{(e) T}=\left[\begin{array}{c}
\mathbf{N} \\
\partial_{1} \mathbf{N} \\
\partial_{2} \mathbf{N}
\end{array}\right]^{T}\left[\begin{array}{cccccc}
\mathbf{0}_{6 \times 1} & \mathbf{0}_{6 \times 1} & \mathbf{0}_{6 \times 1} & \partial_{z} \mathbf{p}_{1} & \partial_{z} \mathbf{p}_{2} & \mathbf{0}_{6 \times 1} \\
\mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \partial_{z} \mathbf{H}_{w} \\
\mathbf{p}_{1} & \mathbf{0}_{6 \times 1} & \mathbf{p}_{2} & \mathbf{0}_{6 \times 1} & \mathbf{0}_{6 \times 1} & \mathbf{0}_{6 \times 1} \\
\mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{H}_{w} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} \\
\mathbf{0}_{6 \times 1} & \mathbf{p}_{1} & \mathbf{p}_{2} & \mathbf{0}_{6 \times 1} & \mathbf{0}_{6 \times 1} & \mathbf{0}_{6 \times 1} \\
\mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{H}_{w} & \mathbf{0}_{3 \times 1}
\end{array}\right] \\
\mathbf{p}_{1}^{T} \equiv\left\{L_{1}(z), 0, L_{2}(z), 0, \mu_{1}^{(k)}(z), 0\right\} ; \mathbf{p}_{2}{ }^{T} \equiv\left\{0, L_{1}(z), 0, L_{2}(z), 0, \mu_{2}^{(k)}(z),\right\} \\
\mathbf{H}_{w}{ }^{T} \equiv\left\{H_{b}^{w}(z), H_{t}^{w}(z), H_{a}^{w}(z)\right\}
\end{gathered}
$$

The stress matrices are defined as

$$
\begin{aligned}
& \mathbf{S}^{(e) T}=\left[\begin{array}{c}
\mathbf{N} \\
\partial_{1} \mathbf{N} \\
\partial_{2} \mathbf{N}
\end{array}\right]^{T}\left[\begin{array}{cccccc}
\mathbf{0}_{3 \times 6} & S_{33}^{(k)} \mathbf{R} \mathbf{A}_{w} & \mathbf{S}_{1}+S_{33}^{(k)} \mathbf{R} \mathbf{A}_{1} & \mathbf{0}_{3 \times 3} & \mathbf{S}_{2}+S_{33}^{(k)} \mathbf{R} \mathbf{A}_{2} & \mathbf{0}_{3 \times 3} \\
\mathbf{T}_{p} & \mathbf{0}_{2 \times 3} & \mathbf{0}_{2 \times 6} & \mathbf{T}_{w 1} & \mathbf{0}_{2 \times 6} & \mathbf{T}_{w 2} \\
\mathbf{0}_{1 \times 6} & \mathbf{A}_{w} & \mathbf{A}_{1} & \mathbf{0}_{1 \times 3} & \mathbf{A}_{2} & \mathbf{0}_{1 \times 3}
\end{array}\right]^{T} \\
& \mathbf{R}^{T} \equiv\left\{R_{11}^{(k)}, R_{22}^{(k)}, R_{12}^{(k)}\right\} ; \mathbf{A}_{w} \equiv\left\{A_{b}^{w}(z), A_{t}^{w}(z), A_{a}^{w}(z),\right\} \\
& \mathbf{T}_{p} \equiv\left[\mathbf{T}_{b}^{u}(z), \mathbf{T}_{t}^{u}(z), \mathbf{T}^{\mu}(z)\right] \\
& \mathbf{T}_{w 1} \equiv\left[\begin{array}{ccc}
\mathbf{T}_{b(1,1)}^{w} & \mathbf{T}_{t(1,1)}^{w} & \mathbf{T}_{a(1,1)}^{w} \\
\mathbf{T}_{b(2,1)}^{w} & \mathbf{T}_{t(2,1)}^{w} & \mathbf{T}_{a(2,1)}^{w}
\end{array}\right] ; \mathbf{T}_{w 2} \equiv\left[\begin{array}{ccc}
\mathbf{T}_{b(1,2)}^{w} & \mathbf{T}_{t(1,2)}^{w} & \mathbf{T}_{a(1,2)}^{w} \\
\mathbf{T}_{b(2,2)}^{w} & \mathbf{T}_{t(2,2)}^{w} & \mathbf{T}_{a(2,2)}^{w}
\end{array}\right] \\
& \mathbf{A}_{1} \equiv\left\{A_{11}^{b}(z), A_{21}^{b}(z), A_{11}^{t}(z), A_{21}^{t}(z), A_{11}^{\mu}(z), A_{21}^{\mu}(z)\right\} \\
& \mathbf{A}_{2} \equiv\left\{A_{12}^{b}(z), A_{22}^{b}(z), A_{12}^{t}(z), A_{22}^{t}(z), A_{12}^{\psi}(z), A_{22}^{\psi}(z)\right\} \\
& \mathbf{S}_{1} \equiv\left[\begin{array}{ccc}
C_{11}^{(k)} & C_{12}^{(k)} & C_{16}^{(k)} \\
C_{12}^{(k)} & C_{22}^{(k)} & C_{26}^{(k)} \\
C_{16}^{(k)} & C_{26}^{(k)} & C_{66}^{(k)}
\end{array}\right]\left[\begin{array}{cccccc}
L_{1}(z) & 0 & L_{2}(z) & 0 & \mu_{1}^{(k)}(z) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_{1}(z) & 0 & L_{2}(z) & 0 & \mu_{2}^{(k)}(z)
\end{array}\right] \\
& \mathbf{S}_{2} \equiv\left[\begin{array}{lll}
C_{11}^{(k)} & C_{12}^{(k)} & C_{16}^{(k)} \\
C_{12}^{(k)} & C_{22}^{(k)} & C_{26}^{(k)} \\
C_{16}^{(k)} & C_{26}^{(k)} & C_{66}^{(k)}
\end{array}\right]\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_{1}(z) & 0 & L_{2}(z) & 0 & \mu_{2}^{(k)}(z) \\
L_{1}(z) & 0 & L_{2}(z) & 0 & \mu_{1}^{(k)}(z) & 0
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{S}_{f}^{(e)}=\left[\begin{array}{ccc}
\mathbf{0}_{3 x 4} & S_{33}^{(k)} \mathbf{R A}^{\mathbf{q z}}(z) & \mathbf{0}_{3 x 4} \\
\mathbf{T}_{p}(z) & \mathbf{0}_{2 \times 2} & \mathbf{T}_{q}(z) \\
\mathbf{0}_{3 x 4} & \mathbf{A}^{\mathbf{q z}}(z) & \mathbf{0}_{3 x 4}
\end{array}\right] \\
& \mathbf{f}_{e}{ }^{T} \equiv\left\{\mathbf{n}_{\mathbf{v}}{ }^{T}, \mathbf{q}_{\mathbf{z}}{ }^{T}, \mathbf{q}^{T}\right\}
\end{aligned}
$$

The matrix involved in the definition of the mass matrix reads as

$$
\mathbf{H}(z)=\left[\begin{array}{ll}
\mathbf{p}_{1}{ }^{T} & \mathbf{0}_{1 \times 3} \\
\mathbf{p}_{2}{ }^{T} & \mathbf{0}_{1 \times 3} \\
\mathbf{0}_{1 \times 6} & \mathbf{H}_{w}{ }^{T}
\end{array}\right]
$$

## Chapter 6

## Analytical results

## 1. Introduction

In this Chapter, for the first time in the open literature, the response prediction capabilities of the RZT model on the undamped free vibration and critical load analyses of panels are assessed. The relevant mismatch in the mechanical properties between two adjacent layers, namely the face and the core, makes the analysis of sandwich panels challenging for every theory. Thus, particular attention is focused on these structures along with widely used cross-ply laminates, both symmetric and non-symmetric. For comparison purposes, the exact Elasticity solution as derived by Pagano [Pagano, 1970] is taken as reference. When not available, high-fidelity FE models or results reported in the open literature play the role of reference solution. Particular attention is focused on the FirstOrder Shear Deformation Theory (FSDT). As explained in Chapter 2, the RZT model assumes the FSDT kinematics as its baseline to which the zigzag contribution is added. From this point of view, it results very interesting the comparison between the RZT and FSDT in order to appreciate the effect of the inclusion of zigzag effect. Moreover, in this context, a comparison between the RZT zigzag function and the Murakami's one [Murakami, 1986] is carried out. Finally, the linear bending and free vibration problems
involving functionally graded sandwich panels are solved in order to investigate the RZT modeling capabilities of these advanced composites.

Later, the same linear bending and free vibration problems solved with the RZT are retaken into consideration in order to assess the capabilities of the mixed RZT model, aka $\mathrm{RZT}^{(\mathrm{m})}$, in enhancing the transverse shear stiffness and stresses prediction with respect to the traditional RZT model. In this context, the comparison of the two transverse shear stresses modeling strategies is performed highlighting the advantages and drawbacks.

Finally, the (3,2)-Mixed Refined Zigzag Theory, quoted as $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$, is assessed on elasto-static problems pertaining the bending of cross-ply panels with two, three and fourlayers, both symmetric and unsymmetric. The test cases are chosen in order to highlight the peculiarities of the model: the capabilities to capture the actual non-linear distribution of in-plane displacements along the thickness of thick laminates, the inclusion of a non constant distribution along the thickness of the transverse displacement and the transverse normal deformability effect, the accurate transverse shear stresses prediction.

In this Chapter, only exact solutions or approximated ones, obtained by using the Rayleigh-Ritz's method are considered, postponing the discussion concerning FE solutions to next Chapter.

Some of the contents of the present Chapter have been item of publications on International Journals [Iurlaro et al., 2014b] and proceedings of International Conferences [Iurlaro et al., 2013b; Iurlaro et al., 2014a,c]. In this Chapter, the already published results are re-taken into consideration and extended.

## 2. RZT assessment

To assess the accuracy of the Refined Zigzag Theory for the analysis of multilayered composite and sandwich structures, both orthotropic and functionally graded, the linear boundary value problem of bending and the linear eigenvalues problems of free vibration and buckling of rectangular plates, both simply supported and clamped (along one or more edges), are considered. The rectangular plates are defined in the Cartesian domain $x_{1} \in[0, a], x_{2} \in[0, b], z \in[-h, h]$. Mechanical material properties, as well as stacking sequences, taken into consideration are listed in Appendix 1.

### 2.1. Linear Bending

The numerical results presented in this section refer to the linear boundary value problem of multilayered composite, both traditional and functionally graded, sandwich plates, fully simply supported and/or clamped along one edge, and subjected to bisinusoidal or a uniform pressure. In the framework of RZT, the governing equations of the problem are Eqs. (2.32).

Problem 1. A rectangular $(b=3 a)$ simply supported plates subjected to bi-sinusoidal transverse pressure, $q\left(x_{1}, x_{2}\right)=q_{0} \sin \left(\pi x_{1} / a\right) \sin \left(\pi x_{2} / b\right)$.

For this problem, an exact Elasticity solution is available and is used as reference in the comparisons. The simply supported boundary conditions read

$$
\begin{align*}
& x_{1}=0, a: u_{2}=w=\theta_{2}=\psi_{2}=N_{1}=M_{1}=M_{1}^{\phi}=0 \\
& x_{2}=0, b: u_{1}=w=\theta_{1}=\psi_{1}=N_{2}=M_{2}=M_{2}^{\phi}=0 \tag{6.1}
\end{align*}
$$

and the exact solution is given by the following trigonometric expansion [Tessler et al., 2010a,b]

$$
\begin{align*}
U_{z} & =W \sin \left(\frac{\pi x_{1}}{a}\right) \sin \left(\frac{\pi x_{2}}{b}\right) \\
\left\{u_{1}, \theta_{1}, \psi_{1}\right\} & =\left\{U_{1}, \Theta_{1}, \Psi_{1}\right\} \cos \left(\frac{\pi x_{1}}{a}\right) \sin \left(\frac{\pi x_{2}}{b}\right)  \tag{6.2}\\
\left\{u_{2}, \theta_{2}, \psi_{2}\right\} & =\left\{U_{2}, \Theta_{2}, \Psi_{2}\right\} \sin \left(\frac{\pi x_{1}}{a}\right) \cos \left(\frac{\pi x_{2}}{b}\right)
\end{align*}
$$

where $\left(U, V, W, \Theta_{1}, \Theta_{2}, \Psi_{1}, \Psi_{2}\right)$ are the unknowns amplitudes of the kinematic variables which are determined from the satisfaction of the equilibrium equations. For comparison purposes, analytic solutions are also obtained using the First-Order Shear Deformation Theory (FSDT) with different shear correction factors: at first, the unit value is used for both factors $k_{1}{ }^{2}$ and $k_{2}{ }^{2}$, then values of the shear correction factors are estimated by extending to the plate case (and assuming the cylindrical bending hypothesis) the procedure proposed in [Raman et al., 1996] in the framework of laminated beams. In order to investigate the influence of the zigzag function, the Murakami's zigzag function is implemented in the framework of a first-order displacement-based zigzag model that, as highlighted in Chapter 2, gives the same governing equations of the RZT. The solutions obtained by means of the first-order zigzag model adopting the Murakami’s zigzag function is quoted as MZZ.

Three cross-ply laminates (laminate L1, L2 and L9; see Table A1.3) are considered and the non-dimensional transverse displacement at the center of the plate, for different span-to-thickness ratio $a / 2 h$, is computed and the results obtained by using the RZT, MZZ and the FSDT are compared with the 3D Elasticity solution (see Tables 1-3).

Results collected in Tables 1-3 show that the FSDT solutions computed by using the unit value for the shear correction factors is very stiff, if compared with the Elasticity solution, above all when the plate is thick. By increasing the span-to-thickness ratio, the FSDT solution converges to the reference due to the reduced influence of the transverse shear strain. By adopting a suitable value of the shear correction factor, the FSDT solution improves the accuracy even in the regime of thick laminates but overestimating the maximum deflection of around $2 \%$ for laminate L1 and $13 \%$ for laminate L2 and L9, when $a / 2 h=8$. The RZT solution matches very well with the Elasticity one over the entire range of span-to-thickness ratios considered with an error of $2 \%$ for laminate $\mathrm{L} 1,0.2 \%$ for laminate L2 and $0.8 \%$ for laminate L9, when $a / 2 h=8$. The MZZ solution behaves according to the laminate considered. Two scenarios are available: the first one (Tables 1 and 2), wherein the MZZ gives the same RZT results, thus providing accurate results. The second one (Table 3), wherein the MZZ results are between the RZT and FSDT ones (with unit shear correction factors), thus underestimating the reference solution. Since the framework of the RZT and the MZZ model differs only for the zigzag function adopted, the reason of the worst behavior of the MZZ model is up to the Murakami's zigzag function. Laminates L1, L2 and L9 give rise to three different situations: laminate L1 (twolayers, unsymmetric), due to the stacking sequence, is almost devoid of zigzag effect (see Figures 6.1-6.2) and the consequence is that the choice of the zigzag function does not affect the results, both in terms of global (maximum transverse displacements) and local response (through-the-thickness distributions of displacements and stresses). Laminate L2 (three-layers, symmetric) originates an in-plane displacements distribution along the thickness that is periodic (that means a recurring alternation of two layers). For this reason, the Murakami's zigzag function ensures accurate results, that fit with the RZT ones. For the unsymmetric stacking sequence (laminate L9), the Murakami's zigzag function, that is the MZZ model, provides results that are not accurate if compared with the reference solution due to the non periodic structure of the stacking sequence. Results for laminate L2 and L9 confirm what Murakami himself stated in his work [Toledano et al., 1987]: "The inclusion of the zig-zag shaped $C^{0}$ function was motivated by the displacement microstructure of periodic laminated composites [..]. Obviously, for arbitrary laminate

Table 1. Problem 1, Laminate L1: normalized maximum deflection, $\bar{w}=\left(10^{2} D_{11} / q_{0} a^{4}\right) w(a / 2, b / 2) ; k^{2}$ is the shear correction factor and $D_{l l}$ is the first element of the bending stiffness matrix.

| $a / h$ | 3D Elasticity | RZT | MZZ | FSDT | FSDT |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  | $k_{1}{ }^{2}=k_{2}{ }^{2}=1$ | $k_{l}{ }^{2}=0.5011$ <br> $k_{2}{ }^{2}=0.1632$ |
| 8 | 2.891 | 2.820 | 2.820 | 2.604 | 2.965 |
| 10 | 2.682 | 2.633 | 2.633 | 2.486 | 2.717 |
| 20 | 2.383 | 2.369 | 2.369 | 2.328 | 2.386 |
| 50 | 2.293 | 2.291 | 2.291 | 2.284 | 2.294 |
| 100 | 2.280 | 2.280 | 2.280 | 2.278 | 2.280 |

Table 2. Problem 1, Laminate L2: normalized maximum deflection, $\bar{w}=\left(10^{2} D_{11} / q_{0} a^{4}\right) w(a / 2, b / 2) ; k^{2}$ is the shear correction factor and $D_{l l}$ is the first element of the bending stiffness matrix.

| $a / h$ | 3D Elasticity | RZT | MZZ | FSDT | FSDT |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  | $k_{l}{ }^{2}=k_{2}{ }^{2}=1$ | $k_{1}{ }^{2}=0.1212$ <br> $k_{2}{ }^{2}=0.3438$ |
| 8 | 4.990 | 4.978 | 4.978 | 1.575 | 5.625 |
| 10 | 3.706 | 3.699 | 3.699 | 1.369 | 3.987 |
| 20 | 1.737 | 1.735 | 1.735 | 1.093 | 1.757 |
| 50 | 1.121 | 1.121 | 1.121 | 1.015 | 1.122 |
| 100 | 1.031 | 1.031 | 1.031 | 1.004 | 1.031 |

Table 3. Problem 1, Laminate L9: normalized maximum deflection, $\bar{w}=\left(10^{2} D_{11} / q_{0} a^{4}\right) w(a / 2, b / 2) ; k^{2}$ is the shear correction factor and $D_{l l}$ is the first element of the bending stiffness matrix.

| $a / h$ | 3D Elasticity | RZT | MZZ | FSDT | FSDT |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  | $k_{l}{ }^{2}=k_{2}{ }^{2}=1$ | $k_{1}{ }^{2}=0.1260$ <br> $k_{2}{ }^{2}=0.1632$ |
| 8 | 3.286 | 3.259 | 2.545 | 1.474 | 3.743 |
| 10 | 2.603 | 2.589 | 2.065 | 1.353 | 2.815 |
| 20 | 1.543 | 1.541 | 1.380 | 1.192 | 1.562 |
| 50 | 1.205 | 1.205 | 1.178 | 1.147 | 1.207 |
| 100 | 1.155 | 1.155 | 1.148 | 1.141 | 1.156 |

configurations, this periodicity is destroyed.". Nevertheless the observation made by Murakami and the results provided by himself in [Toledano et al., 1987], along with the substantial investigation recently carried on by Gherlone [Gherlone, 2013], the use of the Murakami's zigzag function abound in the open-literature (see Chapter 1) even in the case of arbitrary stacking sequence laminates. The Refined Zigzag Theory function, being inspired by the original works made by Di Sciuva, is mechanical-properties dependent, thus is able to accommodate the actual in-plane displacements distribution along the thickness of any laminates, regardless the stacking sequence. On the contrary, the Murakami's zigzag function is mechanical-properties independent and it provides change in slope of in-plane displacements at the interface between two adjacent layers that is always the same in amplitude, but opposite in sign, violating the actual in-plane displacements distribution (see Figures 6.7-6.8, 6.13-6.14).

Figures 6.1-6.6 show a comparison of the through-the-thickness distribution of normalized in-plane displacements and stresses for laminate L1, with a span-to-thickness ratio $a / 2 h=8$.

Due to the stacking sequence, laminate L1 produces a distribution of in-plane displacements along the thickness with an almost vanishing zigzag effect. For this reason, the FSDT provides relatively accurate results, with a slight overestimation of in-plane displacements and underestimation of in-plane normal stresses. The transverse shear stresses (Figures 6.5 and 6.6) are derived by integration of the local equilibrium equations (for this reason are quoted with $(E E)$ ), and fit very well with the Elasticity solution. The RZT and MZZ provide the same accurate results, since in this case, the zigzag effect plays a limited role and the results are not affected by the choice of the zigzag function.

Figures 6.7-6.12 compare the results for an unsymmetric three-layers sandwich plate (laminate L3, see Table A1.3). As the Elasticity solution suggests, the actual distribution of in-plane displacements for a sandwich stacking sequence is strongly affected by the zigzag effect as a consequence of the significant mismatch of mechanical properties between the faces and the core. The requirement of continuity of transverse shear stresses at face-core interface is fulfilled with a sizeable jump in the slope of the first-order derivative with respect to the thickness coordinate of the in-plane displacements. The ESL nature of the FSDT is the reason for a not accurate local response, both in terms of displacements and stresses. The MZZ model is not able to fit with the Elasticity solution as consequence of the Murakami's zigzag function. On the contrary, the RZT results are in agreement with the reference solution, both in terms of displacements and stresses.


Figure 6.1. Problem 1, Laminate L1, $a / 2 h=8$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{1}^{(k)}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{1}^{(k)}$.


Figure 6.2. Problem 1, Laminate L1, $a / 2 h=8$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{2}^{(k)}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{2}^{(k)}$.


Figure 6.3. Problem 1, Laminate L1, $a / 2 h=8$ : through-the-thickness distribution of normalized in-plane normal stress, $\bar{\sigma}_{11}=\left(4 h^{2} / q_{0} a^{2}\right) \sigma_{11}^{(k)}$.


Normalized in-plane normal stress, $\bar{\sigma}_{22}^{(k)}(a / 2, b / 2, z)$

Figure 6.4. Problem 1, Laminate L1, $a / 2 h=8$ : through-the-thickness distribution of normalized in-plane normal stress, $\bar{\sigma}_{22}=\left(4 h^{2} / q_{0} a^{2}\right) \sigma_{22}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(0, b / 2, z)$

Figure 6.5. Problem 1, Laminate L1, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}=\left(2 h / q_{0} a^{2}\right) \tau_{1 z}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{2 z}^{(k)}(a / 2,0, z)$

Figure 6.6. Problem 1, Laminate L1, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{2 z}=\left(2 h / q_{0} a^{2}\right) \tau_{2 z}^{(k)}$.


Figure 6.7. Problem 1, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{1}^{(k)}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{1}^{(k)}$.


Figure 6.8. Problem 1, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{2}^{(k)}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{2}^{(k)}$.


Figure 6.9. Problem 1, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized in-plane normal stress, $\bar{\sigma}_{11}=\left(4 h^{2} / q_{0} a^{2}\right) \sigma_{11}^{(k)}$.


Figure 6.10. Problem 1, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized in-plane normal stress, $\bar{\sigma}_{22}=\left(4 h^{2} / q_{0} a^{2}\right) \sigma_{22}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(0, b / 2, z)$

Figure 6.11. Problem 1, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}=\left(2 h / q_{0} a^{2}\right) \tau_{1 z}^{(k)}$.


Figure 6.12. Problem 1, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{2 z}=\left(2 h / q_{0} a^{2}\right) \tau_{2 z}^{(k)}$.

Figures 6.13-6.16 compare the RZT, MZZ and FSDT solutions with the exact Elasticity one for a symmetric five-layers sandwich plate (laminate L4, see Table A1.3). The actual in-plane displacements distribution along the thickness show a slope change at the interface between face and core, whereas the interface between the two plies of each face are devoid of discontinuity on the first-order derivative with respect to the thickness coordinate. The reason relies on the equal values assumed by the shear moduli of the material used for the faces: in this situation, the in-plane displacement distribution does not exhibit any slope change. From a zigzag model point of view, this behavior can be captured only if the zigzag function takes into account the mechanical properties of the plies; otherwise the result may be not in agreement with the Elasticity solution. In fact, the RZT model is very accurate if compared with the Pagano's solution whereas the MZZ model provides the same results as the FSDT adopting a unit value for the shear correction factors. The use of a not suitable zigzag function leads the variational statement to give a near-zero zigzag amplitude that let the zigzag contribution vanish, thus keeping only the FSDT kinematics alive. For this reason, the MZZ results are the same of the FSDT (with $k_{1}^{2}=k_{2}^{2}=1$ ) ones.

The results discussed in this section reveal the major drawback of the Murakami's zigzag function, that is its inability to provide accurate response (sometimes also with significant errors) for stacking sequences different from the periodic and symmetric ones. When the Murakami's zigzag function is adopted to solve a problem involving arbitrary stacking sequences, the variational principle, that is the Virtual Work Principle, gives a zigzag amplitude that is near-zero thus leading the zigzag contribution of the MZZ model to vanish. As a result, the MZZ model reduces to the FSDT kinematics with a unit shear correction factor. For this reason, the MZZ model and the FSDT ones provide the same, both global and local, results. The RZT model adopts a zigzag function mechanicalproperties dependent, thus is able to accommodate the actual in-plane displacement distributions of any kind of stacking sequences.

Problem 2. A square cantilever plate is subjected to transversal pressure $q\left(x_{1}, x_{2}\right)=q_{0}$.
For this set of load and boundary conditions the exact RZT solution does not exist and an approximate one is computed by using the Rayleigh-Ritz's method. The kinematic variables are approximated in the following way

$$
\begin{equation*}
\left(u_{\alpha}, w, \theta_{\alpha}, \psi_{\alpha}\right)=\sum_{m=1}^{M} \sum_{p=1}^{P}\left(U_{\alpha_{n p}}, W_{m p}, \Theta_{\alpha_{n p}}, \Psi_{\alpha_{n p}}\right) \chi_{m}\left(x_{1}\right) \chi_{p}\left(x_{2}\right) \tag{6.3}
\end{equation*}
$$



Figure 6.13. Problem 1, Laminate $\mathrm{L} 4, a / 2 h=8$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{1}^{(k)}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{1}^{(k)}$.


Figure 6.14. Problem 1, Laminate $\mathrm{L} 4, a / 2 h=8$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{2}^{(k)}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{2}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(0, b / 2, z)$

Figure 6.15. Problem 1, Laminate L4, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}=\left(2 h / q_{0} a^{2}\right) \tau_{1 z}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{2 z}^{(k)}(a / 2,0, z)$

Figure 6.16. Problem 1, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{2 z}=\left(2 h / q_{0} a^{2}\right) \tau_{2 z}^{(k)}$.
where $\chi_{m}\left(x_{1}\right), \chi_{p}\left(x_{2}\right)$ are the Gram-Schmidt polynomials built to satisfy the geometric boundary conditions: for the particular expression of these polynomials refer to [Tessler et al., 2010a,b]. In order to compare the RZT, MZZ and FSDT results, a high-fidelity FEM (MSC/MD-NASTRAN ${ }^{\text {® }}$ ) solution is used. The model is regularly discretized using linearstrain solid elements, HEXA8. There are sixty-five elements along each span direction, five elements through the thickness of the bottom face, eight elements along the top face and fifteen through the core thickness. Analytical FSDT and MZZ solutions are computed as for the RZT one.

Figures 6.17 and 6.18 report the through-the-thickness distribution of the relevant inplane displacement and transverse shear stress: RZT preserves its accuracy also for the clamped boundary condition whereas the MZZ model provides results not accurate as for the simply supported boundary condition (see Figures 6.7-6.12). Even adopting the shear correction factors, the FSDT results overestimates the maximum in-plane displacement and heavily underestimates the maximum transverse shear stress.

Problem 3. A square, simply supported, functionally graded Type A (see Appendix 2) sandwich plate subjected to a bi-sinusoidal pressure, $q\left(x_{1}, x_{2}\right)=q_{0} \sin \left(\pi x_{1} / a\right) \sin \left(\pi x_{2} / b\right)$. Two different values of the grading index, $k$, and several values of $a / 2 h$ are considered.

For this set of load and boundary conditions, the Elasticity solution, as derived by Pagano [Pagano, 1970] and obtained by decomposition of the functionally graded layer with several sub-layers, is available and it is assumed as reference in the comparisons. For comparison purposes, the analytical FSDT solution with appropriate shear correction factors is estimated along with the analytical Third-Order Shear Deformation Theory (TSDT) [Reddy, 2000] one.

Results reported in Table 4 are normalized according to the following relations

$$
\begin{array}{rlrl}
\bar{U}_{1} & =10^{3} \frac{E_{T} U_{1}(0, b / 2, h)}{2 \bar{q}_{0} h(a / 2 h)^{3}} ; & \bar{U}_{z}=10^{3} \frac{E_{T} U_{z}(a / 2, b / 2)}{2 \bar{q}_{0} h(a / 2 h)^{3}} ; \\
\left(\bar{\sigma}_{11}, \bar{\sigma}_{22}\right) & =\frac{\left(\sigma_{11}, \sigma_{22}\right)}{\bar{q}_{0}(a / 2 h)^{2}}(a / 2, b / 2, h) ; & \bar{\tau}_{1 z}=\frac{1}{\bar{q}_{0}(a / 2 h)} \tau_{1 z}(0, b / 2,0)  \tag{6.4}\\
\bar{\tau}_{12}=\frac{1}{\bar{q}_{0}(a / 2 h)^{2}} \tau_{12}(0,0, h) ; & \bar{\tau}_{2 z}=\frac{1}{\bar{q}_{0}(a / 2 h)} \tau_{2 z}(a / 2,0,0)
\end{array}
$$

The RZT is able to provide accurate response predictions (for both displacements and stresses) if compared with the reference solution for all the values of the span-to-thickness ratio and grading index considered. The FSDT with the appropriate shear correction factors


Normalized in-plane displacement, $\bar{U}_{1}^{(k)}(a, b / 2, z)$

Figure 6.17. Problem 2, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{1}^{(k)}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{1}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(a / 10, b / 2, z)$

Figure 6.18. Problem 2, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}=\left(2 h / q_{0} a^{2}\right) \tau_{1 z}^{(k)}$.

Table 4. Problem 3, Type A. Comparison of normalized local responses (see Eqs. (6.4)).The FSDT solution adopts $k_{l}{ }^{2}=k_{2}{ }^{2}=0.0854(k=5)$ and $k_{l}{ }^{2}=k_{2}^{2}=0.3(k=3)$, as shear correction factors.

| $k$ | $a / 2 h$ | Model | $\bar{U}_{z}$ | $\bar{U}_{1}$ | $\bar{\sigma}_{11}$ | $\bar{\sigma}_{22}$ | $\bar{\tau}_{12}$ | $\bar{\tau}_{1 z}$ | $\bar{\tau}_{2 z}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 8 | Elasticity | 9.667 | -0.3045 | 3.641 | 0.342 | -0.226 | 0.212 | 0.069 |
|  |  | RZT | 9.634 | -0.3034 | 3.543 | 0.334 | -0.221 | 0.210 | 0.068 |
|  |  | TSDT | 5.612 | -0.1987 | 2.386 | 0.271 | -0.172 | 0.267 | 0.069 |
|  |  | FSDT | 11.707 | -0.1072 | 1.294 | 0.286 | -0.163 | 0.275 | 0.085 |
|  | 10 | Elasticity | 8.683 | -0.2521 | 2.997 | 0.301 | -0.196 | 0.236 | 0.068 |
|  |  | RZT | 8.671 | -0.2503 | 2.925 | 0.294 | -0.192 | 0.235 | 0.067 |
|  |  | TSDT | 4.956 | -0.1768 | 2.120 | 0.231 | $-0.148$ | 0.282 | 0.062 |
|  |  | FSDT | 9.860 | -0.1119 | 1.341 | 0.262 | -0.152 | 0.281 | 0.078 |
|  | 20 | Elasticity | 6.428 | -0.1704 | 2.035 | 0.185 | -0.124 | 0.291 | 0.051 |
|  |  | RZT | 6.431 | -0.1706 | 1.990 | 0.181 | -0.121 | 0.291 | 0.051 |
|  |  | TSDT | 4.023 | -0.1518 | 1.806 | 0.138 | -0.097 | 0.313 | 0.041 |
|  |  | FSDT | 6.622 | -0.1292 | 1.516 | 0.173 | -0.110 | 0.306 | 0.053 |
| 3 | 8 | Elasticity | 17.344 | -1.027 | 1.671 | 0.155 | -0.102 | 0.281 | 0.060 |
|  |  | RZT | 17.349 | -1.029 | 1.649 | 0.153 | -0.101 | 0.280 | 0.059 |
|  |  | TSDT | 15.305 | -0.914 | 1.486 | 0.142 | -0.094 | 0.303 | 0.058 |
|  |  | FSDT | 18.431 | -0.648 | 1.049 | 0.142 | -0.087 | 0.318 | 0.064 |
|  | 10 | Elasticity | 16.105 | -0.932 | 1.514 | 0.131 | -0.088 | 0.300 | 0.053 |
|  |  | RZT | 16.121 | -0.935 | 1.495 | 0.129 | -0.087 | 0.300 | 0.053 |
|  |  | TSDT | 14.234 | -0.855 | 1.386 | 0.120 | -0.081 | 0.317 | 0.051 |
|  |  | FSDT | 16.723 | -0.676 | 1.088 | 0.122 | -0.077 | 0.326 | 0.056 |
|  | 20 | Elasticity | 15.790 | -0.805 | 1.298 | 0.083 | -0.061 | 0.336 | 0.038 |
|  |  | RZT | 15.804 | -0.806 | 1.282 | 0.082 | -0.060 | 0.336 | 0.038 |
|  |  | TSDT | 14.625 | -0.784 | 1.263 | 0.078 | -0.058 | 0.342 | 0.037 |
|  |  | FSDT | 15.892 | -0.735 | 1.170 | 0.080 | -0.058 | 0.343 | 0.038 |

underestimates the transverse shear stiffness and, as consequence, overestimates the maximum deflection. Moreover, the FSDT underestimates the in-plane displacements and stresses, especially for low values of $a / 2 h$ and high values of grading index. The way to avoid the need of the shear correction factors is to adopt a higher-order kinematics, that is the Third-Order Shear Deformation Theory (TSDT) proposed by Reddy [Reddy, 2000]. Considering the results of Table 4, appears that the TSDT underestimates the maximum deflection, the in-plane displacements and stresses. The error decreases by increasing the span-to-thickness ratio and decreasing the grading index. The transverse shear stresses of Table 4 derive by integration of the equilibrium equations, thus a not accurate prediction of in-plane stresses affects the transverse shear stresses one. For this reason, the RZT is able to provide accurate estimations whereas the FSDT and TSDT models overestimate the shear stresses. Results in Table 8 show that the FSDT errors on the maximum deflection are lower than the TSDT ones (even if the former overestimates the deflection whereas the latter underestimates it) as results of the use of a shear correction factor that is more efficient than the inclusion of a third-order polynomial term in the in-plane displacements approximation.

Figures 6.19-6.24 show a comparison of the through-the-thickness distribution of normalized displacements and stresses for the Type A sandwich plate with $a / 2 h=8$ and a grading index $k=5$.

The behavior of a functionally graded sandwich plate is more complex than that of a traditional multilayered orthotropic composite one: the distribution of the in-plane displacements in the functionally graded face-sheets may exhibit non-linear trends, following in a certain way the distribution of mechanical properties along the functionally graded layer thickness. The FSDT provides through-the-thickness distributions of in-plane displacements and stresses not in agreement with the reference solution. In particular, maximum in-plane normal stresses are heavily underestimated. Due to the inaccuracy in predicting the in-plane normal stresses, the FSDT transverse shear stresses derived by integration of the local equilibrium equations do not fit with the exact solution. Moreover, the error on the in-plane stresses evaluation affects also the estimation of the shear correction factors, which result too low causing an underestimation of the transverse shear stiffness (see Table 4). Adopting a third-order kinematics for the in-plane displacements, the TSDT may reproduce the non-linear distribution of the displacement components. Even if the distribution is not linear, as appears from Figures 6.19 and 6.20, the TSDT is not able to fit with the reference solution and the maximum in-plane displacements and normal


Normalized in-plane displacement, $\bar{U}_{1}^{(k)}(0, b / 2, z)$

Figure 6.19. Problem 3, Type A, $a / 2 h=8, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) in-plane displacement.


Figure 6.20. Problem 3, Type A, $a / 2 h=8, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) in-plane displacement.


Normalized in-plane normal stress, $\bar{\sigma}_{11}^{(k)}(a / 2, b / 2, z)$

Figure 6.21. Problem 3, Type A, $a / 2 h=8, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) in-plane stress.


Normalized in-plane normal stress, $\bar{\sigma}_{22}^{(k)}(a / 2, b / 2, z)$

Figure 6.22. Problem 3, Type A, $a / 2 h=8, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) in-plane stress.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(0, b / 2, z)$

Figure 6.23. Problem 3, Type A, $a / 2 h=8, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) transverse shear stress.


Normalized transverse shear stress, $\bar{\tau}_{2 z}^{(k)}(a / 2,0, z)$

Figure 6.24. Problem 3, Type A, $a / 2 h=8, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) transverse shear stress.
stresses are underestimated. As a consequence, the transverse shear stresses coming from the equilibrium equations are not accurate if compared with the reference. On the contrary, RZT is able to provide accurate response predictions, both in terms of displacements and stresses.

The TSDT and RZT assume, as kinematic baseline, the FSDT; TSDT adds a cubic smeared term to the in-plane displacement approximation, whereas the RZT considers a piecewise linear contribution which becomes a non-linear one due to the actual distribution of the mechanical properties taking place in the functionally graded layer. Thus, while the TSDT adds a non-linear contribution independent of the mechanical properties, the RZT superimposes to the FSDT kinematics a non-linear contribution that depends on the through-the-thickness distribution of transverse shear stiffness moduli. On the contrary, the non-linear term in the TSDT cannot be greater than the third polynomial order. For this reason, the RZT prediction capabilities are superior to those of the TSDT.

The way the refined zigzag function mat be naturally and accurately modified for a wide range of stacking sequences is a remarkable properties of RZT.

Problem 4. A square, cantilevered, functionally graded sandwich plate subjected to a uniform transverse pressure, $q\left(x_{1}, x_{2}\right)=q_{0}$.

For this set of load and boundary conditions, the exact Elasticity solution is not available and a high-fidelity FE model, obtained by using MSC/MD-NASTRAN ${ }^{\circledR}$, is used as reference solution in the comparison. The model is regularly discretized by using the HEXA8, linear-strain element and is comprised of sixty elements along each span direction, three elements through the thickness of homogeneous layer (the core for Type A; the face-sheets for Type B) and twenty elements through the thickness of functionally graded layer (the face-sheets for Type A; the core for Type B), for a total number of 491,172 dof's, for the Type A, and 290,238 dof's, for the Type B. The mechanical properties associated to each element along the functionally graded thickness layers vary according to the grading law, in order to simulate it with a piece-wise constant function.

For this problem, the plate mostly experiences a cylindrical bending: Figures 6.25-6.28 show a comparison of the through-the-thickness distribution of prevalent normalized inplane displacement and transverse shear stress (derived by integration of the equilibrium equations $(E E)$ ), for the functionally graded sandwich plates Type $\mathrm{A}(k=5, a / 2 h=8$, Figures 6.25 and 6.26 ) and Type $B(k=5, a / 2 h=10$, Figures 6.27 and 6.28).


Normalized in-plane displacement, $\bar{U}_{1}^{(k)}(a, b / 2, z)$

Figure 6.25. Problem 4, Type A, $a / 2 h=8, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) in-plane displacement.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(a / 10, b / 2, z)$
Figure 6.26. Problem 4, Type A, $a / 2 h=8, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) transverse shear stress.


Normalized in-plane displacement, $\bar{U}_{1}^{(k)}(a, b / 2, z)$

Figure 6.27. Problem 4, Type B, $a / 2 h=10, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) in-plane displacement.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(a / 10, b / 2, z)$

Figure 6.28. Problem 4, Type B, $a / 2 h=10, k=5$ : through-the-thickness distribution of normalized (see Eqs. (6.4)) transverse shear stress.

The accuracy of the RZT in the response prediction is not affected by changing the load and boundary conditions, since the distributions of in-plane displacement and transverse shear stress fit with the reference solution. As for Problem 3, the FSDT and TSDT solutions still remain inaccurate and no improvements are reached changing the load and boundary conditions.

When Type B functionally graded sandwich plate is considered, the through-thethickness distribution of in-plane displacement (Figure 6.27) provided by FSDT, TSDT and RZT are very accurate. For this functionally graded sandwich plate, the grading core has only the effect of removing the discontinuity at the face-core interface leading to a smooth through-the-thickness distribution of transverse shear stress (Figure 6.28), thus reducing the risk of delamination failure. Each model is able to reproduce accurately the distribution of displacement since the face-to-core shear stiffness ratio is low, leading to a reduced zigzag effect. An approximated estimation of the face-to-core shear stiffness ratio may be performed considering as face stiffness the ceramic shear modulus and as core stiffness the average value estimated by

$$
\begin{equation*}
\bar{G}=\frac{1}{h_{c}} \int_{z_{2}}^{z_{3}}\left\{G_{m}+\left(G_{c}-G_{m}\right) V_{c}(z)\right\} d z \tag{6.5}
\end{equation*}
$$

The face-to-core stiffness ratio estimated as above explained is around 3, thus likely too low to make the zigzag effect pronunced. As a consequence, also models not accounting for the zigzag effect, that is the FSDT and TSDT, are able to provide accurate results.

### 2.2. Free vibrations

In this section, free-vibration analyses are conducted for simply supported and fully clamped square sandwich plates. Frequencies of undamped free vibration are the eigenvalues of Eqs. (2.33) and the related eigenvectors represent the corresponding modal shapes.

Problem 5. A simply supported, cross-ply square sandwich plate (laminate L5, see Table A1.3).

The simply support boundary condition and the cross-ply stacking sequence allow to find an exact solution for the RZT model and, in a similar way, for the FSDT and MZZ model. For this problem, the approximation of the RZT kinematic variables reads as

$$
\begin{align*}
U_{z} & =\sum_{m=1}^{M} \sum_{p=1}^{P} W_{m p} \sin \left(\frac{m \pi x_{1}}{a}\right) \sin \left(\frac{p \pi x_{2}}{b}\right) \sin \left(\omega_{m p} t\right) \\
\left\{u_{1}, \theta_{1}, \psi_{1}\right\} & =\sum_{m=1}^{M} \sum_{p=1}^{P}\left\{U_{1 m p}, \Theta_{1 m p}, \Psi_{1 m p}\right\} \cos \left(\frac{m \pi x_{1}}{a}\right) \sin \left(\frac{p \pi x_{2}}{b}\right) \sin \left(\omega_{m p} t\right)  \tag{6.6}\\
\left\{u_{2}, \theta_{2}, \psi_{2}\right\} & =\sum_{m=1}^{M} \sum_{p=1}^{P}\left\{U_{2 m p}, \Theta_{2 m p}, \Psi_{2 m p}\right\} \sin \left(\frac{m \pi x_{1}}{a}\right) \cos \left(\frac{p \pi x_{2}}{b}\right) \sin \left(\omega_{m p} t\right)
\end{align*}
$$

where $\omega_{m p}$ are the circular frequencies, related to the corresponding natural frequencies, $f_{m p}$, by the simple relation $\omega_{m p}=2 \pi f_{m p}$ and where $m$ and $p$ are the number of half-waves along the $x_{1^{-}}$and $x_{2}$-direction, respectively, of each mode shape. In Table 5, the first six non-dimensional circular frequencies obtained using RZT for two values of the span-tothickness ratio are presented and compared with other solutions available in literature. Solution quoted as LW [Rao et al., 2004a] is obtained using a layer-wise model wherein a cubic expansion in the thickness direction for the three displacement components is assumed and the continuity of transverse stresses at layer interfaces is ensured. The LW model is able to estimate accurate natural frequencies for laminated composite and sandwich plates, therefore its solution can be taken as a reference result in this comparison. Moreover, solution quoted as HSDT [Srinivas et al., 1970] is obtained by means of a higher-order ESL theory which assumes a cubic variation across the thickness for the three displacement components.

Both high-order and first-order displacement-based equivalent single layer models (HSDT and FSDT with $k_{1}{ }^{2}=k_{2}{ }^{2}=1,5 / 6$ ), highly overestimate the natural frequencies. This is due to the difference of mechanical properties between core and faces which causes an overestimation of the stiffness of the plate. When FSDT is used with $k_{1}{ }^{2}$ and $k_{2}{ }^{2}$ evaluated according to [Raman et al., 1996], although an improvement may be observed, frequencies are underestimated. The error is always below $2 \%$ when the plate is thin ( $a / 2 h=100$ ), whereas ranges from $7 \%$ (fundamental frequency) to $30 \%$ ( $6^{\text {th }}$ frequency) when the plate is moderately thick $(a / 2 h=10)$. It is worth to note that the FSDT coupled with suitable value of the shear correction factors, is able to ensure more accurate predictions with respect to the HSDT model. This is due to the sandwich stacking sequence: the ESL models tend to heavily overestimate the transverse shear stiffness and only the FSDT is able, with the aid of very low shear correction factors, to reduce this estimation thus achieving more accurate results. The frequencies predicted by the MZZ model fit with those computed by using the FSDT with $k_{1}{ }^{2}=k_{2}{ }^{2}=1$, as expected. In fact, the L5 stacking sequence (see Table A1.3) is
not a periodic one, thus the Murakami's zigzag function is not suitable for the elastodynamic analysis of this sandwich leading the zigzag contribution to vanish and the FSDT kinematics to survive. This demonstrates that the Murakami's zigzag function can lead to erroneous predictions not only on the static problems, but also for eigenvalues one. On the contrary, the RZT model is able to ensure accurate frequencies estimation (with a maximum error of $0,26 \%$ on the $6^{\text {th }}$ frequency for the thick plate, $a / 2 h=10$ ).

Figure 6.29 shows the contour plots of the first six mode shapes obtained with the RZT and corresponding to the frequencies reported in Table 5, for $a / 2 h=10$.

In order to investigate the influence of the core-to-face thickness ratio on the predictive capabilities of RZT and of other models, the fundamental frequency of laminate L5, $a / 2 h=10$, has been estimated for different values of the core-to-face thickness ratio $t_{d} t_{f}$ (Table 6). The solution for this problem, based on the propagator matrix method and on a semi-analytical solution of a higher-order mixed approach [Rao et al., 2004b], is used as reference and quoted as Exact in Table 6.

The HSDT overestimates the stiffness of the plate leading to high values of the fundamental frequency. The error of HSDT reduces as the core-to-face thickness ratio increases since the plate approaches the behavior of a single-layer plate. When the core-toface thickness ratio is small, the shear correction factors estimation procedure is not effective and FSDT leads to an underestimation of the reference frequency values (more than $30 \%$ for $\left.t_{d} t_{f}=4\right)$. For higher values of $t_{d} / t_{f}$, better results are obtained by means of FSDT. Taking into account the results reported in Tables 5 and 6, the shear correction factors estimation procedure coupled with FSDT, provides accurate frequency estimations for thin sandwich plates or thick laminates with a sizeable value of the core-to-face thickness ratio. Regardless the core-to-face thickness ratio, the MZZ model gives the same results of those obtained with the FSDT model with $k_{1}^{2}=k_{2}^{2}=1$, due to the sandwich stacking sequence. The RZT confirms a very good agreement with the reference solution in the considered range of $t_{d} t_{f}$, thus demonstrating the wide range of applicability of the proposed model for sandwich plates.

Results of Tables 6 and 7 are also useful to assess the effect of adding an adequate zigzag function rather than to expand the polynomial order of an ESL model. In fact, as the results demonstrate, the RZT overcomes the HSDT model in accuracy for every span-tothickness ratio and core-to-face ratio considered, thus supporting the major benefits got by enriching the FSDT kinematics with a zigzag contribution rather than with higher-order

Table 5. Problem 5, Laminate L5, core-to-face thickness ratio, $t_{d} t_{f}=10$ : comparison on the first six nondimensional circular frequencies, $\bar{\omega}_{m p}=\omega_{m p} \sqrt{\left(a^{4} \rho_{f} /(2 h)^{2} E_{2 f}\right)}$, where $\rho_{f}$ and $E_{2 f}$ are the mass density and the transverse Young's modulus of the face, respectively.

|  |  |  |  |  |  | FSDT |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $a / 2 h$ | Mode: $m, p$ | LW | RZT | MZZ | HSDT | $\begin{array}{c}k_{1}^{2}=1 \\ k_{2}{ }^{2}=1\end{array}$ | $\begin{array}{c}k_{1}{ }^{2}=5 / 6 \\ k_{2}{ }^{2}=5 / 6\end{array}$ | $k_{1}{ }^{2}=0.0032$ |
| $k_{2}{ }^{2}=0.0032$ |  |  |  |  |  |  |  |  |$]$.

Table 6. Problem 5, Laminate L5, $a / 2 h=10$ : comparison on the fundamental non-dimensional circular frequency, $\bar{\omega}_{1}=\omega_{1} \sqrt{\left(a^{4} \rho_{f} / h^{2} E_{2 f}\right)}$, where $\rho_{f}$ and $E_{2 f}$ are the mass density and the transverse Young's modulus of the face, respectively.

| $t_{d} t_{f}$ | Exact | RZT | MZZ | HSDT | FSDT $\left(k_{1}{ }^{2}=k_{2}{ }^{2}=1\right)$ | FSDT $\left(k_{l}{ }^{2}=k_{2}{ }^{2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 1.91 | 1.91 | 14.28 | 9.00 | 14.28 | $1.32(0.0017)$ |
| 10 | 1.85 | 1.85 | 14.28 | 4.86 | 14.28 | $1.71(0.0032)$ |
| 20 | 2.13 | 2.13 | 13.30 | 3.14 | 13.30 | $2.09(0.0058)$ |
| 30 | 2.33 | 2.34 | 12.36 | 2.85 | 12.36 | $2.31(0.0084)$ |
| 40 | 2.47 | 2.47 | 11.57 | 2.83 | 11.57 | $2.46(0.0109)$ |
| 50 | 2.57 | 2.57 | 10.90 | 2.86 | 10.90 | $2.56(0.0135)$ |

polynomial terms. This consideration was already highlighted by Di Sciuva and Icardi in [Di Sciuva et al., 2000], wherein the authors discussed the influence of higher-order approximation on the prediction of global responses of sandwich plates.

Problem 6. A fully clamped, cross-ply square sandwich plate (laminate L6).
For this kind of boundary condition, the exact RZT, FSDT and MZZ solution does not exist and an analytic one is computed by using the Rayleigh-Ritz's method. The spatial


Figure 6.29. Problem 5, Laminate L5, $a / 2 h=10$ : contour plots of the first six mode shapes obtained with RZT.


Figure 6.30. Problem 6, Laminate L6, $a / 2 h=5$ : contour plots of the first six mode shapes obtained with RZT.
approximation of the kinematic variables is the same of that in Eq. (6.3) whereas the amplitudes vary cyclically in the time domain.

In Table 7, the first six circular frequencies, computed with different approaches, are compared with the reference solution, cited as 3D FE [Kulkarni et al., 2008], obtained by means of three-dimensional finite element analysis.

The FSDT is able to improve the accuracy by adopting a suitable shear correction factor even though the model leads to the underestimation of the reference frequency values especially for higher-order modes. The error reduces by increasing the span-to-thickness ratio and the results approach the 3D FEM solution. The RZT is able to preserve its remarkable accuracy also for clamped boundary condition, even considering higher modes, ensuring a maximum error around the $4 \%$ in the most challenging case $\left(a / 2 h=5,6^{\text {th }}\right.$ frequency mode). As a result of the use of a not suitable zigzag function, the MZZ results are the same of the FSDT (with unit shear correction factors) ones.

The contour plots of the first six mode shapes, obtained with RZT for $a / 2 h=5$, are represented in Figure 6.30.

Problem 7. A square, cantilevered, functionally graded sandwich plate.
The non-dimensional natural frequencies computed by using the RZT, the FSDT and the TSDT model are compared with those coming from the same high-fidelity FE model used in Problem 4 (Tables 8 and 9). Due to the clamped boundary conditions, the Rayleigh-Ritz's method is employed and the spatial approximation of the kinematic variables is performed by using the Gram-Schmidt polynomials. The number of the polynomials used is indicated for each models.

In Table 8, the FSDT solution adopting the shear correction factor underestimates the frequencies with an error that increases with the mode number. The reason may lie in the value of the shear correction factor: it is computed according to the procedure proposed in [Raman et al, 1996] and extended to the plate case. This procedure is energy-based and requires the estimation of the transverse shear strain energy coming from the integrated transverse shear stresses, that depend on the in-plane ones. A not accurate evaluations of the latter, affects the estimation of the shear correction factor thus leading to erroneous results. The use of a higher-order model, namely the TSDT, is not effective in enhancing the accuracy. In fact, the TSDT overestimates the natural frequencies with a relative error around the $27 \%$ on the first frequency. The $4^{\text {th }}$ mode is a prevalent in-plane one and, as consequence, all models are able to accurately estimate the corresponding natural frequency due to an accurate estimation of the membrane stiffness.

Table 7. Problem 6, Laminate L6, core-to-face thickness ratio, $t_{c} t_{f}=8$ : first six non-dimensional circular frequencies, $\bar{\omega}_{m p}=100 \omega_{m p} a \sqrt{\left(\rho_{c} / E_{1 f}\right)}$, where $\rho_{c}$ is the mass density of the core and $E_{1 f}$ is the longitudinal Young's modulus of the face. $M$ and $P$ are the number of Gram-Schmidt used to ensure converged results for each model.

|  |  |  |  |  |  | $\mathrm{FSDT}_{(\mathrm{M}=\mathrm{P}=10)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a / 2 h$ | Mode: $m, p$ | 3 F FE | $\mathrm{RZT}_{(\mathrm{M}=\mathrm{P}=10)}$ | $\mathrm{MZZ}_{(\mathrm{M}=\mathrm{P}=9)}$ | $\mathrm{FSDT}_{(\mathrm{M}=\mathrm{P}=9)}$ <br> $k_{l}{ }^{2}=k_{2}{ }^{2}=1$ | $k_{1}{ }^{2}=0.0825$ <br> $k_{2}{ }^{2}=0.1446$ |
| 5 | 1,1 | 12.05 | 12.14 | 32.03 | 32.03 | 11.40 |
|  | 2,1 | 18.27 | 18.43 | 50.29 | 50.29 | 16.34 |
|  | 1,2 | 20.57 | 20.77 | 50.81 | 50.81 | 19.53 |
|  | 2,2 | 24.87 | 25.12 | 64.15 | 64.15 | 22.79 |
|  | 3,1 | 26.40 | 26.74 | 72.78 | 72.78 | 22.32 |
|  | 3,2 | 30.64 | 31.88 | 83.03 | 83.03 | 27.41 |
| 10 | 1,1 | 11.22 | 11.26 | 27.72 | 27.72 | 11.13 |
|  | 2,1 | 16.68 | 16.73 | 45.26 | 45.26 | 16.02 |
|  | 1,2 | 18.96 | 19.05 | 45.39 | 45.39 | 18.97 |
|  | 3,1 | 22.71 | 22.80 | 58.06 | 58.06 | 22.08 |
|  | 2,2 | 23.53 | 23.61 | 67.72 | 67.72 | 22.26 |
|  | 3,2 | 28.07 |  | 77.00 | 77.00 | 26.99 |

Table 8. Problem 7, Type A, $a / 2 h=8, k=5$ : first eight non-dimensional frequencies, $\bar{f}=f \sqrt{\left(a^{4} \rho_{c} /(2 h)^{2} E_{T}\right)}$, where $\rho_{c}$ and $E_{T}$ are the mass density and the Young's modulus of the homogeneous core. $M$ and $P$ are the number of Gram-Schmidt polynomials used to ensure convergent results for each model. The shear correction factors are $k_{1}{ }^{2}=k_{2}{ }^{2}=0.085414$.

| MSC/MD-NASTRAN $^{\circledR}$ | RZT $_{(\mathrm{M}=\mathrm{P}=11)}$ | $\mathrm{TSDT}_{(\mathrm{M}=\mathrm{P}=9)}$ | $\mathrm{FSDT}_{(\mathrm{M}=\mathrm{P}=11)}$ |
| :---: | :---: | :---: | :---: |
| 0.465 | 0.466 | 0.591 | 0.394 |
| 0.531 | 0.533 | 0.654 | 0.464 |
| 1.034 | 1.036 | 1.291 | 0.971 |
| 1.345 | 1.344 | 1.345 | 1.344 |
| 1.593 | 1.616 | 2.004 | 1.194 |
| 1.598 | 1.605 | 1.903 | 1.479 |
| 1.673 | 1.696 | 2.105 | 1.282 |
| 1.926 | 1.951 | 2.422 | 1.548 |

The RZT fits very well with the reference solution, with an error around the $1 \%$ for the $8^{\text {th }}$ mode frequency.

In Table 9, the FSDT slightly overestimates the natural frequencies, above all for the high-order modes, with an error increasing with the grading index. Nevertheless, the FSDT appears accurate in an engineering sense, if compared with the reference solution. The TSDT and RZT models fit with the reference, with a slightly better estimation of the former model over the latter one. Similar to Problem 4, the good performances of the ESL models, that is the FSDT and TSDT, is due to the high core-to-face stiffness ratio that makes the zigzag effect not relevant.

### 2.3. Linear buckling

In this section, results concerning the critical buckling load of (i) simply supported sandwich plates subjected to uniform uni-axial compressive load, (ii) fully clamped sandwich plates under uniform bi-axial compressive load, and (iii) plates supported on two edges, clamped on the others and subjected to in-plane shear load, are presented. Buckling loads may be calculated within RZT as eigenvalues of the stability equations, Eqs. (2.34), coupled with suitable homogeneous boundary conditions. The generalized displacement components in Eqs. (2.34) are measured from the state just prior to the occurrence of the buckling.

Problem 8. A square sandwich plate (laminate L7, see Table A1.3), simply supported on all edges and subjected to a uniform uni-axial compressive load, $\bar{N}_{1}$.

Comparison of the critical buckling load, for different values of span-to-thickness ratio, $a / 2 h$, and face-to-overall thickness ratio, $t_{f} / 2 h$, is made. Several models are considered in order to assess the predictive capabilities of RZT (see Table 10). Solution quoted as 3D [Noor et al., 1994] is taken as reference in the comparison: the face-sheets and the core are treated as three dimensional continua and the buckling response is obtained by using the solution procedure suggested by Srinivas and Rao [Srinivas et al., 1970].

The FSDT, with the classical shear correction factors, overestimates the uni-axial buckling load parameter; whereas the results converge to the reference solutions, in an engineering sense, if the shear correction factors are adopted. The RZT increases its accuracy for high values of the span-to-thickness ratio and by increasing the face-tothickness ratio. The MZZ model behaves in a similar manner of the FSDT $\left(k_{1}{ }^{2}=k_{2}{ }^{2}=1\right)$ when the face-to-thickness ratio value is low, whereas it provides results closer to the reference

Table 9. Problem 7, Type B, $a / 2 h=10$ : first five non-dimensional frequencies, $\bar{f}=f \sqrt{\left(a^{4} \rho_{c} /(2 h)^{2} E_{c}\right)}$ where $\rho_{c}$ and $E_{c}$ are the mass density and the Young's modulus of the ceramic phase. $M$ and $P$ are the number of Gram-Schmidt polynomials used to ensure convergent results for each model.

| $k$ | MSC/MD-NASTRAN ${ }^{\circledR}$ | $\mathrm{RZT}_{(\mathrm{M}=\mathrm{P}=12)}$ | $\operatorname{TSDT}_{(\mathrm{M}=\mathrm{P}=10)}$ | $\mathrm{FSDT}_{(\mathrm{M}=\mathrm{P}=11)}$ |
| :---: | :---: | :---: | :---: | :---: |
| 3 |  |  |  | $k_{1}{ }^{2}=k_{2}{ }^{2}=0.704$ |
|  | 0.115 | 0.115 | 0.114 | 0.116 |
|  | $0.271$ | $0.272$ | $0.272$ | 0.278 |
|  | 0.677 | $0.679$ | $0.676$ | 0.693 |
|  | $0.856$ | $0.860$ | $0.869$ | 0.878 |
|  | $0.954$ | $0.960$ | $0.957$ | $0.994$ |
| 5 |  |  |  | $k_{1}{ }^{2}=k_{2}{ }^{2}=0.627$ |
|  | 0.114 | 0.114 | 0.114 | $0.115$ |
|  | 0.268 | 0.269 | 0.269 | 0.276 |
|  | 0.667 | $0.669$ | 0.667 | 0.687 |
|  | 0.844 | 0.848 | 0.847 | 0.869 |
|  | 0.937 | 0.945 | 0.943 | 0.984 |

Table 10. Problem 8, Laminate L7: uni-axial overall buckling load parameter, $\bar{n}_{1}=\bar{N}_{1}^{c r} b^{2} /\left(E_{2 f} h^{3}\right)$ where $\bar{N}_{1}^{c r}$ is the uniform uni-axial critical load and $E_{2 f}$ is the transverse Young's modulus of the face.

|  | $t_{f} / 2 h=0.025$ |  |  |  | $t_{f} / 2 h=0.05$ |  |  |  | $t_{f} / 2 h=0.1$ |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a / 2 h$ | 5 | 10 | 20 | 5 | 10 | 20 | 5 | 10 | 20 |  |  |
| 3D | 1.503 | 2.238 | 2.554 | 2.082 | 3.737 | 4.659 | 2.605 | 5.608 | 7.897 |  |  |
| RZT | 1.539 | 2.263 | 2.566 | 2.115 | 3.765 | 4.681 | 2.628 | 5.633 | 7.921 |  |  |
| MZZ | 1.676 | 2.334 | 2.588 | 2.509 | 4.108 | 4.806 | 3.517 | 6.905 | 8.474 |  |  |
| FSDT |  |  |  |  |  |  |  |  |  |  |  |
| $k_{x}{ }^{2}=k_{y}^{2}=1$ | 1.682 | 2.337 | 2.589 | 2.622 | 4.122 | 4.811 | 4.029 | 6.952 | 8.491 |  |  |
| $k_{x}{ }^{2}=k_{y}{ }^{2}=5 / 6$ | 1.566 | 2.278 | 2.571 | 2.390 | 3.971 | 4.758 | 3.623 | 6.631 | 8.368 |  |  |

$k_{x}^{2}=0.820, k_{y}^{2}=0.782 \quad k_{x}^{2}=0.697, k_{y}^{2}=0.643 \quad k_{x}^{2}=0.541, k_{y}^{2}=0.479$
$\begin{array}{lllllllllll}k_{x}{ }^{2}, k_{y}^{2} & 1.539 & 2.263 & 2.566 & 2.116 & 3.767 & 4.682 & 2.620 & 5.638 & 7.926\end{array}$
solution, with respect to the $\operatorname{FSDT}\left(k_{1}{ }^{2}=k_{2}{ }^{2}=1\right)$, even if the error still remains around the $35 \%\left(t_{f} / 2 h=0.1, a / 2 h=5\right)$.

Problem 9. A fully clamped rectangular sandwich plate (laminate L8, see Table A1.3) under bi-axial compression, $\bar{N}_{2}=0.5 \bar{N}_{1}$.

The same problem has been solved in [Chakrabarti et al., 2007] and, for comparison purposes, the same material and geometrical configuration has been used here. The RZT approximate solution is obtained by using the Rayleigh-Ritz method and the same spatial approximation for the incremental kinematic variables as in Eqs. (6.3). The FSDT and MZZ solutions are computed in a similar manner. Since the RZT, MZZ, and FSDT solutions are approximate, a convergence study was carried out to select the required number of Gram-Schmidt polynomials, i.e. the values of M and P . The critical buckling stress, for different values of the aspect ratio $a / b$, is estimated by means of the above mentioned models and compared with several other solutions available in literature. In particular, two references are considered: a 2D finite element solution, by Khatua and Cheung [Khatua et al., 1973], and one obtained with the Finite Strip Method (FSM), by Yuan and Dawe [Yuan et al., 2001], wherein the core is represented as a three-dimensional solid with quadratic through-the-thickness in-plane displacements and a linear transverse displacement, whereas the face-sheets are modeled as thin plates, i.e. according to the assumptions of the Classical Laminate Theory (CLT).

In Table 11, the critical buckling stresses are compared for the four above stated formulations and the results reported in [Chakrabarti et al., 2007].

The FSDT results, obtained with the classical values for the shear correction factors, overestimate the reference solutions available in literature especially for values of the aspect ratio, $a / b$, lower than 1 . Instead, the use of the shear correction factors improves FSDT critical buckling stress predictions. Due to the stacking sequence, the MZZ model is able to predict critical buckling loads that are in agreement with the published results, for every value of the aspect ratio, $a / b$, considered. The RZT demonstrates its accuracy also in predicting critical buckling loads. Again, since the Yuan and Dawe solution [Yuan et al., 2001] is based on a higher-order kinematics, results in Table 11 demonstrate the ineffectiveness in adopting higher-order models to predict global responses.

Problem 10. A rectangular sandwich plate (laminate L8, see Table A1.3), simply supported on two opposite edges and clamped along the other edges, subjected to a uniform in-plane shear load, $\bar{N}_{12}$.

The same problem has been solved in [Chakrabarti et al., 2007], therefore, for comparison purposes, the same material properties and geometry have been used here. The RZT approximate solution is obtained by using the Rayleigh-Ritz method. In this case, the boundary conditions read as:

$$
\begin{align*}
& x_{1}=0, a: u_{2}^{*}=w^{*}=\theta_{2}^{*}=\psi_{2}^{*}=N_{1}^{*}=M_{1}^{*}=M_{1}^{\phi^{*}}=0 \\
& x_{2}=0, b: u_{1}^{*}=u_{2}^{*}=w^{*}=\theta_{1}^{*}=\theta_{2}^{*}=\psi_{1}^{*}=\psi_{2}^{*}=0 \tag{6.7}
\end{align*}
$$

where the displacements and stress resultants denoted with $(\cdot)^{*}$ are incremental with respect to the pre-buckling state. The incremental kinematic variables are approximated in the following way

$$
\begin{align*}
& \left(u_{1}^{*}, u_{2}^{*}, w^{*}, \theta_{1}^{*}, \theta_{2}^{*}, \psi_{1}^{*}, \psi_{2}^{*}\right)= \\
& \quad \sum_{m=1}^{M} \sum_{p=1}^{P}\left(U_{m p}, V_{m p}, W_{m p}, \Theta_{1 m p}, \Theta_{2 m p}, \Psi_{1 m p}, \Psi_{2 m p}\right) \chi_{m}\left(x_{1}\right) \chi_{p}\left(x_{2}\right) \tag{6.8}
\end{align*}
$$

where $\chi_{m}\left(x_{1}\right)$ and $\chi_{p}\left(x_{2}\right)$ are trigonometric functions and Gram-Schmidt polynomials, respectively, built to satisfy the geometric boundary conditions. In a similar way, the FSDT and MZZ solutions are obtained by means of the Rayleigh-Ritz' s method.

Critical buckling stress values obtained by means of several models and with different values of the aspect ratio $a / b$, are reported in Table 12. Also in this case, the FSDT solution obtained with the classical values of the shear correction factors, overestimates the solution available in literature [Yuan et al., 2001] and the others reported for comparison. When the shear correction factors are computed, FSDT improves and its results approach the reference solutions, even if the solution remains conservative for every value of the aspect ratio considered. Critical buckling stresses obtained with RZT compare favorably with the reference values within the considered range of aspect ratio values. Due to the stacking sequence, the MZZ preserves its accuracy regardless the loading condition, thus providing results in agreement with the published ones.

The contour plots of the buckling mode, obtained with RZT for each value of the aspect ratio considered in Table 12, are represented in Figure 6.31.

Table 11. Problem 9, Laminate L8, $b=0.5969 \mathrm{~m}$ : the critical buckling stress ( $\bar{\sigma}_{c r}=\bar{N}_{1}^{c r} / 2 t_{f}$ in $\mathrm{Nmm}^{-2}$.)

|  | Aspect ratio $(a / b)$ |  |  |
| :--- | :---: | :---: | :---: |
|  | 0.5 | 0.7 | 1.0 |
| Khatua and Cheung | 170.91 | 112.41 | 81.45 |
| Yuan and Dawe | 170.11 | 111.15 | 80.95 |
| $\mathrm{RZT}_{(\mathrm{M}=\mathrm{P}=8)}$ |  | 170.37 | 111.25 |
| $\mathrm{MZZ}_{(\mathrm{M}=\mathrm{P}=5)}$ |  | 170.60 | 111.50 |
| $\mathrm{FSDT}_{(\mathrm{M}=\mathrm{P}=6)}$ | $k_{1}{ }^{2}=k_{2}{ }^{2}=1$ | 220.72 | 129.73 |
|  | $k_{1}{ }^{2}=k_{2}{ }^{2}=5 / 6$ | 220.36 | 129.61 |
|  | $k_{1}{ }^{2}=k_{2}{ }^{2}=0.0264$ | 170.26 | 111.18 |

Table 12. Problem 10, Laminate L8, $b=0.5969 \mathrm{~m}$ : critical buckling stress ( $\bar{\sigma}_{c r}=\bar{N}_{12}^{c r} / 2 t_{f}$ in $\mathrm{Nmm}^{-2}$ ).

|  | Aspect ratio (a/b) |  |  |  |
| :--- | :--- | :---: | :---: | :---: |
|  |  | 0.5 | 0.7 | 1.0 |
| Yuan and Dawe | 256.80 | 172.00 | 134.60 |  |
| $\mathrm{RZT}_{(\mathrm{M}=\mathrm{P}=9)}$ |  | 257.44 | 172.04 | 134.54 |
| $\mathrm{MZZ}_{(\mathrm{M}=\mathrm{P}=9)}$ |  | 257.46 | 172.22 | 134.68 |
| $\mathrm{FSDT}_{(\mathrm{M}=\mathrm{P}=10)}$ | $k_{1}^{2}=k_{2}^{2}=1$ | 336.59 | 208.81 | 158.12 |
|  | $k_{1}^{2}=k_{2}{ }^{2}=5 / 6$ | 336.00 | 208.54 | 157.95 |
|  | $k_{1}{ }^{2}=k_{2}{ }^{2}=0.0264$ | 256.68 | 171.44 | 133.96 |



Figure 6.31. Problem 10, Laminate L8: contour plots of the buckling mode shapes obtained with RZT for each values of the aspect ratio considered in Table 12.

## 3. Mixed Refined Zigzag Theory

The results of the linear bending problems shown in previous section, demonstrate the capabilities of the RZT model to predict accurately the distribution of transverse shear stresses along the thickness coordinate, regardless the type of loads and boundary conditions. The transverse shear stresses plotted till now are obtained by integration of the local equilibrium equations and, for this reason, quoted as $(E E)$, that stands for Equilibrium Equations. As discussed in Chapter 3, the integration of the in-plane stresses may not be very efficient to be implemented in a finite elements framework and the desirable objective is to develop models able to produce accurate constitutive transverse shear stresses, quoted as (CE), since obtained by Constitutive Equations. This motivate the development of the Mixed Refined Zigzag Theory, based on the Reissner Mixed Variational Theorem.

In this paragraph, results of the previous one are taken into consideration again and the discussion on the accuracy of the constitutive transverse shear stresses (CE) provided by the $\mathrm{RZT}^{(\mathrm{m})}$ model is performed. Moreover, the two modeling strategies for the assumed transverse shear stresses, described in Chapter 3, are compared. The Mixed Refined Zigzag Theory adopting the assumed transverse shear stresses coming from integration of the local equilibrium equations is quoted as $\mathrm{RZT}_{1}^{(\mathrm{m})}$ (see Eq. (3.25)), whereas that one employing the assumed transverse shear stresses as approximated by Murakami [Murakami, 1986] (see Eq. (3.14)), is denoted as $\mathrm{RZT}_{2}^{(\mathrm{m})}$.

### 3.1. Linear bending

Herein, problems concerning the elasto-static deformation of sandwich plates, with simply supported or clamped boundary conditions, and subjected to bi-sinusoidal or uniform pressure, are solved.

Problem 11. A simply supported rectangular plate $(b=3 a)$ subjected to a bi-sinusoidal transverse pressure, $q\left(x_{1}, x_{2}\right)=q_{0} \sin \left(\pi x_{1} / a\right) \sin \left(\pi x_{2} / b\right)$.

In Table 13, the maximum transverse (central) deflection of laminates L1 and L3 (see Table A1.3) computed with the RZT and the two $\mathrm{RZT}^{(\mathrm{m})}$ models is compared with the exact Elasticity solution, for several values of the span-to-thickness ratio. For this set of loads and boundary conditions, the exact RZT and $\mathrm{RZT}^{(\mathrm{m})}$ solution exists and is obtained by employing the spatial approximation of Eq. (6.2).

Table 13. Problem 11: normalized maximum (central) deflection, $\bar{w}=\left(10^{2} D_{11} / q_{0} a^{4}\right) w(a / 2, b / 2)$.

| Laminate | $a / 2 h$ | 3D Elasticity | RZT | RZT $_{1}^{(\mathrm{m})}$ | RZT $_{2}^{(\mathrm{m})}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{L 1} 1$ | 8 | 2.546 | 2.512 | 2.547 | 2.545 |
|  | 10 | 2.449 | 2.427 | 2.449 | 2.448 |
|  | 20 | 2.319 | 2.314 | 2.319 | 2.319 |
|  | 50 | 2.283 | 2.282 | 2.283 | 2.283 |
|  | 100 | 2.278 | 2.277 | 2.278 | 2.278 |
| $\mathbf{L 3}$ | 8 | 37.037 | 36.788 | 37.038 | 36.894 |
|  | 10 | 31.701 | 31.578 | 31.701 | 31.629 |
|  | 20 | 16.379 | 16.368 | 16.379 | 16.373 |
|  | 50 | 7.048 | 7.048 | 7.048 | 7.048 |
|  | 100 | 5.280 | 5.280 | 5.280 | 5.280 |



Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(0, b / 2, z)$
Figure 6.32. Problem 11, Laminate L1, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}=\left(2 h / q_{0} a^{2}\right) \tau_{1 z}^{(k)}$.

In the thick plate regime, the mixed approach improves the prediction of the maximum deflection with respect to the displacement-based model, leading to a near-zero error, whereas by increasing $a / 2 h$ the RZT solution approaches the exact one. Even if the differences are paltry, the assumption of transverse shear stresses coming from integration of the local three dimensional equilibrium equations appears to work better than the polynomial assumption strategy.

Figures 6.32 and 6.33 compare the constitutive transverse shear stresses, denoted as $(C E)$, of the RZT model with those of the two mixed models. The exact Elasticity solution is taken as reference in the comparison.

Due to the non fulfillment of the transverse shear stresses continuity at layer interfaces, the RZT model provides piece-wise constant constitutive stresses that violate the contact equilibrium condition. Nevertheless, the RZT model is able to recover the actual distribution of shear stresses by integrating the local equilibrium equations in virtue of the accurate in-plane stresses prediction. The parabolic, vanishing at the top and bottom plate surface and continuous across layer interfaces distribution of constitutive transverse shear stresses is achieved by using the mixed models. By comparison, the $\mathrm{RZT}_{2}^{(\mathrm{m})}$ stresses are significantly less accurate with respect to the $\mathrm{RZT}_{1}^{(\mathrm{m})}$ ones because they follow closely the piece-wise constant distribution of the RZT constitutive shear stresses. This drawback was already highlighted in Auricchio et al. [Auricchio et al., 2003], even if in the FSDT framework. The same authors elucidated the reason of this behavior: when the stationary condition of the variational principle is set, the transverse shear strain coming from the assumed transverse shear stresses are enforced to equal, in a weak form, those coming from the displacement field. Increasing the number of the layers, the variational equation is enforced stronger and stronger, leading the assumed transverse shear stresses to fit with the constitutive ones. This is due to the fact that the number of stress variables involved in the polynomial assumption depends on the number of layers. The same conclusion made by Auricchio [Auricchio et al., 2003] holds true also in this case, allowing to demonstrate the superior predictive capabilities ensured by the novel transverse shear stresses assumption strategy (see Chapter 3).

Problem 12. A cantilever square sandwich plate subjected to a uniform transverse pressure, $q\left(x_{1}, x_{2}\right)=q_{0}$.

For this problem, the exact solution does not exist. To assess the accuracy of $\mathrm{RZT}^{(\mathrm{m})}$, a high-fidelity FE (MSC/MD-NASTRAN ${ }^{\circledR}$ ) solution is used as reference. The model is


Normalized transverse shear stress, $\bar{\tau}_{2 z}^{(k)}(a / 2,0, z)$

Figure 6.33. Problem 11, Laminate L1, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{2 z}=\left(2 h / q_{0} a^{2}\right) \tau_{2 z}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(a / 10, b / 2, z)$

Figure 6.34. Problem 12, Laminate L3, $a / 2 h=8$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}=\left(2 h / q_{0} a^{2}\right) \tau_{1 z}^{(k)}$.
regularly discretized using linear-strain solid elements, HEXA8. There are sixty-five elements along each span direction, five elements through the thickness of the bottom face, eight elements along the top face and fifteen elements along the core thickness.

Figure 6.34 shows the distribution along the thickness of the prevalent constitutive transverse shear stress for different models compared with the FE solution. Also changing the load and boundary conditions, the $\mathrm{RZT}_{1}^{(\mathrm{m})}$ model preserves its remarkable accuracy comparable only with that of the integrated RZT stress, quoted as RZT ( $E E$ ). This case represents a pathological situation for the RZT since its constitutive transverse shear stress heavily overestimates the actual value at the bottom face. The $\mathrm{RZT}_{2}^{(\mathrm{m})}$ distribution, in virtue of the variational equivalence, follows the RZT constitutive stress thus providing an inaccurate evaluation.

Numerical results presented in this paragraph demonstrate that the constitutive transverse shear stresses of the RZT model can be easily improved by developing a mixed model, based on the Reissner Mixed Variational Theorem, wherein the assumed kinematics is that of the RZT and the assumed transverse shear stresses derive by integration of the equilibrium equations, as presented in Chapter 3. The use of the polynomial approximation for the assumed stresses, introduced for the first time by Murakami [Murakami, 1986] and widely adopted in the open literature, leads to errors clearly visible in the local responses (see Figures 6.32-6.34) rather than in the global ones (see Table 13).

### 3.2. Free vibrations

In the present section, the numerical results concerning the free vibration problems of sandwich plates, both simply supported and clamped, and solved by means of the mixed formulation of the RZT, are presented and compared in order to assess the improvement achievable with respect to the displacement-based formulation.

Problem 13. A simply supported, cross-ply square sandwich plate (laminate L5, see Table A1.3).

In Table 14, the non-dimensional free vibration frequencies computed with several models are compared with the reference solution [Rao et al., 2004], quoted as LW. The considerations about the results provided by the MZZ and FSDT models, along with the HSDT [Srinivas et al., 1970] have already been discussed above (see Sect. 2.2), thus herein the attention is focused on the results obtained with the two mixed models. Both mixed RZT models estimate natural frequencies that fit with the RZT

Table 14. Problem 13, Laminate $\mathrm{L} 5, a / 2 h=5, t_{d} t_{f}=10$ : first six non-dimensional circular frequencies, $\bar{\omega}_{m p}=\omega_{m p} \sqrt{\left(a^{4} \rho_{f} /(2 h)^{2} E_{2 f}\right)}$, where $\rho_{f}$ and $E_{2 f}$ are the mass density and the transverse Young's modulus of the face, respectively. The FSDT is computed with ${k_{1}}^{2}=k_{2}{ }^{2}=0.0032$.

| Mode: $m, p$ | LW | RZT | $\mathrm{RZT}_{1}^{(\mathrm{m})}$ | $\mathrm{RZT}_{2}^{(\mathrm{m})}$ | MZZ | HSDT | FSDT |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1,1 | 1.85 | 1.85 | 1.85 | 1.85 | 14.28 | 4.86 | 1.71 |
| 1,2 | 3.22 | 3.23 | 3.22 | 3.22 | 32.46 | 8.02 | 2.75 |
| 2,2 | 4.29 | 4.30 | 4.30 | 4.30 | 44.28 | 10.30 | 3.44 |
| 1,3 | 5.22 | 5.24 | 5.23 | 5.23 | 55.14 | 11.74 | 3.91 |
| 2,3 | 6.09 | 6.12 | 6.11 | 6.11 | 63.11 | 13.47 | 4.42 |
| 3,3 | 7.68 | 7.70 | 7.69 | 7.69 | 77.63 | 16.13 | 5.17 |

Table 15. Problem 13, Laminate L5, $a / 2 h=10$ : fundamental non-dimensional circular frequency, $\bar{\omega}_{1}=\omega_{1} \sqrt{\left(a^{4} \rho_{f} / h^{2} E_{2 f}\right)}$, where $\rho_{f}$ and $E_{2 f}$ are the mass density and the transverse Young's modulus of the face, respectively.

| $t_{d} t_{f}$ | Exact | RZT | $\mathrm{RZT}_{1}^{(\mathrm{m})}$ | $\mathrm{RZT}_{2}^{(\mathrm{m})}$ | MZZ | HSDT | FSDT $\left(k_{l}^{2}=k_{2}^{2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 1.91 | 1.91 | 1.91 | 1.91 | 14.28 | 9.00 | $1.32(0.0017)$ |
| 10 | 1.85 | 1.85 | 1.85 | 1.85 | 14.28 | 4.86 | $1.71(0.0032)$ |
| 20 | 2.13 | 2.13 | 2.13 | 2.13 | 13.30 | 3.14 | $2.09(0.0058)$ |
| 30 | 2.33 | 2.34 | 2.33 | 2.33 | 12.36 | 2.85 | $2.31(0.0084)$ |
| 40 | 2.47 | 2.47 | 2.47 | 2.47 | 11.57 | 2.83 | $2.46(0.0109)$ |
| 50 | 2.57 | 2.57 | 2.56 | 2.56 | 10.90 | 2.86 | $2.56(0.0135)$ |

ones, even if a tendency to improve the estimation appears by moving toward high-order modes.

In order to investigate the effect of the core-to-face thickness ratio, in Table 15 the results on the first natural frequency are compared for different models. The RZT is able to provide accurate results for each core-to-face thickness ratio value, thus the enhancement of the transverse shear stiffness reached with the mixed formulations is not effective. Moreover, no differences between the two mixed models appear.

Problem 14. A fully clamped, cross-ply square sandwich plate (laminate L6, see Table A1.3).

In Table 16 the first six natural frequencies are compared: the use of a mixed RZT formulation enhances the response prediction with differences between the $\mathrm{RZT}_{1}^{(\mathrm{m})}$ and $\mathrm{RZT}_{2}^{(\mathrm{m})}$ model. In fact, as results in Table 16 demonstrate, the $\mathrm{RZT}_{1}^{(\mathrm{m})}$ results are closer to the reference ones, with respect to those estimated with the $\mathrm{RZT}_{2}^{(\mathrm{m})}$ model.

Table 16. Problem 14, Laminate L6, core-to-face thickness ratio, $t_{c} / t_{f}=8$ : first six non-dimensional circular frequencies, $\bar{\omega}_{m p}=100 \omega_{m p} a \sqrt{\left(\rho_{c} / E_{1 f}\right)}$, where $\rho_{c}$ is the mass density of the core and $E_{1 f}$ is the longitudinal Young's modulus of the face. $M$ and $P$ are the number of Gram-Schmidt polynomials used to ensure convergent results for each model. The shear correction factors used are $k_{1}{ }^{2}=0.0825, k_{2}{ }^{2}=0.1446$.

| $a / 2 h$ | Mode: $m, p$ | 3D FE | $\mathrm{RZT}_{(\mathrm{M}=\mathrm{P}=10)}$ | $\mathrm{RZT}_{1}^{(\mathrm{m})}$ <br> $(\mathrm{M}=\mathrm{P}=10)$ | $\mathrm{RZT}_{2}^{(\mathrm{m})}$ <br> $(\mathrm{M}=\mathrm{P}=10)$ | $\mathrm{MZZ}_{(\mathrm{M}=\mathrm{P}=9)}$ | $\mathrm{FSDT}_{(\mathrm{M}=\mathrm{P}=10)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 1,1 | 12.05 | 12.14 | 12.07 | 12.13 | 32.03 | 11.40 |
|  | 2,1 | 18.27 | 18.43 | 18.27 | 18.41 | 50.29 | 16.34 |
|  | 1,2 | 20.57 | 20.77 | 20.63 | 20.74 | 50.81 | 19.53 |
|  | 2,2 | 24.87 | 25.12 | 24.90 | 25.08 | 64.15 | 22.79 |
|  | 3,1 | 26.40 | 26.74 | 26.38 | 26.69 | 72.78 | 22.32 |
|  | 3,2 | 30.64 | 31.88 | 31.49 | 31.83 | 83.03 | 27.41 |
| 10 | 1,1 | 11.22 | 11.26 | 11.23 | 11.26 | 27.72 | 11.13 |
|  | 2,1 | 16.68 | 16.73 | 16.69 | 16.74 | 45.26 | 16.02 |
|  | 1,2 | 18.96 | 19.05 | 19.00 | 19.06 | 45.39 | 18.97 |
|  | 3,1 | 22.71 | 22.80 | 22.74 | 22.81 | 58.06 | 22.08 |
|  | 2,2 | 23.53 | 23.61 | 23.52 | 23.60 | 67.72 | 22.26 |
|  | 3,2 | 28.07 | 28.20 | 28.21 | 28.20 | 77.00 | 26.99 |

## 4. (3,2)-Mixed Refined Zigzag Theory

In this section, the response prediction capabilities of the (3,2)-Mixed Refined Zigzag Theory is assessed on problems concerning the linear bending of multilayered composite and sandwich plates, simply supported and clamped, subjected to bi-sinusoidal or uniform distributed load.

The accuracy of the model in solving static problems involving concentrated loads and dynamic problems will be discussed in the Chapter devoted to the finite element results.

### 4.1. Linear bending

For comparison purposes, the exact Elasticity solution, as derived by Pagano [Pagano, 1970], is taken as reference in the comparison. When not available, a high-fidelity FE model is considered.

Problem 15. A simply supported rectangular $(b=3 a)$ plate subjected to bi-sinusoidal transverse pressure, $q\left(x_{1}, x_{2}\right)=q_{0} \sin \left(\pi x_{1} / a\right) \sin \left(\pi x_{2} / b\right)$, applied at the top surface.

For this set of load and boundary conditions, the exact (3,2)-Mixed Refined Zigzag Theory exists and is obtained by following the approximation of kinematic variables as that in Eq. (6.2). In Figures 6.35-6.42, the through-the-thickness distributions of displacements and stresses obtained by means of the (3,2)-Mixed Refined Zigzag model are compared with the Elasticity solution for a symmetric three-layer thick plate.

As consequence of the laminate thickness $(a / 2 h=5)$, the distribution of in-plane displacements ceases to be piece-wise linear and becomes piecewise non-linear (Figures 6.35-6.36). The (3,2)-Mixed Refined Zigzag model, developed also to account for this response typical of thick laminates, behaves in an efficient way, matching with the Elasticity solution. In addition, the (3,2)-Mixed Refined Zigzag model postulates a nonconstant distribution of the transverse displacement along the thickness that reveals to be in perfect agreement with the reference. Accurate in-plane stresses prediction is also ensured by the model along with a constitutive transverse shear (Figures 6.39-6.40) and normal stress (Figure 6.42).

In order to assess the global response prediction capabilities of the (3,2)-Mixed Refined Zigzag model, especially for thick laminates, the maximum (central) deflection for several span-to-thickness ratios is compared in Tables 17-19.


Figure 6.35. Problem 15, Laminate L10, $a / 2 h=5$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{1}^{(k)}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{1}^{(k)}$.


Figure 6.36. Problem 15, Laminate L10, $a / 2 h=5$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{2}^{(k)}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{2}^{(k)}$.


Figure 6.37. Problem 15, Laminate $\mathrm{L} 10, a / 2 h=5$ : through-the-thickness distribution of normalized transverse displacement, $\bar{U}_{z}=\left(10^{4} D_{11} / q_{0} a^{4}\right) U_{z}$.


Normalized in-plane normal stress, $\bar{\sigma}_{11}^{(k)}(a / 2, b / 2, z)$

Figure 6.38. Problem 15, Laminate L10, $a / 2 h=5$ : through-the-thickness distribution of normalized in-plane normal stress, $\bar{\sigma}_{11}=\left(4 h^{2} / q_{0} a^{2}\right) \sigma_{11}^{(k)}$.


Normalized in-plane normal stress, $\bar{\sigma}_{22}^{(k)}(a / 2, b / 2, z)$

Figure 6.39. Problem 15, Laminate $\mathrm{L} 10, a / 2 h=5$ : through-the-thickness distribution of normalized in-plane normal stress, $\bar{\sigma}_{22}=\left(4 h^{2} / q_{0} a^{2}\right) \sigma_{22}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(0, b / 2, z)$

Figure 6.40. Problem 15, Laminate L10, $a / 2 h=5$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}=\left(2 h / q_{0} a^{2}\right) \tau_{1 z}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{2 z}^{(k)}(a / 2,0, z)$

Figure 6.41. Problem 15, Laminate L10, $a / 2 h=5$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{2 z}=\left(2 h / q_{0} a^{2}\right) \tau_{2 z}^{(k)}$.


Normalized transverse normal stress, $\bar{\sigma}_{z z}^{(k)}(a / 2, b / 2, z)$

Figure 6.42. Problem 15, Laminate $\mathrm{L} 10, a / 2 h=5$ : through-the-thickness distribution of normalized transverse normal stress, $\bar{\sigma}_{z z}=\left(4 h^{2} / q_{0} a^{2}\right) \sigma_{z z}^{(k)}$.

Table 17. Problem 15, L1: normalized maximum (central) deflection, $\bar{w}=\left(10^{2} D_{11} / q_{0} a^{4}\right) w(a / 2, b / 2)$. The shear correction factors are $k_{1}^{2}=0.5011, k_{2}^{2}=0.1632$.

| $a / h$ | 3D Elasticity | RZT $_{(3,2)}^{(\mathrm{m})}$ | RZT | MZZ | FSDT |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 4.368 | 4.460 | 4.161 | 4.161 | 5.026 |
| 6 | 3.310 | 3.279 | 3.197 | 3.197 | 3.500 |
| 8 | 2.891 | 2.854 | 2.820 | 2.820 | 2.965 |
| 10 | 2.682 | 2.652 | 2.633 | 2.633 | 2.717 |
| 20 | 2.383 | 2.373 | 2.369 | 2.369 | 2.386 |
| 50 | 2.293 | 2.292 | 2.291 | 2.291 | 2.294 |
| 100 | 2.280 | 2.280 | 2.280 | 2.280 | 2.280 |

Table 18. Problem 15, L2: normalized maximum (central) deflection, $\bar{w}=\left(10^{2} D_{11} / q_{0} a^{4}\right) w(a / 2, b / 2)$. The shear correction factors are $k_{1}^{2}=0.1212, k_{2}^{2}=0.3438$.

| $a / h$ | 3D Elasticity | RZT $_{(3,2)}^{(\mathrm{m})}$ | RZT | MZZ | FSDT |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 12.238 | 12.212 | 12.139 | 12.139 | 18.523 |
| 6 | 7.352 | 7.346 | 7.325 | 7.325 | 9.082 |
| 8 | 4.990 | 4.988 | 4.978 | 4.978 | 5.625 |
| 10 | 3.706 | 3.705 | 3.699 | 3.699 | 3.987 |
| 20 | 1.737 | 1.737 | 1.735 | 1.735 | 1.757 |
| 50 | 1.121 | 1.121 | 1.121 | 1.121 | 1.122 |
| 100 | 1.031 | 1.031 | 1.031 | 1.031 | 1.031 |

Table 19. Problem 15, L9: normalized maximum (central) deflection, $\bar{w}=\left(10^{2} D_{11} / q_{0} a^{4}\right) w(a / 2, b / 2)$. The shear correction factors are $k_{1}^{2}=0.1260, k_{2}^{2}=0.1632$.

| $a / h$ | 3D Elasticity | RZT $_{(3,2)}^{(\mathrm{m})}$ | RZT | MZZ | FSDT |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 7.190 | 7.145 | 6.938 | 5.808 | 11.303 |
| 6 | 4.540 | 4.542 | 4.471 | 3.504 | 5.724 |
| 8 | 3.286 | 3.292 | 3.259 | 2.545 | 3.743 |
| 10 | 2.603 | 2.608 | 2.589 | 2.065 | 2.815 |
| 20 | 1.543 | 1.544 | 1.541 | 1.380 | 1.562 |
| 50 | 1.205 | 1.206 | 1.205 | 1.178 | 1.207 |
| 100 | 1.155 | 1.155 | 1.155 | 1.148 | 1.156 |

The (3,2)-Mixed Refined Zigzag model improves the global response prediction with respect to the RZT model, especially in the thick plate regime: the maximum errors originated by the RZT solutions, when $a / 2 h=4$, are $4,7 \%, 0.8 \%$ and $3.5 \%$. The (3,2)-Mixed Refined Zigzag model reduces these errors to $2.1 \%, 0.2 \%$ and $0.6 \%$, respectively. Thus, the (3,2)-Mixed Refined Zigzag model is not only efficient in providing accurate local response for thick laminate (see Figures 6.35-6.42) but also in enhancing the global response. When the span-to-thickness ratio decreases, the (3,2)-Mixed Refined Zigzag solution approaches the RZT one and together fit with the reference solution.

## Appendix 1. Mechanical properties of materials and stacking sequences

In the following, the mechanical properties of materials used as face-sheet (Table A1.1) in sandwiches, or simply adopted in multilayered composite plates, are listed along with the mechanical properties (Table A1.2) of materials used as core in sandwiches. Finally, the stacking sequence of laminates considered in the numerical tests are listed ((Table A1.3).

Table A1.1 Mechanical properties of isotropic and orthotropic materials. The Young's moduli, $E_{i}^{(k)}$, and the shear moduli, $G_{i j}^{(k)}$, are expressed in GPa; the density $\rho^{(k)}$ is expressed in $\mathrm{kg} \mathrm{m}^{-3}$.

|  | Orthotropic Materials |  |  |  |  |  |  | Isotropic Materials |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | A | P | Q | $\mathrm{F}_{1}$ | $\mathrm{F}_{2}$ | $\mathrm{F}_{3}$ | $\mathrm{F}_{4}$ |  | $\mathrm{F}_{5}$ | $\mathrm{F}_{6}$ | $\mathrm{F}_{7}$ |
| $E_{1}^{(k)}$ | 157.9 | 5.9 | 525 | 50 | 131 | 19 | 276 |  |  |  |  |
| $E_{2}^{(k)}$ | 9.584 | 10 | 21 | 10 | 10.34 | 1 | 6.9 | $E^{(k)}$ | 50 | 62.5 | 65.5 |
| $E_{3}^{(k)}$ | 9.584 | 10 | 21 | 10 | 10.34 | 1 | 6.9 |  |  |  |  |
| $v_{12}^{(k)}$ | 0.32 | 0.25 | 0.25 | 0.25 | 0.22 | 0.32 | 0.25 |  |  |  |  |
| $v_{13}^{(k)}$ | 0.32 | 0.25 | 0.25 | 0.25 | 0.22 | 0.32 | 0.25 | $v^{(k)}$ | 0.34 | 0.34 | 0.25 |
| $v_{23}^{(k)}$ | 0.49 | 0.25 | 0.25 | 0.25 | 0.49 | 0.49 | 0.3 |  |  |  |  |
| $G_{12}^{(k)}$ | 5.930 | 5.9 | 10.5 | 5 | 6.895 | 0.52 | 6.9 |  |  |  |  |
| $G_{13}^{(k)}$ | 5.930 | 0.2 | 10.5 | 5 | 6.205 | 0.52 | 6.9 |  |  |  |  |
| $G_{23}^{(k)}$ | 3.227 | 0.7 | 10.5 | 5 | 6.895 | 0.338 | 6.9 |  |  |  |  |
| $\rho^{(k)}$ | - | - | - | - | 1627 | - | 681.8 |  |  |  |  |

Table A1.2 Mechanical properties of core materials. The Young's moduli, $E_{i}^{(k)}$, and the shear moduli, $G_{i j}^{(k)}$, are expressed in GPa; the mass density, $\rho^{(k)}$, is expressed in $\mathrm{kg} \mathrm{m}^{-3}$.

| Orthotropic Materials |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | N | $\mathrm{C}_{1}$ | $\mathrm{C}_{2}$ |  | Isotropic Materials |  |
|  |  |  | $\mathrm{C}_{4}$ | $\mathrm{C}_{5}$ |  |  |
| $E_{1}^{(k)}$ | $10^{-5}$ | $3.2 \times 10^{-5}$ | 0.5776 |  |  |  |
| $E_{2}^{(k)}$ | $10^{-5}$ | $2.9 \times 10^{-5}$ | 0.5776 | $E^{(k)}$ | $6.89 \times 10^{-3}$ | negligible |
| $E_{3}^{(k)}$ | $75.85 \times 10^{-3}$ | 0.4 | 0.5776 |  |  |  |
| $v_{12}^{(k)}$ | 0.01 | 0.99 | 0.0025 |  |  |  |
| $v_{13}^{(k)}$ | 0.01 | $3 \times 10^{-5}$ | 0.0025 | $v^{(k)}$ |  |  |
| $v_{23}^{(k)}$ | 0.01 | $3 \times 10^{-5}$ | 0.0025 |  |  |  |
| $G_{12}^{(k)}$ | $22.5 \times 10^{-3}$ | $2.4 \times 10^{-3}$ | 0.1079 |  |  |  |
| $G_{13}^{(k)}$ | $22.5 \times 10^{-3}$ | $7.9 \times 10^{-2}$ | 0.1079 | $G^{(k)}$ |  |  |
| $G_{23}^{(k)}$ | $22.5 \times 10^{-3}$ | $6.6 \times 10^{-2}$ | 0.22215 |  |  |  |
| $\rho^{(k)}$ | - | - | 1000 | $\rho^{(k)}$ | 97 |  |

Table A1.3 Laminate stacking sequences (from bottom to top surface); $t_{c}$ and $t_{f}$ are the core and the single face-sheet thickness, respectively.

| Laminate | Normalized lamina thickness. <br> $h^{(k)} / h$ | Lamina materials | Lamina orientation [ ${ }^{\circ}$ ] |
| :---: | :---: | :---: | :---: |
| L1 | $(0.5 / 0.5)$ | $(\mathrm{A} / \mathrm{A})$ | $(0 / 90)$ |
| L2 | $(0.25 / 0.5 / 0.25)$ | $(\mathrm{A} / \mathrm{A} / \mathrm{A})$ | $(0 / 90 / 0)$ |
| L3 | $(0.1 / 0.7 / 0.2)$ | $(\mathrm{F} 1 / \mathrm{N} / \mathrm{Q})$ | $(0 / \mathrm{Core} / 0)$ |
| L4 | $(0.05 / 0.05 / 0.8 / 0.05 / 0.05)$ | $(\mathrm{F} 1 / \mathrm{F} 1 / \mathrm{N} / \mathrm{F} 1 / \mathrm{F} 1)$ | $\left(0^{\circ} / 90^{\circ} / \mathrm{Core} / 90^{\circ} / 0^{\circ}\right)$ |
| L5 | $(0.5 \mathrm{tf} / 0.5 \mathrm{tf} / \mathrm{tt} / 0.5 \mathrm{tf} / 0.5 \mathrm{tf})$ | $(\mathrm{F} 2 / \mathrm{F} 2 / \mathrm{C} 4 / \mathrm{F} 2 / \mathrm{F} 2)$ | $\left(0^{\circ} / 90^{\circ} / \mathrm{Core} / 0^{\circ} / 90^{\circ}\right)$ |
| L6 | $(0.5 \mathrm{tf} / 0.5 \mathrm{tf} / \mathrm{tc} / 0.5 \mathrm{tf} / 0.5 \mathrm{tf})$ | $(\mathrm{F} 4 / \mathrm{F} 4 / \mathrm{C} 2 / \mathrm{F} 4 / \mathrm{F} 4)$ | $\left(0^{\circ} / 90^{\circ} / \mathrm{Core} / 90^{\circ} / 0^{\circ}\right)$ |
| L7 | $(0.1 \mathrm{tf} / 0.1 \mathrm{tf}) 5 / \mathrm{tc} /(0.1 \mathrm{tf} / 0.1 \mathrm{tf}) 5$ | $\left.(\mathrm{~F} 3 / \mathrm{F} 3)_{5} / \mathrm{C} 1 / \mathrm{F} 3 / \mathrm{F} 3\right)_{5}$ | $(0 / 90) 5 / \mathrm{Core} /(90 / 0) 5$ |
| L8 | $(0.5334 / 4.597 / 0.5334) \mathrm{mm}$ | $(\mathrm{F} 7 / \mathrm{C} 5 / \mathrm{F} 7)$ | $(0 / \mathrm{Core} / 0)$ |
| L9 | $(0.25 / 0.25 / 0.25 / 0.25)$ | $(\mathrm{A} / \mathrm{A} / \mathrm{A} / \mathrm{A})$ | $(0 / 90 / 0 / 90)$ |
| L10 | $(0.25 / 0.5 / 0.25)$ | $(\mathrm{P} / \mathrm{P} / \mathrm{P})$ | $(0 / 90 / 0)$ |

## Appendix 2. Functionally graded sandwich plates: mechanical properties and stacking sequences

Two stacking sequences are taken into consideration (see Figure A2.1), namely: a sandwich plate with functionally graded face-sheets and a homogeneous core (Type A) and a sandwich plate with functionally graded core and homogeneous face-sheets (Type B).

Type A. Three-layer orthotropic functionally graded sandwich plate, with each lamina oriented at $0^{\circ}$, wherein the core is homogeneous whereas the face-sheets are functionally graded. The core and the face-sheets have the same thickness (Figure A2.2). Table A2.1 collects the mechanical properties assumed as the reference ones.

Table A2.1 Reference mechanical properties for Type A sandwich plates.

| $E_{L}$ | $E_{T}$ | $G_{L T}$ | $G_{T T}$ | $\nu_{L T}$ | $\nu_{T T}$ | $\rho$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 174.6 GPa | 6.89 GPa | 3.5 GPa | 1.4 GPa | 0.25 | 0.25 | $10^{3} \mathrm{~kg} / \mathrm{m}^{3}$ |



Figure A2.1 Configuration of functionally graded sandwich plates: (left) Type A, functionally graded facesheets and homogeneous core; (right) Type B, homogeneous face-sheets and functionally graded core.

The gradation of the properties is only for the Young moduli, the shear moduli, and the mass density whereas the Poisson ratios are assumed to be constant throughout the entire thickness. In each layer, the mechanical property $P(z)$ is obtained by the following relation

$$
\begin{equation*}
P(z)=P^{(0)} f(z) \tag{A2.1}
\end{equation*}
$$

where $P^{(0)}$ denotes the value of the corresponding reference mechanical property given in Table A2.1, while the grading law, a piecewise function, is defined as

$$
f(z)=\left\{\begin{array}{lr}
e^{k[(z+h / 3) /(-2 h / 3)]}, & -h \leq z \leq-h / 3  \tag{A2.2}\\
1, & -h / 3 \leq z \leq h / 3 \\
e^{k[(z-h / 3) /(2 h / 3)]}, & h / 3 \leq z \leq h
\end{array}\right.
$$

where $2 h$ is the total laminate thickness and $k$ is the grading index.
Type B. Three-layer functionally graded sandwich plate, wherein the face-sheets are homogeneous and isotropic whereas the core is functionally graded. The span-to-thickness ratio is $a / 2 h=10$. The core, the bottom and the top face-sheet thickness are denoted by $h_{c}$, $h_{b}, h_{t}$ respectively, and the following relations hold: $h_{d} / h_{b}=4 ; h_{d} h_{b}=2$. The mechanical properties are collected in Table A2.2.

The mechanical properties of each layer are derived by the rule of mixture, that is

$$
\begin{equation*}
P(z)=P_{m}+\left(P_{c}-P_{m}\right) V_{c}(z) \tag{A2.3}
\end{equation*}
$$

where $V_{c}(z)$ is the fraction volume distribution of the ceramic phase through the laminate thickness, $P(z)$ is the equivalent mechanical property (i.e., the Young modulus or the mass density) and $P c, P m$ the corresponding mechanical properties value of the ceramic and
metallic phase, respectively (see Table A2.2). The functionally graded sandwich plate Type B features a power-law distribution of the fraction volume of the ceramic phase along the laminate thickness defined as follows

$$
V_{c}(z)=\left\{\begin{array}{ccc}
0 & \text { for } & -h \leq z \leq z_{2}  \tag{A2.4}\\
\left(\frac{z-z_{2}}{z_{3}-z_{2}}\right)^{k} & \text { for } & z_{2} \leq z \leq z_{3} \\
1 & \text { for } & z_{3} \leq z \leq+h
\end{array}\right.
$$

where $z_{2}, z_{3}$ denote the bottom and top core interfaces coordinate (see Figure A2.1) and $k$ the grading index.

Table A2.2 Mechanical properties of the metallic and ceramic phases for Type B sandwich plates.

| Ceramic | Metallic |
| :---: | :---: |
| $E_{c}=360 \mathrm{GPa}$ | $E_{m}=72.5 \mathrm{GPa}$ |
| $v_{c}=0.3$ | $v_{m}=0.3$ |
| $\rho_{c}=3800 \mathrm{~kg} / \mathrm{m}^{3}$ | $\rho_{m}=2707 \mathrm{~kg} / \mathrm{m}^{3}$ |

## Chapter 7

## Finite elements results

## 1. Introduction

This Chapter is devoted to the assessment of the finite elements formulated previously and presented in Chapter 5. Firstly, the novel RZT-based beam element is employed to solve static and free vibrations problems of soft-core sandwich beams. The purpose is to verify the super-convergent nature of the element, with respect to the already developed RZT-based beam elements [Oñate et al., 2010; Gherlone et al., 2011], in static problems, wherein several load and boundary conditions are considered and in free vibrations problems, wherein only cantilever beam is considered.

Secondly, the (3,2)-Mixed Refined Zigzag Theory-based beam and plate elements performances are discussed. The anisoparametric interpolation scheme adopted to develop the shear locking-free beam element originates three kinds of elements, different for the number of nodes/dof's and shape functions (see Chapter 5). The convergence analysis for all the beam elements is performed considering the bending problem of a simply supported sandwich beam subjected to a sinusoidal distributed load, for which an exact solution within the framework of the theory exists. To investigate the effect of the mismatch in the mechanical properties of the adjacent layers, the face-to-core stiffness ratio assumes different values. Once the convergence of the elements is assessed, the element response
capabilities are investigated on problems concerning the bending and the dynamic response to an impact.

The (3,2)-Mixed Refined Zigzag Theory-based plate element is, at first, subjected to a convergence analysis considering the bending problem of a simply supported 4-layer symmetric orthotropic plate subjected to bi-sinusoidal distributed load, for which an exact model solution is available. Later, the response capabilities of the element are assessed on the bending problem of a tapered cantilever sandwich plate, subjected to a point load applied at the tip, and on free vibrations problem of the same structure.

## 2. Exact static stiffness RZT beam element

In Chapter 5, the formulation of a novel RZT beam element has been presented. This element differs from the already developed ones [Oñate et al., 2010; Gherlone et al., 2011] since it employs exact static shape functions, that is, shape functions derived by the exact solution of the homogenous part of the static equilibrium equations. The advantage in developing finite elements adopting exact static shape functions relies on the exact estimation of the element stiffness matrix with respect to analogues elements employing polynomial shape functions, thus resulting in better performances. Exact static stiffness beam elements ensure superior predictive capabilities, with respect to the comparable elements with polynomial shape functions, also in free vibration problems showing a super-convergent behavior.

In this section, the performances of the exact static stiffness RZT beam element, herein quoted as RZT-e, are assessed on static and free vibration problems of sandwich beams and comparisons with the anisoparametric constrained RZT element, developed by Gherlone et al. [Gherlone et al., 2011] and herein quoted as RZT-p, are made.

### 2.1. Static analysis

Problem 1. To assess the performances of the RZT-e beam element in static problems, several load and boundary conditions are considered (see Figures 7.1 and 7.2). The problems solved pertain the elasto-static deformation of symmetric sandwich beams, with two values of the length-to-thickness ratio, $L / 2 h$, and made by Aluminum skins and softcore made by Rohacell ${ }^{\circledR}$ IG31 (see Table 1). The face-to-core thickness ratio is around 0.83 and the core-to-laminate thickness one is around 0.375 .

Table 1. Mechanical material properties.

| Material | $\mathrm{E}[\mathrm{MPa}]$ | $\mathrm{G}[\mathrm{MPa}]$ | $\rho\left[\mathrm{kgm}^{-3}\right]$ |
| :---: | :---: | :---: | :---: |
| Aluminum | 73000 | 28077 | 2700 |
| Rohacell $^{\circledR} \mathrm{IG} 31$ | 31 | 13 | 31 |
| Rohacell $^{\circledR} \mathrm{WF} 110$ | 180 | 70 | 110 |



Figure 7.1. Problem 1: structural configurations for point load case.


Figure 7.2. Problem 1: structural configurations for uniform distributed load case.
Tables 2 and 3 collect results on the non-dimensional maximum deflection, estimated at $x_{l}=L / 2$ for the S-S, C-S and C-C boundary conditions and at $x_{l}=L$ for the C-F configuration. The non-dimensional maximum deflection is defined as $w^{F E M} / w^{E X}$, wherein $w^{E X}$ denotes the analytical RZT solution, obtained as explained in [D'Angelo, 2014], and $w^{\text {FEM }}$ the finite elements one. The convergence analysis is performed using several number of elements, starting from the minimum (one element for the C-F boundary condition and two in other cases) to ten elements along the beam length.

The RZT- $p$ element shows slower convergence ratio for the C-S and C-C boundary conditions both for point and uniform distributed load case. In particular, the solution

Table 2. Problem 1: non-dimensional maximum deflection for point load case (Figure 7.1).

| $n^{(e)}$ | S-S |  |  | C-F |  | C-S |  | C-C |  |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | L/2h | 10 | 20 | 10 | 20 | 10 | 20 | 10 | 20 |
| 1 | RZT-p | - | - | 0.82 | 0.81 | - | - | - | - |
|  | RZT-e | - | - | 1.00 | 1.00 | - | - | - | - |
| 2 | RZT-p | 0.79 | 0.82 | 0.95 | 0.94 | 0.36 | 0.37 | 0.01 | 0.004 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 4 | RZT-p | 0.95 | 0.95 | 0.99 | 0.98 | 0.85 | 0.86 | 0.76 | 0.79 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 6 | RZT-p | 0.98 | 0.98 | 0.99 | 1.00 | 0.93 | 0.94 | 0.90 | 0.91 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 8 | RZT-p | 0.99 | 1.00 | 1.00 | 1.00 | 0.96 | 0.97 | 0.94 | 0.95 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 10 | RZT-p | 1.00 | 1.00 | 1.00 | 1.00 | 0.98 | 0.98 | 0.96 | 0.97 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Table 3. Problem 1: non-dimensional maximum deflection for distributed load case (Figure 7.2).

| $n^{(e)}$ |  | S-S |  | C-F |  | C-S |  | C-C |  |
| :--- | :--- | :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | L/2h | 10 | 20 | 10 | 20 | 10 | 20 | 10 | 20 |
| 1 | RZT- $p$ | - | - | 0.70 | 0.67 | - | - | - | - |
|  | RZT-e | - | - | 1.00 | 1.00 | - | - | - | - |
| 2 | RZT-p | 0.85 | 0.89 | 0.91 | 0.89 | 0.43 | 0.45 | 0.01 | 0.004 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 4 | RZT- $p$ | 0.96 | 0.98 | 0.98 | 0.97 | 0.87 | 0.88 | 0.76 | 0.79 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 6 | RZT-p | 0.98 | 0.99 | 0.99 | 0.99 | 0.94 | 0.95 | 0.90 | 0.91 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 8 | RZT-p | 0.99 | 0.99 | 1.00 | 0.99 | 0.97 | 0.97 | 0.94 | 0.95 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 10 | RZT-p | 0.99 | 1.00 | 1.00 | 1.00 | 0.98 | 0.98 | 0.96 | 0.97 |
|  | RZT-e | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

estimated with two elements, for the C-C configuration, heavily underestimates the analytical solution. In fact, in this case, as consequence of the boundary and simmetry conditions, the unique unconstrained degree of freedom is the transverse displacement at the mid-length that produces an overstiff solution. Doubling the number of elements, the RZT- $p$ element improves of $70 \%$ its accuracy. For both the C-S and the C-C boundary conditions, under point load and uniform distributed one, ten elements are not sufficient to recover the exact solution, whereas an error lower the $1 \%$ is reached for the S-S and C-F configuration with ten elements. Moreover, no substantial differences on the convergence behavior are appreciated by changing the length-to-thickness ratio.

On the contrary, the RZT-e element is able to reach exact solution with only one element for every structural configuration and length-to-thickness ratio considered. This result is in agreement with the element formulation. In fact, the shape functions of the RZT-e element reproduce the solution of the homogeneous part of the static equilibrium equations, that means that are able to recover the exact solution (at nodes and within the element) of beam element subjected to concentrated load at nodes. For this reason, the results in Table 2 are expected. But, as results in Table 3 demonstrate, the RZT-e element is able also to recover the exact nodal solution also in the uniform distributed load cases. This behavior is not unique of the RZT- e element but is proper of every finite elements employing the exact static shape functions, since, as discussed by Reddy [Reddy, 1997], the finite element approximation space is the same as that for the general solutions of the model. It is worth to clarify that for exact nodal solution is implied that the element recovers, only at its nodes and non within the element, the same values predicted by the analytical solution and not the exact Elasticity solution.

As shown in Chapter 5, the RZT- $e$ beam shape functions could be hard to handle, due to the complex structure and the involvement of transcendental functions but they ensure the convergence to the exact nodal solution, in static problems, with only one element, between two consecutive concentrated loads and/or constraints, thus saving the computational cost in large scale analyses of frame structures, for example.

### 2.2. Free vibrations analysis

Problem 2. The exact static shape functions ensure the exact estimation of the element stiffness matrix thus reaching exact nodal solution in static problems, as discussed before. For free vibrations problems, the consistent element mass matrix, along with the exact element stiffness matrix, could provide the element with a super-convergent behavior. In
order to assess this advantage of the RZT- $e$ over the RZT- $p$ element, the first nondimensional flexural free frequencies of the sandwich beam previously considered are compared in Tables 4 and 5. For the free vibrations problem, an analytical RZT solution is still unavailable and for comparison purposes, a high-fidelity FE one is taken as reference. The non-dimensional free frequency is given by $f^{F E M} / f{ }^{R E F}$, wherein $f^{F E M}$ denotes the frequencies computed by using the RZT-e and RZT- $p$ element whereas $f^{\text {REF }}$ denotes the reference solution.

Results in Tables 4 and 5 demonstrate the super-convergent nature of the RZT-e element with respect to the RZT- $p$. In the low frequency regime, the RZT-e element reaches the reference solution (with errors lower than $1 \%$ ) with four/six elements whereas the RZT- $p$ requires a number of elements greater than ten. Moving towards the higher frequency modes, the convergence with RZT-e is reached later but the element preserves its better performances with respect to the RZT- $p$. The difference between the convergence in the low and high frequencies regime is due to the element mass and stiffness matrices estimation. In the low frequency modes, the response is mainly affected by the stiffness matrix, thus an exact estimation of it guarantees very fast convergence. The response in the high frequency regime is mostly governed by the mass matrix, that for the RZT-e element is not exact, as for the RZT- $p$, but better estimated with respect to the RTZ-p since consistent with more accurate shape functions. For this reasons, moving towards the high frequencies modes, the convergence is delayed for the RZT-e but still better than the RZT$p$ one. As for the static problem, no relevant differences in the elements behavior appear by doubling the length-to-thickness ratio and the RZT-e still performs better than the RZT- $p$.

## 3. (3,2)-Mixed Refined Zigzag Theory-based beam element

In this section, the $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$-based beam element formulated in Chapter 5, is assessed on problems pertaining the elasto-static deformation of soft-core sandwich beams, simply supported and subjected to a sinusoidal distributed load applied at the top surface, and dynamic response of a double cantilever beam subjected to an impulsive load. To validate the accuracy of the model in severe conditions, a compliant layer problem, wherein large interlaminar slip occurs, is taken into consideration.

### 3.1. Convergence

A fundamental step in the assessment of a novel finite element is the convergence analysis. This analysis is performed for a simply supported sandwich beam subjected to a
sinusoidal pressure applied at the top surface. For this set of load and boundary condition, the exact (3,2)-Mixed Refined Zigzag Theory solution exists and is obtained by

Table 4. Problem 2: non-dimensional flexural free frequencies, $L / 2 h=10$.

| $n^{(e)}$ |  | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: |
| 2 | RZT-p | 1.074 | 1.611 | 5.023 | 3.341 | 4.516 |
|  | RZT-e | 1.002 | 1.017 | 1.272 | 1.894 | 1.722 |
| 4 | RZT-p | 1.017 | 1.075 | 1.221 | 1.661 | 1.689 |
|  | RZT- $e$ | 1.000 | 1.003 | 1.011 | 1.018 | 1.160 |
| 6 | RZT-p | 1.007 | 1.032 | 1.084 | 1.179 | 1.319 |
|  | RZT-e | 1.000 | 1.001 | 1.003 | 1.009 | 1.019 |
| 8 | RZT-p | 1.004 | 1.018 | 1.045 | 1.093 | 1.161 |
|  | RZT-e | 1.000 | 1.001 | 1.001 | 1.004 | 1.009 |
| 10 | RZT-p | 1.002 | 1.012 | 1.029 | 1.058 | 1.098 |
|  | RZT-e | 1.000 | 1.001 | 1.001 | 1.002 | 1.005 |
| 200 | RZT-p | 1.000 | 1.001 | 1.001 | 1.001 | 1.001 |
|  | RZT-e | 1.000 | 1.001 | 1.001 | 1.001 | 1.001 |

Table 5. Problem 2: non-dimensional flexural free frequencies, $L / 2 h=20$.

| $n^{(e)}$ |  | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: |
| 2 | RZT-p | 1.064 | 1.625 | 7.845 | 7.915 | 7.996 |
|  | RZT-e | 0.994 | 1.022 | 1.375 | 2.092 | 3.051 |
| 4 | RZT- $p$ | 1.009 | 1.072 | 1.183 | 1.620 | 2.994 |
|  | RZT-e | 0.992 | 1.000 | 1.015 | 1.021 | 1.183 |
| 6 | RZT-p | 1.000 | 1.028 | 1.068 | 1.150 | 1.279 |
|  | RZT- $e$ | 0.991 | 0.997 | 1.003 | 1.010 | 1.020 |
| 8 | RZT-p | 0.996 | 1.014 | 1.036 | 1.076 | 1.139 |
|  | RZT-e | 0.991 | 0.996 | 1.000 | 1.003 | 1.008 |
| 10 | RZT-p | 0.994 | 1.007 | 1.022 | 1.047 | 1.083 |
|  | RZT-e | 0.991 | 0.996 | 1.000 | 1.001 | 1.004 |
| 200 | RZT-p | 0.991 | 0.996 | 1.000 | 1.000 | 1.000 |
|  | RZT-e | 0.991 | 0.996 | 1.000 | 1.000 | 1.000 |

approximating the kinematic variables with trigonometric functions able to satisfy the boundary conditions.

In Chapter 5, three (3,2)-Mixed Refined Zigzag Theory-based beam elements are formulated: the first one, denoted with $\Omega_{0}$ is the so-called virgin element, that is the nine-node/fifteen-dof's anisoparametric element with nodes where only extra $w$-degrees of freedom are defined. In order to recover an isoparametric-like configuration, two constrained six-node/twelve-dof's elements, namely the $\Omega_{\eta}$ and the $\Omega_{\gamma}$, where the subscript denotes the quantity on which the constraint is enforced, are developed by condensing out the extra $w$-degrees of freedom.

The analysis is performed considering a symmetric sandwich beam with Aluminum face-sheets and IG31 core (see Table 1). The length-to-thickness ratio is $L / 2 h=10$ and the face-to-core thickness ratio is 0.05 . In order to assess the influence of the face-to-core stiffness ratio, the Young's modulus of the core is scaled with a multiplying factor able to ensure face-to-core stiffness ratio reported in Table 6, wherein the non-dimensional average (in integral sense, along the thickness) maximum deflection, obtained with the three elements, is shown. The non-dimensional deflection is defined as $U_{z}^{F E M} / U_{z}^{E X}$, where $U_{z}^{E X}$ denotes the exact (average, in integral sense) analytical displacement and $U_{z}^{F E M}$ that computed with the finite element approximation.

Results in Table 6 show that elements $\Omega_{0}$ and $\Omega_{\eta}$ converge to the exact solution over the entire range of face-to-core stiffness ratio values considered, with a convergence rate that increases with the heterogeneity of the stacking sequence. The constrained element $\Omega_{\gamma}$ ensures better convergence behavior when the mismatch between the mechanical properties of the face and the core reduces. This is a consequence of the constraint used: the $\Omega_{\eta}$ element enforces the strain measure $\eta$ to be constant along the beam length, whereas the $\Omega_{\gamma}$ works on the average transverse shear strain (for the details, see Chapter 5). The main difference between the two strain measures, $\eta$ and $\gamma$, is that the $\eta$ includes the zigzag amplitude into its definition, whereas $\gamma$ not. For this reason, by increasing the face-to-core thickness ratio, the zigzag deformation of the cross-section becomes relevant and the zigzag amplitude plays an important role. Neglecting it results in an underestimation of the exact beam deflection, as it happens for the $\Omega_{\gamma}$ element. Since elements $\Omega_{0}$ and $\Omega_{\eta}$ predict the same non-dimensional average maximum deflections (see Table 6), it is worth to examine if there are some discrepancies between results provided

Table 6. Non-dimensional maximum deflection: convergence analysis for the $\mathrm{RZT}_{(3,2)}^{(\mathrm{m})}$ beam element.

| $E^{(f)} / E^{(c)}$ | $10^{3}$ | $10^{2}$ | 10 |  |
| :--- | :--- | :--- | :--- | :--- |
| $n^{(e)}$ | Elem. |  |  |  |
| 6 | $\Omega_{0}, \Omega_{\eta}$ | 0.998 | 0.990 | 0.978 |
|  | $\Omega_{\gamma}$ | 0.284 | 0.847 | 0.975 |
| 10 | $\Omega_{0}, \Omega_{\eta}$ | 0.999 | 0.996 | 0.992 |
|  | $\Omega_{\gamma}$ | 0.495 | 0.936 | 0.991 |
| 20 | $\Omega_{0}, \Omega_{\eta}$ | 1 | 1 | 0.998 |
|  | $\Omega_{\gamma}$ | 0.788 | 0.983 | 0.998 |
| 50 | $\Omega_{0}, \Omega_{\eta}$ | 1 | 1 | 1 |
|  | $\Omega_{\gamma}$ | 0.958 | 0.997 | 1 |



Figure 7.3. One and two-elements discretizations of a cantilever beam using $\Omega_{0}$ and $\Omega_{\eta}$ element.

Table 7. Non-dimensional average deflection, $U_{z}^{\Omega_{\eta}} / U_{z}^{\Omega_{0}}$, in different locations along the beam length (see Figure 7.3).

| $n^{(e)}=1$ |  |  | B | A | B | C |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| A | B $=2$ |  |  |  |  |  |
| Tip Load | 0.97 | 0.99 | 0.98 | 0.99 | 1.00 | 1.00 |
| Uniform Load | 0.94 | 0.97 | 0.95 | 0.98 | 0.99 | 1.00 |

by the two elements within the element span. For this reason, the same sandwich beam, now cantilevered at one end, is considered. Two different loading conditions are
considered, that is a point load applied at the free-end and a uniform distributed load, and the problem is solved with $\Omega_{0}$ and $\Omega_{\eta}$ by using one and two-elements discretization (see Figure 7.3). Table 7 compares the non-dimensional average deflection, defined as $U_{z}^{\Omega_{\eta}} / U_{z}^{\Omega_{0}}$, where $U_{z}^{\Omega_{\eta}}$ and $U_{z}^{\Omega_{0}}$ are computed by using the $\Omega_{\eta}$ and the $\Omega_{0}$ element, respectively. Slightly different results are provided inside the elements: these differences are more pronounced for the distributed load case and, for both cases, tend to vanish by increasing the number of elements. In fact, by using two elements, the non-dimensional deflection reaches the unit value both in the middle and at the end node of the second element. Even if not reported, the in-plane displacement and stress distribution along the thickness are the same with both elements; whereas for the transverse normal and shear stress the $\Omega_{\eta}$ element requires a finer discretization to reach the same results as those provided by $\Omega_{0}$.

Due to the better performance of $\Omega_{\eta}$ with respect to $\Omega_{\gamma}$, and thanks to the less number of dof's with respect to $\Omega_{0}$, the $\Omega_{\eta}$ element is employed in next sections.

### 3.2. Static analysis

In this section, results concerning the bending problem of simply supported beams are given. The first problem concerns a soft-core sandwich beam subjected to sinusoidal distributed load applied at the top surface; whereas the second problem investigates the accuracy of the model in reproducing the interlaminar slip between two unidirectional layers.

Problem 3. An unsymmetric sandwich beam made by Aluminum face-sheets and IG31 soft core. The beam length and the length-to-thickness ratio are, respectively, $\mathrm{L}=200 \mathrm{~mm}$ and $L / 2 h=6$. The top face thickness is two-times that of the bottom face and the core thickness is $80 \%$ of the laminate one. The beam is subjected to a sinusoidal distributed load, applied at the top surface, $q\left(x_{1}\right)=q_{o} \sin \left(\pi x_{1} / L\right)$.

For this set of load and boundary conditions, the exact Elasticity solution exists and is assumed as reference in the comparison. Figures 7.4-7.8 show the distribution along the thickness of the in-plane and transverse displacements, in-plane and transverse shear and normal stresses, normalized according to the following relations


Normalized in-plane displacement, $\bar{U}_{1}^{(k)}(L / 4, z)$

Figure 7.4. Problem 3, unsymmetric soft-core sandwich beam, $L / 2 h=6$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{1}^{(k)}=10^{3}\left(8 E_{f} h^{3} / q_{0} L^{4}\right) U_{1}^{(k)}$.


Normalized transverse displacement, $\bar{U}_{z}(L / 4, z)$

Figure 7.5. Problem 3, unsymmetric soft-core sandwich beam, $L / 2 h=6$ : through-the-thickness distribution of normalized transverse displacement, $\bar{U}_{z}=\left(8 E_{f} h^{3} / q_{0} L^{4}\right) U_{z}$.


Figure 7.6. Problem 3, unsymmetric soft-core sandwich beam, $L / 2 h=6$ : through-the-thickness distribution of normalized in-plane stress, $\bar{\sigma}_{11}^{(k)}=\left(4 h^{2} / q_{0} L^{2}\right) \sigma_{11}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(L / 4, z)$

Figure 7.7. Problem 3, unsymmetric soft-core sandwich beam, $L / 2 h=6$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}=\left(2 h / q_{0} L\right) \tau_{1 z}^{(k)}$.


Normalized transverse normal stress, $\bar{\sigma}_{z z}^{(k)}(L / 4, z)$

Figure 7.8. Problem 3, unsymmetric soft-core sandwich beam, $L / 2 h=6$ : through-the-thickness distribution of normalized transverse normal stress, $\sigma_{z z}=\left(2 h / q_{0} L\right) \sigma_{z z}$.


Normalized in-plane displacement, $\bar{U}_{1}^{(k)}(L / 4, z)$

Figure 7.9. Problem 4, compliant layer scale factor $r=10^{-3}, L / 2 h=6$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{1}^{(k)}=10^{3}\left(8 E_{L} h^{3} / q_{0} L^{4}\right) U_{1}^{(k)}$.

$$
\begin{align*}
& \bar{U}_{1}^{(k)}=10^{3} \frac{8 E_{f} h^{3}}{q_{0} L^{4}} U_{1}^{(k)} ; \quad \bar{U}_{z}=\frac{8 E_{f} h^{3}}{q_{0} L^{4}} U_{z} \\
& \bar{\sigma}_{11}^{(k)}=\frac{4 h^{2}}{q_{0} L^{2}} \sigma_{11}^{(k)} ; \quad\left(\bar{\tau}_{1 z}^{(k)}, \bar{\sigma}_{z z}\right)=\frac{2 h}{q_{0} L}\left(\tau_{1 z}^{(k)}, \sigma_{z z}\right) \tag{7.1}
\end{align*}
$$

where $E_{f}$ is the Young's modulus of the face-sheet.
The model is able to provide distribution of the axial displacement that fits with the reference solution over the entire thickness; as consequence, also the distribution on the axial normal stress is accurately reproduced. The assumed transverse shear stress is able to catch the actual distribution provided by the Elasticity solution, whereas the transverse normal stress, due to the smeared-type assumption, slightly deviates from the actual pattern but recovers the exact value at the top and bottom beam surfaces. A similar scenario is valid for the through-the-thickness distribution of transverse displacement: the model response, neglecting a zigzag effect for the transverse displacement, is not able to reproduce the same behavior of the Elasticity solution, affected also by the application of the load only at the top surface, but is able to provide very accurate result of deflection both in the mean sense that in local response, with an error less than $1 \%$ for the displacement at the bottom beam surface.

Problem 4. A simply supported two-layer unidirectional beam made by material A (see Table A1.1, Chapter 6) is subjected to sinusoidal distributed load, $q\left(x_{1}\right)=q_{0} \sin \left(\pi x_{1} / L\right)$. The beam length is $L=220 \mathrm{~mm}$ and the length-to-thickness ratio is $L / 2 h=6$. In order to assess the response prediction capabilities of the model in problem involving large interlaminar slip, a compliant layer is introduced between the two layers, that is a layer with the same mechanical properties of the adjacent ones but with a transverse shear modulus reduced, with respect to the nominal value, of a parameter $r$. The unidirectional layers thickness is $49 \%$ of the laminate one, whereas the compliant layer thickness is $2 \%$ of the entire one.

Figures 7.9-7.12 compare the through-the-thickness distribution of axial displacement and transverse shear stress, normalized according relations defined as in Eq. (7.1) wherein $E_{l}$ is introduced in place of $E_{f}$, with the Elasticity solution. Two different values of the parameter $r$ are taken into consideration. For both values, the results follows accurately the reference solution, that shows a large interlaminar slip as consequence of a reduced transverse shear modulus in the compliant layer. Transverse shear stresses that correctly approach the zero value in the compliant layer are provided by the model and that fit very


Normalized in-plane displacement, $\bar{U}_{1}^{(k)}(L / 4, z)$

Figure 7.10. Problem 4, compliant layer scale factor $r=10^{-4}, L / 2 h=6$ : through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{1}^{(k)}=10^{3}\left(8 E_{L} h^{3} / q_{0} L^{4}\right) U_{1}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(L / 4, z)$

Figure 7.11. Problem 4, compliant layer scale factor $r=10^{-3}, L / 2 h=6$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}=\left(2 h / q_{0} L\right) \tau_{1 z}^{(k)}$.


Normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}(L / 4, z)$

Figure 7.12. Problem 4, compliant layer scale factor $r=10^{-4}, L / 2 h=6$ : through-the-thickness distribution of normalized transverse shear stress, $\bar{\tau}_{1 z}^{(k)}=\left(2 h / q_{0} L\right) \tau_{1 z}^{(k)}$.


Figure 7.13. Problem 5: impact load time history and Fourier transform.
well with the reference solution, a part from a slight difference in the maximum shear stress location.

### 3.3. Dynamic response

Problem 5. In this section, the accuracy of the element in providing accurate dynamic response is assessed on a problem pertaining the response of a double cantilever sandwich beam subjected to an impact occurring in the middle span and on the top surface. The beam is a symmetric soft-core sandwich one, with Aluminum face-sheets and IG31 core. The beam length and width are, respectively, $L=200 \mathrm{~mm}$ and $b=66 \mathrm{~mm}$; whereas the face-to-core thickness and the core-to-laminate thickness ratios are, respectively, equal to 0.05 and 0.91 . For comparison purposes, a high-fidelity FE model, made by linear-elastic, QUAD4 membrane elements ( 36,000 dof's) is taken as reference. The impact force is modeled as a point load varying in time according to the following relation

$$
F(t)=\left\{\begin{array}{cc}
\sin \left(\frac{\pi t}{T}\right) & 0 \leq t \leq T  \tag{7.2}\\
0 & t>T
\end{array}\right.
$$

where $T$ is the contact time. In this section, two contact times are taken into consideration to verify the influence on the model response. In Figure 7.13, the time-histories of the two impact loads along with the Fourier transform are depicted. It is worth to note that, halving the contact time, the frequency content of the impact load doubles (see Fourier transform) thus involving higher-order frequencies mode in the response. For this reason, two different discretizations (number of elements, $n^{(e)}$ ) are used for the solution: $n^{(e)}=56$ for a contact time $\mathrm{T}=1 \mathrm{~ms}, n^{(e)}=76$ for a contact time $\mathrm{T}=0.5 \mathrm{~ms}$. To obtain the model response, an explicit $2^{\text {nd }}$-order Runge-Kutta scheme is adopted. To reduce the computational cost, the reference solution is obtained by approximating the response in the modal space: 150 modes are included in the response to an impact load with $T=1 \mathrm{~ms}$, whereas 200 modes are employed in the solution for an impact load with a contact time $T=$ 0.5 ms .

Figures 7.14 and 7.15 compare the response (in terms of transverse displacement) time histories with the reference solution in three different locations: Figure 7.14 represents the response in the impact point and the opposite one along the thickness direction, that is on the point on the bottom surface; whereas Figure 7.15 shows the response in the point back the impact one and in the point located on the bottom surface and at $x_{I}=L / 4$. Figures 7.14
and 7.15 demonstrate the remarkable accuracy of the model in predict the dynamic response, both qualitatively and quantitatively. The thickness deformation


Figure 7.14. Problem 5: time history response, $L / 2 h=10, T=1 \mathrm{~ms}, n^{(e)}=56$.


Figure 7.15. Problem 5: time history response, $L / 2 h=10, T=1 \mathrm{~ms}, n^{(e)}=56$.
provided by the model fits very well with the reference solution (Figure 7.14) and no propagation delay appears in the model response with respect to the reference (Figure 7.15).

Figures 7.16 and 7.17 show how the response changes (in terms of transverse displacement) by halving the contact time, thus approaching an impulsive load. Also in this case, the model preserves its accuracy. In fact, the response in the impact point and in the opposite point along the thickness direction is in satisfying agreement with the reference solution, a part a slight overestimation of the thickness deformation. Also in this case, no propagation delay appears and the response in a point located $x_{I}=L / 4$ compares favorably with the reference.

## 4. (3,2)-Mixed Refined Zigzag Theory plate element

Herein, the (3,2)-Mixed Refined Zigzag Theory-based plate element is assessed on static and free vibrations problems. Due to the possibility to analyze geometry more complex than a traditional rectangular plate, in this paragraph, the element accuracy is tested on the bending problem concerning the elasto-static deformation of a tapered cantilever soft-core sandwich plate subjected to a tip load and the free vibrations problem of the same structure.

### 4.1. Convergence

In this section, the problem solved analytically in the previous Chapter (problem 15, Chapter 6) is re-taken into consideration to perform the convergence analysis.

Figures 7.18 and 7.19 show the non-dimensional in-plane and transverse displacements and non-dimensional transverse shear stresses for different values of the number of elements of a regular mesh. The non-dimensional displacements and stresses are defined as the ratio between the finite element solution, denoted by the superscript $F E M$, and the analytical solution, quoted with the superscript $E X$. As Figure 7.18 demonstrates, the developed element converges to the analytical solution. The in-plane stresses are not plotted since they are related with the displacement field by means of the linear straindisplacement relations and the material constitutive law. The transverse shear stresses, that are assumed by the model, show convergence to the analytical solution (Figure 7.19) with a very fast ratio.

In order to investigate the effect of an irregular discretization, the convergence analysis is performed also by using an unstructured triangular mesh, automatically generated thanks


Figure 7.16. Problem 5: time history response, $L / 2 h=10, T=0.5 \mathrm{~ms}, n^{(e)}=76$.


Figure 7.17. Problem 5: time history response, $L / 2 h=10, T=0.5 \mathrm{~ms}, n^{(e)}=76$.


Figure 7.18. Convergence analysis, regular mesh: non-dimensional in-plane and transverse displacements in different locations. On the right, the mesh topology with $n^{(e)}=16$.


Figure 7.19. Convergence analysis, regular mesh: non-dimensional transverse shear stresses in different locations. On the right, the mesh topology with $n^{(e)}=16$.
to a Matlab ${ }^{\circledR}$ open-source code (Figure 7.20). In Table 8 the non-dimensional displacements and stresses, evaluated as in Figures 7.18 and 7.19, are reported for different number of elements. To qualify an unstructured grid, the number of elements is not sufficient. The second and third columns of Table 8 list two parameters, the $q_{\text {mean }}$ and $q_{\text {min }}$, that refer to the mean element size and the minimum one in the grid. Greater is the gap between these two parameters and more heterogeneous are the size of the elements. Thus, in Table 8, by increasing the number of elements, it is not possible to indentify always a monotonic increasing or decreasing trend due to the mesh quality parameters. Despite this consideration, the variation is low and the general trend is convergent. This ensure the convergence of the results when geometrically complex problems, wherein a regular mesh is not possible, have to be solved.

### 4.2. Static analysis

Problem 6. A cantilever tapered soft-core sandwich plate with in-plane dimension $a=$ $900 \mathrm{~mm}, b=300 \mathrm{~mm}$ and $c=100 \mathrm{~mm}$ (see Figure 7.21) is subjected to unit transverse point load at the tip. The symmetric stacking sequence is made by Aluminum face-sheets and WF110 core (see Table 1), with a face-to-core thickness ratio equal to 0.125 and a core-tolaminate thickness equal to 0.8 . An unstructured grid made by $n^{(e)}=328$ elements is used to obtain the solution (see Figure 7.21). For comparison purposes, a high-fidelity FE model made with linear-elastic, tetrahedral TET10 elements (77,436 dof's) is assumed as reference in the comparison.

Figures 7.22-7.24 show the through-the-thickness distribution of in-plane and transverse displacements for a point (point A in Figures 7.22-7.24) located at the tip and opposite, along the plate chord, to the loaded point. The displacements are normalized according to the following relations

$$
\begin{equation*}
\bar{U}_{1}^{(k)}=10^{3} \frac{4 E_{f} h^{2}}{F a} U_{1}^{(k)} ; \quad \bar{U}_{z}=\frac{4 E_{f} h^{2}}{F a} U_{z} \tag{7.3}
\end{equation*}
$$

where $E_{f}$ is the face-sheet Young's modulus and $F$ is the applied load.
As results in Figures 7.22-7.24 demonstrate, the results compare favorably with the reference solution even if a slight gap between the two solutions appears for the in-plane displacements along the core thickness. The transverse displacement, although smeared approximated, provide results, as the bottom transverse displacement, with an error less than $1 \%$.

Table 8. Convergence analysis, unstructured mesh: non-dimensional displacements and transverse shear stresses evaluated in different location, as in Figures 7.18 and 7.19.

| $n^{(e)}$ | $q_{\text {mean }}$ | $q_{\text {min }}$ | $U_{1}^{F E M} / U_{1}^{E X}$ | $U_{2}^{F E M} / U_{2}^{E X}$ | $U_{z}^{F E M} / U_{z}^{E X}$ | $\tau_{1 z}^{F E M} / \tau_{1 z}^{E X}$ | $\tau_{2 z}^{F E M} / \tau_{2 z}^{E X}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 69 | 0.901 | 0.758 | 0.96 | 0.97 | 0.86 | 1.22 | 1.02 |
| 92 | 0.891 | 0.574 | 0.97 | 1.11 | 0.96 | 1.04 | 1.12 |
| 114 | 0.906 | 0.784 | 0.97 | 1.00 | 0.96 | 1.04 | 1.05 |
| 120 | 0.929 | 0.775 | 0.96 | 0.99 | 0.96 | 1.06 | 1.06 |
| 122 | 0.948 | 0.834 | 0.99 | 1.02 | 0.96 | 1.04 | 1.03 |
| 340 | 0.905 | 0.696 | 1.00 | 1.03 | 0.98 | 1.01 | 0.96 |
| 388 | 0.946 | 0.771 | 1.00 | 0.99 | 0.99 | 1.02 | 1.01 |



Figure 7.20. Unstructured mesh examples.


Figure 7.21. Problem $6, a / 2 h=8$ : in-plane geometry and unstructured mesh, $n^{(e)}=328$.


Figure 7.22. Problem $6, a / 2 h=8$, point A: through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{1}^{(k)}=10^{3}\left(4 E_{f} h^{2} / F a\right) U_{1}^{(k)}$.


Normalized in-plane displacement, $\bar{U}_{2}^{(k)}$

Figure 7.23. Problem $6, a / 2 h=8$, point A: through-the-thickness distribution of normalized in-plane displacement, $\bar{U}_{2}^{(k)}=10^{3}\left(4 E_{f} h^{2} / F a\right) U_{2}^{(k)}$.


Figure 7.24. Problem 6, $a / 2 h=8$, point A: through-the-thickness distribution of normalized transverse displacement, $\bar{U}_{z}=\left(4 E_{f} h^{2} / F a\right) U_{z}$.


Figure 7.25. Problem 6, $a / 2 h=8$ : static deformation of the bottom plate surface.

Finally, Figure 7.25 depicts the elastic deformation of the bottom plate surface with a rough representation.

### 4.3. Free vibrations analysis

Problem 7. In this section, the same high-fidelity FE model (see Figure 7.26) used to provide reference solution in the static problem, is employed as reference for the free vibrations problem. For the (3,2)-Mixed Refined Zigzag Theory-based plate element solution the unstructured mesh of Figure 7.21 is used.

In Table 9, the first ten frequencies are compared: the (3,2)-Mixed Refined Zigzag Theory results are in perfect agreement with the reference ones. The element is able to catch both the mainly flexural modes, quoted with the superscript $f$ in Table 9 , than the inplane and the coupled torsion/flexural modes, quoted with the superscript $p$ and $f t$, respectively, with high accuracy reaching a maximum error, for the $10^{\text {th }}$ frequency, around the $3 \%$.

Table 9. Problem 7, $a / 2 h=8$ : first ten free frequencies, in Hz . The superscript $f$ denotes a mainly flexural mode, $p$ denotes the in-plane modes and $f t$ a coupled torsion/flexural mode.

| Mode | MSC/MD-NASTRAN | RZT $_{(3,2)}^{(\mathrm{m})}$ |
| :---: | :---: | :---: |
| $1^{f}$ | 3.37 | 3.37 |
| $2^{f}$ | 9.18 | 9.18 |
| $3^{p}$ | 9.84 | 10.02 |
| $4^{4^{t}}$ | 10.84 | 10.97 |
| $5^{f}$ | 16.36 | 16.36 |
| $6^{t t}$ | 23.60 | 23.82 |
| $7^{f}$ | 24.27 | 24.30 |
| $8^{t t}$ | 32.93 | 33.07 |
| $9^{t t}$ | 33.61 | 33.77 |
| $10^{p}$ | 35.13 | 36.13 |



Figure 7.26. High-fidelity FE model: 15,960 linear-elastic tetrahedral TET10 elements.

## Chapter 8

## Experimental assessment

## 1. Introduction

This Chapter is devoted to the discussion of the experimental campaign carried out within the present research activity. The purpose is to provide a set of experimental results useful for a theory assessment. The experimental measures focus on the deflection, via a four-points bending test, and the natural frequencies, via a hammer test, of sandwich beams made by the 7075 Aluminum alloy (Ergal) and a Rohacell ${ }^{\circledR}$ core. Since the measured quantities are global responses, the three models presented in this Thesis provide very similar results and for this reason only the Refined Zigzag Theory (RZT) is assessed. Moreover, to enrich the comparison and to value the benefits ensured by the RZT, the results coming from the Timoshenko's beam theory (TBT), adopting the shear correction factor, are included. In particular, two values for the shear correction factors are considered: the first one is the classical $5 / 6$ value; whereas, the second one is estimated by using the procedure proposed in [Raman et al, 1996].

The whole experimental campaign has been conducted at the LAQ-AERMEC laboratory of the Mechanical and Aerospace Engineering Department of the Politecnico di Torino.

## 2. Specimens

The specimens consist in sandwich beams made by the 7075 Aluminum alloy (Ergal) and Rohacell ${ }^{\circledR}$ core. In order to investigate the effect of the core mechanical properties, two types of core are considered: the IG31 and the WF110. To investigate the effect of the length-to-thickness ratio and the core-to-face thickness ratio, different geometries are provided. The shape of the specimens is also dictated by the test: for a four-points bending test, a simply supported beam is required, whereas for an hammer test, a cantilever boundary condition has to be realized (see Figures 8.1 and 8.2).

The specimens are listed in Table 1 with a nomenclature that contains indications about the core material, the beam length, face thickness and the boundary conditions to which they are subjected. All the dimensions listed in Table 1 have been measured with a caliper in three different positions along the beam span and an average value is provided.

Table 1. Specimens: nomenclature and measured geometry. For reference, see Figures 8.1 and 8.2.

| Specimen | $L_{\text {tot }}$ | $L_{\text {eff }}$ | $b$ | $h_{f}$ | $h_{c}$ | $h_{f} / h_{c}$ | $L / h_{\text {tot }}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| IG31_32_5_SS | 36.10 | 32.10 | 48.43 | 5.00 | 6.00 | 0.83 | 20.00 |
| WF110_32_5_SS | 35.90 | 32.0 | 48.53 | 5.00 | 6.27 | 0.79 | 19.67 |
| IG31_32_5_CF | 42.60 | 32.00 | 48.53 | 5.00 | 6.07 | 0.82 | 19.91 |
| IG31_64_5_CF | 85.30 | 64.00 | 48.40 | 5.00 | 6.00 | 0.83 | 40.00 |
| WF110_32_5_CF | 42.60 | 32.00 | 48.18 | 5.00 | 6.10 | 0.82 | 19.87 |
| WF110_64_5_CF | 85.30 | 64.00 | 48.40 | 5.00 | 6.20 | 0.81 | 39.51 |
| WF110_48_2_CF | 64.00 | 48.00 | 72.13 | 2.00 | 20.13 | 0.09 | 19.89 |
| WF110_96_2_CF | 128.00 | 96.00 | 72.13 | 2.00 | 20.10 | 0.09 | 39.83 |

## 3. Material characterization

In this section, the procedure followed to characterize the mechanical properties of 7075 Aluminum alloy (Ergal) and the Rohacell ${ }^{\circledR}$ IG31 and WF110 are presented along with the results obtained. For reference, in Table 2, the nominal values of the mechanical proprieties are collected.

It is trivial to note that the material characterization is necessary to make the comparison between the experimental and numerical results reasonable.


Figure 8.1. Specimen geometry for the four-points bending test ( $S S$ notation in Table 1).


Figure 8.2. Specimen geometry for the hammer test (CF notation in Table 1).


Figure 8.3. INSTRON machine: A. Console; B. Load Cell; C. Clamps; D. Hydraulic actuator.

Table 2. Material mechanical proprieties: nominal values; the Young's and shear modulus in MPa.

| Material | E | G |
| :---: | :---: | :---: |
| 7075 Aluminum alloy | 73000 | 28077 |
| Rohacell $^{\circledR}$ IG31 | 36 | 13 |
| Rohacell $^{\circledR}$ WF110 | 180 | 70 |

### 3.1. Mass density

The mass density of the Rohacell ${ }^{\circledR}$ IG31 and WF110 are estimated by weighting three core specimens, for each typology, and dividing by the corresponding volume. The core mass density is obtained by averaging the three measures. The Aluminum alloy mass density is obtained by weighting three sandwich beams of those listed in Table 1 and subtracting the mass of the core. The Ergal mass density is obtained by averaging the three measures. The results are collected in Table 3.

Table 3. Measured mass density in $\mathrm{kgm}^{-3}$.

|  | 7075 Aluminum alloy | Rohacell $^{\circledR}$ IG31 | Rohacell $^{\circledR}$ WF110 |
| :---: | :---: | :---: | :---: |
| $\rho$ | 2849 | 36.8 | 112 |

### 3.2. Young's modulus and Poisson's ratio

The Young's modulus and the Poisson's ratio of the Aluminum alloy are characterized by following the standard test methods provided by the ASTM B557M and E 111 norms. The traction load is applied by means of the servo-hydraulic system of the INSTRON 8516 (see Figure 8.3) able to apply a maximum load of 120 kN . One of the longitudinal stresslongitudinal strain curve obtained by the standard test is shown in Figure 8.4, from which one value of the Young's modulus is derived. The Poisson's ratio is evaluated by considering the transverse strain, as prescribed by the standards. Both the Young's modulus and the Poisson' ratio are estimated by averaging the values obtained in three tests.

The Young's and the shear modulus evaluation of the Rohacell ${ }^{\circledR}$ IG31, and similarly of the WF110, is performed into two steps. Firstly, a three-points bending test is carried out using three Rohacell ${ }^{\circledR}$ specimens of each typology, tested twice, with length $L=218 \mathrm{~mm}$, width $b=67 \mathrm{~mm}$ and height $h=20 \mathrm{~mm}$. In Figure 8.5, the experimental set-up used for the three-points bending test is shown: a displacement-control system $(B)$ is fixed to a rigid frame (A); the load cell ( $C$ ) measures the load applied to the specimen for each applied
transverse displacement and two linear displacement transducers, LVDT, ( $D$ and $E$ ) are located at the center of the specimen and at a quarter of its length. The data coming from the load cell and the linear displacement transducers are stored and plotted in Figures 8.6 and 8.7 along with the linear interpolation of each experimental set of measures. To obtain the core Young's and shear moduli, a bi-parametric analysis with a high-fidelity FE model (1,200 linear-elastic QUAD4 membrane elements) is performed. The Young's and shear moduli are changed assuming values within an interval defined by the variation of $\pm 20 \%$ of the nominal values (see Table 2) and the FE solution computed several times in order to obtain the surface as those in Figures 8.8 and 8.9 (for sake of conciseness, only the WF110 are displayed). The surface in Figure 8.8 refers to the deflection at the center of the core specimen while that in Figure 8.9 concerns the deflection at a quarter of the specimen length. The Young's and shear moduli are obtained by finding the couple of values that gives the same experimental value of deflection at each location. By considering Figures 8.8 and 8.9 it is easy to understand that no dependence on the shear modulus appears and the only parameter that can be estimated is the Young's modulus. This is due to the negligible effect of the shear deformation to the total deflection of the specimen.


Figure 8.4. Ergal stress/strain curve obtained by the standard test.


Figure 8.5. Rohacell ${ }^{\circledR}$ three-points bending test: A. rigid frame; B. displacement-control system; C. load cell; D., E. linear displacement transducers.

By performing the same bending analysis with the Timoshenko's beam theory, the same results of the finite element analysis are obtained, with an error lower than $1 \%$. Thus, considering the Timoshenko's beam theory, the deflection $w$, at a given location along the beam axis, of a simply supported beam with a concentrated force at the beam mid-span reads as

$$
\begin{equation*}
w=\alpha \frac{P L^{3}}{E J}+\beta \frac{P L}{k^{2} G A} \tag{8.1}
\end{equation*}
$$

where $P, L, A, J$ are, respectively, the applied load, the beam length, the cross-section area and the moment of inertia. Since Eq. (8.1) is consistent with the Timoshenko's beam theory, a shear correction factor $k^{2}=5 / 6$, adequate for an isotropic single layer, is required to reduce the transverse shear stiffness. The parameters $\alpha$ and $\beta$ depend on the location along the beam axis: for a simply supported beam, they assume values $1 / 48$ and $1 / 4$, respectively, for the central deflection, and $11 / 768$ and $1 / 8$, for the deflection at a quarter of the beam length.

By taking into consideration Eq. (8.1), it is easy to note that the deflection is given by the superposition of two contributions: the first one due to the bending and the second one due to the transverse shear. The ratio between the deflection due to the bending, namely $w_{b}$, and the contribution due to the shear, $w_{s}$, is

$$
\begin{equation*}
w_{s} / w_{b} \sim\left(\frac{h}{L}\right)^{2} \frac{1}{6 k^{2}} \tag{8.2}
\end{equation*}
$$

Due to the specimen dimension, the ratio in Eq. (8.2) reaches an almost vanishing value, about 0.002 . This means that the measured deflection is only due to the bending, that is, the only mechanical property that affects the deflection is the Young's modulus whereas the shear one is ineffective. Thus, from this first step, the Young's modulus of the Rohacell ${ }^{\circledR}$ IG31 and that of the Rohacell ${ }^{\circledR}$ WF110 is obtained by averaging three values, for each foam, obtained by comparing the experimental values of transverse displacement with those coming from the FE model.

The second step is focused on the estimation of the shear modulus. From the above consideration, it results clear that for a homogeneous isotropic specimen the transverse shear deformation effect on the total deformation depends only on the slenderness, $L / h$. A way to augment the shear deformation effect is by increasing the bending-to-shear stiffness ratio. For this reason a sandwich-like stacking sequence wherein the Rohacell ${ }^{\circledR}$ core is bounded by two 7075 Aluminum alloy face-sheets is considered and the results coming from the experimental four-points bending test (see Sect. 4), are used as reference to obtain the core shear modulus. Two specimens are considered: the IG31_32_5_SS and the WF110_32_5_SS, in order to characterize both the polymeric foams employed in the experimental campaign. By using the experimental results (see Sect. 4) in terms of transverse displacement at the center and under the load pin of the sandwich specimen, an updating of a high-fidelity FE model (made by linear-elastic QUAD8 elements: 10 elements along the face-sheets thickness, 12 along the core thickness and 640 along the beam length) is carried out by assuming as Aluminum mechanical proprieties those obtained by the material characterization, whereas, for the Rohacell ${ }^{\circledR}$ core, the Young's modulus coming from the former step is assumed. The Rohacell ${ }^{\circledR}$ shear modulus, instead, assumes different values in the range $\left[0.8 \mathrm{G}_{\mathrm{n}} ; 1.2 \mathrm{G}_{\mathrm{n}}\right]$ where $\mathrm{G}_{\mathrm{n}}$ is the nominal shear modulus, see Table 2. For the IG31_32_5_SS specimen, and similarly for the WF110_32_5_SS, the value of the shear modulus that ensures a match between the experimental displacement and the numerical one, with a maximum error of $1 \%$, is pinpointed (see Figures 8.10 and 8.11). Since the four-points bending test provides two values of displacements in two different positions (see Sect. 4), for each sandwich specimen, the match has been made two times. Thus, the shear modulus for each foam is the average between these two values. In Table 4 the mechanical proprieties are collected.


Figure 8.6. Three-points bending test: force-displacement curve (Rohacell ${ }^{\circledR}$ IG31).


Figure 8.7. Three-points bending test: force-displacement curve (Rohacell ${ }^{\circledR}$ WF110).


Figure 8.8. Deflection (in mm) at the center of the WF110 specimen as function of the Young's and shear moduli (in MPa). Finite elements result with an applied force of 1 N .


Figure 8.9. Deflection (in mm ) at a quarter of the length of the WF110 specimen as function of the Young's and shear moduli (in MPa). Finite elements result with an applied force of 1 N .

It is worth to note that, for the IG31 foam (see Figure 8.10), the values of the shear modulus are not almost identical as for the WF110 foam (see Figure 8.11). Moreover, as shown in Table 4, the standard deviation on the Young's modulus of the IG31 is greater, in terms of percentage with respect to the mean value, of that of WF110, thus giving the idea of a greater dispersion of the mechanical proprieties of the former foam with respect the latter one.

## 4. Four-points bending test

In this section, experimental results concerning the four-points bending test on sandwich beams are presented and compared with the RZT numerical ones.

### 4.1. Experimental set-up

The four-points bending test is performed on the universal testing machine METROCOM (see Figure 8.12) equipped with two linear displacement transducers (HBMLVDT $\pm 2.5 \mathrm{~mm}$ ), a load cell (HBM- Strain Gage Load Cell, 200 kg ) and a load transmission system (two cylinders connected to the load cell by means of a rigid plate). Along with the transverse displacements in two positions along the beam axis, the axial strain is measured by using a strain gage located on the bottom beam face. The load and boundary conditions, as the positions of the LVDTs and the strain gage, are depicted in Figure 8.13. The distance between the two load cylinders is $L_{d}=110.5 \mathrm{~mm}$ and that between the support and the load cylinder is $L_{l}=105.25 \mathrm{~mm}$. As for the material characterization of core, the test is performed in displacement control.

Table 4. Characterized material mechanical proprieties: the Young's and shear modulus in MPa. For the Aluminum alloy and the core shear modulus, due to the reduced number of test, only the mean value is given. For the core Young's modulus, six test have been performed and the standard deviation is given along with the average value.

| Material | E | G |
| :---: | :---: | :---: |
| 7075 Aluminum alloy $_{\text {Rohacell }}$ IG31 | 69570 | 25766 |
| Rohacell $^{\circledR}$ WF110 | $40.3 \pm 4.9$ | 12.4 |
|  | $196 \pm 8.6$ | 65.4 |



Figure 8.10. Rohacell $^{\circledR}$ IG31 shear modulus characterization: updating of the numerical model via experimental results on a four-points bending test of IG31_32_5_SS specimen.


Figure 8.11. Rohacell $^{\circledR}$ WF110 shear modulus characterization: updating of the numerical model via experimental results on a four-points bending test of WF110_32_5_SS specimen.


Figure 8.12. Four-Points Bending Test: experimental set-up.


Figure 8.13. Load and boundary conditions, sensors placement.

### 4.2. Results

The results of the four-points bending test are reported in Figures 8.14-8.17 in terms of transverse displacement at the center of the beam, $w_{m}$, transverse displacement under the load cylinder, $w_{c}$, and axial strain measured at the center of the beam with a strain gage located at the bottom beam surface. The results concerning the IG31_32_5_SS beam are given in Figures 8.14 and 8.15, while those for WF110_32_5_SS beam are shown in Figures 8.16 and 8.17. In both case, the RZT solution has been obtained employing 440 beam anisoparametric constrained element, as formulated in [Gherlone et al., 2011].


Figure 8.14. Four-points bending test: IG31 core, transverse displacements in two locations.


Figure 8.15. Four-points bending test: IG31 core, axial strain at the center of the beam, bottom face.


Figure 8. 16. Four-points bending test: WF110 core, transverse displacements in two locations.


Figure 8.17. Four-points bending test: WF110 core, axial strain at the center of the beam, bottom face.

In both cases, the RZT results fit very well with the experimental values, both in terms of displacement and strain. The error on the transverse displacement valuated at the center is around the $1.7 \%$ whereas that under the load pin is $1.1 \%$, for the IG31_32_5_SS beam. The error on the maximum deflection (at the center of the beam) reduces at $0.8 \%$, for the WF110_32_5_SS beam, while that concerning the deflection under the load remains unchanged. Greater difference between the two beams appears about the axial strain: for the IG31_32_5_SS, the error reaches the $8.9 \%$, whereas it is around the $4.4 \%$ for the WF110_32_5_SS beam.

## 5. Hammer test

In this section, results concerning the experimental evaluation of the first five natural frequencies of sandwich beams are presented. The experimental measure is performed by means of an hammer test [Ewins, 1984]. For this purpose, a cantilever constraint has been realized and the specimens involved in this experimental campaign are those listed in Table 1 with notation $C F$. It is possible to note that the specimens consist in sandwich beams made by both Rohacell ${ }^{\circledR}$ IG31 and WF110 core, in order to investigate the effect of the core stiffness, and are characterized by two nominal values of the face-to-core thickness ratio, $5 / 6$ and $2 / 20$, and two nominal values of the length-to-thickness ratio, 20 and 40 .

### 5.1. Experimental set-up

In the experiment, the natural frequencies are measured by the experimental modal analysis wherein the beam is excited by an impulse hammer (Bruel \& Kjaer 8202 with force transducer 8200) and the acceleration responses are observed by ten accelerometers (Bruel \& Kjaer 4518). The excitation force and the acceleration signal are stored using LMS-Siemens SCADAS III and processed by a workstation to obtain the frequency response function (FRF). The Least Square Complex Exponential algorithm (implemented in LSM-Siemens Test.Lab) was used to estimate the modal data. The beam is cantilevered at one end by two heavy and rigid steel blocks joined each other by bolted connections. In Figure 8.18 the experimental set-up for the IG31_32_5_CF is shown as example. The accelerometers are placed along the beam axis being careful to avoid the coincidence with the nodes of the first five modes (known in advance by means of a FE analysis).

### 5.2. Results

In Tables 5-10, the RZT and Timoshenko's beam theory (TBT) results are given in terms of percentage errors with respect to the experimental values. For reference, the experimental frequency values are listed in Tables 5-10. Moreover, in Figure 8.19 the frequency response function (FRF) obtained for the IG31_32_5_CF is displayed whereas in Figure 8.20, the comparison of the first four modes is depicted.

The TBT results are obtained by using two shear correction factors, $k^{2}$ : the first one is the common used value $5 / 6$, whereas the second one is an ad hoc computed according to the procedure proposed in [Raman et al, 1996]. The TBT employing the $k^{2}=5 / 6$ heavily overestimates the natural frequencies leading to percentage errors that can exceed the $100 \%$. The erroneous estimation of the natural frequencies is consequence of a not suitable shear correction factor that results disproportionate in relation with the stacking sequence, characterized by a sizeable face-to-core stiffness ratio. In fact, the ad hoc estimated shear correction factor is orders of magnitude lower than the $5 / 6$ value.

The adoption of a suitable shear correction factors improves the TBT predictions only for the first natural frequency, whereas the errors on the higher-order mode frequencies still remain noticeable. The higher-order modes, in fact, involve cross-section distortion phenomenon that are not accounted for in the TBT. Figures 8.21 show a zoom of the normal modes obtained by a high-fidelity FE model (made by linear-elastic QUAD4 elements, 699,842 dof's) for the WF110_64_5_CF specimen. As it is easy to notice, the first normal mode is characterized by a cross-section that remains flat whereas the third and fifth normal modes involve cross-section distortion phenomenon not modeled by the TBT kinematic assumption. Moreover, the improvements reachable by adopting the ad hoc shear correction factors depend also on the face-to-core stiffness ratio: for highly heterogeneous sandwiches (Tables 5 and 6) the error on the first natural frequency obtained by the TBT with the ad hoc shear correction factor are $22 \%$ and $10 \%$, according to the beam slenderness. Considerations about the effect on the natural frequencies prediction of the shear correction factors in sandwich-like stacking sequences, by changing the face-to-core stiffness and thickness ratio, are discussed in [Iurlaro et al., 2013a].

The RZT appears substantially more accurate than the TBT adopting the ad hoc shear correction factor. The improvement is noticeable both on the first frequency (Tables 5 and 6) and on the higher-order mode frequencies (Tables 7-10). It is worth to note that the RZT error does not manifest a monotonic trend, contrary to the TBT for which the error
increases by raising the frequency. This is ascribable to the core material properties dispersion that affects in different way the RZT and TBT solution. In particular, attention has to be focused on the shear modulus: in the TBT only one transverse shear stress resultant appears, thus originating one stiffness coefficient related to the shear modulus; whereas, the RZT provides two transverse shear stress resultants thus giving rise to two stiffness coefficients depending on the shear modulus. By increasing the mode order, these stiffness coefficients can participate to the vibration in different way thus making the shear modulus dispersion more or less influential. It is worth to note that the errors obtained by the experimental campaign carried out are in agreement with those obtained by Narita and co-workers [Honda et al., 2013] within an experimental assessment of the RZT for plates.


Figure 8.18. Hammer test: experimental set-up.
Table 5. First five natural frequencies for IG31_32_5_CF specimen.

| Mode | Experimental | RZT | TBT $\left(k^{2}=5 / 6\right)$ | TBT $\left(k^{2}=1.1074 \cdot 10^{-3}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 83.9 | $-5.0 \%$ | $70.8 \%$ | $-22.1 \%$ |
| 2 | 331.5 | $-7.2 \%$ | $159.7 \%$ | $-41.0 \%$ |
| 3 | 772.0 | $-3.9 \%$ | $214.4 \%$ | $-53.3 \%$ |
| 4 | 1409 | $-0.3 \%$ | $172.1 \%$ | $-62.6 \%$ |
| 5 | 2254 | $1.8 \%$ | $115.6 \%$ | $-69.0 \%$ |



Figure 8.19. IG31_32_5 specimen: frequency response function.

Table 6. First five natural frequencies for IG31_64_5_CF specimen.

| Mode | Experimental | RZT | TBT $\left(k^{2}=5 / 6\right)$ | TBT $\left(k^{2}=1.1080 \cdot 10^{-3}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 30.4 | $-4.8 \%$ | $24.8 \%$ | $-10.0 \%$ |
| 2 | 133 | $-14.5 \%$ | $77.3 \%$ | $-30.2 \%$ |
| 3 | 295 | $-15.6 \%$ | $122.6 \%$ | $-39.0 \%$ |
| 4 | 482 | $-13.1 \%$ | $163.6 \%$ | $-46.3 \%$ |
| 5 | 713 | $-10.4 \%$ | $169.7 \%$ | $-52.5 \%$ |

Table 7. First five natural frequencies for WF110_32_5_CF specimen.

| Mode | Experimental | RZT | TBT $\left(k^{2}=5 / 6\right)$ | TBT $\left(k^{2}=5.8227 \cdot 10^{-3}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 111 | $8.0 \%$ | $35.2 \%$ | $3.6 \%$ |
| 2 | 498 | $-3.6 \%$ | $85.4 \%$ | $-18.5 \%$ |
| 3 | 1103 | $-5.3 \%$ | $127.8 \%$ | $-27.5 \%$ |
| 4 | 1797 | $-3.6 \%$ | $112.4 \%$ | $-35.4 \%$ |
| 5 | 2635 | $-1.1 \%$ | $80.0 \%$ | $-42.1 \%$ |



Figure 8.20. First four mode shapes of the IG31_32_5_CF beam: experimental mode shapes are represented at accelerometers' positions only.

Table 8. First five natural frequencies for WF110_64_5_CF specimen.

| Mode | Experimental | RZT | TBT $\left(k^{2}=5 / 6\right)$ | TBT $\left(k^{2}=5.8194 \cdot 10^{-3}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 34.5 | $3.0 \%$ | $10.9 \%$ | $2.3 \%$ |
| 2 | 177 | $-4.8 \%$ | $33.8 \%$ | $-9.6 \%$ |
| 3 | 417 | $-8.5 \%$ | $58.6 \%$ | $-16.7 \%$ |
| 4 | 688 | $-10.0 \%$ | $86.3 \%$ | $-22.1 \%$ |
| 5 | 986 | $-9.9 \%$ | $93.5 \%$ | $-26.6 \%$ |

Table 9. First five natural frequencies for WF110_48_2_CF specimen.

| Mode | Experimental | RZT | TBT $\left(k^{2}=5 / 6\right)$ | TBT $\left(k^{2}=1.5311 \cdot 10^{-2}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 97.6 | $3.0 \%$ | $22.7 \%$ | $2.5 \%$ |
| 2 | 403 | $-3.5 \%$ | $80.1 \%$ | $-5.4 \%$ |
| 3 | 848 | $-6.4 \%$ | $129.7 \%$ | $-8.6 \%$ |
| 4 | 1279 | $-7.6 \%$ | $85.1 \%$ | $-10.3 \%$ |
| 5 | 1715 | $-7.8 \%$ | $111.3 \%$ | $-11.3 \%$ |

Table 10. First five natural frequencies for WF110_96_2_CF specimen.

| Mode | Experimental | RZT | TBT $\left(k^{2}=5 / 6\right)$ | TBT $\left(k^{2}=1.5293 \cdot 10^{-2}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 28.4 | $1.1 \%$ | $6.3 \%$ | $1.0 \%$ |
| 2 | 148 | $-3.2 \%$ | $26.5 \%$ | $-3.8 \%$ |
| 3 | 341 | $-5.3 \%$ | $51.7 \%$ | $-6.1 \%$ |
| 4 | 557 | $-7.2 \%$ | $79.7 \%$ | $-8.3 \%$ |
| 5 | 777 | $-8.3 \%$ | $52.3 \%$ | $-9.6 \%$ |



Figure 8.21. Tip cross-section deformation for the WF110_64_5_CF specimen (MSC/MD-NASTRAN ${ }^{\circledR}$ results): (a) first mode shape; (b) third mode shape; (c) fifth mode shape.

## Summary and conclusions

The research work presented in this Thesis is devoted to the development and assessment of accurate and computationally efficient theories and numerical tools based on these theories. Chapters 2-5 are earmarked to the theoretical developments that serve as basis for the numerical results shown in Chapter 6 and 7 and the experimental ones discussed in Chapter 8.

The cornerstone and the starting point of the activity is the Refined Zigzag Theory, a recently developed zigzag model. Some of the analytical results presented in Chapter 6 are devoted to a further investigation of the Refined Zigzag Theory prediction capabilities over a wide range of problems mainly concerning to the linear bending, free vibrations and buckling loads of sandwich plates, subjected to several load and boundary conditions. In the framework of the Zigzag models, as highlighted in Chapters 1 and 2, the cross-section distortion can be basically modeled in two ways: adopting the Di Sciuva's type zigzag function, that is the Refined Zigzag Theory function, and the Murakami's zigzag function. The problems solved in Chapter 6 constitute also the occasion to compare the prediction capabilities of the Murakami's zigzag function when employed in a first-order zigzag model, that is a model that differs from the Refined Zigzag one only for the zigzag function adopted. This comparison partially covers a lack in the open literature and it is considered, by the author, necessary due to the increasingly interest of the research community in the zigzag models. The comparison involve also the First-Order Shear Deformation Theory, a widely used model that represents the underlying theory of all the finite elements implemented in commercial codes, and higher-order models belonging to the Equivalent Single Layer models class. The results serve to demonstrate that adding a
zigzag contribution to a kinematics is more efficient than enriching the same kinematics with higher-order terms. This observation is sustained by results concerning the global (maximum deflection, natural frequencies and critical buckling loads) and local (through-the-thickness distributions of displacements and stresses) responses of sandwich plates, including also functionally graded layers. With respect to the higher-order models, remarkable accuracy in the global response can also be reached by adopting a suitable shear correction factor in the framework of the First-Order Shear Deformation Theory. Even if accurate in global responses, the First-Order Shear Deformation Theory results unable to fit with the reference solutions when local responses are required. The viable solution is to add a zigzag contribution to the First-Order Shear Deformation Theory kinematics, thus originating a first-order zigzag model. The Refined Zigzag Theory belongs to the first-order zigzag models class, due to its kinematic assumptions, and can be compared with the results obtained by using a first-order zigzag model adopting the Murakami's zigzag function. The numerical results of Chapter 6 demonstrate a superior prediction capabilities of the Refined Zigzag Theory over the First-Order Shear Deformation Theory and the higher-order models in the bending, free vibrations and buckling load problems. When compared with the first-order zigzag model employing the Murakami's zigzag function, three scenarios arise: the two models give the same results; the latter one reaches results between, in terms of accuracy, the Refined Zigzag Theory and the First-Order Shear Deformation one (adopting the shear correction factors); the firstorder Murakami's zigzag model and the First-Order Shear Deformation Theory (without shear correction factors) results are the same. The scenarios depend on the stacking sequence of laminate: for symmetric and periodic ones, the Murakami's zigzag function ensures results equal to those obtained by the Refined Zigzag Theory, both in terms of global and local responses; when the symmetry and periodicity of the stacking sequence are destroyed, the Murakami's zigzag function leads to erroneous results.

The Refined Zigzag Theory is also a suitable model for an efficient $C^{0}$ finite element implementation. As discussed in Chapter 5, beam and plate Refined Zigzag Theory-based element have been already developed. A novel beam element is introduced in Chapter 5, wherein shape functions derived by the solution of the homogeneous static equilibrium equations are employed in the element formulation. Results in Chapter 7 demonstrate a superconvergent behavior of this element with respect to an element based on the same underlying theory, but employing different shape functions, both in static and free vibrations problems.

Consistent with the kinematic assumption, the Refined Zigzag Theory provides piecewise constant constitutive transverse shear stresses along the thickness that are accurate, in an average sense, in each layers but violates the continuity conditions at layer interfaces. The way to improve the constitutive transverse shear stresses prediction is to develop a Mixed Refined Zigzag Theory (Chapter 3) based on the Reissner Mixed Variational Theorem. Results in Chapter 6 demonstrate the improvements achievable by adopting a mixed model, both in terms of through-the-thickness distribution of constitutive transverse shear stresses and in terms of global response predictions (maximum deflection and natural frequencies). The accuracy ensured by the Mixed Refined Zigzag model depends on the assumed transverse shear stress modeling strategy. Two strategies are considered: the first one is an equilibrium-based one, wherein the assumed transverse shear stresses are derived by integration of the local equilibrium equations; the second one is the widely-used, in the open literature, layer-wise polynomial approximation. The first way to assume the transverse shear stresses overcomes in accuracy the second one, as demonstrated and discussed in detail in Chapter 6.

The theoretical developments presented in Chapter 4, are devoted to a novel higherorder model, called (3,2)-Mixed Refined Zigzag Theory, based on the Reissner Mixed Variational Theorem, that arises from the Refined Zigzag Theory. The purpose is to develop a model that accounts for the non-linear piece-wise distribution of displacements along the thickness, the transverse normal deformability effect and the transverse normal stress in thick multilayered composite and sandwich beams/plates. The additional requirement is to develop a higher-order zigzag model involving the lower number of kinematic variables in order to save the computational cost. For this aim, the original kinematic assumption has been condensed out in a reduced one that represents the theoretical basis for the finite element implementations of Chapter 5. The theoretical assessment has been performed on a single bending problem of a thick laminates in Chapter 6, wherein the prediction capabilities of the model in providing accurate through-the-thickness distribution of displacements and stresses is demonstrated. Further investigations on the model predictive capabilities are carried out in Chapter 7, wherein accurate finite elements results concerning static and dynamic problems are presented.

Finally, in Chapter 8, for the first time, an experimental campaign is performed on a series of sandwich beams, with different core, slenderness and core-to-face thickness ratio. Both four-point bending tests and hammer tests are considered and the experimental results compared with those obtained by using the Refined Zigzag Theory and the Timoshenko's
beam theory. The comparison with the experimental results demonstrates great accuracy of the Refined Zigzag theory both in static and free vibration problems. Moreover, the comparison with the Timoshenko's beam theory, wherein the accuracy of the solution is augmented via the adoption of a suitable shear correction factor, demonstrates the superior capabilities of the Refined Zigzag Theory in predicting the first five natural frequencies for highly heterogeneous sandwich beam.

The research activity herein presented, supported by the numerical results, encourages the adoption of the zigzag models in the multilayered composite and sandwich structures. In particular, the work identifies in the Refined Zigzag Theory and the further theoretical developments that originate from it (Mixed Refined Zigzag Theory and (3,2)-Mixed Refined Zigzag Theory), very appealing models in virtue of their accuracy, computational cost and efficient finite element implementations.

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