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A quasilinear differential inclusion for viscous and rate-independent damage systems in non-smooth domains

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Abstract

This paper focuses on (incomplete) rate-independent damage in elastic bodies. Since the driving energy is nonconvex, solutions may have jumps as a function of time, and in this situation it is known that the classical concept of energetic solutions for rate-independent systems may fail to accurately describe the behavior of the system at jumps.

Therefore we resort to the (by now well-established) *vanishing-viscosity* approach to rate-independent processes, and approximate the model by its viscous regularization. In fact, the analysis of the latter PDE system presents remarkable difficulties, due to its highly nonlinear character. We tackle it by combining a *variational* approach to a class of abstract doubly nonlinear evolution equations, with careful regularity estimates tailored to this specific system, relying on a q-Laplacian type gradient regularization of the damage variable. Hence for the viscous problem we conclude the existence of weak solutions, satisfying a suitable energy-dissipation inequality that is the starting point for the vanishing-viscosity analysis. The latter leads to the notion of *(weak) parameterized solution* to our rate-independent system, which encompasses the influence of viscosity in the description of the jump regime.

Keywords: Rate-independent damage evolution, vanishing-viscosity method, arclength reparameterization, time discretization, regularity estimates
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1 Introduction

We analyze the following PDE system for damage evolution

$$-\operatorname{div}(g(z)\mathbb{C}\varepsilon(u+u_D)) = \ell \quad \text{in } \Omega\times(0,T), \tag{1.1a}$$

$$\partial\mathrm{R}_1(z_t) - \operatorname{div}\left((1+|\nabla z|^2)^{\frac{q-2}{2}}\nabla z\right) + f'(z) + \frac{1}{2}g'(z)\mathbb{C}\varepsilon(u+u_D):\varepsilon(u+u_D) \ni 0 \quad \text{in } \Omega\times(0,T). \tag{1.1b}$$

Here, $\Omega\subset\mathbb{R}^d$, with $d\geq 2$, is a bounded Lipschitz domain, occupied by a body subject to damage, $u:\Omega\times[0,T]\to\mathbb{R}^d$ the displacement vector, $\varepsilon(u)$ denoting the symmetrized strain tensor, and $z:\Omega\times[0,T]\to[0,1]$ the damage parameter. Within the approach of Generalized Standard Materials (see also [20] and [14] for stress softening), we model the degradation of the elastic behavior of the body through the internal variable z, which assesses the soundness of the material: for $z(x,t)=1$ ($z(x,t)=0$, respectively) the material is in the undamaged state (in the maximum damaged state), "locally" around $x\in\Omega$ and at time $t\in[0,T]$; the intermediate case $0<z(x,t)<1$ describes partial damage. We consider a gradient regularization for z, which leads to the q-Laplacian operator in (1.1), with $q>d\geq 2$. Rate-independence and unidirectionality of damage evolution stem from the 1-positively homogeneous dissipation potential

$$\mathrm{R}_1:\mathbb{R}\to[0,\infty], \qquad \mathrm{R}_1(\eta)=\begin{cases}\kappa|\eta| & \text{if } \eta\leq 0,\\ \infty & \text{otherwise,}\end{cases}$$

with $\kappa>0$ a given material dependent parameter. R_1 enforces the constraint that $z_t(x,t)\leq 0$ on $\Omega\times(0,T)$; the operator $\partial\mathrm{R}_1:\mathbb{R}\rightrightarrows\mathbb{R}$ is its subdifferential in the sense of convex analysis. Furthermore, $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to(0,\infty)$ are given constitutive functions, $\mathbb{C}=\mathbb{C}(x)$ is the (positive definite, symmetric) x-dependent elasticity tensor, u_D a Dirichlet datum, and ℓ is the external loading. System (1.1) is supplemented with zero Neumann conditions for z on $\partial\Omega$ and with mixed boundary conditions for u on $\partial\Omega=\Gamma_D\cup\Gamma_N$, where Γ_D is a closed subset of $\partial\Omega$ on which Dirichlet boundary conditions are prescribed. For shortness, in this introduction we assume $u_D=0$. We suppose that $g(z)\geq c>0$ for all $z\in\mathbb{R}$: joint with the positive-definiteness of the tensor $\mathbb{C}$, this excludes elliptic degeneracy of equation (1.1a) even in the case of maximal damage, i.e. for $z(x,t)=0$. Namely, here we rule out *complete damage*.

Observe that (1.1a) is the Euler-Lagrange equation for the minimization, with respect to the variable u, of the stored energy functional $\mathcal{E}:[0,T]\times\mathcal{U}\times\mathcal{Z}\to\mathbb{R}$

$$\mathcal{E}(t,u,z):=\frac{1}{q}\int_\Omega(1+|\nabla z|^2)^{\frac{q}{2}}\,\mathrm{d}x+\int_\Omega f(z)\,\mathrm{d}x+\frac{1}{2}\int_\Omega g(z)\mathbb{C}\varepsilon(u):\varepsilon(u)\,\mathrm{d}x-\langle\ell(t),u\rangle_{\mathcal{U}}, \tag{1.2}$$

with the state spaces $\mathcal{U}=\{\,v\in W^{1,2}(\Omega,\mathbb{R}^d)\,;\,v|_{\Gamma_D}=0\,\}$ for u, and $\mathcal{Z}=W^{1,q}(\Omega)$ for z. In fact, in what follows we are going to treat (1.1) as an abstract evolution equation set in the dual space $\mathcal{Z}^*$, viz.

$$\begin{aligned}&\partial\mathcal{R}_1(z'(t))+A_qz(t)+f'(z(t))+\frac{1}{2}g'(z(t))\mathbb{C}\varepsilon(u(t)):\varepsilon(u(t))\ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t\in(0,T),\\ &u(t)\in\operatorname{Argmin}\{\,\mathcal{E}(t,v,z(t))\,;\,v\in\mathcal{U}\,\} \quad \text{for a.a. } t\in(0,T),\end{aligned} \tag{1.3}$$

with $\mathcal{R}_1:L^1(\Omega)\to[0,\infty]$ defined by

$$\mathcal{R}_1(\eta)=\int_\Omega\mathrm{R}_1(\eta(x))\,\mathrm{d}x, \tag{1.4}$$

$\partial\mathcal{R}_1:\mathcal{Z}\rightrightarrows\mathcal{Z}^*$ its (convex analysis) subdifferential, and A_q denoting the q-Laplacian operator $A_qz=-\operatorname{div}(1+|\nabla z|^2)^{\frac{q-2}{2}}\nabla z$ with zero Neumann boundary conditions. Introducing the *reduced* energy $\mathcal{I}:[0,T]\times\mathcal{Z}\to\mathbb{R}$ by $\mathcal{I}(t,z)=\inf_{v\in\mathcal{U}}\mathcal{E}(t,v,z)$, we can further reformulate (1.3) as

$$\partial\mathcal{R}_1(z'(t))+\mathrm{D}_z\mathcal{I}(t,z(t))\ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t\in(0,T), \tag{1.5}$$

where $\mathrm{D}_z\mathcal{I}$ is the Gâteaux derivative of $\mathcal{I}$ w.r.t. z.

Since $\mathcal{R}_1$ has only linear growth and the reduced energy $\mathcal{I}(t,\cdot)$ has no uniform convexity properties, solutions to (1.5) are, in general, only BV-functions of time. This calls for weak, derivative-free solvability concepts for (1.5): first and foremost, the notion of *energetic solution* by Mielke & Theil [44, 39]. For *incomplete* damage, the existence of energetic solutions to a version of (1.3) was established for $q > d$ in [43], and extended to $q > 1$ in [54]; in [53] the case of a BV-regularization (i.e. $q = 1$) was analyzed, whereas in [17] a model without gradient terms for z was investigated by means of an energetic-type solution concept relaxed via Young measures. We also refer to [3] for the study of a model undergoing damage and fracture, while in [21] the evolution of damage in a material that can be in two configurations at the microscale is examined via the concept of *threshold solution*.

Over the last years, it has been realized that the description of rate-independent evolution resulting from the global stability condition of the energetic solution concept does not seem to be mechanically feasible in the case of a nonconvex driving energy. Indeed, in order to satisfy the global stability, energetic solutions may change instantaneously in a very drastic way, jumping into very far-apart energetic configurations (see, for instance, [32, Ex. 6.3], [40, Ex. 1], as well as the characterization of energetic solutions to one-dimensional rate-independent systems provided in [49]). This observation has motivated the introduction of alternative weak solution notions. A well-established approach for deriving a concept which accurately describes the behavior of the solution at jumps is taking the *vanishing-viscosity* limit in the *viscous* approximation of a given rate-independent system. Starting from the seminal paper [16], this technique has by now been thoroughly developed both for abstract rate-independent systems [40, 41, 45], and in the applications to fracture [55, 32, 34, 36], and to plasticity [4, 12, 13, 19].

Following on the analysis initiated in [33], in this paper we develop this approach for the damage system (1.3), and accordingly consider its viscous regularization

$$\partial\mathcal{R}_1(z'(t)) + \epsilon z'(t) + A_q z(t) + f'(z(t)) + \frac{1}{2}g'(z(t))\mathbb{C}\varepsilon(u(t)) : \varepsilon(u(t)) \ni 0 \text{ in } \mathcal{Z}^* \text{ for a.a. } t \in (0,T), \quad (1.6)$$

with $u(t) \in \mathrm{Argmin}\{\, \mathcal{E}(t, v, z(t))\,;\, v \in \mathcal{U}\,\}$ for almost all $t \in (0,T)$. Observe that (1.6) rewrites as

$$\partial\mathcal{R}_\epsilon(z'(t)) + \mathrm{D}_z\mathcal{I}(t, z(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0,T), \quad (1.7)$$

with $\mathcal{R}_\epsilon(\eta) := \mathcal{R}_1(\eta) + \frac{\epsilon}{2}\|\eta\|^2_{L^2(\Omega)}$. In fact, the analysis of (1.6) is itself fraught with analytical difficulties. In what follows, we briefly hint at them, and illustrate our approach and existence result, Theorem 3.5, for the Cauchy problem associated with (1.6). We then describe the vanishing-viscosity analysis of (1.6).

The viscous problem: mathematical difficulties and existing results The most evident difficulty attached to the analysis of (1.6) is the presence of the quadratic term $g'(z(t))\mathbb{C}\varepsilon(u(t)){:}\varepsilon(u(t))$. The basic energy estimate for (1.6) provides a (uniform w.r.t. time) $W^{1,2}(\Omega;\mathbb{R}^d)$-bound for u which, even assuming $|g'(z)| \le C$, only gives an $L^1(\Omega)$-estimate for $g'(z)\mathbb{C}\varepsilon(u) : \varepsilon(u)$. Therefore, it is necessary to enhance the spatial regularity of u, which requires performing enhanced regularity estimates on (1.6). The latter issue poses further difficulties due to the *doubly nonlinear* character of (1.6), because of the simultaneous presence of the nonlinear q-Laplacian operator A_q, and of the multivalued operator $\partial\mathcal{R}_1 : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$.

Last but not least, since the domain of $\mathcal{R}_1$ is not the whole space $\mathcal{Z}$, $\partial\mathcal{R}_1$ is an *unbounded* operator. This rules out the possibility of deriving bounds for $\mathrm{D}_z\mathcal{I}$ by comparison arguments in (1.7). Since the term $f'(z)$ contributing to $\mathrm{D}_z\mathcal{I}$ may be considered of lower order under suitable assumptions on f, the problem boils down to deriving further estimates for $A_q z$ and, again, for the quadratic term $g'(z)\mathbb{C}\varepsilon(u) : \varepsilon(u)$.

All of these difficulties are reflected in the results available in the literature on damage problems, starting from the first, pioneering paper on the viscous system (1.6), viz. [7]. There, the Laplace operator (i.e., $q = 2$) was considered but a gradient regularizing term $A_2 z_t$ was also added, enabling the authors to derive enhanced estimates on z by resorting to elliptic regularity results, valid under suitable smoothness assumptions on the domain Ω. The latter are also at the core of the analysis developed in the paper [8], where the flow rule for z (with $q = 2$) is coupled with a parabolic equation for u, in the context of linear viscoelasticity. The authors of [8] exploit the available estimates on the viscous term $\varepsilon(u_t)$, and elliptic regularity arguments on u, in order to test (1.6) by $\partial_t(A_2 z + f'(z))$. This allows them to estimate the term $A_2 z$ and to gain enhanced spatial regularity for z, again by elliptic regularity. Refined estimates combined with regularity assumptions on the domain Ω are crucial also in [6], extending the analysis to a temperature-dependent model.

In the recent [27], different techniques have been adopted to analyze models coupling damage with phase separation processes in elastic bodies (see also [28]). Also in [27], a q-Laplacian regularization with $q > d$ (d being the space dimension) is used in order to ensure $\mathrm{C}^0(\overline{\Omega})$-regularity for z. Because of the complexity of the overall system for damage and phase separation, and because of the triply nonlinear character of the equation for the damage parameter (featuring the q-Laplacian and the multivalued operators $\partial\mathcal{R}_1$ and $\partial I_{[0,1]}$), the authors are able to prove existence only for a weak solution notion.

The viscous problem: our results Our aim is to analyze (1.3) and its viscous approximation (1.6) under *minimal regularity* assumptions on Ω. This is meaningful in view of the applications to engineering problems, where the spatial domain occupied by the elastic body is usually far from being of class C^2. Therefore, we have to apply refined elliptic regularity results to enhance the spatial regularity of u.

Let us motivate the choice of $q > d$ for the q-Laplacian operator A_q. Since the damage variable z enters into the coefficients of the operator of linear elasticity $-\operatorname{div}(g(z)\mathbb{C}\varepsilon(u))$, there is an intimate relation between the regularity of z and the regularity of the displacements u. In our analysis we rely on the fact that $u \in W^{1,p}(\Omega)$ with $p > d$. Such a regularity property can be achieved for the solutions of linear elliptic systems on nonsmooth domains with mixed boundary conditions (under certain geometric conditions), assuming that the coefficients are at least uniformly continuous on $\overline{\Omega}$. This is in particular guaranteed, if $z \in W^{1,q}(\Omega)$ with $q > d$, see Section 2.2 for details. However, if $q = 2$, i.e. A_q coincides with the standard Laplacian, then the coefficient $g(z)\mathbb{C}$ belongs to $L^\infty(\Omega) \cap H^1(\Omega)$. In contrast to the case of scalar elliptic equations, for linear elliptic systems this regularity of the coefficients in general does not imply that solutions are continuous. This is highlighted in the three-dimensional example in [46], with coefficients from $L^\infty(\Omega) \cap H^1(\Omega)$ leading to weak solutions u that do not belong to $\mathrm{C}^0(\overline{\Omega})$ and hence also not to $W^{1,p}(\Omega)$ with $p > d$. For this reason in the present paper we assume that $q > d$.

In the same spirit, in [33] we chose the fractional s-Laplacian operator A_s, on the Sobolev-Slobodeckij space $W^{s,2}(\Omega)$, with $s \geq \frac{d}{2}$, in place of the q-Laplacian. Note that for the case $d = 2$ the analysis performed in [33] deals with the standard Laplacian for z, so the choice of a "pure" s-Laplacian operator was made for space dimension $d \geq 3$. The q-Laplacian is, however, a more physically justifiable regularization than the nonlocal operator A_s, which fact has motivated the present study.

Relying on the spatial continuity of z, we obtain the regularity result which lies at the core of our analysis, viz. Lemma 2.3 asserting that, under suitable conditions on the data u_D and ℓ, $\exists\, p_* > d$ such that $\|u\|_{W^{1,p_*}(\Omega;\mathbb{R}^d)} \leq C$. Its proof is based on regularity results for elliptic systems with constant (or smooth) coefficients, combined with an iteration argument drawn from [5]. Let us stress that the regularity results which we invoke allow for elliptic operators with changing boundary conditions and, more importantly, for nonconvex, *nonsmooth* polyhedral domains, see Example 2.4 later on.

The higher integrability estimate for u enables us to improve the regularity of z which results from the sole basic energy estimate. In particular, (formally) differentiating (1.7) and testing it by z', we enhance the spatial regularity of z' by deducing the *mixed estimate*

$$\int_0^T \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} |\nabla z'(t)|^2 \, \mathrm{d}x \, \mathrm{d}r < \infty. \tag{1.8}$$

All these calculations are made rigorous on the time-discretization scheme with which we approximate (1.7): discrete solutions $(z_k^\tau)_{k=0}^N$, with $\tau > 0$ a constant time-step, are constructed via the time-incremental minimization scheme

$$z_{k+1}^\tau \in \mathrm{Argmin}\{\, \mathcal{I}(t_{k+1}^\tau, z) + \tau \mathcal{R}_\epsilon \left(\frac{z - z_k^\tau}{\tau} \right) \; ; \; z \in \mathcal{Z} \,\}. \tag{1.9}$$

A crucial ingredient for passing to the limit as $\tau \to 0$ is to obtain suitable estimates for the family $(A_q \overline{z}_\tau)_\tau$ ($\overline{z}_\tau$ denoting the piecewise constant interpolant of the values (z_τ^k)). Indeed, its weak convergence cannot be solely deduced from estimates for $(\overline{z}_\tau)_\tau$ in the space $\mathcal{Z} = W^{1,q}(\Omega)$, due to the nonlinear character of A_q. This is in its own right a challenging feature of the problem investigated here: the linear operator A_s considerably simplified the existence proof for (3.2), in [33]. In fact, after Lemma 2.3, the second milestone of our analysis is Theorem 4.4: based on a careful difference quotient argument, for the discrete solutions to (1.9) it ensures

$$\forall\, 0 < \beta < \tfrac{1}{q}\left(1 - \tfrac{d}{q}\right) \; \exists\, C_\beta > 0 \; \forall\, \tau > 0 \; \forall t \in (0,T] : \quad \|\overline{z}_\tau(t)\|_{W^{1+\beta,q}(\Omega)}^q \leq C_\beta (1 + \epsilon \, \|\hat{z}_\tau'(t)\|_{L^2(\Omega)}). \tag{1.10}$$

Estimate (1.10) yields $W^{1,q}(\Omega)$-compactness for $(\overline{z}_\tau)_\tau$ and thus allows us to take the limit of the term $A_q \overline{z}_\tau$. Indeed, for the limit passage as $\tau \to 0$ we adopt a *variational* approach: instead of passing to the limit directly in the discrete version of (1.5), we take the limit of the associated *discrete energy inequality*, cf. (6.4) ahead. With suitable compactness and lower semicontinuity arguments, we deduce that there exists a limit curve $z \in L^{2q}(0,T; W^{1+\beta,q}(\Omega)) \cap W^{1,2}(0,T; W^{1,2}(\Omega))$, with $z \in [0,1]$ a.e. in $\Omega \times (0,T)$, for which the mixed estimate (1.8) holds and fulfilling the energy inequality associated with (1.7), viz.

$$\int_s^t \mathcal{R}_\epsilon(z'(r)) \, \mathrm{d}r + \int_s^t \mathcal{R}_\epsilon^*(-\mathrm{D}_z \mathcal{I}(r, z(r))) \, \mathrm{d}r + \mathcal{I}(t, z(t)) \leq \mathcal{I}(s, z(s)) + \int_s^t \partial_t \mathcal{I}(r, z(r)) \, \mathrm{d}r. \tag{1.11}$$

for all $0 \leq s \leq t \leq T$, with $\mathcal{R}_\epsilon^*$ the Fenchel-Moreau conjugate of $\mathcal{R}_\epsilon$ with respect to the $\mathcal{Z}$-$\mathcal{Z}^*$-duality. We also prove in Theorem 3.2 that, along the limit curve z a chain-rule formula is valid, viz.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I}(t, z(t)) - \partial_t \mathcal{I}(t, z(t)) = & \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t) \cdot \nabla z'(t) \, \mathrm{d}x \\ & + \int_\Omega (f'(z(t)) + \frac{1}{2} g'(z(t)) \mathbb{C}\varepsilon(u(t)) \colon \varepsilon(u(t))) z'(t) \, \mathrm{d}x \quad \text{for a.a.}\, t \in (0,T). \end{aligned} \tag{1.12}$$

A key ingredient for (1.12) is (1.8) guaranteeing

that the first integral on the right-hand side of (1.12) is well defined. With (1.12), in Proposition 3.3 we show that the energy inequality (1.11) is equivalent to

$$\begin{aligned} \mathcal{R}_\epsilon(w) - \mathcal{R}_\epsilon(z'(t)) \geq \langle -A_q z(t), w \rangle_{\mathcal{Z}} + & \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t) \cdot \nabla z'(t) \, \mathrm{d}x \\ - & \int_\Omega (f'(z(t)) + \frac{1}{2} g'(z(t)) \mathbb{C}\varepsilon(u(t)) \colon \varepsilon(u(t)))(w - z'(t)) \quad \text{for all } w \in \mathcal{Z} \end{aligned} \tag{1.13}$$

for almost all $t \in (0,T)$. This variational inequality defines our notion of weak solution for the viscous doubly nonlinear equation (1.6), cf. Definition 3.1; the existence Theorem 3.5, follows by the above arguments.

Observe that, as soon as we can interpret the terms on the r.h.s. of (1.13) as the duality product $\langle -A_q z - f'(z) - \frac{1}{2} g'(z)\mathbb{C}\varepsilon(u) \colon \varepsilon(u), w - z' \rangle_{\mathcal{Z}}$, then (1.13) is in fact equivalent to the subdifferential inclusion (1.6). In Sec. 3.1 the relation of our weak solution concept for (1.6) to the usual subdifferential formulation (1.7) is discussed at length, also in connection with the chain rule (1.12), and with the failure of the energy inequality (1.11) to hold as an equality.

The vanishing-viscosity analysis As in [33], for passing to the limit in (1.7) as $\epsilon \to 0$ we adopt the *reparameterization technique* from [16], which leads to a notion of solution for the rate-independent system (1.5), encompassing a finer description of the energetic behavior of the system jumps. The underlying philosophy is that, at jumps the vanishing-viscosity solutions to (1.5) follow a path which is reminiscent of the viscous approximation. To reveal this, one has to go over to an extended state space and study the limiting behavior of the sequence $(\tilde{t}_\epsilon, \tilde{z}_\epsilon)_\epsilon$ as $\epsilon \downarrow 0$, with $\tilde{z}_\epsilon = z_\epsilon \circ \tilde{t}_\epsilon$ a suitable reparameterization of a family $(z_\epsilon)_\epsilon$ of *weak* solutions (in the sense of (1.13)) to (1.7). The choice of this reparameterization is related to the key BV-estimate $\sup_{\epsilon>0} \int_0^T \|z'_\epsilon(t)\|_{L^2(\Omega)}\,\mathrm{d}t \leq C$ for viscous solutions to (1.7).

In Theorem 7.4 we prove that, up to a subsequence, the curves $(\tilde{t}_\epsilon, \tilde{z}_\epsilon)$ converge to a pair $(\tilde{t}, \tilde{z}) : [0, S] \to [0, T] \times \mathcal{Z}$, termed *weak parameterized solution*, which fulfills the parameterized energy inequality

$$\int_{s_1}^{s_2} \widetilde{\mathcal{M}}_0(\tilde{t}'(r), \tilde{z}'(r), -\mathrm{D}_z\mathcal{I}(\tilde{t}(r), \tilde{z}(r)))\,\mathrm{d}r + \mathcal{I}(\tilde{t}(s_2), \tilde{z}(s_2)) \leq \mathcal{I}(\tilde{t}(s_1), \tilde{z}(s_1)) + \int_{s_1}^{s_2} \partial_t\mathcal{I}(\tilde{t}(r), \tilde{z}(r))\tilde{t}'(r)\mathrm{d}r \tag{1.14}$$

for all $0 \leq s_1 \leq s_2 \leq S$. In (1.14), the term

$$\widetilde{\mathcal{M}}_0(\tilde{t}', \tilde{z}', -\mathrm{D}_z\mathcal{I}(\tilde{t}, \tilde{z})) = \begin{cases} \mathcal{R}_1(\tilde{z}') + I_{\partial\mathcal{R}_1(0)}(-\mathrm{D}_z\mathcal{I}(\tilde{t}, \tilde{z})) & \text{if } \tilde{t}' > 0, \\ \mathcal{R}_1(\tilde{z}') + \|\tilde{z}'\|_{L^2(\Omega)}\,\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}, \tilde{z}), \partial\mathcal{R}_1(0)) & \text{if } \tilde{t}' = 0, \end{cases}$$

$(\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}, \tilde{z}), \partial\mathcal{R}_1(0))$ denoting the $L^2(\Omega)$-distance of $-\mathrm{D}_z\mathcal{I}(\tilde{t}, \tilde{z})$ from $\partial\mathcal{R}_1(0))$ enforces the *local stability* condition $-\mathrm{D}_z\mathcal{I}(\tilde{t}, \tilde{z}) \in \partial\mathcal{R}_1(0)$ in the case of purely rate-independent evolution, i.e. when $\tilde{t}' > 0$. When the (slow) external time, encoded by the function $\tilde{t}$, is frozen, the system jumps. Then, the system may switch to a *viscous* regime. We refer to Sec. 7.2 for further details on this.

Concluding remarks Let us finally comment on the structure of the q-Laplace operator A_q. Our analysis relies on the fact that A_q is non-degenerate, meaning that $A_q(z) = -\operatorname{div}(\delta + |\nabla z|^2)^{\frac{q-2}{2}} \nabla z$ with $\delta > 0$. With $\delta > 0$ it is possible to derive the spatial regularity estimate (1.10), which in turn is at the core of our proof of existence of a family of viscous solutions $(z_\epsilon)_{\epsilon>0}$ to (1.7). Furthermore, (1.10) turns out to hold, in a suitable form, for $(z_\epsilon)_{\epsilon>0}$, uniformly w.r. to ϵ. This crucial bound allows us to conclude that for almost all $s \in (0, S)$ $\widetilde{z}_\epsilon(s) \to \widetilde{z}(s)$ strongly in $\mathcal{Z}$, which is exploited when passing to the limit in the expression $\mathrm{D}_z\mathcal{I}(\widetilde{t}_\epsilon, \widetilde{z}_\epsilon)$.

The chain of arguments does not work if one considers the degenerate q-Laplacian with $\delta = 0$ and it is not clear whether standard monotonicity tricks for monotone operators would do the job. Clearly, instead of the particular A_q, one could consider general monotone potential operators for which the associated energy densities satisfy the monotonicity and convexity estimates (2.3)–(2.4).

Plan of the paper In Section 2 we specify all assumptions, prove the regularity Lemma 2.3, and collect all properties of the reduced energy $\mathcal{I}$. In Section 3, we introduce and motivate our notion of *weak solution* for the Cauchy problem associated with the viscous equation (1.6), state Theorem 3.5 (=existence of weak solutions and a priori estimates uniform w.r. to the viscosity parameter ϵ), and prove

Theorem 3.2, providing the chain rule (1.12). In Section 4 we set up the time-discretization scheme for (1.7) and prove the higher differentiability result yielding (1.10). Section 5 is devoted to the proof of a series of a priori estimates on the discrete solutions, most of which uniform both w.r.t. τ *and* ϵ. In particular, the discrete version of the BV-estimate $\sup_{\epsilon>0}\int_0^T \|z_\epsilon'(t)\|_{L^2(\Omega)}\,\mathrm{d}t \leq C$ is derived. We prove Theorem 3.5 by passing to the limit as $\tau \to 0$ in the time discrete scheme, also exploiting Young measure techniques which are recapped in the Appendix. Finally, in Section 7 we develop the vanishing-viscosity analysis of (1.7).

2 Preliminaries

2.1 Set-up

Notation For a given Banach space X, we shall denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X, and, if X is a Hilbert space, we shall use the symbol $(\cdot,\cdot)_X$ for its scalar product. For matrices $A, B \in \mathbb{R}^{m\times d}$ the inner product is defined by $A : B = \mathrm{tr}(B^\top A) = \sum_{i=1}^m \sum_{j=1}^d a_{ij}b_{ij}$.

Let $d \geq 3$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a closed Dirichlet boundary $\Gamma_D \subset \partial\Omega$ and Neumann boundary $\Gamma_N = \partial\Omega\backslash\Gamma_D$. Further assumptions on the regularity of Ω and on Γ_D will be specified in Sections 2.2 and 3.1 (cf. ($\mathrm{A}_\Omega 1$) and ($\mathrm{A}_\Omega 2$)). The letter Q shall stand for the space-time cylinder $\Omega \times (0,T)$. The following function spaces and notation shall be used for $\sigma \geq 0$, $p \in [1,\infty]$:

- $W^{\sigma,p}(\Omega)$ Sobolev-Slobodeckij spaces,
- $W^{1,p}_{\Gamma_D}(\Omega) := \{\, u \in W^{1,p}(\Omega)\,;\, u\big|_{\Gamma_D} = 0 \,\}$ and $W^{-1,p}_{\Gamma_D}(\Omega) := \big(W^{1,p'}_{\Gamma_D}(\Omega)\big)^*$ the dual space, $\frac{1}{p}+\frac{1}{p'} = 1$.

We shall denote by $u : \Omega \to \mathbb{R}^d$ the displacement, and by $z : \Omega \to \mathbb{R}$ the (scalar) damage variable. The corresponding state spaces are

$$\mathcal{U} := \{\, v \in W^{1,2}(\Omega,\mathbb{R}^d)\,;\, v\big|_{\Gamma_D} = 0 \,\} = W^{1,2}_{\Gamma_D}(\Omega,\mathbb{R}^d), \qquad \mathcal{Z} := W^{1,q}(\Omega), \tag{2.1}$$

with $q > d$. On the space $\mathcal{Z}$ the q-Laplacian operator is defined as follows

$$A_q : \mathcal{Z} \to \mathcal{Z}^*, \quad \langle A_q(z), v\rangle_{\mathcal{Z}} := \int_\Omega (1+|\nabla z|^2)^{\frac{q-2}{2}}\nabla z\cdot\nabla v\,\mathrm{d}x \quad \text{for } z, v \in \mathcal{Z}.$$

Useful inequalities We collect here some inequalities which shall be extensively used in the following. First of all, for $p_* > d$

$$\forall\,\rho > 0\ \exists\, C_\rho > 0\ \forall\, z \in W^{1,2}(\Omega) : \qquad \|z\|_{L^{2p_*/(p_*-2)}(\Omega)} \leq \rho\|z\|_{W^{1,2}(\Omega)} + C_\rho\|z\|_{L^2(\Omega)}. \tag{2.2}$$

This follows from the the compact and continuous embeddings $W^{1,2}(\Omega) \Subset L^{2p_*/(p_*-2)}(\Omega) \subset L^2(\Omega)$ (due to $p_* > d$), on account of [51, Lemma 8].

Secondly, let us recall that with $G_q(A) = \frac{1}{q}(1+|A|^2)^{\frac{q}{2}}$ for $A \in \mathbb{R}^d$, as a consequence of [23, Lemma 8.3] the following monotonicity and convexity estimates are valid for $q \geq 2$ and A, $B \in \mathbb{R}^d$,

$$(\mathrm{D}G_q(A) - \mathrm{D}G_q(B))\cdot(A-B) \geq c_q(1+|A|^2+|B|^2)^{\frac{q-2}{2}}\,|A-B|^2 \tag{2.3}$$

$$\begin{aligned} G_q(B) - G_q(A) - \mathrm{D}G_q(A)\cdot(B-A) &\geq c_q(1+|A|^2+|B|^2)^{\frac{q-2}{2}}\,|A-B|^2 \\ &\geq \widetilde{c}_q(|A-B|^2 + |A-B|^q). \end{aligned} \tag{2.4}$$

Observe that (2.3) implies for all $z_1, z_2 \in \mathcal{Z}$:

$$\langle A_q z_1 - A_q z_2, z_1 - z_2 \rangle_{\mathcal{Z}} \geq c_q \int_\Omega (1 + |\nabla z_1|^2 + |\nabla z_2|^2)^{\frac{q-2}{2}} |\nabla(z_1 - z_2)|^2 \, \mathrm{d}x. \tag{2.5}$$

Moreover, for all z_1, z_2 and $w \in \mathcal{Z}$

$$\big| \langle A_q z_1 - A_q z_2, w \rangle_{\mathcal{Z}} \big| \leq c \int_\Omega (1 + |\nabla z_1|^2 + |\nabla z_2|^2)^{\frac{q-2}{2}} |\nabla(z_1 - z_2)| \, |\nabla w| \, \mathrm{d}x. \tag{2.6}$$

The energy functional The energy $\mathcal{E} = \mathcal{E}(t, u, z)$ consists of two contributions. The first one, $\mathcal{I}_1$, only depends on the damage variable. The second one, $\mathcal{E}_2 = \mathcal{E}_2(t, u, z)$, is given by the sum of an elastic energy of the type $\int_\Omega g(z) W(\varepsilon(x, u + u_D(t))) \, \mathrm{d}x$ with u_D a Dirichlet datum, and of the external loading term.

Assumption 2.1. *We consider*

$$\mathcal{I}_1 : \mathcal{Z} \to \mathbb{R} \ \ \textit{defined by} \ \ \mathcal{I}_1(z) := \mathcal{I}_q(z) + \int_\Omega f(z) \, \mathrm{d}x \ \ \textit{with} \ \mathcal{I}_q(z) := \frac{1}{q} \int_\Omega (1 + |\nabla z|^2)^{\frac{q}{2}} \, \mathrm{d}x, \ \ q > d, \tag{2.7}$$

and f fulfilling

$$f \in \mathrm{C}^2(\mathbb{R}), \quad \textit{such that} \quad \exists\, K_1, K_2 > 0 \quad \forall\, x \in \mathbb{R} : \quad f(x) \geq K_1 |x| - K_2. \tag{2.8}$$

A typical choice for f is $f(z) = z^2$, see [22]. As for $\mathcal{E}_2$, linearly elastic materials are considered with an elastic energy density $W(x, \eta) = \frac{1}{2}\mathbb{C}(x)\eta : \eta$, for $\eta \in \mathbb{R}^{d\times d}_{sym}$ and almost every $x \in \Omega$. Hereafter, we shall suppose for the elasticity tensor that

$$\mathbb{C} \in \mathrm{C}^0_{\mathrm{lip}}(\Omega, \mathrm{Lin}(\mathbb{R}^{d\times d}_{sym}, \mathbb{R}^{d\times d}_{sym})) \ \textit{with} \ \mathbb{C}(x)\xi_1 : \xi_2 = \mathbb{C}(x)\xi_2 : \xi_1 \ \textit{for all} \ x \in \Omega, \xi_i \in \mathbb{R}^{d\times d}_{sym}, \tag{2.9a}$$

$$\exists\, \gamma_0 > 0 \quad \textit{for all} \ \xi \in \mathbb{R}^{d\times d}_{sym} \ \textit{and almost all} \ x \in \Omega : \quad \mathbb{C}(x)\xi : \xi \geq \gamma_0 |\xi|^2 . \tag{2.9b}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a further constitutive function such that

$$g \in \mathrm{C}^2(\mathbb{R}), \ \textit{with} \ g' \in L^\infty(\mathbb{R}), \quad \textit{and} \ \exists\, \gamma_1, \gamma_2 > 0 \ : \ \forall\, z \in \mathbb{R} \ : \ \gamma_1 \leq g(z) \leq \gamma_2. \tag{2.10}$$

Then, we take the elastic energy

$$\mathcal{E}_2 : [0, T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R} \ \ \textit{defined by} \ \ \mathcal{E}_2(t, u, z) := \int_\Omega g(z) W(x, \varepsilon(u + u_D(t))) \, \mathrm{d}x - \ \langle \ell(t), u \rangle_{\mathcal{U}} \tag{2.11}$$

where $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetrized strain tensor and $\ell \in \mathrm{C}^0([0, T], \mathcal{U}^*)$ an external loading (cf. (2.22) later on for further requirements).

For $u \in \mathcal{U}$ and $z \in \mathcal{Z}$ the stored energy is then defined as

$$\mathcal{E}(t, u, z) = \mathcal{I}_1(z) + \mathcal{E}_2(t, u, z). \tag{2.12}$$

Reduced energy Minimizing the $\mathcal{E}$ with respect to the displacements we obtain the reduced energy

$$\mathcal{I} : [0, T] \times \mathcal{Z} \to \mathbb{R} \ \text{ given by } \ \mathcal{I}(t, z) = \mathcal{I}_1(z) + \mathcal{I}_2(t, z) \text{ with } \mathcal{I}_2(t, z) = \inf\{\, \mathcal{E}_2(t, v, z) \,;\, v \in \mathcal{U} \,\}. \tag{2.13}$$

Remark 2.2. As already mentioned in the introduction, our model does not allow for *complete* damage: this is reflected in the coercivity (2.9b), and in the strict positivity (2.10) of the constitutive function g.

The Lipschitz continuity (2.9a) of the coefficient matrix $\mathbb{C}$, as well as the smoothness of g, shall be exploited in the forthcoming Lemma 2.3, providing higher integrability for $\varepsilon(u)$. For proving this result, which will be at the core of all the subsequent analysis, we have to stay with a *quadratic* elastic energy.

Relying on these regularity properties for $\mathbb{C}$ and g, we are also going to prove higher differentiability for z, cf. the crucial Theorem 4.4 later on.

Nonetheless, let us stress that significant damage models fall within the scope of the above conditions: for example, the Ambrosio-Tortorelli model, whose quasistatic evolution was discussed in [22], as an approximation of the Francfort-Marigo model [18] for crack propagation. Observe that, in the energy functional considered for the rate-independent model of [22], the index in (2.7) is $q = 2$. Instead, in the more recent [5], which deals with the (metric) *gradient flow* of the Ambrosio-Tortorelli functional, it is assumed that $q > d$ like in the present setting.

2.2 Geometric assumptions and regularity of the displacement field

For the analysis of the time-dependent damage model higher integrability properties of the gradients of the displacement field are needed, and hence the domain Ω and the data should be more regular than stated above.

With $\mathbb{C}$ as in (2.9a)–(2.9b), g from (2.10) and $z \in W^{1,q}(\Omega)$ let $L, L_{g(z)} : W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d) \to W^{-1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ be the operators associated with the bilinear forms describing linear elasticity, i.e.

$$\forall u, v \in W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d): \qquad \langle Lu, v\rangle := \int_\Omega \mathbb{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x, \quad \langle L_{g(z)}u, v\rangle := \int_\Omega g(z)\mathbb{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x. \quad (2.14)$$

A good compromise between the smoothness needed for our analysis and nevertheless allowing for polyhedral domains and changing boundary conditions is formulated in the following

Assumption on the domain

($\mathrm{A}_\Omega 1$) $\Omega \subset \mathbb{R}^d$ is a bounded domain, and Ω and $\Gamma_D \subset \partial\Omega$ (Γ_D is closed and with positive measure) are chosen in such a way that the following two conditions are satisfied:

(i) The spaces $W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$, $p \in (1,\infty)$, form an interpolation scale.

(ii) There exists $p_* > d$ such that for all $p \in [2, p_*]$ the operator $L : W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d) \to W^{-1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ is an isomorphism.

For an abstract definition of interpolation scales we refer to [56], while in Example 2.4 here below we present nonsmooth, nonconvex domains with mixed boundary conditions satisfying ($\mathrm{A}_\Omega 1$).

Observe that the isomorphism property stated in ($\mathrm{A}_\Omega 1$) is also valid for all $p \in [p_*', 2]$, and that the operator norms are uniformly bounded, i.e. with $\mathcal{X}_p := W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ and $\mathcal{Y}_p := W^{-1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ it holds (denoting by $\|L\|_{\mathcal{X}\to\mathcal{Y}}$ the norm of an operator $L : \mathcal{X} \to \mathcal{Y}$)

$$\sup_{p\in[p_*', p_*]} \|L\|_{\mathcal{X}_p\to\mathcal{Y}_p} + \left\|L^{-1}\right\|_{\mathcal{Y}_p\to\mathcal{X}_p} =: c_{p_*} < \infty. \quad (2.15)$$

Lemma 2.3 plays a key role in the subsequent analysis and relies on an iteration argument from [5].

Lemma 2.3. *Let ($\mathrm{A}_\Omega 1$) be satisfied, g as in* (2.10) *and $q > d$. Let furthermore $p_* > d$ be chosen according to ($\mathrm{A}_\Omega 1$) and let $k_* \in \mathbb{N}$ be the smallest number with $k_* > \frac{dq}{2(q-d)}$. Then for all $p \in [p_*', p_*]$ and* *all $z \in W^{1,q}(\Omega)$ the operator $L_{g(z)} : W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d) \to W^{-1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ is an isomorphism. Moreover, there exists a constant $c_{q,p_*} > 0$ such that for all $z \in W^{1,q}(\Omega)$ and all $p \in [p'_*, p_*]$ it holds*

$$\left\| L^{-1}_{g(z)} \right\|_{\mathcal{Y}_p \to \mathcal{X}_p} \le c_{q,p_*} (1 + \|\nabla z\|_{L^q(\Omega)})^{k_* \frac{p_* |p-2|}{p(p_*-2)}} . \tag{2.16}$$

Observe that

$$\sup_{p \in [p'_*, p_*]} \frac{p_* |p-2|}{p(p_*-2)} \le 1 . \tag{2.17}$$

Proof. It is sufficient to prove the lemma for $p = p_*$. The other assertions follow with interpolation and duality arguments. The proof extends the recursion argument from [5], where it is carried out for smooth domains and $W^{2,2}(\Omega;\mathbb{R}^d)$, to the $W^{1,p}(\Omega;\mathbb{R}^d)$-setting and to domains satisfying ($\mathrm{A}_\Omega 1$).

Let $p_* > 2$, $q > d$ and k_* be chosen as stated in Lemma 2.3. Define q_* via the relation

$$p_* = \frac{2dq_*}{dq_* + 2k_*(d-q_*)}, \quad \text{i.e.} \quad q_* = \frac{2k_* p_* d}{p_*(2k_* - d) + 2d}.$$

Observe that $q_* \in (d, q]$. Clearly, $\mathcal{Y}_p \subset W^{-1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ if $p \ge 2$. Moreover, for all $z \in W^{1,q}(\Omega)$ the functions $g(z), g(z)^{-1}$ are multiplicators for the spaces $W^{-1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ and the following estimate is valid: there exists a constant $c > 0$ such that for all $p \in [2, p_*]$, all $z \in W^{1,q}(\Omega)$ and all $b \in \mathcal{Y}_p$ it holds

$$\left\| g(z)^{-1} b \right\|_{\mathcal{Y}_p} \le c(1 + \|\nabla z\|_{L^q(\Omega)}) \|b\|_{\mathcal{Y}_p} . \tag{2.18}$$

For $b \in \mathcal{Y}_{p_*}$, let $u \in W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ be the unique function satisfying for all $v \in W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d)$

$$\langle L_{g(z)} u, v \rangle = \int_\Omega g(z) \mathbb{C} \varepsilon(u) : \varepsilon(v) \,\mathrm{d}x = \langle b, v \rangle_{W^{1,2}(\Omega;\mathbb{R}^d)} . \tag{2.19}$$

Due to the multiplier property of $g(z)$, using the product rule and choosing $v = \frac{1}{g(z)} \widetilde{v}$ in (2.19), it follows that for all $\widetilde{v} \in W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ we have

$$\begin{aligned} \int_\Omega \mathbb{C}\varepsilon(u) : \varepsilon(\widetilde{v}) \,\mathrm{d}x &= \left\langle \frac{1}{g(z)} b, \widetilde{v} \right\rangle_{W^{1,2}(\Omega;\mathbb{R}^d)} + \int_\Omega \frac{g'(z)}{g(z)} \mathbb{C}\varepsilon(u) \nabla z \cdot \widetilde{v} \,\mathrm{d}x \\ &=: \left\langle \frac{1}{g(z)} b, \widetilde{v} \right\rangle_{W^{1,2}(\Omega;\mathbb{R}^d)} + \int_\Omega h(u,z) \cdot \widetilde{v} \,\mathrm{d}x. \end{aligned} \tag{2.20}$$

Since $\frac{1}{g(z)} b \in \mathcal{Y}_{p_*}$ with estimate (2.18) (for $p = p_*$), in the following we only have to concentrate on the term $h(u,z)$, which belongs to $L^{\alpha_0}(\Omega)$ with $\alpha_0 = \frac{2q_*}{2+q_*} \in (1,2)$. Hence, by embedding it follows that $h(z,u) \in L^{\alpha_0}(\Omega) \subset \mathcal{Y}_{p_1}$ with $p_1 = \frac{2q_* d}{dq_* - 2(q_* - d)}$. Observe that $2 < p_1 \le p_*$. Thanks to assumption ($\mathrm{A}_\Omega 1$) and estimate (2.15) it follows from (2.20) that $u \in W^{1,p_1}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ with

$$\|u\|_{W^{1,p_1}(\Omega)} \le \widetilde{c} \Big(1 + \|\nabla z\|_{L^q(\Omega)}\Big) \Big(\|u\|_{W^{1,2}(\Omega)} + \|b\|_{\mathcal{Y}_{p_1}}\Big) \le c \Big(1 + \|\nabla z\|_{L^q(\Omega)}\Big) \|b\|_{\mathcal{Y}_{p_1}} ,$$

where the last inequality ensues from the estimate due to the Lax-Milgram lemma, applied to the solution u of (2.19). The constants $\widetilde{c}, c$ are independent of z and p_1.

We now iterate this argument. Assume that $u \in W^{1,p_k}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ for some $p_k \in [2, p_*)$. Then, again by embedding, we have $h(z,u) \in L^{\alpha_k}(\Omega)$ with $\alpha_k = \frac{p_k q_*}{p_k + q_*}$, and $L^{\alpha_k}(\Omega) \subset \mathcal{Y}_{p_{k+1}}$ with

$$p_{k+1} = \frac{dp_k q_*}{dq_* + p_k(d - q_*)} \equiv \frac{2dq_*}{dq_* + 2(k+1)(d-q_*)} . \tag{2.21}$$

The last identity follows by induction starting with $p_0 = 2$. This argument can be repeated as long as $k < k_*$, since for these values of k it holds $p_k < p_{k+1} \le p_*$. Observe that $p_{k_*} = p_*$. This implies that $L^{\alpha_{k_*-1}}(\Omega) \subset \mathcal{Y}_{p_*}$ and hence, again by ($\mathrm{A}_\Omega 1$), we have $u \in W^{1,p_*}_{\Gamma_D}(\Omega; \mathbb{R}^d)$. The estimate for u follows recursively, namely

$$
\begin{aligned}
\|u\|_{W^{1,p_*}(\Omega;\mathbb{R}^d)} &\le c\left((1 + \|\nabla z\|_{L^q(\Omega)}) \|b\|_{\mathcal{Y}_{p_*}} + \|u\|_{W^{1,p_{k_*-1}}(\Omega)} \|\nabla z\|_{L^{q_*}(\Omega)}\right) \\
&\le c_{k_*} (1 + \|\nabla z\|_{L^q(\Omega)})^{k_*} \|b\|_{\mathcal{Y}_{p_*}} .
\end{aligned}
$$

The remaining norm estimates in Lemma 2.3 follow from interpolation theory. □

Example 2.4. A sufficient condition such that the interpolation scale property ($\mathrm{A}_\Omega 1$)(i) is satisfied is to assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary and that the boundary sets Γ_D and Γ_N are regular in the sense of Gröger viz., loosely speaking, that the hypersurface separating Γ_D and Γ_N is Lipschitz, see [25, 24] for more details. A more general geometric setting is characterized in [26].

In order to obtain also the isomorphism property ($\mathrm{A}_\Omega 1$)(ii), one can apply the regularity theory for linear elliptic systems in polyhedral domains. Sufficient conditions on the geometry of Ω, the Dirichlet and the Neumann boundaries can be identified for instance with the help of [38, Theorem 7.1] (applied for $\vec{\beta} = 0$, $\vec{\delta} = 0$, $l = 1$ in the notation of [38, Section 7]).

For example, for the Lamé-operator (i.e. linear isotropic elasticity) sufficient conditions for ($\mathrm{A}_\Omega 1$) to hold are the following: $d = 3$, $\Omega \subset \mathbb{R}^3$ is a bounded domain of polyhedral type (see [38, Section 7.1]), and on each face either Dirichlet boundary conditions or Neumann boundary conditions are prescribed. Furthermore, the interior opening angles along Dirichlet-Dirichlet and along Neumann-Neumann edges are less than 2π (i.e. no cracks), and the interior opening angle along Dirichlet-Neumann edges is less than or equal to π (more general situations are possible). Then the singular exponents along the edges of the polyhedron Ω satisfy the conditions required in [38, Theorem 7.1] in order to allow for $p_* > 3$ in ($\mathrm{A}_\Omega 1$). We refer for example to [47] for estimates of the singular exponents along edges in different geometric settings. Concerning the singular exponents associated with the vertices of the polyhedron, one has to guarantee that the strip $\{ \lambda \in \mathbb{C}\,;\, -\frac{1}{2} < \mathrm{R}e\ \lambda \le 0 \}$ contains at most the singular exponent $\lambda = 0$. In the case of pure Dirichlet conditions in the vicinity of a given vertex there is no further geometric restriction in order to guarantee this property, [37]. In case of pure Neumann conditions in the vicinity of a given vertex one has to assume that the boundary locally is the graph of a function that is positively homogeneous of degree one, [35]. In case of mixed boundary conditions in the vicinity of a vertex, a sufficient condition is to assume that the domain is convex in the vicinity of the vertex and that at most one face belongs to the Dirichlet boundary or that at most one face belongs to the Neumann boundary (see [47] for a more general condition, an example is illustrated in Figure 1(ii)). We refer to [29] for an overview on the literature on estimates for the corner and edge singularities associated with the Laplace- and Lamé-operator on three-dimensional polyhedral domains. Clearly, ($\mathrm{A}_\Omega 1$) as well as Lemma 2.3 can be extended to coefficient matrices $\mathbb{C}$ with piecewise constant entries if certain geometric conditions are satisfied. The Fichera corner plotted in Figure 1 is an example for a nonconvex, nonsmooth domain with mixed boundary conditions that is admissible with respect to assumption ($\mathrm{A}_\Omega 1$), in connection with the Lamé-operator.

2.3 Properties of the energy functional

In what follows, we prove the continuity and differentiability properties of $\mathcal{I}$ needed for our analysis. The following results shall also provide fine estimates for $|\partial_t \mathcal{I}|$ and for suitable norms of $\mathrm{D}_z\mathcal{I}$, in terms of

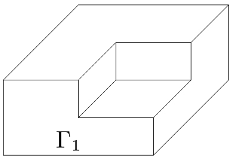

Figure 1: Admissible domain if for example: (i) Dirichlet-conditions on the bottom plane and Neumann conditions on the remaining part of $\partial\Omega$ or (ii) $\Gamma_D = \Gamma_1$ and Neumann conditions on the rest.

quantities which continuously depend on $\|z\|_{\mathcal{Z}}$, and which are therefore bounded on sublevels of $\mathcal{I}$.

Hereafter, we shall work under these additional conditions on the data ℓ and u_D:

Assumption 2.5. *We require that*

$$\ell \in \mathrm{C}^{1,1}([0,T]; W^{-1,p_*}_{\Gamma_D}(\Omega;\mathbb{R}^d)), \quad u_D \in \mathrm{C}^{1,1}([0,T]; W^{1,p_*}(\Omega;\mathbb{R}^d)) \ \ \textit{with } p_* > d \textit{ from } (\mathrm{A}_\Omega 1). \tag{2.22}$$

From now on, to shorten the notation we introduce for $z_1, z_2 \in \mathcal{Z}$ and k_* as in Lemma 2.3 the quantity

$$P(z_1, z_2) := (1 + \|\nabla z_1\|_{L^q(\Omega)} + \|\nabla z_2\|_{L^q(\Omega)})^{k_*}. \tag{2.23}$$

Our first result is based on Lemma 2.3.

Lemma 2.6 (Existence of minimizers and their regularity).
Under Assumptions 2.1, 2.5, and (A_Ω1), for every $(t,z) \in [0,T] \times \mathcal{Z}$ there exists a unique $u_{\min}(t,z) \in \mathcal{U}$, which minimizes $\mathcal{E}(t,z,\cdot)$.

Moreover, there exist $c_0 > 0$ such that for all $p \in [p_*', p_*]$ and $(t,z) \in [0,T] \times \mathcal{Z}$ it holds that $u_{\min}(t,z) \in W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$, and

$$\|u_{\min}(t,z)\|_{W^{1,p}(\Omega;\mathbb{R}^d)} \le c_0 P(z,0)^{\frac{p_*|p-2|}{p(p_*-2)}} \big(\|\ell(t)\|_{W^{-1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)} + \|u_D(t)\|_{W^{1,p}(\Omega;\mathbb{R}^d)} \big), \tag{2.24}$$

with P as in (2.23)*, and p_* the exponent from (A_Ω1)(ii). Furthermore, the following coercivity inequality for $\mathcal{I}$ is valid: There exist constants $c_1, c_2 > 0$ such that for all $(t,z) \in [0,T] \times \mathcal{Z}$ it holds*

$$\mathcal{I}(t,z) \ge c_1 \big(\|\nabla z\|^q_{L^q(\Omega)} + \|z\|_{L^1(\Omega)} + \|u_{\min}(t,z)\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)} \big) - c_2. \tag{2.25}$$

Observe that, on the right-hand side of (2.24) the dependence on $\|z\|_{\mathcal{Z}}$ of the quantity which bounds $\|u_{\min}(t,z)\|_{W^{1,p}(\Omega;\mathbb{R}^d)}$ is very explicitly displayed. In particular, observe that for $p = 2$ we have no dependence on $\|z\|_{\mathcal{Z}}$ as $P(z,0)^0 = 1$, while for the extreme case $p = p_*$ we have $P(z,0)$, cf. (2.17).

Lemma 2.7 (Continuous dependence on the data).
Under Assumptions 2.1, 2.5, and (A_Ω1), there exists a constant $c_3 > 0$ such that for all ℓ and u_D as in (2.22)*, all $t_1, t_2 \in [0,T]$ and all $\widetilde{p} \in [p_*', p_*)$ it holds with $r = p_*\widetilde{p}(p_* - \widetilde{p})^{-1}$ and all $z_1, z_2 \in \mathcal{Z}$*

$$\begin{aligned} &\|u_{\min}(t_1,z_1) - u_{\min}(t_2,z_2)\|_{W^{1,\widetilde{p}}(\Omega;\mathbb{R}^d)} \\ &\le c_3 \big(|t_1 - t_2| + \|z_1 - z_2\|_{L^r(\Omega)} \big) P(z_1,z_2)^2 \big(\|\ell\|_{\mathrm{C}^1([0,T];W^{-1,p_*}_{\Gamma_D}(\Omega;\mathbb{R}^d))} + \|u_D\|_{\mathrm{C}^1([0,T];W^{1,p_*}(\Omega;\mathbb{R}^d))} \big). \end{aligned} \tag{2.26}$$

Remark 2.8. Observe that for $\widetilde{p} \in [p_*', p_*)$ we have $r = p_*\widetilde{p}(p_* - \widetilde{p})^{-1} \in [\frac{p_*}{p_*-2}, \infty)$, and r is strictly increasing with respect to $\widetilde{p}$. In particular, for $\widetilde{p} = 3p_*(3+p_*)^{-1}$ we have $r = 3$ and for $\widetilde{p} := 6p_*(6+p_*)^{-1}$ we have $r = 6$.

Proof. For $i = 1, 2$, let $u_i := u_{\min}(t_i, z_i) \in W^{1,p_*}(\Omega;\mathbb{R}^d)$, with p_* from $(\mathrm{A}_\Omega 1)$. From the corresponding Euler-Lagrange equations written for u_i, $i = 1, 2$, with algebraic manipulations we obtain that $u_1 - u_2$ satisfies for all $v \in W^{1,p_*}_{\Gamma_D}(\Omega;\mathbb{R}^d)$

$$\begin{aligned}\int_\Omega g(z_1)\mathbb{C}\varepsilon(u_1 - u_2) \colon \varepsilon(v)\,\mathrm{d}x &= \int_\Omega \big(g(z_2) - g(z_1)\big)\mathbb{C}\varepsilon(u_2) \colon \varepsilon(v)\,\mathrm{d}x \\ &- \int_\Omega \big(g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\big) \colon \varepsilon(v)\,\mathrm{d}x + \langle \ell(t_1) - \ell(t_2), v\rangle_{\mathcal{U}}\,.\end{aligned} \tag{2.27}$$

Hence, by density and Lemma 2.3, the function $u_1 - u_2$ fulfills for all $\widetilde{p} \in [p_*', p_*]$ and all $v \in W^{1,\widetilde{p}'}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ the relation

$$\int_\Omega g(z_1)\mathbb{C}\varepsilon(u_1 - u_2) : \varepsilon(v)\,\mathrm{d}x = \langle \tilde{\ell}_{1,2}, v\rangle_{W^{1,\widetilde{p}'}_{\Gamma_D}(\Omega;\mathbb{R}^d)},$$

where $\tilde{\ell}_{1,2} \in W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ subsumes the terms on the right-hand side of (2.27). Therefore, (2.16) gives $\|u_1 - u_2\|_{W^{1,\widetilde{p}}(\Omega;\mathbb{R}^d)} \le c_0 P(z_1, 0) \left\|\tilde{\ell}_{1,2}\right\|_{W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega;\mathbb{R}^d)}$, whence we deduce the estimate

$$\begin{aligned}\|u_1 - u_2\|_{W^{1,\widetilde{p}}(\Omega;\mathbb{R}^d)} \le c_0 P(z_1, 0)\big(\, &\|\ell(t_1) - \ell(t_2)\|_{W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega;\mathbb{R}^d)} + \|(g(z_1) - g(z_2))\mathbb{C}\varepsilon(u_2)\|_{L^{\widetilde{p}}(\Omega;\mathbb{R}^d)} \\ &+ \|g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\|_{L^{\widetilde{p}}(\Omega;\mathbb{R}^d)}\big).\end{aligned} \tag{2.28}$$

Now, the Lipschitz continuity of g and Hölder's inequality imply that

$$\begin{aligned}\|(g(z_1) - g(z_2))\mathbb{C}\varepsilon(u_2)\|_{L^{\widetilde{p}}(\Omega;\mathbb{R}^d)} &\le C\,\|z_1 - z_2\|_{L^r(\Omega)}\,\|\varepsilon(u_2)\|_{L^{p_*}(\Omega;\mathbb{R}^d)} \\ &\le C' P(z_2, 0)\,\|z_1 - z_2\|_{L^r(\Omega)}\end{aligned} \tag{2.29}$$

with $r = p_*\widetilde{p}(p_* - \widetilde{p})^{-1}$, where the second inequality follows from condition (2.22) and from estimate (2.24). We use (2.29) to estimate the second term on the right-hand side of (2.28). In a similar way the third summand is treated, where we use again (2.22). □

For the proof of Lemma 2.9 on the differentiability in t, the calculations are similar to those in [33, Lemma 2.3], taking into account estimates (2.24) and (2.26). Therefore we choose not to detail them.

Lemma 2.9 (Differentiability and growth w.r. to time).
Under Assumptions 2.1, 2.5, and $(\mathrm{A}_\Omega 1)$, for every $z \in \mathcal{Z}$ the map $t \mapsto \mathcal{I}(t, z)$ is in $\mathrm{C}^1([0, T];\mathbb{R})$ *with*

$$\partial_t\mathcal{I}(t, z) = \int_\Omega g(z)\mathbb{C}\varepsilon(u_{\min}(t, z) + u_D(t)) \colon \varepsilon(\dot{u}_D(t))\,\mathrm{d}x - \langle \dot{\ell}(t), u_{\min}(t, z)\rangle_{W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d)}. \tag{2.30}$$

Moreover, there exists a constant $c_4 > 0$ such that for all $t \in [0, T]$, $z \in \mathcal{Z}$ and u_D, ℓ with (2.22) *we have*

$$|\partial_t\mathcal{I}(t, z)| \le c_4\big(\,\|u_D\|^2_{\mathrm{C}^1([0,T];W^{1,2}(\Omega;\mathbb{R}^d))} + \|\ell\|^2_{\mathrm{C}^1([0,T];W^{-1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d))}\big). \tag{2.31}$$

Finally, there exists a constant $c_5 > 0$ depending on $\|\ell\|_{\mathrm{C}^{1,1}([0,T];W^{-1,p_*}_{\Gamma_D}(\Omega;\mathbb{R}^d))}$ and $\|u_D\|_{\mathrm{C}^{1,1}([0,T];W^{1,p_*}(\Omega))}$ such that for all $r \in [\frac{p_*}{p_*-2}, \infty)$, for all $t_i \in [0, T]$ and $z_i \in \mathcal{Z}$ we have

$$|\partial_t\mathcal{I}(t_1, z_1) - \partial_t\mathcal{I}(t_2, z_2)| \le c_5 P(z_1, z_2)^2\big(\,|t_1 - t_2| + \|z_1 - z_2\|_{L^r(\Omega)}\big). \tag{2.32}$$

The differentiability of $\mathcal{I}$ with respect to z will be studied in the $\mathcal{Z} - \mathcal{Z}^*$ duality. In particular, $\mathrm{D}_z\mathcal{I}(t, \cdot) : \mathcal{Z} \to \mathcal{Z}^*$ shall denote the Gâteaux-differential of the functional $\mathcal{I}(t, \cdot)$. We have the following result, whose proof is completely analogous to the proof of [33, Lemma 2.4].

Lemma 2.10 (Gâteaux-differentiability).
Under Assumptions 2.1, 2.5, and ($A_\Omega 1$), for all $t \in [0,T]$ the functional $\mathcal{I}(t,\cdot) : \mathcal{Z} \to \mathbb{R}$ is Gâteaux-differentiable at all $z \in \mathcal{Z}$, and for all $\eta \in \mathcal{Z}$ we have

$$\langle \mathrm{D}_z\mathcal{I}(t,z), \eta\rangle_{\mathcal{Z}} = \langle A_q z, \eta\rangle_{\mathcal{Z}} + \int_\Omega f'(z)\eta \,\mathrm{d}x + \int_\Omega g'(z)\widetilde{W}(t, \nabla u_{\min}(t,z))\eta \,\mathrm{d}x, \tag{2.33}$$

where we use the abbreviation $\widetilde{W}(t,\nabla v) = W(\nabla v + \nabla u_D(t)) = \frac{1}{2}\mathbb{C}\varepsilon(v+u_D(t)){:}\varepsilon(v+u_D(t))$. In particular, the following estimate holds with a constant c_6 depending on the data ℓ, u_D, but independent of t and z:

$$\forall\,(t,z) \in [0,T]\times\mathcal{Z}: \quad \|\mathrm{D}_z\mathcal{I}(t,z)\|_{\mathcal{Z}^*} \le c_6\left(\|z\|_{\mathcal{Z}}^{q-1} + \|f'(z)\|_{L^\infty(\Omega)} + 1\right). \tag{2.34}$$

We define

$$\widetilde{\mathcal{I}}(t,z) := \mathcal{I}_2(t,z) + \int_\Omega f(z)\,\mathrm{d}x \quad \text{for all } (t,z) \in [0,T]\times\mathcal{Z} \tag{2.35}$$

with $\mathcal{I}_2$ from (2.13), as the part of the reduced energy collecting all lower order terms. Accordingly, $\mathrm{D}_z\mathcal{I}$ from (2.33) decomposes as

$$\mathrm{D}_z\mathcal{I}(t,z) = A_q z + \mathrm{D}_z\widetilde{\mathcal{I}}(t,z) \quad \text{for all } (t,z)\in[0,T]\times\mathcal{Z}. \tag{2.36}$$

In Lemma 2.12 below we prove that the maps $(t,z) \mapsto \widetilde{\mathcal{I}}(t,z(t))$ and that $(t,z)\mapsto \mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$ are Lipschitz continuous w.r.t. a suitable *Lebesgue* norm. In view of this, and in order to emphasize that, in (2.36), $\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$ is a lower order term w.r.t. $A_q z$, from now on we shall often resort to the following

Notation 2.11 (Abuse of notation for $\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$). In view of (2.33), the term $\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)$ can be identified with an element of $L^\mu(\Omega)$ for some $\mu \ge 1$. The quantity $\left\|\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)\right\|_{L^\mu(\Omega)}$ will be interpreted in this sense, and with the symbol $\mathrm{D}_z\widetilde{\mathcal{I}}$, we shall denote both the derivative of $\widetilde{\mathcal{I}}$ as an operator and the corresponding density in $L^1(\Omega)$. Accordingly, for a given $v \in L^{\mu'}(\Omega)$ we shall write $\int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z)v\,\mathrm{d}x$ in place of $\langle \mathrm{D}_z\widetilde{\mathcal{I}}(t,z), v\rangle_{L^{\mu'}(\Omega)}$.

For $h \in \mathrm{C}^0(\mathbb{R})$ and $z_1, z_2 \in \mathcal{Z}$ let

$$C_h(z_1,z_2) = \max\{\,|h(s)|;\ \ |s| \le \|z_1\|_{L^\infty(\Omega)} + \|z_2\|_{L^\infty(\Omega)}\,\}. \tag{2.37}$$

This notation will be used along the proof of the following lemma.

Lemma 2.12 (Local Lipschitz continuity of $\widetilde{\mathcal{I}}$ and $\mathrm{D}_z\widetilde{\mathcal{I}}$).
Under Assumptions 2.1, 2.5, and ($A_\Omega 1$), there exists a constant $c_7 > 0$ depending on $\|\ell\|_{\mathrm{C}^{1,1}([0,T];W^{1,-p_*}_{\Gamma_D}(\Omega;\mathbb{R}^d))}$ and $\|u_D\|_{\mathrm{C}^{1,1}([0,T];W^{1,p_*}_{\Gamma_D}(\Omega;\mathbb{R}^d))}$ such that for all $t_i \in [0,T]$ and all $z_i \in \mathcal{Z}$ it holds

$$\left|\widetilde{\mathcal{I}}(t_1,z_1) - \widetilde{\mathcal{I}}(t_2,z_2)\right| \le c_7(C_{f'}(z_1,z_2) + P(z_1,z_2)^2)\left(|t_1-t_2| + \|z_1-z_2\|_{L^{2p_*/(p_*-2)}(\Omega)}\right), \tag{2.38}$$

with $C_{f'}(z_1,z_2)$ as in (2.37) (corresponding to $h(x) = f'(x)$). Further, for every $\mu \in [1,p_*/2)$,

$$\begin{aligned}\left\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\right\|_{L^\mu(\Omega)} &\le c_7\big(C_{f''}(z_1,z_2) \\ &\quad + (1 + C_{g''}(z_1,z_2))P(z_1,z_2)^3\big)\big(\,|t_1-t_2| + \|z_1-z_2\|_{L^r(\Omega)}\,\big),\end{aligned} \tag{2.39}$$

where $r = p_*\mu(p_*-2\mu)^{-1}$, and for $\mu \in [1,p_*/2]$,

$$\forall\,(t,z)\in[0,T]\times\mathcal{Z}: \qquad \|\mathrm{D}_z\widetilde{\mathcal{I}}(t,z)\|_{L^\mu(\Omega)} \le c_7(1 + \|f'(z)\|_{L^\infty(\Omega)} + P(z,0)^2). \tag{2.40}$$

Proof. In order to prove estimate (2.38), with elementary calculations we observe that

$$\begin{aligned}\left|\widetilde{\mathcal{I}}(t_1,z_1)-\widetilde{\mathcal{I}}(t_2,z_2)\right| &\leq \int_\Omega |f(z_1)-f(z_2)|\,\mathrm{d}x+\int_\Omega |g(z_1)-g(z_2)||\widetilde{W}(t_1,\nabla u_1)|\,\mathrm{d}x\\ &\quad+\int_\Omega |g(z_2)||\widetilde{W}(t_1,\nabla u_1)-\widetilde{W}(t_2,\nabla u_2)|\,\mathrm{d}x+|\langle \ell(t_1)-\ell(t_2),u_1\rangle_{\mathcal{U}}|\\ &\quad+|\langle \ell(t_2),u_1-u_2\rangle_{\mathcal{U}}| \doteq I_1+I_2+I_3+I_4+I_5,\end{aligned}$$

where $u_i := u_{\min}(t_i,z_i)\in W^{1,p_*}(\Omega;\mathbb{R}^d)$ for $i=1,2$. Since $f\in \mathrm{C}^1(\mathbb{R})$ (cf. (2.8)),

$$I_1 \leq C_{f'}(z_1,z_2)\|z_1-z_2\|_{L^1(\Omega)}. \tag{2.41}$$

Moreover, using (2.9a), (2.10), (2.22), and the Hölder inequality,

$$\begin{aligned}I_2 &\leq \|g(z_1)-g(z_2)\|_{L^{p_*/(p_*-2)}(\Omega)}\|\widetilde{W}(t_1,\nabla u_1)\|_{L^{p_*/2}(\Omega)} \leq C\|z_1-z_2\|_{L^{p_*/(p_*-2)}(\Omega)}\|u_1\|^2_{W^{1,p_*}(\Omega;\mathbb{R}^d)}\\ &\leq CP(z_1,0)^2\|z_1-z_2\|_{L^{p_*/(p_*-2)}(\Omega)}\end{aligned}$$

where the constant C also incorporates the data and the last inequality follows from (2.24) (with $p=p_*$). Analogously,

$$\begin{aligned}I_3 &\leq C\int_\Omega |g(z_2)||\varepsilon(u_1+u_D(t_1))+\varepsilon(u_2+u_D(t_2))||\varepsilon(u_1+u_D(t_1))-\varepsilon(u_2+u_D(t_2))|\,\mathrm{d}x\\ &\leq C(\|u_1+u_2\|_{W^{1,2}(\Omega;\mathbb{R}^d)}+1)(\|u_1-u_2\|_{W^{1,2}(\Omega;\mathbb{R}^d)}+\|u_D(t_1)-u_D(t_2)\|_{W^{1,2}(\Omega;\mathbb{R}^d)})\\ &\leq CP(z_1,z_2)^2(|t_1-t_2|+\|z_1-z_2\|_{L^{2p_*/(p_*-2)}(\Omega)})\end{aligned}$$

due to (2.10) and (2.22) and, for the last inequality, to (2.24) (with $p=2$), and (2.26) with $p=2$, whence $r=2p_*/(p_*-2)$. Finally,

$$\begin{aligned}I_4 &\leq \|\ell(t_1)-\ell(t_2)\|_{W^{-1,p_*}(\Omega;\mathbb{R}^d)}\|u_1\|_{W^{1,p_*}(\Omega;\mathbb{R}^d)} \leq C|t_1-t_2|P(z_1,z_2),\\ I_5 &\leq \|\ell(t_2)\|_{W^{-1,2}(\Omega;\mathbb{R}^d)}\|u_1-u_2\|_{W^{1,2}(\Omega;\mathbb{R}^d)} \leq CP(z_1,z_2)^2(|t_1-t_2|+\|z_1-z_2\|_{L^{2p_*/(p_*-2)}(\Omega)})\end{aligned}$$

where the first estimate is due to (2.22) and (2.24), and the second one follows from $\ell\in \mathrm{C}^0([0,T];W^{-1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d))$ and again (2.26). Collecting the above calculations, we conclude (2.38).

Since f' is locally Lipschitz, for the proof of (2.39) we confine ourselves to investigating the properties of $\mathrm{D}_z\mathcal{I}_2$, given by (2.13). Let $\mu\in[1,p_*/2)$. We have

$$\begin{aligned}&\|\mathrm{D}_z\mathcal{I}_2(t_1,z_1)-\mathrm{D}_z\mathcal{I}_2(t_2,z_2)\|_{L^\mu(\Omega)}\\ &\quad\leq C_{g''}(z_1,z_2)\,\|z_1-z_2\|_{L^r(\Omega)}\left\|\widetilde{W}(t_1,\nabla u_1)\right\|_{L^{p_*/2}(\Omega)}+c\left\|\widetilde{W}(t_1,\nabla u_1)-\widetilde{W}(t_2,\nabla u_2)\right\|_{L^\mu(\Omega)}\\ &\quad\leq C(1+C_{g''}(z_1,z_2))P(z_1,z_2)^3\big(\,|t_1-t_2|+\|z_1-z_2\|_{L^r(\Omega)}\,\big)\end{aligned}$$

and C depends on the data ℓ and u_D. Indeed, the first inequality follows from the form of $\mathrm{D}_z\mathcal{I}_2$ (cf. (2.33)). The second one ensues from (2.10) and the Hölder inequality for the first term, which is then estimated by means of (2.24) Lemma 2.6. For the term $\left\|\widetilde{W}(t_1,\nabla u_1)-\widetilde{W}(t_2,\nabla u_2)\right\|_{L^\mu(\Omega)}$, we again use the Hölder inequality. Ultimately, this leads us to estimate the quantity $\|u_1+u_2\|_{W^{1,p_*}(\Omega;\mathbb{R}^d)}$, for which we use (2.24), and the quantity $\|u_1-u_2\|_{W^{1,\widetilde{p}}(\Omega;\mathbb{R}^d)}$ with $\widetilde{p}=\mu p_*/(p_*-\mu)$, for which we use (2.26) (observe that $\mu\leq \frac{p_*}{2}$), with $r=p_*\mu/(p_*-2\mu)$ (indeed, $r=p_*\widetilde{p}/(p_*-\widetilde{p})$). With completely analogous calculations, we prove (2.40). □

From (2.39) we deduce estimate (2.42) below, which occurs in several of the calculations in Sec. 5.

Corollary 2.13. *Under Assumptions 2.1, 2.5, and ($A_\Omega 1$), for every $w \in \mathcal{Z}$ there holds*

$$\begin{aligned}&\left|\langle \mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2), w\rangle_{\mathcal{Z}}\right| \\ &\le c_7(C_{f''}(z_1,z_2) + (1+C_{g''}(z_1,z_2))P(z_1,z_2)^3)(|t_1-t_2| + \|z_1-z_2\|_{L^{2p_*/(p_*-2)}(\Omega)})\|w\|_{L^{2p_*/(p_*-2)}(\Omega)}.\end{aligned} \tag{2.42}$$

Proof. To check (2.42), we use the Hölder inequality and estimate

$$\left|\langle \mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2), w\rangle_{\mathcal{Z}}\right| \le \|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\|_{L^{2p_*/(p_*+2)}(\Omega)}\|w\|_{L^{2p_*/(p_*-2)}(\Omega)},$$

(cf. Notation 2.11), which, with (2.39) for $\mu = 2p_*/(p_*+2)$ and $r = 2p_*/(p_*-2)$, implies (2.42). □

We also have the following monotonicity property for $\mathrm{D}_z\mathcal{I}$.

Corollary 2.14.
Under Assumptions 2.1, 2.5, and ($A_\Omega 1$), there exist constants c_8, $c_9 > 0$ such that for all $t \in [0,T]$ and $z_i \in \mathcal{Z}$, $i = 1,2$, we have

$$\|z_1-z_2\|^2_{L^2(\Omega)} + \langle \mathrm{D}_z\mathcal{I}(t,z_1) - \mathrm{D}_z\mathcal{I}(t,z_2), z_1-z_2\rangle_{\mathcal{Z}} \ge c_8\|z_1-z_2\|^2_{W^{1,2}(\Omega)} - c_9\|z_1-z_2\|^2_{L^2(\Omega)}\,. \tag{2.43}$$

Proof. We observe that by (2.5) and (2.42) there holds

$$\begin{aligned}&\|z_1-z_2\|^2_{L^2(\Omega)} + \langle \mathrm{D}_z\mathcal{I}(t,z_1) - \mathrm{D}_z\mathcal{I}(t,z_2), z_1-z_2\rangle_{\mathcal{Z}} \\ &= \|z_1-z_2\|^2_{L^2(\Omega)} + \langle A_q z_1 - A_q z_2, z_1-z_2\rangle_{\mathcal{Z}} + \langle \mathrm{D}_z\widetilde{\mathcal{I}}(t,z_1) - \mathrm{D}_z\widetilde{\mathcal{I}}(t,z_2), z_1-z_2\rangle_{\mathcal{Z}} \\ &\ge \|z_1-z_2\|^2_{L^2(\Omega)} + c_q\int_\Omega (1+|\nabla z_1|^2+|\nabla z_2|^2)^{\frac{q-2}{2}}|\nabla z_1 - \nabla z_2|^2\,\mathrm{d}x - c\|z_1-z_2\|^2_{L^{2p_*/(p_*-2)}(\Omega)}\end{aligned}$$

where $\widetilde{\mathcal{I}}$ is defined as in (2.35). Then, (2.43) follows upon using (2.2). □

Corollary 2.14 implies that the functional $z \mapsto \mathcal{I}(t,z)$ is λ-convex w.r.t. the $L^2(\Omega)$-norm for some $\lambda \in \mathbb{R}$:

$$\exists\,\lambda\in\mathbb{R}\;\forall\, z_0,\, z_1\in\mathcal{Z}\;\forall\,\theta\in(0,1):\quad \mathcal{I}(t,z_\theta) \le (1-\theta)\mathcal{I}(t,z_0) + \theta\mathcal{I}(t,z_1) - \frac{\lambda\theta(1-\theta)}{2}\|z_0-z_1\|^2_{L^2(\Omega)}\,.$$

However, this property does not automatically guarantee the validity of the chain rule for $\mathcal{I}$, cf. the discussion at the beginning of Sec. 3.1. As a summary of the previous lemmata we obtain

Corollary 2.15 (Fréchet differentiability of $\mathcal{I}$)**.**
Under Assumptions 2.1, 2.5, and ($A_\Omega 1$), the functional $\mathcal{I}$ is Fréchet differentiable on $[0,T]\times\mathcal{Z}$ and

$$t_n \to t \text{ and } z_n \to z \text{ strongly in } \mathcal{Z} \text{ implies } \mathrm{D}_z\mathcal{I}(t_n,z_n) \to \mathrm{D}_z\mathcal{I}(t,z) \text{ strongly in } \mathcal{Z}^*. \tag{2.44}$$

Furthermore, $t_n \to t$ and $z_n \rightharpoonup z$ weakly in $\mathcal{Z}$ implies

$$\begin{aligned}&\liminf_{n\to\infty}\mathcal{I}(t_n,z_n) \ge \mathcal{I}(t,z), \quad \widetilde{\mathcal{I}}(t_n,z_n) \to \widetilde{\mathcal{I}}(t,z), \quad \partial_t\mathcal{I}(t_n,z_n) \to \partial_t\mathcal{I}(t,z), \\ &\mathrm{D}_z\widetilde{\mathcal{I}}(t_n,z_n) \to \mathrm{D}_z\widetilde{\mathcal{I}}(t,z) \text{ strongly in } \mathcal{Z}^*.\end{aligned} \tag{2.45}$$

Proof. Observe that $z_n \to z$ in $\mathcal{Z}$ implies $\mathrm{D}_z\mathcal{I}_q(z_n) \to \mathrm{D}_z\mathcal{I}_q(t,z)$ in $\mathcal{Z}^*$. Therefore, in view of Lemma 2.12, the Gâteaux-differential $\mathrm{D}_z\mathcal{I}$ fulfills (2.44), which yields that $\mathcal{I}$ is Fréchet differentiable. The continuity property (2.45) of $\partial_t\mathcal{I}$ and $\mathrm{D}_z\widetilde{\mathcal{I}}$ is an immediate consequence of estimates (2.32) and (2.39), and of the compact embedding $\mathcal{Z} \Subset L^r(\Omega)$ for all $1 \le r \le \infty$. □

3 The viscous problem

We now address the analysis of the viscous L^2-regularization of (1.6) of the rate-independent system (1.3). To this aim, we introduce the *viscous* dissipation potential

$$\mathcal{R}_\epsilon = \mathcal{R}_1 + \mathcal{R}_{2,\epsilon} \quad \text{with } \mathcal{R}_{2,\epsilon}(\eta) = \frac{\epsilon}{2} \|\eta\|^2_{L^2(\Omega)}, \tag{3.1}$$

with $\mathcal{R}_1$ from (1.4). We denote by $\partial\mathcal{R}_\epsilon : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ its subdifferential (in the sense of convex analysis), in the duality between $\mathcal{Z}^*$ and $\mathcal{Z}$, and consider *viscous* doubly nonlinear evolution equation

$$\partial\mathcal{R}_\epsilon(z'(t)) + \mathrm{D}_z\mathcal{I}(t, z(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0, T), \tag{3.2}$$

with the initial condition, featuring $z_0 \in \mathcal{Z}$,

$$z(0) = z_0. \tag{3.3}$$

It follows from [2, Cor. IV.6] that $\partial\mathcal{R}_\epsilon(\eta) = \partial\mathcal{R}_1(\eta) + \epsilon\eta$ for all $\eta \in \mathcal{Z}$. Thus, also taking into account formula (2.33) for $\mathrm{D}_z\mathcal{I}$, we see that (3.2) translates into

$$\partial\mathcal{R}_1(z'(t)) + \epsilon z'(t) + A_q(z(t)) + f'(z(t)) + g'(z(t))\widetilde{W}(t, \nabla u_{\min}(t, z(t))) \ni 0 \text{ in } \mathcal{Z}^* \text{ for a.a. } t \in (0, T). \tag{3.4}$$

3.1 Weak solutions: definition and existence result

We are going to prove an existence result for a suitable weak solution notion for the Cauchy problem associated with (3.2). Before defining such a concept, let us explain why we do not treat (3.2) as a pointwise-in-time differential inclusion in $\mathcal{Z}^*$. Indeed, (3.2) is equivalent to $-A_q z(t) - \mathrm{D}_z\widetilde{\mathcal{I}}(t, z(t)) \in \partial\mathcal{R}_\epsilon(z'(t))$ for almost all $t \in (0, T)$ with $\widetilde{\mathcal{I}}$ from (2.35), viz.

$$\mathcal{R}_\epsilon(w) - \mathcal{R}_\epsilon(z'(t)) \geq \langle -A_q z(t) - \mathrm{D}_z\widetilde{\mathcal{I}}(t, z(t)), w - z'(t)\rangle_{\mathcal{Z}} \quad \text{for all } w \in \mathcal{Z}, \quad \text{for a.a. } t \in (0, T). \tag{3.5}$$

In fact, (3.5) implies the information that $z'(t) \in \mathcal{Z}$ for almost all $t \in (0, T)$. However, as we are going to show in what follows, the best spatial regularity for $z'(t)$ we can obtain is $z'(t) \in W^{1,2}(\Omega)$, which is less than $z'(t) \in \mathcal{Z}$. For achieving the latter, given a sequence of approximate solutions $(z_k)_k$ to (3.2) (in our case, constructed by time-discretization), one would need a $\mathcal{Z}$-estimate for $(z_k')_k$, uniform w.r.t. $k \in \mathbb{N}$. This seems to be out of reach, due to the doubly nonlinear character of (3.2), and in particular to the fact that the multivalued, unbounded operator $\partial\mathcal{R}_\epsilon$ acts on $z'(t)$.

Another possibility is to interpret the duality pairing $\langle A_q z(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(t, z(t)), w - z'(t)\rangle_{\mathcal{Z}}$ as a pairing between Lebesgue spaces. For this it is necessary that $z'(t) \in L^\sigma(\Omega)$ and $A_q z(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(t, z(t)) \in L^{\sigma'}(\Omega)$ for some $\sigma \in [1, \infty)$. This boils down to proving that $A_q z(t) \in L^{\sigma'}(\Omega)$, since the term $\mathrm{D}_z\widetilde{\mathcal{I}}(t, z(t))$ may be considered of lower order due to (2.39). Indeed, $\int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t, z(t))z'(t)\, \mathrm{d}x$ makes sense thanks to (3.7). However, an estimate for $(A_q z_k)_k$ in $L^{\sigma'}(\Omega)$ (for a sequence of approximate solutions $(z_k)_k$), is out of grasp in the present context. Only for $\sigma = 2$ it would be possible to estimate $(A_q z_k)_k$ in $L^\infty(0, T; L^2(\Omega))$, by testing an approximate version of (3.2) by the quantity $\partial_t(A_q z_k + f'(z_k))$. This is by now standard in the analysis of doubly nonlinear equations of the type (3.2) and dates back to [9]. Nonetheless, to carry out the calculations attached to this test, one has to exploit elliptic regularity results for u, which hold in smooth domains, while in this paper we aim to work under minimal regularity requirements on Ω.

Because of these reasons, we need to resort to the weak solution concept in Definition 3.1 below, where for general $q > d$ the duality pairing $\langle A_q z(t), z'(t)\rangle_{\mathcal{Z}}$ is replaced by the quantity

$$\int_\Omega (1 + |\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t) \cdot \nabla z'(t)\, \mathrm{d}x, \tag{3.6}$$

which in fact coincides with $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_q(z(t))$ for almost all $t \in (0,T)$, cf. (3.20) below.

Definition 3.1 (Weak solution)**.** We say that

$$z \in L^\infty(0,T;W^{1,q}(\Omega)) \cap W^{1,2}(0,T;W^{1,2}(\Omega)) \tag{3.7}$$

fulfilling

$$\int_0^T \int_\Omega (1+|\nabla z(r)|^2)^{\frac{q-2}{2}} |\nabla z'(r)|^2 \,\mathrm{d}x\,\mathrm{d}r < \infty\,, \tag{3.8}$$

is a *weak solution* of (3.2), if it complies with the variational inequality

$$\begin{aligned}\mathcal{R}_\epsilon(w) - \mathcal{R}_\epsilon(z'(t)) \geq \langle -A_q z(t), w\rangle_{\mathcal{Z}} + \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t)\cdot\nabla z'(t)\,\mathrm{d}x \\ - \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))(w-z'(t))\,\mathrm{d}x \quad \text{for all } w\in\mathcal{Z} \qquad \text{for a.a.}\, t\in(0,T)\,.\end{aligned} \tag{3.9}$$

Observe that $\int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t)\cdot\nabla z'(t)\,\mathrm{d}x$ is well defined as soon as z fulfills (3.8), cf. (3.24) below. Hereafter, we shall refer to (3.8) as "mixed estimate", for it involves both z and z'. In fact, (3.8) shall result from the a priori estimates on the time-discretization of (3.2), contained in Lemma 5.3.

The regularity (3.8) also guarantees the validity of the following *chain-rule* formula

Theorem 3.2. *Under Assumptions 2.1, 2.5, and* ($\mathrm{A}_\Omega 1$)*, for every curve z fulfilling* (3.7) *and* (3.8)

1. *the map $t \mapsto \mathcal{I}(t,z(t))$ is absolutely continuous on* $(0,T)$*;*
2. *the following chain-rule formula is valid:*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t,z(t)) - \partial_t\mathcal{I}(t,z(t)) \\ &= \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t)\cdot\nabla z'(t)\,\mathrm{d}x + \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))z'(t)\,\mathrm{d}x \quad \textit{for a.a.}\, t\in(0,T),\end{aligned} \tag{3.10}$$

where for the second equality we refer to Notation 2.11.

We postpone the proof of this result to Sec. 3.2, and point out that as a consequence of formula (3.10) the variational inequality in (3.9) is *equivalent* to the energy inequality associated with (3.2). The latter inequality involves the Fenchel-Moreau convex conjugate $\mathcal{R}_\epsilon^*$ taken in the $\mathcal{Z}-\mathcal{Z}^*$ duality, and defined by $\mathcal{R}_\epsilon^*(\xi) = \sup\{\,\langle\xi,w\rangle_{\mathcal{Z}} - \mathcal{R}_\epsilon(w)\,:\, w\in\mathcal{Z}\}$. In (7.2) we give the explicit formula for $\mathcal{R}_\epsilon^*$.

Proposition 3.3. *Under Assumptions 2.1, 2.5, and* ($\mathrm{A}_\Omega 1$)*, a curve z fulfilling* (3.7) *and* (3.8) *is a weak solution of* (3.2) *in the sense of Def. 3.1 if and only if it fulfills for all $0\leq s\leq t\leq T$ the energy inequality*

$$\int_s^t \mathcal{R}_\epsilon(z'(r))\,\mathrm{d}r + \int_s^t \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(r,z(r)))\,\mathrm{d}r + \mathcal{I}(t,z(t)) \leq \mathcal{I}(s,z(s)) + \int_s^t \partial_t\mathcal{I}(r,z(r))\,\mathrm{d}r. \tag{3.11}$$

Proof. Taking into account that $w\in\mathcal{Z}$ is arbitrary, (3.9) rephrases as

$$\begin{aligned}\mathcal{R}_\epsilon(z'(t)) + \sup_{w\in\mathcal{Z}}\Big(-\langle A_q z(t), w\rangle_{\mathcal{Z}} - \langle \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t)), w\rangle_{\mathcal{Z}} - \mathcal{R}_\epsilon(w)\Big) \\ + \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \nabla z(t)\cdot\nabla z'(t)\,\mathrm{d}x + \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))z'(t)\,\mathrm{d}x \leq 0 \qquad \text{for a.a.}\, t\in(0,T).\end{aligned}$$

In view of the definition of $\mathcal{R}_\epsilon^*$ and the chain-rule formula (3.10), the above inequality is equivalent to

$$\mathcal{R}_\epsilon(z'(t)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(t,z(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t,z(t)) \leq \partial_t\mathcal{I}(t,z(t)) \qquad \text{for a.a.}\, t\in(0,T),$$

i.e. (3.11) upon integrating in time. $\square$

Remark 3.4 (Failure of energy identity). It remains an open problem to improve (3.11) to an energy identity. This would result from the following chain of inequalities

$$\begin{aligned}
&\int_s^t \mathcal{R}_\epsilon(z'(r))\,\mathrm{d}r + \int_s^t \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(r,z(r)))\,\mathrm{d}r \\
&\le \mathcal{I}(s,z(s)) - \mathcal{I}(t,z(t)) + \int_s^t \partial_t\mathcal{I}(r,z(r))\,\mathrm{d}r \\
&\overset{(1)}{=} -\int_s^t\int_\Omega (1+|\nabla z(r)|^2)^{\frac{q-2}{2}}\nabla z(r)\cdot\nabla z'(r)\,\mathrm{d}x\,\mathrm{d}r - \int_s^t\int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(r,z(r))z'(r)\,\mathrm{d}x\,\mathrm{d}r \\
&\overset{(2,?)}{=} -\int_s^t \langle \mathrm{D}_z\mathcal{I}(r,z(r)), z'(r)\rangle_{\mathcal{Z}}\,\mathrm{d}r \\
&\overset{(3)}{\le} \int_s^t \mathcal{R}_\epsilon(z'(r))\,\mathrm{d}r + \int_s^t \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(r,z(r)))\,\mathrm{d}r\,.
\end{aligned}$$

While (1) follows from an integrated version of (3.10) on the right-hand side of (3.11) and (3) from an elementary convex analysis inequality, $(2,?)$ implies the information that $z'(t)\in\mathcal{Z}$ for almost all $t\in(0,T)$, which is not at our disposal. Observe that, with this argument we would also conclude that z fulfills the subdifferential inclusion (3.2), cf. the proofs of [42, Thm. 4.4], [33, Thm. 3.1]. Therefore, the validity of (3.2) and of the related energy identity is at the moment open for general $q>d$.

We are now in the position of stating our existence result for the Cauchy problem associated with (3.2). In fact, we need to impose a further, natural condition on the domain Ω. This is exploited in the proof of fine spatial regularity estimates on the discrete solutions, which lead to the enhanced regularity (3.13) below for z, and will enable us to pass to the limit in the time-discretization scheme of (3.2).

Theorem 3.5 (Existence of weak solutions, $\epsilon>0$)**.** *Under Assumptions 2.1, 2.5, and (A_Ω1), suppose in addition that*

(A_Ω2) $\Omega\subset\mathbb{R}^d$ is a bounded domain and satisfies the uniform cone condition.

Suppose that the initial datum $z_0\in\mathcal{Z}$ also fulfills

$$\mathrm{D}_z\mathcal{I}(0,z_0)\in L^2(\Omega). \tag{3.12}$$

Then,

1. *for every $\epsilon>0$ there exists a weak solution (in the sense of Definition 3.1) $z_\epsilon\in L^\infty(0,T;W^{1,q}(\Omega))\cap W^{1,2}(0,T;W^{1,2}(\Omega))$ to the Cauchy problem* (3.2)*–*(3.3)*, fulfilling* (3.8) *as well as the enhanced regularity*
$$z_\epsilon\in L^{2q}(0,T;W^{1+\beta,q}(\Omega)) \qquad \text{for every } \beta\in\left[0,\frac{1}{q}\Big(1-\frac{d}{q}\Big)\right). \tag{3.13}$$
If in addition f and g comply with (4.3) *(cf. Proposition 4.1 ahead) and if $z_0\in[0,1]$, then $z_\epsilon(t,x)\in[0,1]$ for all $(t,x)\in[0,T]\times\Omega$.*

2. *There exists a family of viscous solutions $(z_\epsilon)_{\epsilon>0}$ and constants $C_0, C_\beta>0$ such that the following*

estimates hold uniformly *w.r.t.* ϵ

$$\sup_{\epsilon>0} \|z_\epsilon\|_{W^{1,1}(0,T;L^2(\Omega))} \leq C_0, \tag{3.14}$$

$$\sup_{\epsilon>0} \|z_\epsilon\|_{L^{2q}(0,T;W^{1+\beta,q}(\Omega))\cap L^\infty(0,T;W^{1,q}(\Omega))} \leq C_\beta \qquad \textit{for every } \beta \in \left[0, \frac{1}{q}\Big(1-\frac{d}{q}\Big)\right), \tag{3.15}$$

$$\sup_{\epsilon>0} \int_0^T \|z_\epsilon(t)\|^q_{W^{1+\beta,q}(\Omega)} \|z'_\epsilon(t)\|_{L^2(\Omega)} \,\mathrm{d}t \leq C_\beta \qquad \textit{for every } \beta \in \left[0, \frac{1}{q}\Big(1-\frac{d}{q}\Big)\right), \tag{3.16}$$

$$\sup_{\epsilon>0} \int_0^T \left(\int_\Omega (1+|\nabla z_\epsilon(t)|^2)^{\frac{q-2}{2}} |\nabla z'_\epsilon(t)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \,\mathrm{d}t \leq C_0. \tag{3.17}$$

The *proof* (see Section 6) relies on the time-discretization analysis performed in Section 4 and on the a priori estimates provided in Section 5.

The uniform w.r.t. ϵ estimates (3.14)–(3.17) are the starting point for the vanishing-viscosity analysis in Section 7. We prove them in Section 5 arguing on the time-discretization of (3.2) and thus deduce them only for those viscous solutions z_ϵ to (3.2), which arise in the limit of the time-discretization scheme of Sec. 4. The additional condition (3.12) on the initial datum z_0 is needed in order to prove the enhanced regularity estimates for z, as well as the uniform discrete $W^{1,1}$-estimate (see Sections 5.3 and 5.4).

A discussion on the interpretation of weak solutions For $\xi \in W^{1,q}(\Omega)$ let

$$\|\xi\|_{\nabla z(t)} := \left(\|\xi\|^2_{L^2(\Omega)} + \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} |\nabla \xi|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \tag{3.18}$$

and define $\mathcal{V}_{\nabla z(t)}(\Omega) := \overline{\mathcal{Z}}^{\|\cdot\|_{\nabla z(t)}}$. Observe that the set $\mathcal{Z}_- := \{\, z \in \mathcal{Z}\,;\, z \leq 0 \,\}$ is dense in $\mathcal{V}_{\nabla z(t),-} := \{\, v \in \mathcal{V}_{\nabla z(t)}\,;\, v \leq 0 \,\}$. This implies that the conjugate functional of $\mathcal{R}_\epsilon$ calculated with respect to the $\mathcal{Z}-\mathcal{Z}^*$ duality (which in this context we denote by $\mathcal{R}_\epsilon^{*\mathcal{Z}}$), and the conjugate functional $\mathcal{R}_\epsilon^{*\mathcal{V}_{\nabla z(t)}}$ with respect to the $\mathcal{V}_{\nabla z(t)} - \mathcal{V}^*_{\nabla z(t)}$ duality, coincide on $\mathcal{V}^*_{\nabla z(t)}$. Now, let $z \in L^\infty(0,T;W^{1,q}(\Omega)) \cap W^{1,2}(0,T;W^{1,2}(\Omega))$ be a weak solution to the Cauchy problem (3.2)–(3.3) in the sense of Definition 3.1, with the enhanced regularity (3.13), and assume in addition that $z'(t) \in \mathcal{V}_{\nabla z(t)}$ for almost all $t \in (0,T)$. As it will be discussed below this is not a trivial assumption, and at the moment it is open whether the solution z satisfies this assumption at all.

Now we can verify directly relying on Section 2.3 that $\mathrm{D}_z\mathcal{I}(t,z(t)) \in \mathcal{V}^*_{\nabla z(t)}$. Having this, with the additional assumption that $z'(t) \in \mathcal{V}_{\nabla z(t)}$ for almost all $t \in (0,T)$, from the local version of (3.11) in combination with the chain rule (3.10) and the Young-Fenchel inequality for conjugate functionals we deduce that for almost all $t \in (0,T)$ it holds

$$\begin{aligned}\mathcal{R}_\epsilon(z'(t)) + \mathcal{R}_\epsilon^{*\mathcal{Z}}(-\mathrm{D}_z\mathcal{I}(t,z(t))) &\leq \langle -\mathrm{D}_z\mathcal{I}(t,z(t)), z'(t)\rangle_{\mathcal{V}_{\nabla z(t)}} \\ &\leq \mathcal{R}_\epsilon(z'(t)) + \mathcal{R}_\epsilon^{*\mathcal{V}_{\nabla z(t)}}(-\mathrm{D}_z\mathcal{I}(t,z(t))) = \mathcal{R}_\epsilon(z'(t)) + \mathcal{R}_\epsilon^{*\mathcal{Z}}(-\mathrm{D}_z\mathcal{I}(t,z(t))).\end{aligned}$$

Hence, for almost all $t \in (0,T)$ the inclusion $0 \in \partial\mathcal{R}_\epsilon(z'(t)) + \mathrm{D}_z\mathcal{I}(t,z(t))$ is satisfied in the $\mathcal{V}_{\nabla z(t)} - \mathcal{V}^*_{\nabla z(t)}$ duality and in (3.11) we have equality instead of an inequality.

However, proving that $z'(t) \in \mathcal{V}_{\nabla z(t)}$ is at the moment an open problem: Due to the mixed estimate, for almost all t the function $z'(t)$ belongs to the Banach space $\mathcal{W}_{\nabla z(t)} := \{\, v \in H^1(\Omega)\,;\, \|v\|_{\mathcal{V}_{\nabla z(t)}} < \infty \,\}$. If the weight $\omega(t) := (1+|\nabla z(t)|^2)^{\frac{q-2}{2}}$ can be shown to be a Muckenhoupt weight, then the spaces $\mathcal{V}_{\nabla z(t)}$ and $\mathcal{W}_{\nabla z(t)}$ coincide, see for instance [10]. However, we do not see how to deduce this property for our solution. Another possibility would be to prove directly from the construction of the solutions (via a time-incremental procedure), that $z(t) \in \mathcal{V}_{\nabla z(t)}$. But also this is not clear for us.

3.2 Proof of the chain rule of Theorem 3.2

Recalling the decomposition $\mathcal{I}(t,z) = \mathcal{I}_q(z) + \widetilde{\mathcal{I}}(t,z)$, we separately address the chain-rule properties of the functionals $\mathcal{I}_q$ and $\widetilde{\mathcal{I}}$. As for the latter, we observe that the Fréchet differentiability stated in Corollary 2.15 allows us to conclude the validity of the chain-rule formula (3.19), only if the curve z is in $W^{1,1}([0,T];\mathcal{Z})$, which is not granted by (3.7) and (3.8). In the proof of Proposition 3.6 below, we in fact exploit the finer estimates on $\widetilde{\mathcal{I}}$ and $\mathrm{D}_z\widetilde{\mathcal{I}}$ provided by Lemma 2.12, and combine them with the regularity (3.7) for z. Note that the *mixed estimate* (3.8) is not needed.

Proposition 3.6. *Under Assumptions 2.1, 2.5, and ($A_\Omega 1$), for every curve z fulfilling* (3.7) *the map $t \mapsto \widetilde{\mathcal{I}}(t,z(t))$ is absolutely continuous on* $(0,T)$ *and there holds (cf. Notation 2.11)*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{I}}(t,z(t)) - \partial_t\widetilde{\mathcal{I}}(t,z(t)) = \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))z'(t)\,\mathrm{d}x \qquad \text{for a.a. } t\in(0,T). \tag{3.19}$$

Proof. For any fixed $z \in L^\infty(0,T;W^{1,q}(\Omega)) \cap W^{1,2}(0,T;W^{1,2}(\Omega))$, the map $t \mapsto \widetilde{\mathcal{I}}(t,z(t))$ is absolutely continuous on $[0,T]$: indeed, it follows from (2.38) that

$$\left|\widetilde{\mathcal{I}}(t,z(t)) - \widetilde{\mathcal{I}}(s,z(s))\right| \le C(|t-s| + \|z(t)-z(s)\|_{L^{2p_*/(p_*-2)}}) \le C(|t-s| + \|z(t)-z(s)\|_{W^{1,2}(\Omega)})$$

where the last inequality follows from the continuous embedding $W^{1,2}(\Omega) \subset L^{2p_*/(p_*-2)}(\Omega)$, cf. (2.2). We now prove the chain-rule formula (3.19). The integral on the right-hand side of (3.19) is well defined since $\mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t)) \in L^\mu(\Omega)$ and $z'(t) \in L^{\mu'}(\Omega)$ with $\mu = 2p_*/(p_*+2)$ and $\mu' = 2p_*/(p_*-2)$, cf. Lemma 2.12. In fact, since $L^\mu(\Omega) \subset W^{1,2}(\Omega)^*$, it follows that $\mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))$ can be identified with an element in $W^{1,2}(\Omega)^*$ for a.a. $t\in(0,T)$. We fix $t\in(0,T)$, out of a negligible set, such that $\exists\, \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{I}}(t,z(t))$, and compute

$$\begin{aligned}
&h^{-1}(\widetilde{\mathcal{I}}(t+h,z(t+h)) - \widetilde{\mathcal{I}}(t,z(t)))\\
&\quad = h^{-1}(\widetilde{\mathcal{I}}(t+h,z(t+h)) - \widetilde{\mathcal{I}}(t,z(t+h))) + h^{-1}(\widetilde{\mathcal{I}}(t,z(t+h)) - \widetilde{\mathcal{I}}(t,z(t)))\\
&\quad = \frac{1}{h}\int_t^{t+h} \partial_t\widetilde{\mathcal{I}}(s,z(t+h))\,\mathrm{d}s\\
&\qquad + \frac{1}{h}\int_\Omega\int_0^1 \mathrm{D}_z\widetilde{\mathcal{I}}(t,(1-\theta)z(t)+\theta z(t+h))(z(t+h)-z(t))\,\mathrm{d}\theta\,\mathrm{d}x \doteq I_h^1 + I_h^2.
\end{aligned}$$

We have that

$$I_h^1 = \frac{1}{h}\int_t^{t+h} \partial_t\widetilde{\mathcal{I}}(s,z(t))\,\mathrm{d}s + \frac{1}{h}\int_t^{t+h}\Big(\partial_t\widetilde{\mathcal{I}}(s,z(t+h)) - \partial_t\widetilde{\mathcal{I}}(s,z(t))\Big)\,\mathrm{d}s.$$

The first term on the right-hand side converges to $\partial_t\widetilde{\mathcal{I}}(s,z(t))$ as $h\to0$, while the second one tends to zero in view of (2.32) and of the fact that $z\in \mathrm{C}^0([0,T];\mathrm{C}^0(\overline{\Omega}))$ by interpolation in (3.7). To take the limit as $h\to 0$ of I_h^2, we first of all observe that for almost all $t\in(0,T)$ $\frac{z(t+h)-z(t)}{h} \to z'(t)$ in $W^{1,2}(\Omega)\subset L^{2p_*/(p_*-2)}(\Omega)$ as $h\to0$, due to $z'\in L^2(0,T;W^{1,2}(\Omega))$ (cf., e.g., [51, Lemma 5, Sect. 5]). Moreover, in view of (2.39), the family $j_h(t,\cdot) := \int_0^1 \mathrm{D}_z\widetilde{\mathcal{I}}(t,(1-\theta)z(t,\cdot)+\theta z(t+h,\cdot))\,\mathrm{d}\theta$ converges to $j(t,\cdot) := \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t,\cdot))$ in $L^{2p_*/(p_*+2)}(\Omega)$ as $h\to0$. Hence, $I_h^2 \to \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(t,z(t))z'(t)\,\mathrm{d}x$ as $h\to0$, and (3.19) follows. $\square$

For the functional $\mathcal{I}_q$, we have the following result.

Proposition 3.7. *For every curve z fulfilling* (3.7) *and* (3.8) *the map $t\mapsto\mathcal{I}_q(z(t))$ is absolutely continuous on* $(0,T)$ *and there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_q(z(t)) = \int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}}\nabla z(t)\cdot\nabla z'(t)\,\mathrm{d}x \qquad \text{for a.a. } t\in(0,T). \tag{3.20}$$

We are going to deduce Proposition 3.7 from applying the result below to $F := \nabla z$.

Lemma 3.8. *Define* $\mathcal{G}_q : L^q(\Omega;\mathbb{R}^d) \to [0,\infty)$ *by* $\mathcal{G}_q(F) := \int_\Omega G_q(F(x))\,\mathrm{d}x = \frac{1}{q}\int_\Omega (1+|F(x)|^2)^{\frac{q}{2}}\,\mathrm{d}x$. *If* $F \in L^\infty(0,T;L^q(\Omega;\mathbb{R}^d)) \cap W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^d))$ *fulfills*

$$\int_0^T \int_\Omega (1+|F|^2)^{\frac{q-2}{2}} |F_t|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{3.21}$$

then the map $t \mapsto \mathcal{G}_q(F(t))$ *is absolutely continuous on* $(0,T)$*, and there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_q(F(t)) = \int_\Omega (1+|F(t)|^2)^{\frac{q-2}{2}} F(t)\cdot F_t(t)\,\mathrm{d}x \qquad \text{for a.a. } t \in (0,T)\,. \tag{3.22}$$

Proof. We split the proof in three claims.

Claim 1 *There holds for all* $0 \le s \le t \le T$ *and for almost all* $x \in \Omega$

$$G_q(F(t,x)) - G_q(F(s,x)) = \int_s^t (1+|F(r,x)|^2)^{\frac{q-2}{2}} F(r,x)\cdot F_t(r,x)\,\mathrm{d}r\,. \tag{3.23}$$

Indeed, (3.23) follows from integrating in time the chain rule $\frac{\mathrm{d}}{\mathrm{d}t}G_q(F(t,x)) = (1+|F(r,x)|^2)^{\frac{q-2}{2}}F(r,x)\cdot F_t(r,x)$ at fixed x, which in turn ensues from applying (3.26) below with $\eta(t) = F(t,x)$ (here x is fixed outside a negligible set) and $\varphi = G_q$. Indeed, (3.21) and the fact that $F \in L^\infty(0,T;L^q(\Omega;\mathbb{R}^d))$ guarantee

$$\begin{aligned} &(t,x) \mapsto (1+|F(t,x)|^2)^{\frac{q-2}{4}} F_t(t,x) \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\ &(t,x) \mapsto (1+|F(t,x)|^2)^{\frac{q-2}{4}} F(t,x) \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)). \end{aligned} \tag{3.24}$$

Hence, by the properties of Bochner integrals we have for almost all $x\in\Omega$ that $t \mapsto (1+|F(t,x)|^2)^{\frac{q-2}{4}}F_t(t,x) \in L^2(0,T)$ and $t \mapsto (1+|F(t,x)|^2)^{\frac{q-2}{4}}F(t,x) \in L^2(0,T)$, therefore $t \mapsto |(1+|F(t,x)|^2)^{\frac{q-2}{2}}F(t,x)||F_t(t,x)|$ is in $L^1(0,T)$ for almost all $x \in \Omega$, and we can apply Lemma 3.9.

Claim 2 *the map* $t \mapsto \mathcal{G}_q(F(t))$ *is absolutely continuous on* $[0,T]$.
Indeed, integrating w.r.t. $x \in \Omega$ formula (3.23) we find

$$\mathcal{G}_q(F(t)) - \mathcal{G}_q(F(s)) = \int_s^t \int_\Omega (1+|F(r,x)|^2)^{\frac{q-2}{2}} F(r,x)\cdot F_t(r,x)\,\mathrm{d}r\,\mathrm{d}x \quad \text{for all } 0 \le s \le t \le T. \tag{3.25}$$

We use this to estimate the difference $|\mathcal{G}_q(F(t)) - \mathcal{G}_q(F(s))|$. In view of (3.24) (and the properties of Bochner integrals), the map $t \mapsto \int_\Omega (1+|F(t,x)|^2)^{\frac{q-2}{2}}F(t,x)\cdot F_t(t,x)\,\mathrm{d}x \in L^1(0,T)$, therefore the absolute continuity of $t \mapsto \mathcal{G}_q(F(t))$ follows from (3.25) and the absolute continuity property of the Lebesgue integral.

Claim 3 *formula* (3.22) *holds.* Let us fix t outside a negligible set such that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(F(t))$ exists as limit of the difference quotient. Writing (3.25) at t and $t+h$ yields

$$\frac{1}{h}\left(\mathcal{G}_q(F(t+h)) - \mathcal{G}_q(F(t))\right) = \frac{1}{h}\int_t^{t+h}\int_\Omega (1+|F(r,x)|^2)^{\frac{q-2}{2}}F(r,x)\cdot F_t(r,x)\,\mathrm{d}r\,\mathrm{d}x.$$

Then, it remains to observe that as $h \to 0$

$$\frac{1}{h}\int_t^{t+h}\int_\Omega (1+|F(r,x)|^2)^{\frac{q-2}{2}}F(r,x)\cdot F_t(r,x)\,\mathrm{d}r\,\mathrm{d}x \to \int_\Omega (1+|F(t,x)|^2)^{\frac{q-2}{2}}F(t,x)\cdot F_t(t,x)\,\mathrm{d}x.$$

This is true for almost all $t \in (0,T)$ thanks to the Lebesgue point property of the map $t \mapsto \int_\Omega (1+|F(t,x)|^2)^{\frac{q-2}{2}}F(t,x)\cdot F_t(t,x)\,\mathrm{d}x \in L^1(0,T)$. □

We conclude by stating, for the sake of completeness, the following auxiliary result.

Lemma 3.9. *Given $\varphi \in \mathrm{C}^1(\mathbb{R}^d;\mathbb{R})$, for every $\eta \in W^{1,2}(0,T;\mathbb{R}^d)$ such that $t \mapsto |\nabla\varphi(\eta(t))||\eta'(t)| \in L^1(0,T)$ the map $t \mapsto \varphi(\eta(t))$ is absolutely continuous on $(0,T)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\eta(t)) = \nabla\varphi(\eta(t)) \cdot \eta'(t) \quad \textit{for a.a.}\, t \in (0,T). \tag{3.26}$$

Proof. The absolute continuity property can be shown by arguing in the very same way as in the proof of [1, Thm. 1.2.5, page 28]. The chain-rule formula follows from classical arguments. □

4 Time-discretization for the viscous problem

We consider the following time-discrete incremental minimization problem: Given $\epsilon > 0$, $z_0 \in \mathcal{Z}$ and a uniform partition $\{0 = t_0^\tau < \ldots < t_N^\tau = T\}$ of the time interval $[0,T]$ with fineness $\tau = t_{k+1}^\tau - t_k^\tau = T/N$ (cf. Remark 5.7 ahead), the elements $(z_k^\tau)_{0\le k\le N}$ are determined through $z_0^\tau = z_0$ and

$$z_{k+1}^\tau \in \mathrm{Argmin}\{\, \mathcal{I}(t_{k+1}^\tau, z) + \tau\mathcal{R}_\epsilon\left(\frac{z - z_k^\tau}{\tau}\right) \,;\, z \in \mathcal{Z}\,\}. \tag{4.1}$$

The existence of minimizers can be checked via the direct method in the calculus of variations, thanks to the properties of the reduced energy $\mathcal{I}$ formulated in Section 2.2. It follows from the representation formula for $\partial\mathcal{R}_\epsilon$ (cf. [2, Cor. IV.6]), that, any family $\{z_1^\tau, \ldots, z_N^\tau\} \subset \mathcal{Z}$ of minimizers of the incremental problem (4.1) satisfies for all $k \in \{0, \ldots, N-1\}$ the *discrete Euler-Lagrange* equation

$$\partial\mathcal{R}_1\left(\frac{z_{k+1}^\tau - z_k^\tau}{\tau}\right) + \epsilon\frac{z_{k+1}^\tau - z_k^\tau}{\tau} + \mathrm{D}_z\mathcal{I}(t_{k+1}^\tau, z_{k+1}^\tau) \ni 0 \qquad \text{in } \mathcal{Z}^*. \tag{4.2}$$

Proposition 4.1. *Under Assumptions 2.1, 2.5, and (A_Ω1), for τ sufficiently small the minimum problem* (4.1) *admits a unique solution. Suppose in addition that f and g comply with the following condition*

$$f(0) \le f(z), \quad g(0) \le g(z) \quad \textit{for all}\ z \le 0, \tag{4.3}$$

and that the initial datum z_0 fulfills $z_0(x) \in [0,1]$ for almost all $x \in \Omega$. Then, the minimizer z_k^τ from (4.1) *also fulfills $z_k^\tau(x) \in [0,1]$ for almost all $x \in \Omega$.*

The proof of uniqueness follows from standard arguments, exploiting estimate (2.43) from Corollary 2.14. The property $z_k^\tau(x) \in [0,1]$ is standard, as well, (see e.g. [33, Prop. 4.5]).

Notation 4.2. The following piecewise constant and piecewise linear interpolation functions will be used:

$$\overline{z}_\tau(t) = z_{k+1}^\tau \text{ for } t \in (t_k^\tau, t_{k+1}^\tau], \ \ \underline{z}_\tau(t) = z_k^\tau \text{ for } t \in [t_k^\tau, t_{k+1}^\tau), \ \ \hat{z}_\tau(t) = z_k^\tau + \frac{t - t_k^\tau}{\tau}(z_{k+1}^\tau - z_k^\tau) \text{ for } t \in [t_k^\tau, t_{k+1}^\tau].$$

Furthermore, we shall use the notation

$$\begin{array}{lll}
\tau(r) & = \tau & \text{for } r \in (t_k^\tau, t_{k+1}^\tau), \\
\bar{t}_\tau(r) & = t_{k+1}^\tau & \text{for } r \in (t_k^\tau, t_{k+1}^\tau], \\
\underline{t}_\tau(r) & = t_k^\tau & \text{for } r \in [t_k^\tau, t_{k+1}^\tau), \\
\overline{u}_\tau(r) & = u_{\min}(\bar{t}_\tau(r), \overline{z}_\tau(r)) & \text{for } r \in (t_k^\tau, t_{k+1}^\tau], \\
\underline{u}_\tau(r) & = u_{\min}(\underline{t}_\tau(r), \underline{z}_\tau(r)) & \text{for } r \in [t_k^\tau, t_{k+1}^\tau), \\
\hat{u}_\tau(r) & = \underline{u}_\tau(r) + \frac{r - \underline{t}_\tau(r)}{\tau}(\overline{u}_\tau(r) - \underline{u}_\tau(r)) & \text{for } r \in [t_k^\tau, t_{k+1}^\tau].
\end{array}$$

Clearly,

$$\overline{t}_\tau(t),\ \underline{t}_\tau(t) \to t \quad \text{as } \tau \to 0 \text{ for all } t \in [0,T]. \tag{4.4}$$

Moreover, for any given function b which is piecewise constant on the intervals (t_i^τ, t_{i+1}^τ) we set

$$\triangle_{\tau(r)} b(r) = b(r) - b(s) \text{ for } r \in (t_k^\tau, t_{k+1}^\tau) \text{ and } s \in (t_{k-1}^\tau, t_k^\tau).$$

With the above notation, (4.2) can be reformulated in $\mathcal{Z}^*$ as

$$\partial\mathcal{R}_1\left(\hat{z}'_\tau(t)\right) + \epsilon\hat{z}'_\tau(t) + \mathrm{D}_z\mathcal{I}(\overline{t}_\tau(t), \overline{z}_\tau(t)) \ni 0 \quad \text{for a.a. } t \in (0,T), \quad \text{viz.} \tag{4.5}$$

$$\begin{cases} \overline{\omega}_\tau(t) + \epsilon\hat{z}'_\tau(t) + A_q\overline{z}_\tau(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_\tau(t)) = 0, \\ \overline{\omega}_\tau(t) \in \partial\mathcal{R}_1\left(\hat{z}'_\tau(t)\right) \end{cases} \quad \text{for a.a. } t \in (0,T). \tag{4.6}$$

Notation 4.3. In what follows, we will denote most of the positive constants occurring in the calculations by the symbols c, C', whose meaning may vary even within the same line. Furthermore, the symbols I_i, S_i, F_i, $i = 0, 1, \ldots$, will be used as abbreviations for several integral terms appearing in the various estimates: we warn the reader that we will not be self-consistent with the numbering, so that, for instance, I_1 will appear several times with different meanings.

4.1 Global higher differentiability of the time-discrete damage variable

In this section we derive the higher differentiability of the solutions z_k^τ of the time-incremental minimization problem (4.1), Theorem 4.4. The proof relies on a difference quotient argument in the spirit of [50, 15, 31] and requires the additional condition $(\mathrm{A}_\Omega 2)$ on Ω stated in Theorem 3.5.

We address the higher differentiability of minimizers for (4.1) in a more general context. In particular, in view of future developments we deal with an L^α-viscosity term instead of L^2-viscosity. Therefore, let $q > d$, $p > 2$. For $z, \zeta \in W^{1,q}(\Omega)$, $w \in W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$, $\tau > 0$, $\epsilon \geq 0$ and $\alpha \geq 2$ we define

$$\mathcal{F}(z;\alpha,\tau,\epsilon,\zeta,w) := \int_\Omega \frac{1}{2}g(z)\mathbb{C}\varepsilon(w) : \varepsilon(w) + f(z) + \frac{1}{q}(1 + |\nabla z|^2)^{\frac{q}{2}}\,\mathrm{d}x + \mathcal{R}_1(z-\zeta) + \frac{\epsilon\tau}{\alpha}\left\|\frac{z-\zeta}{\tau}\right\|^\alpha_{L^\alpha(\Omega)} \tag{4.7}$$

with $\mathcal{R}_1$ from (1.4). The time-incremental minimization problem (4.1) can be rewritten as

$$z_{k+1}^\tau \in \mathrm{Argmin}\{\, \mathcal{F}(z; 2, \tau, \epsilon, z_k^\tau, u_{\min}(t_{k+1}^\tau, z) + u_D(t_{k+1}^\tau))\,;\ z \in \mathcal{Z}\,\}. \tag{4.8}$$

Theorem 4.4 (Spatial differentiability of the damage variable)**.** *Under Assumptions 2.1, $(\mathrm{A}_\Omega 1)$, and $(\mathrm{A}_\Omega 2)$, suppose further that $w \in W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)$ for some $p \geq 2$, and that $\tau > 0$, $\epsilon \geq 0$, $\alpha \geq 2$ and $q > d$.*

Let $z \in \mathcal{Z} = W^{1,q}(\Omega)$ be a minimizer of $\mathcal{F}(\cdot;\alpha,\tau,\epsilon,\zeta,w)$ over $\mathcal{Z}$. Then for all $0 \leq \beta < \frac{1}{q}\left(1 - \frac{d}{q}\right)$ we have $z \in W^{1+\beta,q}(\Omega)$. Moreover, there exists a constant $c_\beta > 0$ such that

$$\|z\|_{W^{1+\beta,q}(\Omega)} \leq c_\beta(1 + \|z\|_{W^{1,q}(\Omega)})\left(1 + \|f'(z)\|^{\frac{1}{q}}_{L^\infty(\Omega)} + \|w\|^{\frac{2}{q}}_{W^{1,p}(\Omega;\mathbb{R}^d)} + \epsilon^{\frac{1}{q}}\left\|\frac{z-\zeta}{\tau}\right\|^{\frac{\alpha-1}{q}}_{L^\alpha(\Omega)}\right), \tag{4.9}$$

and the constant c_β is independent of $\alpha, \epsilon, \tau, z, w$ and ζ.

Remark 4.5. For $\epsilon = 0$, Theorem 4.4 yields a regularity result for global energetic solutions associated with the energy $\mathcal{I}(\cdot,\cdot)$ from (2.13) and the dissipation potential $\mathcal{R}_1$. Indeed, let $z : [0,T] \to \mathcal{Z}$ be a global energetic solution associated with $\mathcal{I}$ and $\mathcal{R}_1$. The stability condition, that is satisfied by global energetic solutions, implies that for all $t \in [0,T]$ the function $z(t)$ minimizes $\mathcal{F}(\cdot; 2, 1, 0, z(t), u_{\min}(t, z(t)))$. Hence, by Theorem 4.4, for all $t \in [0,T]$ it holds $z(t) \in W^{1+\beta,q}(\Omega)$ with $\sup_{t\in[0,T]}\|z(t)\|_{W^{1+\beta,q}(\Omega)} < \infty$. We refer to [43, 54] for the analysis of damage models in the context of global energetic solutions.

Proof of Theorem 4.4. The proof relies on a difference quotient argument. Since spatially shifted versions of the minimizer z of $\mathcal{F}$ not necessarily lie below the function ζ, we also have to shift the function z in "vertical" direction.

Let $\Omega \subset \mathbb{R}^d$ satisfy $(\mathrm{A}_\Omega 2)$. Let $x_0 \in \partial\Omega$ be arbitrary and choose $e \in \mathbb{R}^d$ with $|e| = 1$ in such a way that there exist constants $R, h_0 > 0$ such that for all $y \in \overline{\Omega} \cap B_R(x_0)$ and all $0 < h \leq h_0$ we have $y + he \in \Omega$. Since Ω satisfies the uniform cone condition it is possible to find a basis of $\mathbb{R}^d$ such that every basis vector has this property.

Let $\varphi \in \mathrm{C}_0^\infty(B_R(x_0))$ be a cut-off function with $0 \leq \varphi \leq 1$ and $\varphi\big|_{B_{R/2}(x_0)} \equiv 1$. Further, let us define the transformation $T_h : \mathbb{R}^d \to \mathbb{R}^d$ by $T_h(x) := x + h\varphi(x)e$. If $h \in [0, h_0]$ is small enough, this mapping is an isomorphism with $T_h(\Omega) \subset \Omega$ and it coincides with the identity outside of the ball $B_R(x_0)$. For $w \in W^{1,p}_{\Gamma_D}(\Omega; \mathbb{R}^d)$ and $\zeta \in W^{1,q}(\Omega)$ let

$$z \in \operatorname{Argmin}\{\, \mathcal{F}(\widetilde{z}; \alpha, \tau, \epsilon, \zeta, w)\,;\, \widetilde{z} \in \mathcal{Z}\,\}. \tag{4.10}$$

From the definition of $\mathcal{R}_1$ it follows that $z \leq \zeta$ almost everywhere in Ω. Moreover, since $q > d$, we have $z \in \mathrm{C}^{0,\gamma}(\overline{\Omega})$ with $\gamma = 1 - \frac{d}{q} > 0$. For $h > 0$ let $\delta_h := h^\gamma \left\| z \right\|_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} \leq ch^\gamma \left\| z \right\|_{W^{1,q}(\Omega)}$. Observe that

$$|z(x) - z(T_h(x))| \leq \delta_h \qquad \text{for all } x \in \Omega. \tag{4.11}$$

Hence, the function $z_h(x) := z(T_h(x)) - \delta_h$ is an admissible test function for the minimization problem (4.10) in the sense that $\mathcal{R}_1(z_h - \zeta)$ is finite. Indeed, $\zeta(x) - z_h(x) = \zeta(x) - z(x) + (z(x) - z(T_h(x)) + \delta_h) \geq 0$ for all $x \in \Omega$. Since z is a minimizer, for all $\widetilde{z} \in W^{1,q}(\Omega)$ it satisfies the variational inequality

$$\begin{aligned} &\mathcal{R}_1(\widetilde{z} - \zeta) - \mathcal{R}_1(z - \zeta) \\ &\quad \geq -\langle A_q z, \widetilde{z} - z\rangle_{\mathcal{Z}} - \int_\Omega \big(\frac{1}{2} g'(z)\mathbb{C}\varepsilon(w) : \varepsilon(w) + f'(z)\big)(\widetilde{z} - z)\,\mathrm{d}x - \epsilon \int_\Omega \left|\frac{z-\zeta}{\tau}\right|^{\alpha-2} \frac{z-\zeta}{\tau}(\widetilde{z} - z)\,\mathrm{d}x. \end{aligned} \tag{4.12}$$

With the special choice $\widetilde{z} = z_h$, due the definition of $\mathcal{R}_1$ this variational inequality rewrites as

$$\begin{aligned} -\int_\Omega (1 + |\nabla z|^2)^{\frac{q-2}{2}} \nabla z \cdot \nabla(z \circ T_h - z)\,\mathrm{d}x \leq \int_\Omega \rho(z - z_h)\,\mathrm{d}x + \int_\Omega \big(\tfrac{1}{2} g'(z)\mathbb{C}\varepsilon(w) : \varepsilon(w) + f'(z)\big)(z_h - z)\,\mathrm{d}x \\ + \epsilon \int_\Omega \left|\frac{z-\zeta}{\tau}\right|^{\alpha-2} \frac{z-\zeta}{\tau}(z_h - z)\,\mathrm{d}x. \end{aligned}$$

Now we apply inequality (2.4) with $a = \nabla z$ and $b = \nabla(z \circ T_h)$, and setting $\triangle_h z := z \circ T_h - z$ we thus obtain the estimate (recall that $G_q(A) = \frac{1}{q}(1 + |A|^2)^{\frac{q}{2}}$)

$$\begin{aligned} c_q \int_\Omega (1 + |\nabla z|^2 + |\nabla z_h|^2)^{\frac{q-2}{2}} \left|\nabla \triangle_h z\right|^2 \,\mathrm{d}x &\leq \int_\Omega G_q(\nabla(z \circ T_h)) - G_q(\nabla z)\,\mathrm{d}x \\ &+ \int_\Omega \rho\left|\triangle_h z - \delta_h\right| \,\mathrm{d}x + \int_\Omega \big(g'(z)\mathbb{C}\varepsilon(w) : \varepsilon(w) + f'(z)\big)(z_h - z)\,\mathrm{d}x \\ &+ \epsilon \int_\Omega \left|\frac{z-\zeta}{\tau}\right|^{\alpha-2} \frac{z-\zeta}{\tau}(z_h - z)\,\mathrm{d}x =: S_1 + S_2 + S_3 + S_4. \end{aligned}$$

The goal is to show that there exists a $\beta \in (0, 1)$ such that the right-hand side can be estimated by ch^β. This estimate then implies that $z\big|_{B_{R/2}(x_0)}$ belongs to the Nikolskii space $\mathcal{N}^{1+\frac{\beta}{2},2}(\Omega) \cap \mathcal{N}^{1+\frac{\beta}{q},q}(\Omega)$, which is continuously embedded in $W^{1+\frac{\beta}{2}-\delta,2}(\Omega) \cap W^{1+\frac{\beta}{q}-\delta,q}(\Omega)$ for all $\delta > 0$.

Since by assumption we have $z \in W^{1,q}(\Omega)$, the term S_2 can be estimated as

$$S_2 \le \int_\Omega \rho \, |\triangle_h z| \, \mathrm{d}x + \delta_h \, |\Omega| \le c(|h| + |h|^\gamma) \left(1 + \|z\|_{W^{1,q}(\Omega)}\right),$$

and the constant c depends on Ω and the chosen cut-off-function, but is independent of h and z.

Taking into account (2.8) the second part of S_3 can be estimated as follows

$$\int_\Omega |f'(z)| \, |z_h - z| \, \mathrm{d}x \le c \, \|f'(z)\|_{L^\infty(\Omega)} \, \|z\|_{W^{1,q}(\Omega)} \, (|h|^\gamma + |h|).$$

Hölder's inequality applied to the first component of S_3 yields

$$\int_\Omega g'(z) \mathbb{C}\varepsilon(w) : \varepsilon(w)(z_h - z) \, \mathrm{d}x \le c \, \|w\|^2_{W^{1,p}(\Omega;\mathbb{R}^d)} \, (\|\triangle_h z\|_{L^{\frac{p}{p-2}}(\Omega)} + \delta_h),$$

where we have used that $p \ge 2$. By (4.11), the term in brackets on the right-hand side is bounded by $c\delta_h \le \widetilde{c} \, |h|^\gamma \, \|z\|_{W^{1,q}(\Omega)}$. Putting together these estimates we obtain

$$|S_3| \le c(|h| + |h|^\gamma) \, \|z\|_{W^{1,q}(\Omega)} \, (\|f'(z)\|_{L^\infty(\Omega)} + \|w\|^2_{W^{1,p}(\Omega;\mathbb{R}^d)})$$

and the constant c is independent of h, z, w.

In a similar way we obtain for S_4, applying again Hölder's inequality,

$$|S_4| \le c\epsilon \left\| \frac{z - \zeta}{\tau} \right\|^{\alpha - 1}_{L^\alpha(\Omega)} \left(\|\triangle_h z\|_{L^\alpha(\Omega)} + \delta_h \right) \le c\epsilon \left\| \frac{z - \zeta}{\tau} \right\|^{\alpha - 1}_{L^\alpha(\Omega)} (|h| + |h|^\gamma) \, \|z\|_{W^{1,q}(\Omega)} .$$

It remains to estimate S_1. Here we use an argument that relies on a change of variables in the first term (cf. [15, 50, 30]): With $y = T_h(x)$ it follows

$$\int_\Omega G_q(\nabla z(T_h(x)) \nabla T_h(x)) \, \mathrm{d}x = \int_{T_h(\Omega)} G_q(\nabla z(y) \nabla T_h(T_h^{-1}(y))) \det \nabla_y T_h^{-1}(y) \, \mathrm{d}y$$

Observe that due to the special choice of the vector e it holds $T_h(\Omega) \subset \Omega$ for $0 \le h < h_0$. Hence, since $G_q(\nabla z) \ge 0$ almost everywhere, we arrive at

$$S_1 \le \int_{T_h(\Omega)} G_q(\nabla z(y)(\nabla T_h^{-1}(y))^{-1}) \det \nabla T_h^{-1}(y) \, \mathrm{d}y - \int_{T_h(\Omega)} G_q(\nabla z) \, \mathrm{d}x.$$

Elementary calculations (based on the fact that $\det \nabla T_h^{-1} \sim (1 - h \, \|\varphi\|_{\mathrm{C}^1(\overline{\Omega})})$ and a Taylor expansion of G_q) show that S_1 can be further estimated by $S_1 \le c \, |h| \, (1 + \|z\|^q_{W^{1,q}(\Omega)})$. Again, the constant c is independent of h and z. Collecting all estimates we finally arrive at

$$\begin{aligned} &\int_\Omega (1 + |\nabla z|^2 + |\nabla z_h|^2)^{\frac{q-2}{2}} \, |\nabla \triangle_h z|^2 \, \mathrm{d}x \\ &\quad \le c(|h| + |h|^\gamma)(1 + \|z\|^q_{W^{1,q}(\Omega)}) \left(1 + \|f'(z)\|_{L^\infty(\Omega)} + \|w\|^2_{W^{1,p}(\Omega;\mathbb{R}^d)} + \epsilon \left\| \frac{z - \zeta}{\tau} \right\|^{\alpha - 1}_{L^\alpha(\Omega)} \right). \end{aligned}$$

Since $x_0 \in \partial\Omega$ was chosen arbitrarily, after covering $\overline{\Omega}$ with a finite number of balls $B_{R_{x_0}}(x_0)$ we finally obtain that $z \in \mathcal{N}^{1+\frac{\gamma}{q},q}(\Omega)$ with

$$\|z\|_{\mathcal{N}^{1+\frac{\gamma}{q},q}(\Omega)} \le c(1 + \|z\|_{W^{1,q}(\Omega)}) \left(1 + \|f'(z)\|^{\frac{1}{q}}_{L^\infty(\Omega)} + \|w\|^{\frac{2}{q}}_{W^{1,p}(\Omega;\mathbb{R}^d)} + \epsilon^{\frac{1}{q}} \left\| \frac{z - \zeta}{\tau} \right\|^{\frac{\alpha-1}{q}}_{L^\alpha(\Omega)} \right),$$

and the constant c is independent of $\alpha, \epsilon, \tau, z, w$ and ζ. □

5 A priori estimates

This section is devoted to deriving for the approximate solutions $(\overline{z}_\tau, \hat{z}_\tau, \overline{u}_\tau, \hat{u}_\tau)$ constructed from the time-incremental minimization problem (4.1) a number of a priori estimates, uniform w.r.t. $\tau > 0$. These will allow us to pass to the limit in the approximate differential inclusion (4.5) and conclude the existence of weak viscous solutions to (the Cauchy problem for) (3.2). In view of the vanishing-viscosity analysis in Sec. 7, in the following we will specify which estimates are, in addition, uniform w.r.t. $\epsilon > 0$. However, for notational simplicity we shall omit to indicate the dependence of the interpolants $(\overline{z}_\tau, \hat{z}_\tau, \overline{u}_\tau, \hat{u}_\tau)$ on ϵ.

5.1 Energy estimate

We start by stating the basic energy estimate derived from the time-incremental minimization (4.1). It holds uniformly with respect to τ and $\epsilon > 0$.

Lemma 5.1. *Under Assumptions 2.1, 2.5, and (A$_\Omega$1), for every $z_0 \in \mathcal{Z}$ there exists a constant $C_1 > 0$ such that for all $\tau > 0$ and $\epsilon > 0$ there holds*

$$\sup_{t\in[0,T]} \mathcal{I}(\bar{t}_\tau(t), \overline{z}_\tau(t)) + \int_0^T \mathcal{R}_\epsilon(\hat{z}_\tau'(r))\,\mathrm{d}r \le C_1, \tag{5.1}$$

$$\sup_{t\in[0,T]} \|\overline{z}_\tau(t)\|_{W^{1,q}(\Omega)} + \sup_{t\in[0,T]} \|\hat{z}_\tau(t)\|_{W^{1,q}(\Omega)} \le C_1. \tag{5.2}$$

$$\|\overline{u}_\tau\|_{L^\infty(0,T;W^{1,p_*}(\Omega;\mathbb{R}^d))} \le C_1. \tag{5.3}$$

Proof. From (4.1) (with competitor $z = z_k^\tau$) we deduce

$$\mathcal{I}(t_{k+1}^\tau, z_{k+1}^\tau) + \tau_k \mathcal{R}_\epsilon\left(\frac{z_{k+1}^\tau - z_k^\tau}{\tau_k}\right) \le \mathcal{I}(t_{k+1}^\tau, z_k^\tau) = \mathcal{I}(t_k^\tau, z_k^\tau) + \int_{t_k^\tau}^{t_{k+1}^\tau} \partial_t \mathcal{I}(s, z_k^\tau)\,\mathrm{d}s. \tag{5.4}$$

Then, we observe that $\sup_{t\in[0,T]} |\partial_t \mathcal{I}(t, z_k^\tau)| \le C$ thanks to (2.31) in Lemma 2.9. Hence, (5.1) follows upon adding (5.4) up for $k = 0, \ldots, N-1$. Observe that (5.1) yields (5.2) thanks to (2.25) in Lemma 2.6 and the Poincaré inequality. Finally, (5.3) follows from (5.2) via estimate (2.24). $\square$

5.2 Higher spatial differentiability for the damage variable

Theorem 4.4 yields an enhanced differentiability estimate for $\overline{z}_\tau$ and $\hat{z}_\tau$, uniform w.r.t. τ and ϵ.

Lemma 5.2. *Under Assumptions 2.1, 2.5, (A$_\Omega$1) and (A$_\Omega$2), for every $\beta \in [0, \frac{1}{q}(1 - \frac{d}{q}))$, for every $z_0 \in \mathcal{Z}$ it holds*

$$\overline{z}_\tau(t),\ \hat{z}_\tau(t) \in W^{1+\beta,q}(\Omega) \quad \text{for all } \tau > 0 \text{ and all } t \in (0,T].$$

Moreover, for all $\beta \in [0, \frac{1}{q}(1 - \frac{d}{q}))$ there exists a constant $C_2 > 0$ such that for all $\tau > 0$ and $\epsilon > 0$ there holds

$$\|\overline{z}_\tau\|_{L^{2q}(0,T;W^{1+\beta,q}(\Omega))} + \|\hat{z}_\tau\|_{L^{2q}(0,T;W^{1+\beta,q}(\Omega))} \le C_2\,. \tag{5.5}$$

Proof. Applying Theorem 4.4 with $\alpha = 2$, $\zeta = z_k^\tau$, $w = u_{\min}(t_{k+1}^\tau, z_{k+1}^\tau) + u_D(t_{k+1}^\tau)$ and $p = p_*$, we find

$$\begin{aligned}\left\|z_{k+1}^\tau\right\|_{W^{1+\beta,q}(\Omega)} \le & c_\beta(1 + \|z_k^\tau\|_{W^{1,q}(\Omega)}) \\ & \times \left(1 + \left\|u_{\min}(t_{k+1}^\tau, z_{k+1}^\tau) + u_D(t_{k+1}^\tau)\right\|_{W^{1,p_*}(\Omega;\mathbb{R}^d)}^{\frac{1}{q}} + \epsilon^{\frac{1}{q}} \left\|\frac{z_{k+1}^\tau - z_k^\tau}{\tau}\right\|_{L^2(\Omega)}^{\frac{1}{q}}\right),\end{aligned} \tag{5.6}$$

with c_β independent of τ *and* ϵ. Taking into account the previously proved uniform estimates (5.2) and (5.3) for z_k^τ and $u_{\min}(t_{k+1}^\tau, z_{k+1}^\tau)$, we then have $\|\overline{z}_\tau(t)\|^{2q}_{W^{1+\beta,q}(\Omega)} \le C\left(1+\epsilon^2 \|\hat{z}'_\tau(t)\|^2_{L^2(\Omega)}\right)$ for a.a. $t$in $(0,T)$. Then, (5.5) follows from integrating the above estimate on $(0,T)$ and using (5.1). □

5.3 Enhanced temporal regularity estimates

The proof of the enhanced regularity estimates (5.7) and (5.8) below relies on the higher regularity for z_0 guaranteed by (3.12), i.e., $\mathrm{D}_z\mathcal{I}(0,z_0) \in L^2(\Omega)$. We also provide estimate (5.9), later used in the proof of Lemma 5.5, cf. (5.30) below. Observe that the bounds in (5.7)–(5.9) might explode as $\epsilon \to 0$.

Lemma 5.3. *Under Assumptions 2.1, 2.5, (A_Ω1) and (A_Ω2), for every $z_0 \in \mathcal{Z}$ fulfilling* (3.12) *and for every $\epsilon > 0$ there exists a constant $C_3 = C_3(\epsilon) > 0$, with $C_3(\epsilon) \to \infty$ as $\epsilon \to 0$, such that for all $\tau > 0$*

$$\int_0^T \int_\Omega (1+|\nabla \hat{z}_\tau(r)|^2)^{\frac{q-2}{2}} |\nabla \hat{z}'_\tau(r)|^2 \,\mathrm{d}x\,\mathrm{d}r \le C_3(\epsilon), \tag{5.7}$$

$$\epsilon \, \|\hat{z}'_\tau\|^2_{L^\infty(0,T;L^2(\Omega))} \le C_3(\epsilon), \tag{5.8}$$

$$\epsilon \left\| \hat{z}'_\tau\left(\frac{t_1^\tau}{2}\right)\right\|_{L^2(\Omega)} \le C_{3,1}\big(1+\|D_z\mathcal{I}(0,z_0)\|_{L^2(\Omega)}\big)\exp\left(C_{3,2}\tau/\epsilon\right), \tag{5.9}$$

where t_1^τ is the first non-zero node of the partition of $[0,T]$, and $C_{3,1}$, $C_{3,2}$ do not depend on ϵ or τ.

Proof. For $t \in (t_k^\tau, t_{k+1}^\tau)$ we define $\overline{h}_\tau(t) := \epsilon \hat{z}'_\tau(t) + A_q\overline{z}_\tau(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_\tau(t))$. Hence, relation (4.5) is equivalent to $-\overline{h}_\tau(t) \in \partial\mathcal{R}_1(\hat{z}'_\tau(t))$ for $t \in (t_k^\tau, t_{k+1}^\tau)$. Since for convex, 1-homogeneous functionals Ψ it holds $\partial\Psi(v) \subset \partial\Psi(0)$ as well as $\partial\Psi(v) = \{\,\eta\,;\, \Psi(v) = \langle \eta, v\rangle\,\}$, we deduce

$$\forall\, t \in (t_k^\tau, t_{k+1}^\tau) \qquad\qquad -\mathcal{R}_1(\hat{z}'_\tau(t)) = \langle \overline{h}_\tau(t), \hat{z}'_\tau(t)\rangle_{\mathcal{Z}}, \tag{5.10}$$

$$\forall\, r \in [0,T]\backslash\{t_0^\tau,\dots,t_N^\tau\} \qquad\qquad \mathcal{R}_1(\hat{z}'_\tau(t)) \ge \langle -\overline{h}_\tau(r), \hat{z}'_\tau(t)\rangle_{\mathcal{Z}}. \tag{5.11}$$

Adding both relations, it follows $0 \ge \tau^{-1}\langle \overline{h}_\tau(\rho) - \overline{h}_\tau(\sigma), \hat{z}'_\tau(\rho)\rangle_{\mathcal{Z}}$ with $\rho \in (t_i^\tau, t_{i+1}^\tau)$ and $\sigma \in (t_{i-1}^\tau, t_i^\tau)$, which can be rewritten as (for $1 \le i \le N-1$)

$$\begin{aligned}\epsilon\tau^{-1} \underbrace{\langle \hat{z}'_\tau(\rho) - \hat{z}'_\tau(\sigma), \hat{z}'_\tau(\rho)\rangle_{L^2(\Omega)}}_{=I_1} &+ \underbrace{\tau^{-1}\,\langle A_q\overline{z}_\tau(\rho) - A_q\overline{z}_\tau(\sigma), \hat{z}'_\tau(\rho)\rangle_{\mathcal{Z}}}_{=I_2} \\ &\le \underbrace{-\tau^{-1}\,\langle \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_\tau(\rho)) - \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(\sigma), \overline{z}_\tau(\sigma)), \hat{z}'_\tau(\rho)\rangle_{\mathcal{Z}}}_{=I_3}\ .\end{aligned} \tag{5.12}$$

Now, we observe that $I_1 \ge \frac{1}{2}\int_\Omega \left(|\hat{z}'_\tau(\rho)|^2 - |\hat{z}'_\tau(\sigma)|^2\right)\,\mathrm{d}x$ whereas, relying on inequality (2.3), we find

$$I_2 \ge c\int_\Omega (1+|\nabla\overline{z}_\tau(\rho)|^2 + |\nabla\overline{z}_\tau(\sigma)|^2)^{\frac{q-2}{2}}|\nabla\hat{z}'_\tau(\rho)|^2\,\mathrm{d}x \ge c\int_\Omega (1+|\nabla\hat{z}_\tau(\rho)|^2)^{\frac{q-2}{2}}|\nabla\hat{z}'_\tau(\rho)|^2\,\mathrm{d}x, \tag{5.13}$$

where the second inequality is due to the fact that $|\nabla\hat{z}_\tau(\rho)|^2 \le 2|\nabla\overline{z}_\tau(\rho)|^2 + 2|\nabla\overline{z}_\tau(\sigma)|^2$. Finally, relying on estimate (2.42), we obtain

$$|I_3| \le C(1+\|\hat{z}'_\tau(\rho)\|_{L^{2p_*/(p_*-2)}(\Omega)})\|\hat{z}'_\tau(\rho)\|_{L^{2p_*/(p_*-2)}(\Omega)}\,. \tag{5.14}$$

All in all, inserting the above calculation in (5.12) and multiplying by τ we find

$$\begin{aligned}&\frac{\epsilon}{2}\,\|\hat{z}'_\tau(\rho)\|^2_{L^2(\Omega)} + \tau C\int_\Omega \left(1+|\nabla\hat{z}_\tau(\rho)|^2\right)^{\frac{q-2}{2}}|\nabla\hat{z}'_\tau(\rho)|^2\,\mathrm{d}x \\ &\qquad \le \frac{\epsilon}{2}\,\|\hat{z}'_\tau(\sigma)\|^2_{L^2(\Omega)} + \tau C(1+\|\hat{z}'_\tau(\rho)\|_{L^{2p_*/(p_*-2)}(\Omega)})\,\|\hat{z}'_\tau(\rho)\|_{L^{2p_*/(p_*-2)}(\Omega)}\,.\end{aligned} \tag{5.15}$$

Hence, taking the sum with respect to $\rho \in (t_i^\tau, t_{i+1}^\tau)$ of (5.15) on the time interval (t_0, t), with $t_0 \in (0, t_1^\tau)$ and $t \in (t_k^\tau, t_{k+1}^\tau)$, and using Young's inequality, we arrive at

$$\frac{\epsilon}{2} \|\hat{z}_\tau'(t)\|_{L^2(\Omega)}^2 + C \int_{t_1^\tau}^{\bar{t}_\tau(t)} \int_\Omega \left(1 + |\nabla \hat{z}_\tau(\rho)|^2\right)^{\frac{q-2}{2}} |\nabla \hat{z}_\tau'(\rho)|^2 \, \mathrm{d}x \, \mathrm{d}\rho$$
$$\leq \frac{\epsilon}{2} \|\hat{z}_\tau'(t_0)\|_{L^2(\Omega)}^2 + C \int_{t_1^\tau}^{\bar{t}_\tau(t)} (1 + \|\hat{z}_\tau'(\rho)\|_{L^{2p_*/(p_*-2)}(\Omega)}^2) \, \mathrm{d}\rho. \tag{5.16}$$

For the first time step with $t_0 \in (0, t_1^\tau)$ we obtain from (5.10):

$$0 = \mathcal{R}_1(\hat{z}_\tau'(t_0)) + \langle \bar{h}_\tau(t_0), \hat{z}_\tau'(t_0)\rangle_{\mathcal{Z}} \geq \epsilon \|\hat{z}_\tau'(t_0)\|_{L^2(\Omega)}^2 + \langle \mathrm{D}_z \mathcal{I}(t_1^\tau, z_1^\tau), \hat{z}_\tau'(t_0)\rangle_{\mathcal{Z}}. \tag{5.17}$$

With $\mathrm{D}_z\mathcal{I}(t_1^\tau, z_1^\tau) = \mathrm{D}\mathcal{I}_q(z_1^\tau) - \mathrm{D}\mathcal{I}_q(z_0) + \mathrm{D}_z\widetilde{\mathcal{I}}(t_1^\tau, z_1^\tau) - \mathrm{D}_z\widetilde{\mathcal{I}}(0, z_0) + \mathrm{D}_z\mathcal{I}(0, z_0)$, Young's inequality, (3.12) and similar arguments as for I_2 and I_3 from above, we find

$$\epsilon \|\hat{z}_\tau'(t_0)\|_{L^2(\Omega)}^2 + c\tau \int_\Omega (1 + |\nabla \hat{z}_\tau(t_0)|^2)^{\frac{q-2}{2}} |\nabla \hat{z}_\tau'(t_0)|^2 \, \mathrm{d}x$$
$$\leq -\langle \mathrm{D}_z\mathcal{I}(0, z_0), \hat{z}_\tau'(t_0)\rangle_{\mathcal{Z}} + \langle \mathrm{D}_z\widetilde{\mathcal{I}}(0, z_0) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_1^\tau, z_1^\tau), \hat{z}_\tau'(t_0)\rangle_{\mathcal{Z}}$$
$$\leq \frac{\epsilon}{2} \|\hat{z}_\tau'(t_0)\|_{L^2(\Omega)}^2 + \epsilon^{-1} \|\mathrm{D}_z\mathcal{I}(0, z_0)\|_{L^2(\Omega)}^2 + c\tau(1 + \|\hat{z}_\tau'(t_0)\|_{L^{2p_*/(p_*-2)}(\Omega)}^2).$$

We sum the above estimate with (5.16). Adding the term $\int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(\rho)\|_{L^2(\Omega)}^2 \, \mathrm{d}\rho$ to both sides of the resulting inequality, we obtain

$$\frac{\epsilon}{2} \|\hat{z}_\tau'(t)\|_{L^2(\Omega)}^2 + C_1 \int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(\rho)\|_{L^2(\Omega)}^2 \, \mathrm{d}\rho + C_1 \int_0^{\bar{t}_\tau(t)} \int_\Omega (1 + |\nabla \hat{z}_\tau(\rho)|^2)^{\frac{q-2}{2}} |\nabla \hat{z}'(\rho)|^2 \, \mathrm{d}x \, \mathrm{d}\rho$$
$$\leq \epsilon^{-1} \|\mathrm{D}_z\mathcal{I}(0, z_0)\|_{L^2(\Omega)}^2 + C_1 \int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(\rho)\|_{L^2(\Omega)}^2 \, \mathrm{d}\rho + C_2 \int_0^{\bar{t}_\tau(t)} (1 + \|\hat{z}_\tau'(\rho)\|_{L^{2p_*/(p_*-2)}(\Omega)}^2) \, \mathrm{d}\rho$$
$$\leq C + \epsilon^{-1} \|\mathrm{D}_z\mathcal{I}(0, z_0)\|_{L^2(\Omega)}^2 + C \int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(\rho)\|_{L^2(\Omega)}^2 \, \mathrm{d}\rho + \frac{C_1}{4} \int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(\rho)\|_{W^{1,2}(\Omega)}^2 \, \mathrm{d}\rho, \tag{5.18}$$

where for the last inequality we have applied estimate (2.2) to absorb $\frac{C_1}{4} \int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(\rho)\|_{W^{1,2}(\Omega)}^2 \, \mathrm{d}\rho$ into the corresponding term on the left-hand side. Now with the Gronwall inequality, we conclude that for all $t \in [0, T] \backslash \{t_0^\tau, \ldots, t_N^\tau\}$

$$\epsilon \|\hat{z}_\tau'(t)\|_{L^2(\Omega)}^2 \leq \left(C' + \frac{1}{2\epsilon} \|\mathrm{D}_z\mathcal{I}(0, z_0)\|_{L^2(\Omega)}^2 \right) \exp(C\bar{t}_\tau(t)/\epsilon), \tag{5.19}$$

from which we derive (5.8), (5.9) and (5.7). □

Lemma 5.4. *Under Assumptions 2.1, 2.5, (A_Ω1), and (A_Ω2), for every $z_0 \in \mathcal{Z}$ such that* (3.12) *is valid there exists a constant $C_4 = C_4(\epsilon) > 0$, with $C_4(\epsilon) \to \infty$ as $\epsilon \to 0$, such that for all $\tau > 0$ there holds*

$$\|\hat{u}_\tau\|_{W^{1,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^d))} \leq C_4(\epsilon). \tag{5.20}$$

Proof. Lemma 2.7 with $\widetilde{p} = 2$ and $r = 2p_*(p_* - 2)^{-1}$ implies for all $t \in (0, T) \backslash \{t_0^\tau, \ldots, t_N^\tau\}$ that $\|\hat{u}_\tau'(t)\|_{W^{1,2}(\Omega)} \leq c(1 + \|\hat{z}_\tau'(t)\|_{L^r(\Omega)})$. Since $W^{1,2}(\Omega) \subset L^r(\Omega)$, the claim of Lemma 5.4 follows with (5.3) and (5.7). □

5.4 A uniform discrete BV-estimate

The following estimates will be used to pass to the vanishing-viscosity limit $\epsilon \to 0$ and therefore are uniform both w.r.t. τ *and* w.r.t. ϵ.

Lemma 5.5. *Under Assumptions 2.1, 2.5, (A_Ω1) and (A_Ω2), for every $z_0 \in \mathcal{Z}$ such that* (3.12) *is valid there exists a constant $C_5 > 0$ such that for all $\tau > 0$* and *$\epsilon > 0$ with $\tau \le 2\epsilon$ there holds*

$$\int_0^T \|\hat{z}_\tau'(t)\|_{W^{1,2}(\Omega)} \,\mathrm{d}t \le C_5, \tag{5.21}$$

$$\int_0^T \|\hat{z}_\tau'(t)\|_{L^2(\Omega)} \,\mathrm{d}t + \int_0^T \left(\int_\Omega (1+|\nabla \hat{z}_\tau(r)|^2)^{\frac{q-2}{2}} |\nabla \hat{z}_\tau'(r)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \mathrm{d}r \le C_5. \tag{5.22}$$

Note that, in comparison with the previous (5.7), formula (5.22) has an L^1-character, in the sense that it can be rewritten as

$$\|\mathrm{M}_\tau\|_{L^1(0,T)} \le C \quad \text{with} \quad \mathrm{M}_\tau(t) := \left(\|\hat{z}_\tau'(t)\|^2_{L^2(\Omega)} + \int_\Omega (1+|\nabla \hat{z}_\tau(r)|^2)^{\frac{q-2}{2}} |\nabla \hat{z}_\tau'(t)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{5.23}$$

This reflects the fact that (5.22) is invariant under time rescalings.

Proof. We start from (5.12), written for $\rho = m_k$ and $\sigma = m_{k-1}$, where $m_k := \frac{1}{2}(t^\tau_{k-1} + t^\tau_k)$ and $k \in \{2, \dots, N\}$. Adding the term $\|\hat{z}_\tau'(m_k)\|^2_{L^2(\Omega)}$ on both sides and taking into account that $\overline{z}_\tau(m_{k-1}) = \underline{z}_\tau(m_k)$, we obtain

$$\begin{aligned}\frac{\epsilon}{\tau} \langle \hat{z}_\tau'(m_k) - \hat{z}_\tau'(m_{k-1}), \hat{z}_\tau'(m_k) \rangle_{L^2(\Omega)} + \tau^{-1} \langle A_q \overline{z}_\tau(m_k) - A_q \underline{z}_\tau(m_k), \hat{z}_\tau'(m_k) \rangle_{\mathcal{Z}} + \|\hat{z}_\tau'(m_k)\|^2_{L^2(\Omega)} \\ \le -\tau^{-1} \langle \mathrm{D}_z \widetilde{\mathcal{I}}(t^\tau_k, \overline{z}_\tau(m_k)) - \mathrm{D}_z \widetilde{\mathcal{I}}(t^\tau_{k-1}, \underline{z}_\tau(m_k)), \hat{z}_\tau'(m_k) \rangle_{\mathcal{Z}} + \|\hat{z}_\tau'(m_k)\|^2_{L^2(\Omega)}\,,\end{aligned} \tag{5.24}$$

where $\widetilde{\mathcal{I}}$ is defined as in (2.35). For the first time step we start with (5.17) and find

$$\begin{aligned}&\frac{\epsilon}{\tau} \|\hat{z}_\tau'(m_1)\|^2_{L^2(\Omega)} + \tau^{-1} \langle A_q \overline{z}_\tau(m_1) - A_q \underline{z}_\tau(m_1), \hat{z}_\tau'(m_1) \rangle_{\mathcal{Z}} + \|\hat{z}_\tau'(m_1)\|^2_{L^2(\Omega)} \\ &\quad \le -\tau^{-1} \langle \mathrm{D}_z \mathcal{I}(0, z_0), \hat{z}_\tau'(m_1) \rangle_{\mathcal{Z}} - \tau^{-1} \langle \mathrm{D}_z \widetilde{\mathcal{I}}(t^\tau_1, \overline{z}_\tau(m_1)) - \mathrm{D}_z \widetilde{\mathcal{I}}(t^\tau_0, \underline{z}_\tau(m_1)), \hat{z}_\tau'(m_1) \rangle_{\mathcal{Z}} + \|\hat{z}_\tau'(m_1)\|^2_{L^2(\Omega)}\,,\end{aligned} \tag{5.25}$$

where we used the fact that $(0, z_0) = (t^\tau_0, \underline{z}_\tau(m_1))$. Hence, with $\hat{z}_\tau'(m_0) := 0$, for all $k \in \{1, \dots, N\}$ we have

$$\begin{aligned}&\frac{\epsilon}{\tau} \langle \hat{z}_\tau'(m_k) - \hat{z}_\tau'(m_{k-1}), \hat{z}_\tau'(m_k) \rangle_{L^2(\Omega)} + \tau^{-1} \langle A_q \overline{z}_\tau(m_k) - A_q \underline{z}_\tau(m_k), \hat{z}_\tau'(m_k) \rangle_{\mathcal{Z}} + \|\hat{z}_\tau'(m_k)\|^2_{L^2(\Omega)} \\ &\le -\frac{1}{\tau} \langle \mathrm{D}_z \widetilde{\mathcal{I}}(t^\tau_k, \overline{z}_\tau(m_k)) - \mathrm{D}_z \widetilde{\mathcal{I}}(t^\tau_{k-1}, \underline{z}_\tau(m_k)), \hat{z}_\tau'(m_k) \rangle_{\mathcal{Z}} + \|\hat{z}_\tau'(m_k)\|^2_{L^2(\Omega)} + \frac{\delta_{1,k}}{\tau} \left| \langle \mathrm{D}_z \mathcal{I}(0, z_0), \hat{z}_\tau'(m_1) \rangle_{\mathcal{Z}} \right|,\end{aligned} \tag{5.26}$$

with the Kronecker symbol $\delta_{i,j}$. Thanks to estimate (2.5) and the fact that $|\nabla \hat{z}_\tau(m_k)|^2 \le 2|\nabla \overline{z}_\tau(m_k)|^2 + 2|\nabla \overline{z}_\tau(m_{k-1})|^2$, the left-hand side of (5.26) can be bounded by

$$\text{L.H.S.} \ge \frac{\epsilon}{2\tau} \|\hat{z}_\tau'(m_k)\|_{L^2(\Omega)} \left(\|\hat{z}_\tau'(m_k)\|_{L^2(\Omega)} - \|\hat{z}_\tau'(m_{k-1})\|_{L^2(\Omega)} \right) + \mathrm{M}_k^2, \tag{5.27}$$

where we use the abbreviation (cf. notation (5.23))

$$\mathrm{M}_k^2 := c_q \int_\Omega (1+|\nabla \hat{z}_\tau(m_k)|^2)^{\frac{q-2}{2}} |\nabla \hat{z}_\tau'(m_k)|^2 \,\mathrm{d}x + \|\hat{z}_\tau'(m_k)\|^2_{L^2(\Omega)}$$

with a constant $c_q \in (0,1]$. Using estimate (5.14) for the first term on the right-hand side of (5.26), the fact that $W^{1,2}(\Omega)$ is compactly embedded in $L^{2p_*/p_*-2}(\Omega) \subset L^2(\Omega) \subset L^1(\Omega)$, the Gagliardo-Nirenberg estimate $\|\zeta\|_{L^{2p_*/p_*-2}(\Omega)} \le c\|\zeta\|_{L^1(\Omega)}^{\theta}\|\zeta\|_{W^{1,2}(\Omega)}^{1-\theta}$ with suitable $\theta \in (0,1)$ and Young's inequality, the right-hand side of (5.26) can be bounded as follows (see the proof of [33, Proposition 4.3])

$$
\begin{aligned}
\text{R.H.S.} &\le \frac{c_q}{2}\|\hat z_\tau'(m_k)\|_{W^{1,2}(\Omega)}^2 + C\left(1+\|\hat z_\tau'(m_k)\|_{L^1(\Omega)}\mathcal{R}_1(\hat z_\tau'(m_k))\right) + \delta_{1,k}\tau^{-1}\left|\langle \mathrm{D}_z\mathcal{I}(0,z_0),\hat z_\tau'(m_1)\rangle_{\mathcal{Z}}\right| \\
&\le \frac{1}{2}\mathrm{M}_k^2 + C\left(1+\|\hat z_\tau'(m_k)\|_{L^2(\Omega)}\mathcal{R}_1(\hat z_\tau'(m_k)) + \delta_{1,k}\tau^{-1}\left|\langle \mathrm{D}_z\mathcal{I}(0,z_0),\hat z_\tau'(m_1)\rangle_{\mathcal{Z}}\right|\right).
\end{aligned}
$$

Hence, estimate (5.26) yields for $k \in \{1,\dots,N\}$

$$
\begin{aligned}
&\frac{\epsilon}{2\tau}\|\hat z_\tau'(m_k)\|_{L^2(\Omega)}\left(\|\hat z_\tau'(m_k)\|_{L^2(\Omega)} - \|\hat z_\tau'(m_{k-1})\|_{L^2(\Omega)}\right) + \frac{1}{2}\mathrm{M}_k^2 \\
&\qquad\le C\left(1+\|\hat z_\tau'(m_k)\|_{L^2(\Omega)}\mathcal{R}_1(\hat z_\tau'(m_k)) + \delta_{1,k}\tau^{-1}\left|\langle \mathrm{D}_z\mathcal{I}(0,z_0),\hat z_\tau'(m_1)\rangle_{\mathcal{Z}}\right|\right),
\end{aligned}
$$

where the constant C is independent of τ, k and ϵ. Multiplying this inequality by $4\tau/\epsilon$ and taking into account that $\mathrm{M}_k^2 \ge \|\hat z_\tau'(m_k)\|_{L^2(\Omega)}^2$ we arrive at

$$
\begin{aligned}
&2\|\hat z_\tau'(m_k)\|_{L^2(\Omega)}\left(\|\hat z_\tau'(m_k)\|_{L^2(\Omega)} - \|\hat z_\tau'(m_{k-1})\|_{L^2(\Omega)}\right) + \frac{\tau}{\epsilon}\|\hat z_\tau'(m_k)\|_{L^2(\Omega)}^2 + \frac{\tau}{\epsilon}\mathrm{M}_k^2 \\
&\le \frac{4\tau C}{\epsilon} + \frac{4\tau C}{\epsilon}\|\hat z_\tau'(m_k)\|_{L^2(\Omega)}\mathcal{R}_1(\hat z_\tau'(m_k)) + 4C\frac{\delta_{1,k}}{\epsilon}\left|\langle \mathrm{D}_z\mathcal{I}(0,z_0),\hat z_\tau'(m_1)\rangle_{\mathcal{Z}}\right|,
\end{aligned}
\tag{5.28}
$$

which is valid for all $1 \le k \le N$. We define now for $1 \le k \le N$

$$
a_k = \|\hat z_\tau'(m_k)\|_{L^2(\Omega)}, \quad c^2 = 8C, \quad \mu^2 = \epsilon\left|\langle \mathrm{D}_z\mathcal{I}(0,z_0),\hat z_\tau'(m_1)\rangle\right|, \quad r_k = 2C\mathcal{R}_1(\hat z_\tau'(m_k)), \quad \gamma = \frac{\tau}{2\epsilon},
$$

so that, with $a_0 = 0$, (5.28) can be rewritten as

$$
2a_k(a_k - a_{k-1}) + 2\gamma a_k^2 + 2\gamma\mathrm{M}_k^2 \le c^2\gamma\left(1+\tfrac{\delta_{1,k}}{\tau\epsilon}\mu^2\right) + 4\gamma a_k r_k, \tag{5.29}
$$

which holds for $1 \le k \le N$. With Lemma B.1 we arrive at

$$
\sum_{k=1}^{N}\tau\mathrm{M}_k \le C\left(T + \mu + \sum_{k=1}^{N}\tau\mathcal{R}_1(\hat z_\tau'(m_k))\right), \tag{5.30}
$$

where we used that, here, $a_0 = 0$. Thanks to (5.9) and assumption (3.12), μ is uniformly bounded w.r.t. ϵ and we conclude that (5.22) and therefore (5.21) hold. □

For later use we pin down a crucial consequence of the higher differentiability estimate (5.6) for $\overline{z}_\tau$, and of the uniform $W^{1,1}(0,T;L^2(\Omega))$-estimate for $\hat z_\tau$, combined with (5.1).

Lemma 5.6. *Under Assumptions 2.1, 2.5, ($\mathrm{A}_\Omega 1$) and ($\mathrm{A}_\Omega 2$), for every $z_0 \in \mathcal{Z}$ with* (3.12), *for all $\beta \in [0, \frac{1}{q}(1-\frac{d}{q}))$ there exists a constant $C_6 > 0$ such that for all $\tau > 0$ and $\epsilon > 0$ there holds*

$$
\int_0^T \|\overline{z}_\tau(t)\|_{W^{1+\beta,q}(\Omega)}^q \|\hat z_\tau'(t)\|_{L^2(\Omega)}\,\mathrm{d}t \le C_6\,. \tag{5.31}
$$

Proof. From (5.6), again taking into account the previously proved uniform estimates (5.2) and (5.3) for z_k^τ and $u_{\min}(t_{k+1}^\tau, z_{k+1}^\tau)$, we also gather $\|z_{k+1}^\tau\|_{W^{1+\beta,q}(\Omega)}^q \le C(1+\epsilon\|(z_{k+1}^\tau - z_k^\tau)/\tau\|_{L^2(\Omega)})$, whence

$$
\|z_{k+1}^\tau\|_{W^{1+\beta,q}(\Omega)}^q \left\|\frac{z_{k+1}^\tau - z_k^\tau}{\tau}\right\|_{L^2(\Omega)} \le C\left(\left\|\frac{z_{k+1}^\tau - z_k^\tau}{\tau}\right\|_{L^2(\Omega)} + \epsilon\left\|\frac{z_{k+1}^\tau - z_k^\tau}{\tau}\right\|_{L^2(\Omega)}^2\right). \tag{5.32}
$$

Then, (5.31) follows by integrating (5.32) in time, taking into account the basic energy estimate (5.1) as well as estimate (5.21). □

Remark 5.7. Observe that the a priori estimates from Lemmas 5.1–5.4 could be obtained also in the case of a time-discretization scheme with *variable* time step $\tau_k = t_{k+1}^\tau - t_k^\tau$, with fineness $\tau = \sup_{0\le k\le N}(t_{k+1}^\tau - t_k^\tau)$. Accordingly, part 1 of Theorem 3.5 could be extended to the variable time step framework, like in [33]. The reason why we have confined ourselves to a *constant* time step is in fact related to the validity of some calculations in the proof of Lemma 5.5.

6 Proof of Theorem 3.5 on the existence of viscous solutions

In this section, $\epsilon > 0$ is fixed and the limit as τ tends to zero is discussed. In order to pass to the limit in the time-discretization scheme of the viscous problem, as in [33] we are going to adopt a *variational* approach, along the lines of [42]. Namely, instead of taking the limit of the discrete subdifferential inclusion (4.5), we shall pass to the limit in the discrete energy inequality (6.1) derived in Lemma 6.1 below. Observe that, one of the peculiarities of this problem is that we have not used inequality (6.1) to deduce the basic energy estimates for the approximate solutions like it could be expected. In fact, the last remainder term on the right-hand side of (6.1) prevents us from doing so. Instead, relying on the a priori estimates obtained in Section 5 and on suitable compactness arguments (see the forthcoming Prop. 6.2), we are going to show that this remainder tends to zero, cf. (6.8) ahead).

Lemma 6.1 (Discrete energy inequality).
Under Assumptions 2.1, 2.5, and ($\mathrm{A}_\Omega 1$), the discrete solutions of (4.5) *satisfy the* discrete energy inequality *for all* $0 \le s \le t \le T$

$$
\begin{aligned}
&\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \big(\mathcal{R}_\epsilon(\hat{z}_\tau'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r), \overline{z}_\tau(r)))\big)\,\mathrm{d}r + \mathcal{I}(t, \hat{z}_\tau(t)) \\
&\le \mathcal{I}(s, \hat{z}_\tau(s)) + \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \partial_t \mathcal{I}(r, \hat{z}_\tau(r))\,\mathrm{d}r \\
&+ C \sup_{t\in[0,T]} \|\overline{z}_\tau(t) - \hat{z}_\tau(t)\|_{L^{2p_*/(p_*-2)}(\Omega)} \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \big(|\overline{t}_\tau(r) - r| + \|\overline{z}_\tau(r) - \hat{z}_\tau(r)\|_{L^{2p_*/(p_*-2)}(\Omega)}\big)\,\mathrm{d}r.
\end{aligned}
\tag{6.1}
$$

Proof. From (4.5) and as a consequence of the Fenchel-Moreau theorem we get

$$
\mathcal{R}_\epsilon\left(\hat{z}_\tau'(r)\right) + \mathcal{R}_\epsilon^*\left(-\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r), \overline{z}_\tau(r))\right) = \langle -\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r), \overline{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}} \qquad \text{for a.a. } r \in (0,T)\,. \tag{6.2}
$$

On the other hand, since $\hat{z}_\tau \in \mathrm{C}^0_{\mathrm{lip}}([0,T];\mathcal{Z})$, the *standard* chain rule yields $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(r,\hat{z}_\tau(r)) = \partial_t\mathcal{I}(r,\hat{z}_\tau(r)) + \langle \mathrm{D}_z\mathcal{I}(r,\hat{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}}$ for a.a. $r \in (0,T)$. Thus the right-hand side of (6.2) can be rewritten as

$$
\begin{aligned}
&\langle \mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r), \overline{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}} \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{I}(r, \hat{z}_\tau(r)) - \partial_t\mathcal{I}(r, \hat{z}_\tau(r)) + \langle \mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r), \overline{z}_\tau(r)) - \mathrm{D}_z\mathcal{I}(r, \hat{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\,.
\end{aligned}
\tag{6.3}
$$

Then, combining (6.2) and (6.3) and integrating on the interval $(\underline{t}_\tau(s), \overline{t}_\tau(t))$ we get

$$
\begin{aligned}
&\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \big(\mathcal{R}_\epsilon(\hat{z}_\tau')(r) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r), \overline{z}_\tau(r)))\big)\,\mathrm{d}r + \mathcal{I}(t, \hat{z}_\tau(t)) \\
&= \mathcal{I}(s, \hat{z}_\tau(s)) + \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \partial_t\mathcal{I}(r, \hat{z}_\tau(r))\,\mathrm{d}r - \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \langle \mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r), \overline{z}_\tau(r)) - \mathrm{D}_z\mathcal{I}(r, \hat{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\,\mathrm{d}r\,.
\end{aligned}
\tag{6.4}
$$

Let us estimate now the last term on the right-hand side:

$$\begin{aligned}
&\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \langle \mathrm{D}_z\mathcal{I}(\overline{t}_\tau(r),\overline{z}_\tau(r)) - \mathrm{D}_z\mathcal{I}(r,\hat{z}_\tau(r)),\hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\ \mathrm{d}r\\
&= \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \langle A_q\overline{z}_\tau(r) - A_q\hat{z}_\tau(r),\hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\ \mathrm{d}r + \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \langle \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(r),\overline{z}_\tau(r)) - \mathrm{D}_z\widetilde{\mathcal{I}}(r,\hat{z}_\tau(r)),\hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\ \mathrm{d}r\\
&=: F_1 + F_2.
\end{aligned}$$

Now, from the definition of $\hat{z}_\tau$ and (2.5) it follows that $F_1 \geq 0$. To estimate F_2, we use (2.42) from Corollary 2.13 and, observing that $P(\overline{z}_\tau,\hat{z}_\tau)$ (for $P(z_1,z_2)$ defined as in (2.23)) is bounded uniformly in τ thanks to (5.2), we get

$$\begin{aligned}
|F_2| &\leq C \textstyle\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} (|(\overline{t}_\tau(r)-r| + \|\overline{z}_\tau(r)-\hat{z}_\tau(r)\|_{L^{2p_*/(p_*-2)}(\Omega)})\|\overline{z}_\tau(r)-\hat{z}_\tau(r)\|_{L^{2p_*/(p_*-2)}(\Omega)}\ \mathrm{d}r\\
&\leq C\sup_{t\in[0,T]}\|\overline{z}_\tau(t)-\hat{z}_\tau(t)\|_{L^{2p_*/(p_*-2)}(\Omega)} \textstyle\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} (|(\overline{t}_\tau(r)-r| + \|\overline{z}_\tau(r)-\hat{z}_\tau(r)\|_{L^{2p_*/(p_*-2)}(\Omega)})\ \mathrm{d}r,
\end{aligned}$$

which together with (6.4) and the fact that $-F_1 \leq 0$ gives (6.1). □

As a consequence of the a priori estimates of Sec. 5, we have the following result.

Proposition 6.2 (Compactness). *Under Assumptions 2.1, 2.5, (A_Ω1), and (A_Ω2), for every $z_0 \in \mathcal{Z}$ with* (3.12) *and for every sequence of time-steps $(\tau_j)_j$ tending to 0 there exist a (not-relabeled) subsequence and $z \in L^\infty(0,T;\mathcal{Z}) \cap W^{1,2}(0,T;W^{1,2}(\Omega))$ fulfilling the mixed estimate* (3.8)*, as well as the enhanced regularity* (3.13)*, and such that the following convergences hold: for all $\beta \in [0,\frac{1}{q}(1-\frac{d}{q}))$*

$$\overline{z}_{\tau_j},\ \hat{z}_{\tau_j} \overset{*}{\rightharpoonup} z \qquad \text{in } L^{2q}(0,T;W^{1+\beta,q}(\Omega)) \cap L^\infty(0,T;\mathcal{Z}), \tag{6.5}$$

$$\hat{z}_{\tau_j} \rightharpoonup z \qquad \text{in } W^{1,2}(0,T;W^{1,2}(\Omega)), \tag{6.6}$$

$$\hat{z}_{\tau_j} \to z \qquad \text{strongly in } L^{2q}(0,T;W^{1+\beta,q}(\Omega)), \tag{6.7}$$

$$\sup_{t\in[0,T]} \left\|\overline{z}_{\tau_j}(t) - \hat{z}_{\tau_j}(t)\right\|_{W^{1,2}(\Omega)} \leq C(\epsilon)\sqrt{\tau_j} \tag{6.8}$$

$$\sup_{t\in[0,T]} \left\|\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_j}(t),\overline{z}_{\tau_j}(t)) - \mathrm{D}_z\mathcal{I}(t,\hat{z}_{\tau_j}(t))\right\|_{\mathcal{Z}^*} \leq C(\epsilon)\sqrt{\tau_j}. \tag{6.9}$$

Therefore, (6.7), (6.8) *and* (6.9) *imply*

$$\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau_j}(t),\overline{z}_{\tau_j}(t)) \to \mathrm{D}_z\mathcal{I}(t,z(t)) \qquad \text{strongly in } \mathcal{Z}^* \text{ for a.a. } t\in(0,T). \tag{6.10}$$

Moreover,

$$\hat{z}_{\tau_j}(t) \rightharpoonup z(t) \quad \text{in } \mathcal{Z} \qquad\qquad \text{for all } t\in[0,T], \tag{6.11}$$

$$\mathcal{I}(t,\hat{z}_{\tau_j}(t)) \to \mathcal{I}(t,z(t)) \qquad\qquad \text{for almost all } t\in(0,T). \tag{6.12}$$

Proof. Convergences (6.5)–(6.6) are a straightforward consequence of estimates (5.2), (5.5), and (5.7) via the Banach selection principle.

Estimate (5.5) implies that $\hat{z}_{\tau_j} \rightharpoonup z$ in $L^{2q}(0,T;W^{1+\beta,q}(\Omega))$ for every $\beta \in [0,\frac{1}{q}(1-\frac{d}{q}))$. This fact, together with (6.6) and [51, Corollary 4] yields the strong convergence (6.7) due to the compact embedding $W^{1+\beta_1,q}(\Omega) \subset W^{1+\beta_2,q}(\Omega)$ for $\beta_1 > \beta_2$. Now, (6.8) follows from the bound $\|\hat{z}_\tau'\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C$ (cf. estimates (5.7) and (5.8)).

In order to prove estimate (6.9), we notice that for every $w \in \mathcal{Z}$

$$\begin{aligned} & |\langle \mathrm{D}_z \mathcal{I}(\bar{t}_{\tau_j}(t), \bar{z}_{\tau_j}(t)) - \mathrm{D}_z \mathcal{I}(t, \hat{z}_{\tau_j}(t)), w \rangle_{\mathcal{Z}} | \\ & \leq |\langle A_q \bar{z}_{\tau_j}(t) - A_q \hat{z}_{\tau_j}(t), w \rangle_{\mathcal{Z}} | + |\langle \mathrm{D}_z \widetilde{\mathcal{I}}(\bar{t}_{\tau_j}(t), \bar{z}_{\tau_j}(t)) - \mathrm{D}_z \widetilde{\mathcal{I}}(t, \hat{z}_{\tau_j}(t)), w \rangle_{\mathcal{Z}} | =: F_1 + F_2. \end{aligned} \tag{6.13}$$

By estimate (2.6) and a careful application of the Hölder inequality with $\frac{1}{2} + \frac{q-2}{2q} + \frac{1}{q} = 1$

$$\begin{aligned} F_1 \leq & C \left(\int_\Omega (1 + |\nabla \bar{z}_{\tau_j}(t)|^2 + |\nabla \hat{z}_{\tau_j}(t)|^2)^{\frac{q-2}{2}} |\nabla(\bar{z}_{\tau_j}(t) - \hat{z}_{\tau_j}(t))|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\ & \times (1 + \|\bar{z}_{\tau_j}(t)\|_{W^{1,q}(\Omega)} + \|\hat{z}_{\tau_j}(t)\|_{W^{1,q}(\Omega)})^{\frac{q-2}{2}} \|\nabla w\|_{L^q(\Omega)}. \end{aligned}$$

Therefore, by using the energy estimate (5.2) we obtain

$$\begin{aligned} \|A_q \bar{z}_{\tau_j}(t) - A_q \hat{z}_{\tau_j}(t)\|_{\mathcal{Z}^*} & \leq C \left(\int_\Omega (1 + |\nabla \bar{z}_{\tau_j}(t)|^2 + |\nabla \underline{z}_{\tau_j}(t)|^2)^{\frac{q-2}{2}} \tau^2 |\nabla \hat{z}'_{\tau_j}(t)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\ & \leq C\sqrt{\tau} \int_0^T \left(\int_\Omega (1 + |\nabla \bar{z}_{\tau_j}(t)|^2 + |\nabla \underline{z}_{\tau_j}(t)|^2)^{\frac{q-2}{2}} |\nabla \hat{z}'_{\tau_j}(t)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \, \mathrm{d}t \leq C\sqrt{\tau C_3(\epsilon)}, \end{aligned}$$

where for the second estimate we have used the Hölder inequality and (5.7) for the last one. All in all,

$$\|A_q \bar{z}_{\tau_j}(t) - A_q \hat{z}_{\tau_j}(t)\|_{\mathcal{Z}^*} \leq C(\epsilon)\sqrt{\tau}. \tag{6.14}$$

Now we estimate F_2. By Corollary 2.13 and the embedding of $W^{1,2}(\Omega)$ in $L^{2p_*/(p_*-2)}(\Omega)$

$$\begin{aligned} F_2 & \leq C(|\bar{t}_{\tau_j}(t) - t| + \|\bar{z}_{\tau_j}(t) - \hat{z}_{\tau_j}(t)\|_{L^{2p_*/(p_*-2)}(\Omega)}) \|w\|_{L^{2p_*/(p_*-2)}(\Omega)} \\ & \leq C(\tau + \|\bar{z}_{\tau_j}(t) - \hat{z}_{\tau_j}(t)\|_{W^{1,2}(\Omega)}) \|w\|_{L^{2p_*/(p_*-2)}(\Omega)}. \end{aligned}$$

Therefore, taking into account (6.8) we get

$$\|\mathrm{D}_z \widetilde{\mathcal{I}}(\bar{t}_{\tau_j}(t), \bar{z}_{\tau_j}(t)) - \mathrm{D}_z \widetilde{\mathcal{I}}(t, \hat{z}_{\tau_j}(t))\|_{\mathcal{Z}^*} \leq C(\tau + \sqrt{\tau}), \tag{6.15}$$

and (6.13)–(6.15) give (6.9).

Now, from (6.7) it follows that $\hat{z}_{\tau_j}(t) \to z(t)$ strongly in $W^{1+\beta,q}(\Omega))$ for a.a. $t \in (0,T)$. Thus, by (2.44) in Corollary 2.15, $\mathrm{D}_z \mathcal{I}(t, \hat{z}_{\tau_j}(t)) \to \mathrm{D}_z \mathcal{I}(t, z(t))$ strongly in $\mathcal{Z}^*$ for a.a. $t \in (0,T)$. This, together with (6.9) yields (6.10).

The mixed estimate (3.8) follows from estimate (5.7) by lower semicontinuity of the functional $(A, B) \mapsto \int_0^T \int_\Omega (1 + |A|^2)^{\frac{q-2}{2}} |B|^2 \, \mathrm{d}x \, \mathrm{d}t$, which is convex in B, observing that (6.6) implies $\nabla \hat{z}'_{\tau_j} \rightharpoonup \nabla z'$ in $L^2((0,T) \times \Omega)$ and that (6.7) implies $\nabla \hat{z}_{\tau_j} \to \nabla z$ in $L^1((0,T) \times \Omega)$ (see e.g. [11, Theorem 3.23]).

Convergence (6.11) follows from the fact that $L^\infty(0,T;\mathcal{Z}) \cap W^{1,2}(0,T;W^{1,2}(\Omega))$ is compactly embedded in $\mathrm{C}^0([0,T];\mathcal{X})$ for every $\mathcal{X}$ such that $\mathcal{Z} \Subset \mathcal{X} \subset W^{1,2}(\Omega)$ (cf., e.g., [51]), combined with the estimate $\sup_{j \in \mathbb{N}} \sup_{t \in [0,T]} \|\hat{z}_{\tau_j}\|_{W^{1,q}(\Omega)} \leq C$, cf. (5.2).

Finally, from (6.7) we get pointwise convergence in $W^{1+\beta,q}(\Omega))$ for a.a. t. Then, the continuity of $z \mapsto \mathcal{I}(t,z)$ ensues (6.12). □

The convergences (6.5)–(6.12) are sufficient to pass to the limit in the time-discretization scheme, and conclude the existence of a weak solution (in the sense of Def. 3.1), to the Cauchy problem (3.2)–(3.3). In order to deduce by lower semicontinuity arguments the uniform w.r.t. ϵ-estimates (3.14)–(3.17) for *any* family of solutions (z_ε) arising from the time-discretization procedure of Sec. 4, additional compactness arguments are needed, which we develop in the forthcoming Lemma 6.3. We postpone its statement and proof after the proof of Theorem 3.5.

Proof of Theorem 3.5. For fixed $\epsilon > 0$ let $(\tau_j)_{j\in\mathbb{N}}$ be a sequence along which the convergences in Proposition 6.2 are valid. Proposition 6.2 also ensures that, for the limit curve z fulfills the mixed estimate (3.8) holds.

First of all, we pass to the limit in the discrete energy inequality (6.1). Thanks to convergence (6.11), for all $t \in [0,T]$ it holds that $\liminf_{j\to\infty} \mathcal{I}(t, \hat{z}_{\tau_j}(t)) \geq \mathcal{I}(t, z(t))$ while, from (6.12) $\mathcal{I}(s, \hat{z}_{\tau_j}(s)) \to \mathcal{I}(s, z(s))$ for a.a. $s \in (0,T)$. The convergence of the term involving $\partial_t \mathcal{I}$ is an immediate consequence of the convergence stated in (6.7), taking into account the continuity properties of $\partial_t \mathcal{I}$ (see estimate (2.32) in Lemma 2.9). Due to (6.10) and the lower semicontinuity of $\mathcal{R}^*_\epsilon$ we conclude that

$$\liminf_{\tau_j} \int_{\underline{t}_{\tau_j}(s)}^{\bar{t}_{\tau_j}(t)} \mathcal{R}^*_\epsilon(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau_j}(r), \bar{z}_{\tau_j}(r)))\,\mathrm{d}r \geq \int_s^t \mathcal{R}^*_\epsilon(-\mathrm{D}_z\mathcal{I}(r, z(r)))\,\mathrm{d}r.$$

Similarly, from (6.6), by lower semicontinuity it follows that $\liminf_{\tau_j} \int_{\underline{t}_{\tau_j}(s)}^{\bar{t}_{\tau_j}(t)} \mathcal{R}_\epsilon(\hat{z}'_{\tau_j}(r))\,\mathrm{d}r \geq \int_s^t \mathcal{R}_\epsilon(z'(r))\,\mathrm{d}r$. Moreover, the remainder term on the right-hand side of (6.1) tends to zero thanks to (6.8) and the embedding of $W^{1,2}(\Omega)$ in $L^{2p_*/(p_*-2)}(\Omega)$. Altogether we arrive at the energy inequality

$$\int_s^t (\mathcal{R}_\epsilon(z'(r)) + \mathcal{R}^*_\epsilon(-\mathrm{D}_z\mathcal{I}(r, z(r)))\ \mathrm{d}r + \mathcal{I}(t, z(t)) \leq \mathcal{I}(s, z(s)) + \int_s^t \partial_t\mathcal{I}(r, z(r))\ \mathrm{d}r, \tag{6.16}$$

for all $t \in [0,T]$, for $s = 0$, and for almost all $0 < s < t$.

We now check that (6.16) holds for all $0 \leq s \leq t$. Let $s_n \nearrow s$ be a sequence of points for which (6.16) is satisfied. Thus,

$$\begin{aligned}\int_{s_n}^s (\mathcal{R}_\epsilon(z'(r)) + \mathcal{R}^*_\epsilon(-\mathrm{D}_z\mathcal{I}(r, z(r)))\ \mathrm{d}r &\leq \mathcal{I}(s_n, z(s_n)) - \mathcal{I}(s, z(s)) + \int_{s_n}^s \partial_t\mathcal{I}(r, z(r))\ \mathrm{d}r \\ = -\int_{s_n}^s \int_\Omega (1 + |\nabla z(r)|^2)^{(q-2)/2} \nabla z(r) \cdot \nabla z'(r)\ \mathrm{d}x\ \mathrm{d}r &- \int_{s_n}^s \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(r, z(r)) z'(r)\ \mathrm{d}x\ \mathrm{d}r\end{aligned}$$

where the equality follows by an integrated-in-time version of the chain-rule formula (3.10). Passing to the limit as $s_n \nearrow s$ and using the absolute continuity of the Lebesgue integral, from the second inequality we derive $\mathcal{I}(s_n, z(s_n)) \to \mathcal{I}(s, z(s))$, and therefore we obtain (3.11) for all s and t. Thanks to Proposition 3.3, we conclude that z is a weak solution (in the sense of Def. 3.9), to the Cauchy problem (3.2)–(3.3).

Estimates (3.14)–(3.17) follow from Lemma 6.3 below. □

The proof of the following Lemma exploits Young measure tools, which we recall in Appendix A.

Lemma 6.3. *Under Assumptions 2.1, 2.5, (A$_\Omega$1) and (A$_\Omega$2), for every $z_0 \in \mathcal{Z}$ such that $\mathrm{D}_z\mathcal{I}(0, z_0) \in L^2(\Omega)$ and for every $\epsilon > 0$ estimates* (3.14)–(3.15) *hold. In addition, z from Proposition 6.2 also fulfills*

$$\int_0^T \|z(t)\|^q_{W^{1+\beta-\delta,q}(\Omega)} \|z'(t)\|_{L^2(\Omega)}\ \mathrm{d}t \leq \liminf_{j\to 0} \int_0^T \|\bar{z}_{\tau_j}(t)\|^q_{W^{1+\beta,q}(\Omega)} \|\hat{z}'_{\tau_j}(t)\|_{L^2(\Omega)}\ \mathrm{d}t \tag{6.17}$$

for all $\beta \in [0, \frac{1}{q}(1 - \frac{d}{q}))$, and

$$\int_0^T \left(\int_\Omega (1 + |\nabla z(t)|^2)^{\frac{q-2}{2}} |\nabla z'(t)|^2\ \mathrm{d}x \right)^{\frac{1}{2}} \mathrm{d}t \leq \liminf_{j\to 0} \int_0^T \left(\int_\Omega (1 + |\nabla \hat{z}_{\tau_j}(t)|^2)^{\frac{q-2}{2}} |\nabla \hat{z}'_{\tau_j}(t)|^2\ \mathrm{d}x \right)^{\frac{1}{2}} \mathrm{d}t. \tag{6.18}$$

As a consequence, estimates (3.16)–(3.17) *hold.*

Proof. Estimates (3.14)–(3.15) follow from (5.1), (5.2), and (5.5) by convergences (6.5)–(6.6) and lower semicontinuity arguments.

In order to prove inequalities (6.17) and (6.18), we resort to a Young measure argument, based on A. Indeed, we can apply the compactness theorem A.2, with the space $V = W^{1+\beta,q}(\Omega) \times W^{1,2}(\Omega)$, to the sequence $(\overline{z}_{\tau_j}, \hat{z}'_{\tau_j})_j$, bounded in $L^2(0,T; W^{1+\beta,q}(\Omega) \times W^{1,2}(\Omega))$ for all $\beta \in [0, \frac{1}{q}(1-\frac{d}{q}))$. Therefore, up to a not relabeled subsequence, $(\overline{z}_{\tau_j}, \hat{z}'_{\tau_j})_j$ admits a limiting Young measure $\boldsymbol{\mu} = \{\mu_t\}_{t\in(0,T)} \in \mathscr{Y}(0,T; W^{1+\beta,q}(\Omega) \times W^{1,2}(\Omega))$, such that for almost all $t \in (0,T)$ the measure μ_t is concentrated on the limit points of $(\overline{z}_{\tau_j}(t), \hat{z}'_{\tau_j}(t))_j$, w.r.t. the $W^{1+\beta,q}(\Omega) \times W^{1,2}(\Omega)$-weak topology. Now, by (6.7)–(6.8) we have that $\overline{z}_{\tau_j}(t) \to z(t)$ strongly in $W^{1,q}(\Omega)$. Therefore, denoting by π_1 the projection operator $(z,v) \in W^{1+\beta,q}(\Omega) \times W^{1,2}(\Omega) \mapsto z \in W^{1,q}(\Omega)$, it is immediate to check that the projection measure $(\pi_1)_\sharp(\mu_t)$ coincides with the Dirac delta $\delta_{z(t)}$. With a disintegration argument we in fact see that μ_t is of the form $\delta_{z(t)} \otimes \nu_t$, and that the parameterized measure $\{\nu_t\}_{t\in(0,T)}$ fulfills

$$\int_{W^{1,2}(\Omega)} v \, \mathrm{d}\nu_t(v) = z'(t) \qquad \text{for almost all } t \in (0,T). \tag{6.19}$$

Then, the lim inf-inequality (A.2) with the normal integrand $\mathcal{H}(t,(z,v)) := \|z\|^q_{W^{1+\beta,q}(\Omega)} \|v\|_{L^2(\Omega)}$ yields

$$\begin{aligned}
&\liminf_{j\to 0} \int_0^T \|\overline{z}_{\tau_j}(t)\|^q_{W^{1+\beta,q}(\Omega))} \|\hat{z}'_{\tau_j}(t)\|_{L^2(\Omega)} \, \mathrm{d}t \\
&\geq \int_0^T \iint_{W^{1+\beta,q}(\Omega)\times W^{1,2}(\Omega)} \|z\|^q_{W^{1+\beta,q}(\Omega))} \|v\|_{L^2(\Omega)} \, \mathrm{d}(\delta_{z(t)} \otimes \nu_t)(z,v) \, \mathrm{d}t \\
&\geq \int_0^T \|z(t)\|^q_{W^{1+\beta,q}(\Omega))} \left\| \int_{W^{1,2}(\Omega)} v \, \mathrm{d}\nu_t(v) \right\|_{L^2(\Omega)} = \int_0^T \|z(t)\|^q_{W^{1+\beta,q}(\Omega)} \|z'(t)\|_{L^2(\Omega)} \, \mathrm{d}t
\end{aligned}$$

where the second estimate is due to Jensen's inequality and the last equality to (6.19). This gives (6.17).

As for (6.18), we now consider the sequence of gradients $(\nabla \hat{z}_{\tau_j}, \nabla \hat{z}'_{\tau_j})_j$, bounded in $L^2(0,T; L^2(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^d))$. Relying on Theorem A.2, we associate with $(\nabla \hat{z}_{\tau_j}, \nabla \hat{z}'_{\tau_j})_j$ its limiting Young measure $\widetilde{\boldsymbol{\mu}} = \{\widetilde{\mu}_t\}_{t\in(0,T)} \in \mathscr{Y}(0,T; L^2(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^d))$, concentrated on the set of the weak-$L^2(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^d)$ limit points of $(\nabla \hat{z}_{\tau_j}, \nabla \hat{z}'_{\tau_j})_j$. On account of the strong convergence (6.7), arguing as in the above lines we conclude that $\widetilde{\boldsymbol{\mu}}_t = \delta_{\nabla z(t)} \otimes \widetilde{\nu}_t$ for almost all $t \in (0,T)$, with $\{\widetilde{\nu}_t\}_{t\in(0,T)}$ satisfying

$$\int_{L^2(\Omega;\mathbb{R}^d)} B \, \mathrm{d}\widetilde{\nu}_t(B) = \nabla z'(t) \qquad \text{for almost all } t \in (0,T). \tag{6.20}$$

Therefore, inequality (A.2) with the normal integrand $\mathcal{H}(t,(A,B)) := \left(\int_\Omega (1+|A|^2)^{\frac{q-2}{2}} |B|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$ yields

$$\begin{aligned}
&\liminf_{j\to 0} \int_0^T \left(\int_\Omega (1+|\nabla \hat{z}_{\tau_j}(t)|^2)^{\frac{q-2}{2}} |\nabla z'_{\tau_j}(t)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \, \mathrm{d}t \\
&\geq \int_0^T \iint_{L^2(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^d)} \left(\int_\Omega (1+|A|^2)^{\frac{q-2}{2}} |B|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \, \mathrm{d}(\delta_{\nabla z(t)} \otimes \widetilde{\nu}_t)(A,B) \, \mathrm{d}t \\
&= \int_0^T \int_{L^2(\Omega;\mathbb{R}^d)} \left(\int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} |B|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \, \mathrm{d}\widetilde{\nu}_t(B) \, \mathrm{d}t \\
&\geq \int_0^T \left(\int_\Omega (1+|\nabla z(t)|^2)^{\frac{q-2}{2}} \left| \int_{L^2(\Omega;\mathbb{R}^d)} B \, \mathrm{d}\widetilde{\nu}_t(B) \right|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \, \mathrm{d}t
\end{aligned}$$

where the latter estimate again follows from Jensen's inequality. Then, in view of (6.20), (6.18) ensues.

Estimates (3.16)–(3.17) are then a consequence of (6.17) and (6.18), combined with the bounds (5.31) and (5.22), respectively. $\square$

7 Vanishing viscosity limit

Throughout this section, we shall work with a family $(z_\epsilon)_\epsilon \subset L^\infty(0,T;W^{1,q}(\Omega)) \cap W^{1,2}(0,T;W^{1,2}(\Omega))$ of *weak* solutions (in the sense of Definition 3.1), to the ϵ-viscous Cauchy problem (3.2)–(3.3). We shall suppose that for $(z_\epsilon)_\epsilon$ the following estimates, uniform w.r.t. the parameter ϵ, are valid:

$$\sup_{\epsilon>0} \|z_\epsilon\|_{W^{1,1}(0,T;L^2(\Omega))} \leq C, \tag{7.1a}$$

$$\sup_{\epsilon>0} \|z_\epsilon\|_{L^{2q}(0,T,W^{1+\beta,q}(\Omega))\cap L^\infty(0,T;W^{1,q}(\Omega))} \leq C, \tag{7.1b}$$

$$\sup_{\epsilon>0} \int_0^T \|z_\epsilon(t)\|^q_{W^{1+\beta,q}(\Omega)} \|z'_\epsilon(t)\|_{L^2(\Omega)} \,\mathrm{d}t \leq C, \qquad \text{for every } \beta \in \left[0, \frac{1}{q}(1-\frac{d}{q})\right), \tag{7.1c}$$

$$\sup_{\epsilon>0} \int_0^T \left(\int_\Omega (1+|\nabla z_\epsilon(t)|^2)^{\frac{q-2}{2}} |\nabla z'_\epsilon(t)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}t \leq C. \tag{7.1d}$$

The existence of solutions $(z_\epsilon)_\epsilon$ fulfilling (7.1) is ensured by Theorem 3.5, under the condition that the initial datum $z_0 \in \mathcal{Z}$ also fulfills $\mathrm{D}_z\mathcal{I}(0,z_0) \in L^2(\Omega)$.

In what follows, we shall reparameterize the curves $(z_\epsilon)_\epsilon$ by their $L^2(\Omega)$-arclength, and study the asymptotic behavior of the reparameterized trajectories as $\epsilon \to 0$. This leads (cf. Theorem 7.4 below) to the notion of *weak parameterized* solution to the rate-independent damage system (1.3), which we introduce in Definition 7.2.

7.1 Weak parameterized solutions

The starting point for the passage of the vanishing-viscosity limit is the energy inequality (3.11), which lies at the core of the notion of *weak* solutions to the viscous problem. Taking into account the definition of $\mathcal{R}_\epsilon$, and the fact that $\mathcal{R}_\epsilon^*$ is given by (cf. [33, Lemma 3.1])

$$\mathcal{R}_\epsilon^*(\sigma) = \frac{1}{\epsilon} \min_{\mu\in\partial\mathcal{R}_1(0)} \widetilde{\mathcal{R}}_2(\sigma-\mu), \qquad \text{with } \widetilde{\mathcal{R}}_2(\sigma) := \begin{cases} \frac{1}{2}\|\sigma\|^2_{L^2(\Omega)} & \text{if } \sigma \in L^2(\Omega), \\ \infty & \text{if } \sigma \in \mathcal{Z}^*\backslash L^2(\Omega), \end{cases} \tag{7.2}$$

thanks to the inf-sup convolution formula, inequality (3.11) rephrases as

$$\begin{aligned} &\int_s^t \mathcal{R}_1(z'_\epsilon(r)) + \frac{\epsilon}{2}\|z'_\epsilon(r)\|^2_{L^2(\Omega)} \,\mathrm{d}r + \int_s^t \frac{1}{\epsilon} \min_{\mu\in\partial\mathcal{R}_1(0)} \widetilde{\mathcal{R}}_2(-\mathrm{D}_z\mathcal{I}(r,z_\epsilon(r))-\mu) \,\mathrm{d}r + \mathcal{I}(t,z_\epsilon(t)) \\ &\qquad \leq \mathcal{I}(s,z_\epsilon(s)) + \int_s^t \partial_t\mathcal{I}(r,z_\epsilon(r)) \,\mathrm{d}r \qquad \text{for all } 0 \leq s \leq t \leq T. \end{aligned} \tag{7.3}$$

Now, for every $\epsilon > 0$ we consider the $L^2(\Omega)$-arclength parameterization of the curve z_ϵ, viz.

$$s_\epsilon(t) = t + \int_0^t \|z'_\epsilon(r)\|_{L^2(\Omega)} \,\mathrm{d}r. \tag{7.4}$$

Let $S_\epsilon = s_\epsilon(T)$: it follows from (7.1a) that $\sup_{\epsilon>0} S_\epsilon < \infty$. We introduce the functions $\tilde{t}_\epsilon : [0,S_\epsilon] \to [0,T]$ and $\tilde{z}_\epsilon : [0,S_\epsilon] \to \mathcal{Z}$

$$\tilde{t}_\epsilon(s) := s_\epsilon^{-1}(s), \quad \tilde{z}_\epsilon(s) := z_\epsilon(\tilde{t}_\epsilon(s)) \tag{7.5}$$

fulfilling the *normalization condition*

$$\tilde{t}'_\epsilon(s) + \|\tilde{z}'_\epsilon(s)\|_{L^2(\Omega)} = 1 \quad \text{for a.a. } s \in (0,S_\epsilon) \tag{7.6}$$

and study the limiting behavior as $\epsilon \to 0$ of the parameterized trajectories $\{ (\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s))\,;\, s \in [0, S_\epsilon] \}$. Since $\sup_{\epsilon>0} S_\epsilon < \infty$, up to a subsequence, $S_\epsilon \to S$ as $\epsilon \to 0$, with $S \geq T$ (the latter inequality follows from the fact that $s_\epsilon(t) \geq t$). With no loss of generality, we may consider the parameterized trajectories to be defined on the fixed time interval $[0, S]$.

From the energy inequality (7.3) we deduce that the parameterized trajectories $(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s))_{s\in[0,S]}$ fulfill

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} \Big(\mathcal{R}_1(\tilde{z}_\epsilon'(s)) + \frac{\epsilon}{2\tilde{t}_\epsilon'(s)} \|\tilde{z}_\epsilon'(s)\|_{L^2(\Omega)}^2 + \frac{\tilde{t}_\epsilon'(s)}{2\epsilon} \mathrm{d}_2^2(-\mathrm{D}_z\mathcal{I}(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s)), \partial\mathcal{R}_1(0))\Big)\, \mathrm{d}s + \mathcal{I}(\tilde{t}_\epsilon(\sigma_2), \tilde{z}_\epsilon(\sigma_2)) \\ \leq \mathcal{I}(\tilde{t}_\epsilon(\sigma_1), \tilde{z}_\epsilon(\sigma_1)) + \int_{\sigma_1}^{\sigma_2} \partial_t\mathcal{I}(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s))\tilde{t}_\epsilon'(s)\, \mathrm{d}s \quad \forall\, (\sigma_1, \sigma_2) \subset [0, S],\end{aligned} \tag{7.7}$$

where we have used the short-hand notation $\mathrm{d}_2(\xi, \partial\mathcal{R}_1(0)) := \min_{\mu\in\partial\mathcal{R}_1(0)} \sqrt{2\widetilde{\mathcal{R}}_2(\xi-\mu)}$. Upon introducing the functional (cf. [40, Sec. 3.2])

$$\mathcal{M}_\epsilon : (0, \infty) \times L^2(\Omega) \times [0, \infty) \to [0, \infty], \quad \mathcal{M}_\epsilon(\alpha, v, \zeta) := \mathcal{R}_1(v) + \frac{\epsilon}{2\alpha} \|v\|_{L^2(\Omega)}^2 + \frac{\alpha}{2\epsilon}\zeta^2, \tag{7.8}$$

the above inequality rephrases as

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} \mathcal{M}_\epsilon(\tilde{t}_\epsilon'(s), \tilde{z}_\epsilon'(s), \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s)), \partial\mathcal{R}_1(0)))\, \mathrm{d}s + \mathcal{I}(\tilde{t}_\epsilon(\sigma_2), \tilde{z}_\epsilon(\sigma_2)) \\ \leq \mathcal{I}(\tilde{t}_\epsilon(\sigma_1), \tilde{z}_\epsilon(\sigma_1)) + \int_{\sigma_1}^{\sigma_2} \partial_t\mathcal{I}(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s))\tilde{t}_\epsilon'(s)\, \mathrm{d}s \quad \forall\, (\sigma_1, \sigma_2) \subset [0, S].\end{aligned} \tag{7.9}$$

We will pass to the limit as $\epsilon \to 0$ in (7.9). For this, we shall rely on the following Γ-convergence/lower semicontinuity result, [40, Lemma 3.1] (cf. also [33, Lemma 5.1]).

Lemma 7.1. *Extend the functional $\mathcal{M}_\epsilon$ (7.8) to $[0, +\infty) \times L^2(\Omega) \times [0, \infty)$ via*

$$\mathcal{M}_\epsilon(0, v, \zeta) := \begin{cases} 0 & \text{for } v = 0 \text{ and } \zeta \in [0, +\infty)\,, \\ \infty & \text{for } v \in L^2(\Omega)\backslash\{0\} \text{ and } \zeta \in [0, +\infty)\,. \end{cases}$$

Define $\mathcal{M}_0 : [0, \infty) \times L^2(\Omega) \times [0, \infty) \to [0, \infty]$ by

$$\mathcal{M}_0(\alpha, v, \zeta) := \begin{cases} \mathcal{R}_1(v) + \zeta \|v\|_{L^2(\Omega)} & \text{if } \alpha = 0, \\ \mathcal{R}_1(v) + \mathrm{I}_0(\zeta) & \text{if } \alpha > 0, \end{cases} \tag{7.10}$$

where I_0 denotes the indicator function of the singleton $\{0\}$. Then,
(A) $\mathcal{M}_\varepsilon$ Γ-converges to $\mathcal{M}_0$ on $[0, \infty) \times L^2(\Omega) \times [0, \infty)$ w.r. to the strong-weak-strong topology. (B) If $\alpha_\varepsilon \rightharpoonup \bar{\alpha}$ in $L^1(a, b)$, $v_\varepsilon \rightharpoonup \bar{v}$ in $L^1(a, b; L^2(\Omega))$, and $\liminf_{\varepsilon\to0} \zeta_\varepsilon(s) \geq \bar{\zeta}(s)$ for a.a. $s \in (a, b)$, then

$$\int_a^b \mathcal{M}_0(\bar{\alpha}(s), \bar{v}(s), \bar{\zeta}(s))\; \mathrm{d}s \leq \liminf_{\varepsilon\to0} \int_a^b \mathcal{M}_\varepsilon(\alpha_\varepsilon(s), v_\varepsilon(s), \zeta_\varepsilon(s))\; \mathrm{d}s\,.$$

We now introduce the notion of solution which arises from passing to the limit as $\epsilon \to 0$ in (7.9).

Definition 7.2 (Weak parameterized solutions)**.** A pair $(\tilde{t}, \tilde{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0, S]; [0, T] \times L^2(\Omega))$ is a *weak parameterized solution* of the rate-independent damage system (1.3), if it satisfies the *energy inequality*

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} \mathcal{M}_0(\tilde{t}'(s), \tilde{z}'(s), \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}(s), \tilde{z}(s)), \partial\mathcal{R}_1(0)))\, \mathrm{d}s + \mathcal{I}(\tilde{t}(\sigma_2), \tilde{z}(\sigma_2)) \\ \leq \mathcal{I}(\tilde{t}(\sigma_1), \tilde{z}(\sigma_1)) + \int_{\sigma_1}^{\sigma_2} \partial_t\mathcal{I}(\tilde{t}(s), \tilde{z}(s))\tilde{t}'(s)\; \mathrm{d}s \quad \text{for all } 0 \leq \sigma_1 \leq \sigma_2 \leq S.\end{aligned} \tag{7.11}$$

We say that a weak parameterized solution $(\tilde{t},\tilde{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times L^2(\Omega))$ is *non-degenerate* if it fulfills

$$\tilde{t}'(s) + \|\tilde{z}'(s)\|_{L^2(\Omega)} > 0 \quad \text{for a.a. } s \in (0,S). \tag{7.12}$$

Recall that the chain rule provided by Theorem 3.2 is a key ingredient for getting further insight into the notion of weak solution to the viscous system from Def. 3.9. Indeed, it is by a chain-rule argument that we can show that the pointwise variational inequality (3.9) is equivalent to the energy inequality (3.11). Likewise, the following result, which the parameterized counterpart to the chain rule of Theorem 3.2, shall enable us to obtain a differential characterization of the notion of weak parameterized solution in terms of the energy inequality (7.11). Indeed, Prop. 7.3 shall be exploited in the proof of Prop. 7.6.

Proposition 7.3. *Under Assumptions 2.1, 2.5, and (A$_\Omega$1), let $(\tilde{t},\tilde{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times L^2(\Omega))$ fulfill in addition*

$$\tilde{z} \in L^\infty(0,S;W^{1,q}(\Omega)), \tag{7.13}$$

$$\int_0^S \mathrm{M}(s)\,\mathrm{d}s < \infty \qquad \textit{with } \mathrm{M}(s) := \left(\int_\Omega (1+|\nabla\tilde{z}(s)|^2)^{\frac{q-2}{2}}|\nabla\tilde{z}'(s)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{7.14}$$

Then, the map $s \mapsto \mathcal{I}(\tilde{t}(s),\tilde{z}(s))$ is absolutely continuous on $(0,S)$, and the following chain-rule formula is valid:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\tilde{t}(s),\tilde{z}(s)) - \partial_s\mathcal{I}(\tilde{t}(s),\tilde{z}(s))\tilde{t}'(s) = &\int_\Omega (1+|\nabla\tilde{z}(s)|^2)^{\frac{q-2}{2}}\nabla\tilde{z}(s)\cdot\nabla\tilde{z}'(s)\,\mathrm{d}x \\ &+ \int_\Omega \mathrm{D}_z\widetilde{\mathcal{I}}(\tilde{t}(s),\tilde{z}(s))\tilde{z}'(s)\,\mathrm{d}x\,.\end{aligned} \tag{7.15}$$

Proof. From (7.13) and (7.14) we deduce with the Hölder inequality that

$$\begin{aligned}&\int_0^S\int_\Omega \left|(1+|\nabla\tilde{z}(s)|^2)^{\frac{q-2}{2}}\nabla\tilde{z}(s)\cdot\nabla\tilde{z}'(s)\right|\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^S\int_\Omega \left|(1+|\nabla\tilde{z}(s)|^2)^{\frac{q-2}{4}}\nabla\tilde{z}'(s)\right|\left|(1+|\nabla\tilde{z}(s)|^2)^{\frac{q-2}{4}}\nabla\tilde{z}(s)\right|\,\mathrm{d}x\,\mathrm{d}s \\ &\le \int_0^S \mathrm{M}(s)\|(1+|\nabla\tilde{z}(s)|^2)^{\frac{q-2}{4}}\nabla\tilde{z}(s)\|_{L^2(\Omega)}\,\mathrm{d}s \le c\int_0^S \mathrm{M}(s)(1+\|\nabla\tilde{z}(s)\|^{\frac{q}{2}}_{L^q(\Omega)})\,\mathrm{d}s < \infty,\end{aligned}$$

where the last estimate relies on (7.13). Now we can argue as in the proof of Theorem 3.2 to deduce that $(\tilde{t},\tilde{z})$ fulfill the parameterized version of the chain rule (7.15). □

7.2 The vanishing-viscosity result

We are now in the position of stating and proving our main vanishing-viscosity result.

Theorem 7.4. *Under Assumptions 2.1, 2.5, (A$_\Omega$1), and (A$_\Omega$2), let $(z_\epsilon)_\epsilon$ be a family of weak solutions (according to Definition 3.1), in $L^\infty(0,T;W^{1,q}(\Omega))\cap W^{1,2}(0,T;W^{1,2}(\Omega))$, to the ϵ-viscous Cauchy problem* (3.2)–(3.3). *Suppose that the estimates* (7.1) *are valid for $(z_\epsilon)_\epsilon$, and let $(\tilde{t}_\epsilon,\tilde{z}_\epsilon)_{\epsilon>0} \subset \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times L^2(\Omega))$ be defined by* (7.5).

Then, for every sequence $\epsilon_n \searrow 0$ there exist a pair $(\tilde{t},\tilde{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times L^2(\Omega))$, such that $\tilde{z}$ has the regularity

$$\tilde{z} \in L^q(0,S;W^{1+\beta,q}(\Omega)) \cap L^\infty(0,S;W^{1,q}(\Omega)) \qquad \textit{for every } \beta \in \left[0,\frac{1}{q}\big(1-\frac{d}{q}\big)\right), \tag{7.16}$$

and a (not-relabeled) subsequence such that

$$\begin{aligned} &(\tilde{t}_{\epsilon_n}, \tilde{z}_{\epsilon_n}) \overset{*}{\rightharpoonup} (\tilde{t}, \tilde{z}) \text{ in } W^{1,\infty}(0,S;[0,T]\times L^2(\Omega)), \\ &\tilde{t}_{\epsilon_n} \to \tilde{t} \text{ in } \mathrm{C}^0([0,S];[0,T]), \quad \tilde{z}_{\epsilon_n}(s) \rightharpoonup \tilde{z}(s) \text{ in } L^2(\Omega) \text{ for all } s\in[0,S], \end{aligned} \tag{7.17}$$

and $(\tilde{t}, \tilde{z})$ is a weak parameterized solution of the rate-independent damage system (1.3)*, fulfilling*

$$\hat{t}'(s) + \|\hat{z}'(s)\|_{L^2(\Omega)} \leq 1 \quad \text{for a.a. } s\in(0,S). \tag{7.18}$$

Furthermore, $\tilde{z}$ fulfills (7.14)*.*

For the proof, we will rely on the following a priori estimates for the parameterized solutions

Lemma 7.5. *Under Assumptions 2.1, 2.5, (A_Ω1), and (A_Ω2), let $(z_\epsilon)_\epsilon$ be a family of weak solutions (according to Definition 3.1), in $L^\infty(0,T;W^{1,q}(\Omega))\cap W^{1,2}(0,T;W^{1,2}(\Omega))$, to the ϵ-viscous Cauchy problem* (3.2)–(3.3)*. Suppose that $(z_\epsilon)_\epsilon$ satisfy* (7.1)*. Then*

$$\sup_{\epsilon>0} \|\tilde{z}_\epsilon\|_{W^{1,\infty}(0,S;L^2(\Omega))} \leq C, \tag{7.19a}$$

$$\sup_{\epsilon>0} \|\tilde{z}_\epsilon\|_{L^{\infty}(0,S;W^{1,q}(\Omega))} \leq C, \tag{7.19b}$$

$$\sup_{\epsilon>0} \|\tilde{z}_\epsilon\|_{L^{q}(0,S;W^{1+\beta,q}(\Omega))} \leq C_\beta \qquad \text{for every } \beta \in \left[0, \frac{1}{q}(1-\frac{d}{q})\right), \tag{7.19c}$$

$$\sup_{\epsilon>0} \int_0^S \left(\int_\Omega (1+|\nabla \tilde{z}_\epsilon(s)|^2)^{\frac{q-2}{2}} |\nabla \tilde{z}_\epsilon'(s)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \,\mathrm{d}s \leq C. \tag{7.19d}$$

Moreover, there holds

$$\sup_{\epsilon>0} \int_0^S \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s)), \partial\mathcal{R}_1(0))\,\mathrm{d}s \leq C. \tag{7.20}$$

Proof. Estimates (7.19a)–(7.19b) are trivial consequences of (7.1a) and (7.1b). It can be easily checked that (7.19c) ensues from (7.1b) and (7.1c) via reparameterization. Moreover, since (7.1d) essentially has a L^1-character (cf. (5.23)), it is preserved by the reparameterization in (7.19d).

Finally, as a consequence of (7.7) and the uniform bounds (7.19), we have

$$\begin{aligned} C &\geq \int_0^S \Big(\frac{\epsilon}{2\tilde{t}_\epsilon'(s)} \|\tilde{z}_\epsilon'(s)\|^2_{L^2(\Omega)} + \frac{\tilde{t}_\epsilon'(s)}{2\epsilon} \mathrm{d}_2^2(-\mathrm{D}_z\mathcal{I}(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s)), \partial\mathcal{R}_1(0)) \Big)\,\mathrm{d}s \\ &\geq \int_{\{\, s\in(0,S)\,;\, \tilde{t}_\epsilon'(s)\leq\delta \,\}} \|\tilde{z}_\epsilon'(s)\|_{L^2(\Omega)}\, \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s)), \partial\mathcal{R}_1(0))\,\mathrm{d}s \\ &\quad + \int_{\{\, s\in(0,S)\,;\, \tilde{t}_\epsilon'(s)>\delta \,\}} \frac{\delta}{2\epsilon} \mathrm{d}_2^2(-\mathrm{D}_z\mathcal{I}(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s)), \partial\mathcal{R}_1(0))\,\mathrm{d}s \end{aligned}$$

for arbitrary $\delta\in(0,1)$. Due to the normalization condition (7.6) the first term on the right-hand side satisfies $\|\tilde{z}_\epsilon'(s)\|_{L^2(\Omega)} = 1-\tilde{t}_\epsilon'(s) \geq 1-\delta$. Hence, with $\theta_\epsilon(s) := \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}_\epsilon(s), \tilde{z}_\epsilon(s)), \partial\mathcal{R}_1(0))$ we obtain

$$\begin{aligned} \int_0^S \theta_\epsilon(s)\,\mathrm{d}s &\leq \int_{\{\, s\in(0,S)\,;\, \theta_\epsilon(s)\geq 1, \tilde{t}_\epsilon'(s)\leq\delta \,\}} \theta_\epsilon(s)\,\mathrm{d}s + \int_{\{\, s\in(0,S)\,;\, \theta_\epsilon(s)\geq 1, \tilde{t}_\epsilon'(s)>\delta \,\}} \theta_\epsilon(s)^2\,\mathrm{d}s + \int_{\{\, s\,;\, \theta_\epsilon(s)<1 \,\}} 1\,\mathrm{d}s \\ &\leq C((1-\delta)^{-1} + 2\epsilon\delta^{-1}) + S, \end{aligned}$$

which is (7.20). □

Relying on the above result, we now develop the

Proof of Theorem 7.4. From the normalization condition (7.6), we deduce that there exists a parameterized curve $(\tilde{t},\tilde{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times L^2(\Omega))$ such that convergences (7.17) hold along some subsequence. Further, from estimates (7.19) it follows (possibly after extracting a further subsequence) that

$$\widetilde{z}_\epsilon \overset{*}{\rightharpoonup} \widetilde{z} \quad \text{ in } L^\infty(0,S;W^{1,q}(\Omega)), \tag{7.21}$$

$$\widetilde{z}_\epsilon(s) \to \widetilde{z}(s) \text{ uniformly in } \mathcal{X} \text{ for all } W^{1,q}(\Omega) \Subset \mathcal{X} \subset L^2(\Omega) \text{ and all } s \in [0,S], \tag{7.22}$$

$$\widetilde{z}_\epsilon \to \widetilde{z} \text{ strongly in } L^q(0,S;W^{1+\beta,q}(\Omega)) \text{ for all } \beta \in \left[0, \frac{1}{q}\big(1-\frac{d}{q}\big)\right). \tag{7.23}$$

(7.22) is a consequence of [51, Cor. 5, Sect. 8] together with (7.19a) and (7.19b), while (7.23) follows from (7.19a) and (7.19c) with [51, Cor. 9, Sect. 10]. Arguing as in the proof of Theorem 3.5 and relying on Corollary 2.15, we find that

$$\begin{aligned} \lim_{n\to\infty} \mathcal{I}(\tilde{t}_{\epsilon_n}(s),\tilde{z}_{\epsilon_n}(s)) = \mathcal{I}(\tilde{t}(s),\tilde{z}(s)), \quad \mathrm{D}_z\mathcal{I}(\tilde{t}_{\epsilon_n}(s),\tilde{z}_{\epsilon_n}(s)) \to \mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)) \text{ strongly (!) in } \mathcal{Z}^*, \\ \partial_t\mathcal{I}(\tilde{t}_{\epsilon_n}(s),\tilde{z}_{\epsilon_n}(s)) \to \partial_t\mathcal{I}(\tilde{t}(s),\tilde{z}(s)) \quad \text{in } L^1(0,S). \end{aligned} \tag{7.24}$$

for almost all $s \in (0,S)$. Now, (7.18) follows by taking the limit as $\epsilon_n \to 0$ in (7.6), with a trivial lower semicontinuity argument. We then apply Lemma 7.1. Estimate (7.20) guarantees that for a.a. $s \in (0,S)$ $\liminf_{\epsilon_n\to 0} \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}_{\epsilon_n}(s),\tilde{z}_{\epsilon_n}(s)),\partial\mathcal{R}_1(0)) < \infty$. In view of (7.24), we have that, for all $0 \le \sigma_1 \le \sigma_2 \le S$

$$\begin{aligned} \liminf_{\epsilon_n\to 0} \int_{\sigma_1}^{\sigma_2} \mathcal{M}_{\epsilon_n}(\tilde{t}'_{\epsilon_n}(s),\tilde{z}'_{\epsilon_n}(s),\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}_{\epsilon_n}(s),\tilde{z}_{\epsilon_n}(s)),\partial\mathcal{R}_1(0)))\,\mathrm{d}s \\ \ge \int_{\sigma_1}^{\sigma_2} \mathcal{M}_0(\tilde{t}'(s),\tilde{z}'(s),\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)),\partial\mathcal{R}_1(0)))\,\mathrm{d}s. \end{aligned}$$

Then, combining (7.17) and (7.24), and using that $\tilde{z}_\epsilon(0) = z_\epsilon(0) = z_0$ for all $\epsilon > 0$, we pass to the limit in (7.9) for all $\sigma_2 \in [0,S]$, for $\sigma_1 = 0$, and for almost all $0 < \sigma_1 < \sigma_2$ such that the convergences in (7.24) are valid. We thus find that the pair $(\tilde{t},\tilde{z})$ satisfies (7.11) for all $\sigma_2 \in [0,S]$, for $\sigma_1 = 0$, and for almost all $0 < \sigma_1 < \sigma_2$.

With the same Young measure argument as in the proof of Lemma 6.3, it follows that the limit function $\tilde{z}$ satisfies the *mixed estimate* (7.14). Applying the chain rule (7.3), we then conclude in the same way as at the end of the proof of Theorem 3.5 that the energy inequality in fact holds for *all* $0 \le \sigma_1 \le \sigma_2 \le S$. $\square$

Differential characterization of (non-degenerate) weak parameterized solutions Following the lines of [41, Prop. 5.3,Cor. 5.4] and of [33, Prop. 5.1], we now aim to provide a characterization of weak parameterized solutions as solutions of a suitable subdifferential inclusion. Loosely speaking, the latter should reflect two evolutionary regimes for the damage system, namely

- rate-independent evolution when $\tilde{t}' > 0$ (and $\tilde{z}' \neq 0$)
- (possibly) *viscous* evolution when $\tilde{t}' = 0$ (and $\tilde{z}' \neq 0$).

We have to interpret Proposition 7.6 below in this spirit: for $\tilde{t}' > 0$, the variational inequality (7.25) is a weak formulation of the *rate-independent* subdifferential inclusion $\partial\mathcal{R}_1(\tilde{z}'(s)) + \mathrm{D}_z\mathcal{I}(s,\tilde{z}(s)) \ni 0$ for a.a. $s \in (0,S)$. For $\tilde{t}' = 0$, $\tilde{z}' \neq 0$ follows from the non-degeneracy condition. The system may be subject to *viscous* dissipation. This *viscous* regime is seen as a jump in the (slow) external time scale, encoded by the time function $\tilde{t}$, which is frozen. Indeed, the variational inequality (7.27) is a (very) weak form of the *viscous* $\partial\mathcal{R}_1(\tilde{z}'(s)) + \lambda(s)\tilde{z}'(s) + \mathrm{D}_z\mathcal{I}(s,\tilde{z}(s)) \ni 0$ for a.a. $s \in (0,S)$ (with $\lambda : (0,S) \to [0,+\infty)$).

Proposition 7.6 (Differential characterization)**.** *Under Assumptions 2.1, 2.5, ($A_\Omega 1$), and ($A_\Omega 2$), let* $(\tilde{t}, \tilde{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times L^2(\Omega))$ *be a* non-degenerate *parameterized weak solution of* (1.3) *with* (7.14)*, then*

1. *If* $\tilde{t}'(s) > 0$*, then for every* $w \in \mathcal{Z}$

$$\begin{aligned} \mathcal{R}_1(w) - \mathcal{R}_1(\tilde{z}'(s)) \geq &-\int_\Omega (1+|\nabla \tilde{z}(s)|^2)^{\frac{q-2}{2}} \nabla \tilde{z}(s) \cdot (\nabla w - \nabla \tilde{z}'(s))\,\mathrm{d}x \\ &- \int_\Omega \mathrm{D}_z \widetilde{\mathcal{I}}(\tilde{t}(s),\tilde{z}(s))(w - \tilde{z}'(s))\,\mathrm{d}x \end{aligned} \tag{7.25}$$

2. *If* $\tilde{t}'(s) = 0$*, then*

$$\begin{aligned} &\mathcal{R}_1(\tilde{z}'(s)) + \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)), \partial\mathcal{R}_1(0))\|\tilde{z}'(s)\|_{L^2(\Omega)} \\ &\leq -\int_\Omega (1+|\nabla \tilde{z}(s)|^2)^{\frac{q-2}{2}} \nabla \tilde{z}(s) \cdot \nabla \tilde{z}'(s)\,\mathrm{d}x - \int_\Omega \mathrm{D}_z \widetilde{\mathcal{I}}(\tilde{t}(s),\tilde{z}(s))\tilde{z}'(s)\,\mathrm{d}x. \end{aligned} \tag{7.26}$$

   As a consequence, for every $w \in \mathcal{Z}$

$$\begin{aligned} &\mathcal{R}_1(w) - \mathcal{R}_1(\tilde{z}'(s)) \geq \langle \mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)) + \eta(s), w - \tilde{z}'(s)\rangle_{L^2(\Omega)} \\ &-\int_\Omega (1+|\nabla \tilde{z}(s)|^2)^{\frac{q-2}{2}} \nabla \tilde{z}(s) \cdot (\nabla w - \nabla \tilde{z}'(s))\,\mathrm{d}x - \int_\Omega \mathrm{D}_z \widetilde{\mathcal{I}}(\tilde{t}(s),\tilde{z}(s))(w - \tilde{z}'(s))\,\mathrm{d}x, \end{aligned} \tag{7.27}$$

   where $\eta(s) \in \partial\mathcal{R}_1(0)$ *is such that*

$$\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)), \partial\mathcal{R}_1(0)) = \| - \mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)) - \eta(s)\|_{L^2(\Omega)}. \tag{7.28}$$

Observe that, in view of Notation 2.11 we could replace the duality pairings on the right-hand sides of (7.25) and (7.27) by $\int_\Omega \mathrm{D}_z \widetilde{\mathcal{I}}(\tilde{t}(s),\tilde{z}(s))(w - \tilde{z}'(s))\,\mathrm{d}x$.

Proof. We differentiate (7.11) w.r.t. time and get for a.a. $s \in (0,S)$

$$\begin{aligned} &\mathcal{M}_0(\tilde{t}'(s), \tilde{z}'(s), \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)), \partial\mathcal{R}_1(0))) \leq -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\tilde{t}(s),\tilde{z}(s)) + \partial_t\mathcal{I}(\tilde{t}(s),\tilde{z}(s))\tilde{t}'(s) \\ &= -\int_\Omega (1+|\nabla \tilde{z}(s)|^2)^{\frac{q-2}{2}} \nabla \tilde{z}(s) \cdot \nabla \tilde{z}'(s)\,\mathrm{d}x - \int_\Omega \mathrm{D}_z \widetilde{\mathcal{I}}(\tilde{t}(s),\tilde{z}(s))\tilde{z}'(s)\,\mathrm{d}x, \end{aligned} \tag{7.29}$$

where the second equality follows from the parameterized chain rule (7.15). Now, according to the definition (7.10) of $\mathcal{M}_0$ we distinguish between two cases.

If $\tilde{t}'(s) > 0$, then (7.29) yields

$$\begin{aligned} &\mathcal{R}_1(\tilde{z}'(s)) + I_0(\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)), \partial\mathcal{R}_1(0))) \\ &\leq -\int_\Omega (1+|\nabla \tilde{z}(s)|^2)^{\frac{q-2}{2}} \nabla \tilde{z}(s) \cdot \nabla \tilde{z}'(s)\,\mathrm{d}x - \int_\Omega \mathrm{D}_z \widetilde{\mathcal{I}}(\tilde{t}(s),\tilde{z}(s))\tilde{z}'(s)\,\mathrm{d}x. \end{aligned} \tag{7.30}$$

Thus $\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)), \partial\mathcal{R}_1(0)) = 0$, so that $-\mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)) \in \partial\mathcal{R}_1(0) \subset \mathcal{Z}^*$ which implies that for every $w \in \mathcal{Z}$

$$\mathcal{R}_1(w) \geq \langle -\mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)), w\rangle_{\mathcal{Z}}\,. \tag{7.31}$$

Adding (7.30) and (7.31) we get (7.25).

If $\tilde{t}'(s) = 0$, then from (7.29) together with (7.10) we deduce (7.26). Let now $\eta \in \partial\mathcal{R}_1(0)$ as in (7.28). Then, for every $w \in \mathcal{Z}$ there holds $\mathcal{R}_1(w) \geq \langle \eta, w\rangle_{\mathcal{Z}}$ which, together with (7.26) (upon adding and subtracting $\langle \mathrm{D}_z\mathcal{I}(\tilde{t}(s),\tilde{z}(s)), w\rangle_{\mathcal{Z}}$ on the right-hand side) provides (7.27). $\square$

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A Young measure tools

Here, we provide a minimal aside on Young measures with values in infinite-dimensional reflexive Banach spaces (see e.g. [57]). Theorem A.2 is an extension to weak topologies of the so-called *Fundamental Theorem of Young measures.*

We denote by $\mathscr{L}_{(0,T)}$ the σ-algebra of the Lebesgue measurable subsets of the interval $(0,T)$ and, given a reflexive Banach space V, by $\mathscr{B}(V)$ its Borel σ-algebra. We use the symbol $\otimes$ for product σ-algebras. We consider the space V endowed with the *weak* topology, and say that a $\mathscr{L}_{(0,T)} \otimes \mathscr{B}(V)$–measurable functional $\mathcal{H} : (0,T) \times V \to (-\infty,+\infty]$ is a *weakly-normal integrand* if for a.a. $t \in (0,T)$ the map $w \mapsto \mathcal{H}(t,w)$ is sequentially lower semicontinuous on V w.r.t. the weak topology. We denote by $\mathscr{M}(0,T;V)$ the set of all $\mathscr{L}_{(0,T)}$-measurable functions $y : (0,T) \to V$.

Definition A.1 ((Time-dependent) Young measures). A *Young measure* in the space V is a family $\boldsymbol{\mu} := \{\mu_t\}_{t\in(0,T)}$ of Borel probability measures on V such that the map on $(0,T)$, $t \mapsto \mu_t(B)$ is $\mathscr{L}_{(0,T)}$-measurable for all $B \in \mathscr{B}(V)$. We denote by $\mathscr{Y}(0,T;V)$ the set of all Young measures in V.

We are now in the position of recalling the following compactness result, which was proved in [48, Thm. 3.2] (see also [52, Thm. 4.2]).

Theorem A.2. *Let $1 \le p \le \infty$ and let $(w_n) \subset L^p(0,T;V)$ be a bounded sequence. Then, there exists a subsequence (w_{n_k}) and a Young measure $\boldsymbol{\mu} = \{\mu_t\}_{t\in(0,T)} \in \mathscr{Y}(0,T;V)$ such that for a.a. $t \in (0,T)$*

$$\mu_t \text{ is concentrated on the set } L(t) := \bigcap_{l=1}^{\infty} \overline{\{w_{n_k}(t) \;:\; k \ge l\}}^{\text{weak}} \tag{A.1}$$

of the limit points of the sequence $(w_{n_k}(t))$ with respect to the weak topology of V and, for every weakly-normal integrand $\mathcal{H} : (0,T) \times V \to (-\infty,+\infty]$ such that the sequence $t \mapsto \mathcal{H}^-(t,w_{n_k}(t))$ is uniformly integrable ($\mathcal{H}^-$ denoting the negative part of $\mathcal{H}$), there holds

$$\liminf_{k\to\infty} \int_0^T \mathcal{H}(t,w_{n_k}(t))\,\mathrm{d}t \ge \int_0^T \int_V \mathcal{H}(t,w)\,\mathrm{d}\mu_t(w)\,\mathrm{d}t\,. \tag{A.2}$$

As a consequence, setting $\mathrm{w}(t) := \int_V w\,\mathrm{d}\mu_t(w)$ for a.a. $t \in (0,T)$, there holds $w_{n_k} \rightharpoonup \mathrm{w}$ in $L^p(0,T;V)$, with $\rightharpoonup$ replaced by $\overset{*}{\rightharpoonup}$ if $p = \infty$.

B An abstract discrete estimate

Lemma B.1. *Let $\{a_k\}_{k=0}^N$, $\{M_k\}_{k=1}^N$, $\{r_k\}_{k=1}^N$, μ and c be non-negative numbers, $\epsilon, \tau > 0$ with $\gamma := \tau/(2\epsilon) \leq 1$ and $N \in \mathbb{N}$, $N\tau = T$. Assume that for $1 \leq k \leq N$ it holds*

$$2a_k(a_k - a_{k-1}) + 2\gamma a_k^2 + 2\gamma M_k^2 \leq c^2\gamma(1 + \tfrac{\delta_{1,k}}{\tau\epsilon}\mu^2) + 4\gamma a_k r_k, \tag{B.1}$$

where $\delta_{1,k}$ is the Kronecker symbol. Then there exists a constant $C = C(c, N\tau) > 0$ not depending on any of the other above quantities such that

$$\sum_{k=1}^N \tau M_k \leq C\left(T + \epsilon a_0 + \mu + \sum_{k=1}^N \tau r_k\right). \tag{B.2}$$

The proof of this lemma is a discrete version of the calculations in Section 3.4 in the preprint version of [45] and a slightly modified version of [33, (4.36)–(4.47)] with the additional term involving μ and starting with the summation from $k = 1$ instead of $k = 2$ in [33]. We give here a short sketch, often referring to [33] and not elaborating all intermediate steps.

Proof. With $b_i = \sqrt{2\gamma}M_i$, $c_i^2 = c^2\gamma(1 + \frac{\delta_{i,1}}{\tau\epsilon}\mu^2)$, $d_i = 2\gamma r_i$, (B.1) can be rewritten as

$$2a_i(a_i - a_{i-1}) + b_i^2 + 2\gamma a_i^2 \leq c_i^2 + 2a_i d_i$$

for $1 \leq i \leq N$. Hence, by the discrete Gronwall estimate [33, Lemma 4.1, (4.51)] we find for all $n \geq 1$:

$$\left(\sum_{k=1}^n (1+\gamma)^{2(k-n)-1} b_k^2\right)^{1/2} \leq \left((1+\gamma)^{-2n}a_0^2 + \sum_{k=1}^n (1+\gamma)^{2(k-n)-1}c_k^2\right)^{1/2} + \sqrt{8}\sum_{k=1}^n (1+\gamma)^{k-n-1}\gamma r_k. \tag{B.3}$$

As in [33, (4.38)] one estimates

$$\sum_{k=1}^n (1+\gamma)^{2(k-n)-1}c_k^2 \leq c^2(1 + \frac{\gamma}{\epsilon\tau}(1+\gamma)^{1-2n}\mu^2). \tag{B.4}$$

Observe that $\gamma(1+\gamma)/(\epsilon\tau) \leq \epsilon^{-2}$. With the Hölder inequality and (B.3) one obtains

$$\begin{aligned}\sum_{k=1}^n (1+\gamma)^{2(k-n)-1}\gamma M_k &\leq \left(\sum_{k=1}^n (1+\gamma)^{2(k-n)-1}\gamma\right)^{1/2}\left(\sum_{k=1}^n (1+\gamma)^{2(k-n)-1}\gamma M_k^2\right)^{1/2} \\ &\leq C\left(1 + (1+\gamma)^{-n}(a_0 + \epsilon^{-1}\mu) + \sum_{k=1}^n \gamma(1+\gamma)^{k-n-1}r_k\right),\end{aligned} \tag{B.5}$$

which corresponds to [33, (4.41)]. Multiplication with τ and summation with respect to $n \in \{1, \dots, N\}$ yields

$$\sum_{n=1}^N \tau \sum_{k=1}^n \gamma(1+\gamma)^{2(k-n)-1}M_k \leq C\sum_{n=1}^N \tau\left(1 + (1+\gamma)^{-n}(a_0 + \epsilon^{-1}\mu) + \sum_{k=1}^n \gamma(1+\gamma)^{k-n-1}r_k\right). \tag{B.6}$$

We now change the order of summation and obtain as in [33, (4.43)]

$$\sum_{n=1}^N \tau \sum_{k=1}^n \gamma(1+\gamma)^{2(k-n)-1}M_k = \sum_{k=1}^N \tau M_k \frac{1+\gamma}{2+\gamma}(1+\gamma)(1 - (1+\gamma)^{-2(N-k+1)}). \tag{B.7}$$

With similar calculations for the term with r_k on the right-hand side of (B.6) and taking into account that $\sum_{n=1}^{N} \tau(1+\gamma)^{-n} \le 2\epsilon$ we finally deduce from (B.6)

$$\Big(\sum_{k=1}^{N} \tau M_k\Big) - \sum_{k=1}^{N} \tau M_k (1+\gamma)^{2(k-N-1)} \le C\Big(N\tau + \epsilon a_0 + \mu + \sum_{k=1}^{N} \tau r_k\Big).$$

Estimating the second term on the left hand side with the aid of (B.5), we finally arrive at (B.2). $\square$

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