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Abstract

We give a necessary and sufficient smoothness condition for the scheme parameterizing the $n-$dimensional representations of a finitely generated associative algebra over an algebraically closed field of characteristic zero. In particular, our result implies that the points $M \in \mathrm{Rep}_A^n(k)$ satisfying $\mathrm{Ext}_A^2(M, M) = 0$ are regular. This generalizes well-known results on finite-dimensional algebras to finitely generated algebras.

1 Introduction

Let A be a finitely generated associative $k-$algebra with k an algebraically closed field. Let $V_n(A)$ be the commutative $k-$algebra representing the functor from commutative algebras to sets

$$\mathcal{C}_k \to \mathsf{Set} : B \mapsto \mathrm{Hom}_{\mathcal{N}_k}(A, M_n(B))$$

of the $n-$dimensional representations of A over B, (see Section 2.2). The scheme Rep_A^n of the linear representations of dimension n of A is defined to be $\mathrm{Spec}\, V_n(A)$.

In this paper we study the smoothness of the scheme Rep_A^n.

It is well-known that if A is formally smooth then Rep_A^n is smooth (see [11, Proposition 19.1.4.] and [16, Proposition 6.3.]). If A is finite-dimensional then it

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is formally smooth if and only if it is hereditary (see Theorem 4.6) and, therefore Rep_A^n is smooth for all n if and only if A is hereditary (see [2, Proposition 1]).

For infinite-dimensional algebras the picture is more complex, e.g. there are hereditary algebras which are not formally smooth (see Theorem 4.7). It is therefore interesting to find other sufficient (or necessary) conditions on A which ensure Rep_A^n to be smooth.

Let M be an $A-$module in $\mathrm{Rep}_A^n(k)$. It is well-known that the linear space $\mathrm{Ext}^2_A(M,M)$ contains the obstructions in extending the infinitesimal deformations of M to the formal ones. For this reason an algebra A such that $\mathrm{Ext}^2_A(M,M)=0$, for all $M\in\mathrm{Rep}_A^n(k)$ and $n\geq 1$, will be called *finitely unobstructed*.

It has been proved by Geiss and de la Peña (see [8, 9]) that, when A is finite-dimensional, finitely unobstructed implies that Rep_A^n is smooth.

We underline that any hereditary algebra is finitely unobstructed but the converse is not true, e.g. the universal enveloping algebra of a finite-dimensional semisimple Lie algebra is finitely unobstructed but not hereditary if the dimension of the underlying Lie algebra is greater than one.

The proof given in [8, 9] is based on the analysis of the local geometry of Rep_A^n, and it specifically relies on the upper semicontinuity of certain dimension functions arising from the bar resolution of A. As we observe in the last section of this paper, their approach remains valid if one assumes that A is finitely presented or bimodule coherent.

We follow here a different path, namely, we study the smoothness problem via the adjunction

$$\mathrm{Hom}_{\mathcal{C}_k}(V_n(A),B)\xrightarrow{\cong}\mathrm{Hom}_{\mathcal{N}_k}(A,M_n(B)) \tag{1.1}$$

The adjunction (1.1) allows us to use the Harrison cohomology of $V_n(A)$ instead of the Hochschild cohomology of A. The Harrison cohomology of a commutative $k-$algebra is the symmetric part of its Hochschild cohomology, and it has been proved by Harrison [14] that an affine ring R is regular if and only if its second Harrison cohomology vanishes.

This is our main result.

Theorem. *Let A be a finitely generated $k-$algebra, let $f:V_n(A)\to k$ be a $k-$algebra map and let $\rho:A\to M_n(k)$ be the algebra map that corresponds to f through the adjunction above. Then there is a linear embedding of* $\mathrm{Harr}^2(V_n(A),{}_fk)$ *into* $H^2(A,{}_\rho M_n(k)_\rho)$. *As a consequence,* $M\in\mathrm{Rep}_A^n$ *is a regular point whenever* $\mathrm{Ext}^2_A(M,M)=0$.

We have thus extended the known results on smoothness to infinite-dimensional finitely generated algebras.

We remark that the above embedding is not an isomorphism in general. We give a counterexample by using 2-Calabi Yau algebras (Remark 3.5).

The paper goes as follows.

In paragraph 2.2 we recall the definition of Rep_A^n as the scheme parameterizing the $n-$dimensional representations of A.

In Section 3 we recall the Harrison cohomology which may be seen as the commutative version of the Hochschild cohomology. We prove that the regularity of a point in $\mathrm{Rep}_A^n = \mathrm{Spec}(V_n(A))$ is equivalent to the vanishing of $\mathrm{Harr}^2(V_n(A), {}_f k)$, for the $k-$algebra map $f : V_n(A) \to k$ associated to the point (see Theorem 3.3). Then Theorem 3.4 shows that there is a linear embedding of $\mathrm{Harr}^2(V_n(A), {}_f k)$ into $H^2(A, {}_\rho M_n(k)_\rho)$ and as a consequence, that $M \in \mathrm{Rep}_A^n$ is a regular point whenever $\mathrm{Ext}_A^2(M, M) = 0$.

Then, as said before, by using 2-Calabi Yau algebras, we exhibit an example which shows that the above embedding is not an isomorphism.

In Section 4 we present a list of examples and applications of the aforementioned results. To this aim, we first recall the notions of formally smooth and hereditary algebra. We mention the known result on the smoothness of Rep_A^n when A is formally smooth or hereditary to compare the notions of formally smoothness, hereditary, finitely unobstructed and we stress the difference between the finite and the infinite-dimensional case.

Afterward, we give the definition of finitely unobstructed algebra and we prove that if A is finitely unobstructed then Rep_A^n is smooth (see Corollary 4.2).

Then we produce examples of finitely unobstructed algebras (neither hereditary nor formally smooth) whose associate representation scheme is smooth (see Example 4.12).

In Section 5 we study the relationships between the deformation theory of $M \in \mathrm{Rep}_A^n(k)$, in the sense of Gerstenhaber, Geiss and de la Peña , and the deformation theory of $V_n(A)$ as usually defined in algebraic geometry.

In particular, by using the adjunction (1.1), we will see that there are no obstructions to the integrability of the infinitesimal deformations of M if and only if $\mathrm{Harr}^2(V_n(A), {}_f k) = 0$. Motivated by this fact we formulate the following conjecture.

Conjecture 1.1. *The image of the embedding* $\mathrm{Harr}^2(V_n(A), {}_f k) \hookrightarrow \mathrm{Ext}^2(M, M)$ *contains the subspace of* $\mathrm{Ext}^2(M, M)$ *of the obstructions to integrate the infinitesimal deformations of* M.

As a bonus, we further show that the approach to this smoothness problem developed for A finite-dimensional in [8, 9]) works as well if A is finitely unobstructed and finitely presented or bimodule coherent.

2 Preliminaries

2.1 Notations

Unless otherwise stated we adopt the following notations:

- k is an algebraically closed field;
- $F = k\{x_1, \ldots, x_m\}$ is the associative free $k-$algebra on m letters;

- $A \cong F/J$ is a finitely generated associative $k-$algebra;
- $\mathcal{N}_-$, $\mathcal{C}_-$ and Set denote the categories of $-$algebras, commutative $-$algebras and sets, respectively;
- The term "$A-$module" indicates a left $A-$module. The categories of left $-$modules is denoted by -Mod. The full subcategory of modules having finite dimension over k will be denoted by -Mod_f;
- We write $\mathrm{Hom}_{\mathcal{A}}(B, C)$ for the morphisms from an object B to C in a category $\mathcal{A}$. If $\mathcal{A} = A$-Mod, then we will write $\mathrm{Hom}_A(-,-)$;
- A^{op} is the opposite algebra of A and $A^e := A \otimes A^{\mathrm{op}}$ is the envelope of A. It is an $A-$bimodule and a $k-$algebra. One can identify the category of the $A-$bimodules with A^e-Mod and we will do it thoroughly this paper;
- $\mathrm{Ext}^i_-(,)$ denotes the Ext groups on the category -Mod;
- $\mathrm{H}^i(A, -)$ is the Hochschild cohomology with coefficients in A^e-Mod.

2.2 The scheme of $n-$dimensional representations

Denote by $M_n(B)$ the full ring of $n \times n$ matrices over B, with B a ring. If $f : B \to C$ is a ring homomorphism we denote by $M_n(f) : M_n(B) \to M_n(C)$ the homomorphism induced on matrices.

Definition 2.1. Let $A \in \mathcal{N}_k$, $B \in \mathcal{C}_k$. By an *n-dimensional representation of* A over B we mean a homomorphism of $k-$algebras $\rho : A \to M_n(B)$.

It is clear that this is equivalent to give an $A-$module structure on B^n. The assignment $B \mapsto \mathrm{Hom}_{\mathcal{C}_k}(A, M_n(B))$ defines a covariant functor

$$\mathcal{C}_k \longrightarrow \mathsf{Set}.$$

which is represented by a commutative $k-$algebra $V_n(A)$.

Lemma 2.2. *[19, Ch.4, §1] For all $A \in \mathcal{N}_k$ and $\rho : A \to M_n(B)$ a linear representation, there exist $V_n(A) \in \mathcal{C}_k$ and a representation $\eta_A : A \to M_n(V_n(A))$ such that $\rho \mapsto M_n(\rho) \circ \eta_A$ gives an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}_k}(V_n(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{N}_k}(A, M_n(B)) \tag{2.1}$$

for all $B \in \mathcal{C}_k$.

If $A = F$, one has that $V_n(F) := k[\xi_{lij}]$, the polynomial ring in variables $\{\xi_{lij} : i, j = 1, \dots, n, l = 1, \dots, m\}$ over k. If $A = F/J$ finitely generated $k-$algebra, one defines $V_n(A) := k[\xi_{lij}]/I$ where I is the ideal of $V_n(F)$ generated by the $n \times n$ entries of $f(\xi_1, ..., \xi_m)$, f runs over the elements of J and ξ_l is the matrix (ξ_{lij}). Therefore $V_n(A)$ is an affine ring (i.e. a finitely generated algebra with identity) when A is a finitely generated $k-$algebra.

Definition 2.3. We write Rep_A^n to denote $\mathrm{Spec}\, V_n(A)$. It is considered as a $k-$scheme. The map

$$\eta_A : A \to M_n(V_n(A)), \quad a_l \longmapsto \xi_l^A := (\xi_{lij} + I).$$

is called *the universal n-dimensional representation.*

Examples 2.4. (i) By construction, if $A = F$, then $\mathrm{Rep}_F^n(k) = M_n(k)^m$. If $A = F/J$, the $B-$points of Rep_A^n can be described as follows:

$$\mathrm{Rep}_A^n(B) = \{(X_1, \ldots, X_m) \in M_n(B)^m \;:\; f(X_1, \ldots, X_m) = 0 \text{ for all } f \in J\};$$

(ii) If $A = \mathbb{C}[x, y]$, $\mathrm{Rep}_A^n(\mathbb{C}) = \{(M_1, M_2) \in M_2(\mathbb{C})^2, M_1M_2 = M_2M_1\}$ is the *commuting scheme*, see [21].

Remark 2.5. Note that Rep_A^n may be quite complicated. It is not reduced in general and it seems to be hopeless to describe the coordinate ring of its reduced structure. The scheme Rep_A^n is also known as *the scheme of n-dimensional A-modules.*

3 The main result

We proof our main result using Harrison cohomology. Given a commutative ring R and an $R-$module N, we denote by $\mathrm{Harr}^*(R, N)$ the Harrison cohomology group i.e. the group $\mathcal{E}^*(R, N)$ introduced in [14]. Harrison cohomology can be seen as a commutative version of Hochschild cohomology. We here just recall some basic facts. For further details the reader is referred to [23, section 9.3], where the second commutative Hochschild cohomology group is denoted by $\mathrm{H}_s^2(R, N)$ (the subscript "s" stands for "symmetric").

Consider an arbitrary $B \in \mathcal{N}_k$ and a $B-$bimodule M. Then $\mathrm{H}^2(B, M)$ is the quotient of the group of $2-$cocycles by the group of $2-$coboundaries. Denote by $[\omega]$ the class for a $2-$cocycle ω. A Hochschild extension of B by M is a $k-$algebra E together with an algebra map $p : E \to B$ such that $\ker(p)$ is an ideal of square-zero and an $R-$bimodule isomorphism of M with $\ker(p)$. The latter condition makes sense as, by the square-zero assumption, $\ker(p)$ becomes a $B-$bimodule where, for every $b \in B$ and $x \in \ker(p)$, the left action is defined by $b \cdot x := ex$, where e is any element in $p^{-1}(b)$, and similarly on the right. Note also that, since k is a field (not just a commutative ring as in [23, Chapter 9]) then the exact sequence

$$0 \longrightarrow M \xrightarrow{\;i\;} E \xrightarrow{\;p\;} B \longrightarrow 0 \tag{3.1}$$

is $k-$split i.e. there is a $k-$linear map $\sigma : B \to E$ such that $p \circ \sigma = \mathrm{Id}_B$. The isomorphism between M and $\ker(p)$ rereads as $i(mb) = i(m)\sigma(b)$ and $i(bm) = \sigma(b)i(m)$. If there is another Hochschild extension

$$0 \longrightarrow M \xrightarrow{\;i'\;} E' \xrightarrow{\;p'\;} B \longrightarrow 0 \tag{3.2}$$

we say that the extensions (3.1) and (3.2) are equivalent if there is a $k-$algebra map $f : E \to E'$ such that $f \circ i = i'$ and $p' \circ f = p$ (note that such a map is necessarily bijective by the short 5-Lemma). A Hochschild extension as in (3.1) is called trivial whenever there is σ as above which is an algebra map.

Given a $2-$cocycle $\omega : B \otimes B \to M$, we can define the $k-$algebra $B \oplus_\omega M$ to be the $k-$vector space $B \oplus M$ endowed with multiplication and unit

$$(b, m) \cdot (b', m') := (bb', mb' + bm' - \omega(b, b')), \quad 1_{B \oplus_\omega M} = (1_B, \omega(1_B, 1_B)).$$

Then $B \oplus_\omega M$ is a Hochschild extension of B by M with respect to the canonical projection $p : B \oplus_\omega M \to B$. Conversely, for any Hochschild extension of B by M as in (3.1), one sees that the map $\theta : B \otimes B \to E : a \otimes b \mapsto \sigma(ab) - \sigma(a)\sigma(b)$ is in the kernel of p so that there is a unique map $\omega : B \otimes B \to M$ such that $i \circ \omega = \theta$ and ω comes out to be a $2-$cocycle. The argument above yields a bijective correspondence between the elements of $\mathrm{H}^2(B, M)$ and the set of equivalence classes of Hochschild extension of B by M (see e.g. [23, Theorem 9.3.1]). As a consequence a Hochschild extension is trivial if and only if the corresponding $2-$cocycle is a $2-$coboundary.

Consider now $B \in \mathcal{C}_k$ and a left $B-$module M. Regard M as a symmetric $B-$bimodule (i.e. $mb = bm$ for every $b \in B$ and $m \in M$). The construction above adapts to the commutative case minding that the commutative Hochschild extension has E commutative and a commutative $2-$cocycles is further required to be symmetric i.e. $\omega(b \otimes b') = \omega(b' \otimes b)$ for all $b, b' \in B$. Then one defines

$$\mathrm{Harr}^2(B, M) := \{[\omega] \in \mathrm{H}^2(B, M) \mid \omega \text{ is symmetric}\}.$$

We still have a bijection between the elements of $\mathrm{Harr}^2(B, M)$ and the set of equivalence classes of commutative Hochschild extensions of B by M (see e.g. [23, Theorem 9.3.1.1]). As a consequence a commutative Hochschild extension is trivial if and only if the corresponding commutative $2-$cocycle is a $2-$coboundary.

The following standard result establishes a link between $\mathrm{Ext}^i_A(M, M)$ and the Hochschild cohomology of A with coefficients in $\mathrm{End}_k(M)$.

Theorem 3.1. *[3, Corollary 4.4.] We have*

$$\mathrm{Ext}^i_A(M, M) \cong \mathrm{H}^i(A, \mathrm{End}_k(M)).$$

For every algebra map $f : B \to A$ and $N \in A\text{-}\mathsf{Mod}$, denote by ${}_fN$ the corresponding left $B-$module structure on N. A similar notation is used on the right. In particular, if $N \in A^e\text{-}\mathsf{Mod}$, the notation ${}_fN_f$ means that N is regarded as a $B-$module via f.

Proposition 3.2. *The following assertions are equivalent for $A \in \mathcal{N}_k$ and for every $M \in A^e\text{-}\mathsf{Mod}$:*

1. $\mathrm{H}^2(A, M) = 0$;

2. *Let $f : A \to B$ be an algebra map and let $p : E \to B$ be a Hochschild extension of $B \in \mathcal{N}_k$ with kernel N such that ${}_fN_f = M$ (here N is endowed with the canonical $B-$bimodule structure described above). Then f has a lifting i.e. there is an algebra map $\overline{f} : A \to E$ such that $p \circ \overline{f} = f$.*

$$\begin{tikzcd}
 & & & A \arrow[d, "f"] \arrow[dl, dotted, "\overline{f}"'] & \\
0 \arrow[r] & N \arrow[r, "i"] & E \arrow[r, shift left, "p"] & B \arrow[l, dotted, shift left, "\sigma"] \arrow[r] & 0
\end{tikzcd}$$

Proof. The proof is the same of [23, Proposition 9.3.3] for our specific M. However, we recall a different proof of (1) implies (2) that will be needed in the proof of Theorem 3.4. Let $\omega : B \otimes B \to N$ be the Hochschild 2−cocycle associated to the Hochschild extension E of B by M. Then $\overline{\omega} := \omega \circ (f \otimes f) : A \otimes A \to {}_fN_f = M$ is a Hochschild 2−cocycle so that we can consider the Hochschild extension $A \oplus_{\overline{\omega}} M$ of A by M. Since $\overline{\omega}$ is, by assumption, a 2−coboundary, then the latter extension is trivial i.e. there is an algebra map $s : A \to A \oplus_{\overline{\omega}} M$ which is a right inverse of the canonical projection. Composing s with the algebra map $A \oplus_{\overline{\omega}} M \to E : (a, m) \mapsto \sigma f(a) + i(m)$ yields the required map $\overline{f}$. □

Let R be a commutative noetherian ring. Recall that a point $p \in \operatorname{Spec} R$ is *regular* if the localization R_p of R at p is a regular local ring i.e. $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim R_p$, where $\mathfrak{m}$ is the unique maximal ideal of R_p and $\dim R_p$ is its Krull dimension. The ring R is said to be *regular* if the localization at every prime ideal is a regular local ring.
The following result is a variant of [14, Corollary 20].

Theorem 3.3. *Let $f : V_n(A) \to k$ be a $k-$point of Rep_A^n. Then f is a regular point of $\mathrm{Rep}_A^n(k)$ if and only if $\mathrm{Harr}^2(V_n(A), {}_fk) = 0$.*

Proof. Set $\mathfrak{m} := \ker(f)$ and $R := V_n(A)$. Note that k is a perfect field, as it is algebraically closed. Moreover, since A is f.g., then R is an affine ring (as observed after Lemma 2.2) and hence we can apply [14, Corollary 20] to get that f is regular if and only if $\mathrm{Harr}^2(R, R/\mathfrak{m}) = 0$. We conclude by observing that ${}_fk = \mathrm{im} f \cong R/\ker f = R/\mathfrak{m}$ as left $R-$modules. □

Theorem 3.4. *Let A be a f.g. $k-$algebra, let $f : V_n(A) \to k$ be a $k-$algebra map and let $\rho : A \to M_n(k)$ be the algebra map that corresponds to f through (2.1). Then there is a linear embedding of $\mathrm{Harr}^2(V_n(A), {}_fk)$ into $H^2(A, {}_\rho M_n(k)_\rho)$. As a consequence, $M \in \mathrm{Rep}_A^n$ is a regular point whenever $\mathrm{Ext}_A^2(M, M) = 0$.*

Proof. Each $M \in \mathrm{Rep}_A^n$ is of the form $M \cong {}_\rho(k^n)$ for some $\rho : A \to M_n(k)$ as in the statement. By Theorem 3.1, we have $\mathrm{Ext}_A^2(M, M) \cong H^2(A, \mathrm{End}_k(M)) \cong H^2(A, {}_\rho M_n(k)_\rho)$. Thus the last assertion of the statement follows by Theorem 3.3 once proved the embedding of $\mathrm{Harr}^2(V_n(A), {}_fk)$ into $H^2(A, {}_\rho M_n(k)_\rho)$. Let us construct it explicitly. The idea of the proof of this fact is inspired by [11, Proposition 19.1.4] where the functor $M_n(-)$ is applied to a commutative extension with nilpotent kernel. Set $B := V_n(A)$ and let $\omega : B \otimes B \to {}_f k$ be a Harrison $2-$cocycle. Consider the Hochschild extension associated to ω

$$0 \longrightarrow {}_f k \xrightarrow{i} B_\omega \underset{\sigma}{\overset{p}{\rightleftarrows}} B \longrightarrow 0 \tag{3.3}$$

where, for brevity, we set $B_\omega := B \oplus_\omega k$. Set $S := M_n(k)$ and apply the exact functor $S \otimes (-)$ to (3.3) to obtain the Hochschild extension

$$0 \longrightarrow S \otimes {}_f k \xrightarrow{S\otimes i} S \otimes B_\omega \underset{S\otimes\sigma}{\overset{S\otimes p}{\rightleftarrows}} S \otimes B \longrightarrow 0 \tag{3.4}$$

Here $S \otimes {}_f k$ is a bimodule over $S \otimes B$ via $(s \otimes b)(s' \otimes l)(s'' \otimes b'') = ss's'' \otimes blb'' = ss's'' \otimes f(b)lf(b'')$, for every $s, s', s'' \in S, l \in k, b, b'' \in B$. Now, let E be either ${}_f k$, B_ω or B and apply the canonical isomorphism $S \otimes E \to M_n(E) : (k_{ij}) \otimes e \mapsto (k_{ij}e)$ to (3.4) to obtain the Hochschild extension

$$0 \longrightarrow N \xrightarrow{i_n} M_n(B_\omega) \underset{\sigma_n}{\overset{p_n}{\rightleftarrows}} M_n(B) \longrightarrow 0 \tag{3.5}$$

where we set $p_n := M_n(p), \sigma_n = M_n(\sigma)$, $i_n := M_n(i)$ and N is $M_n(k)$ regarded as a bimodule over $M_n(B)$ via $(b_{is})(l_{tj}) = (\sum_s b_{is}l_{sj})$ and $(l_{tj})(b_{is}) = (\sum_j l_{tj}b_{js})$ for every $(b_{is}) \in M_n(B)$ and $(l_{tj}) \in M_n(k)$. Thus

$$(b_{is})(l_{tj}) = (\sum_s b_{is}l_{sj}) = (\sum_s f(b_{is})l_{sj}) = f_n((b_{is})) \cdot (l_{tj})$$

where $f_n := M_n(f)$ i.e. $N = {}_{f_n}(M_n(k))$. Let $\eta = \eta_A : A \to M_n(V_n(A)) = M_n(B)$ be the universal $n-$dimensional representation of Definition 2.3. Hence ${}_\eta N = {}_\eta({}_{f_n}(M_n(k))) = {}_{(f_n\circ\eta)}M_n(k) = {}_\rho M_n(k)$, where we used that $f_n \circ \eta = M_n(f) \circ \eta = \rho$ which holds by definition of ρ. A similar argument applies to the right so that we get ${}_\eta N_\eta = {}_\rho M_n(k)_\rho$. Let $\omega_n : M_n(B) \otimes M_n(B) \to N$ be the Hochschild $2-$cocycle associated to the Hochschild extension (3.5). Then $\overline{\omega_n} := \omega_n \circ (\eta \otimes \eta)$ is a Hochschild $2-$cocycle so that we can consider the assignment

$$\alpha : \mathrm{Harr}^2(V_n(A), {}_f k) \to H^2(A, {}_\eta N_\eta) : [\omega] \mapsto [\overline{\omega_n}].$$

This is a well-defined map. In fact, if $[\omega] = 0$, then we can choose σ to be an algebra map from the very beginning and hence σ_n is an algebra map so that $\omega_n = 0$. Suppose $\alpha([\omega]) = 0$. Then $\overline{\omega_n}$ is a $2-$coboundary. This condition guarantees, by the proof of Proposition 3.2, that there is a $k-$algebra map $\overline{\lambda} : A \to M_n(B_\omega)$ such that $p_n \circ \overline{\lambda} = \eta_A$. This map corresponds, via (2.1), to an algebra map $\lambda : B \to B_\omega$ such that $p \circ \lambda = \mathrm{Id}_B$. This means that the Hochschild extension (3.3) is trivial whence $[\omega] = 0$. Thus α is injective. □

Remark 3.5. The map $\mathrm{Harr}^2(V_n(A), {}_f k) \hookrightarrow H^2(A, {}_\rho M_n(k)_\rho)$ is not an isomorphism in general. Furthermore the condition $\mathrm{Ext}^2_A(M, M) = 0$ is not necessarily

satisfied by regular points in Rep_A^n. There is indeed the following counterexample.
Let A be a $2-$Calabi Yau algebra, see [12, Definition 3.2.3] for details. It has been proven by Bocklandt that such an algebra has simple modules and that these modules are regular points in Rep_A^n (see [1, Section 7.1]). Therefore, for a simple $M \in \mathrm{Rep}_A^n(k)$ one has $\mathrm{Harr}^2(V_n(A), {}_f k) = 0$. On the other hand, since A is $2-$Calabi Yau, one has $\mathrm{Ext}_A^2(M, M) \cong \mathrm{Ext}_A^0(M, M) \cong \mathrm{End}_A(M) \cong k$, for all $M \in \mathrm{Rep}_A^n$.

4 Examples and Applications

Next aim is to introduce and investigate the notion of finitely unobstructed algebra. We will give several examples of such algebras. Moreover we will analyze the relationship between finitely unobstructed, formally smooth and hereditary algebras to better understand the influence of the structure of A on the smoothness of Rep_A^n.

4.1 Finitely unobstructed algebras

Definition 4.1. Let A be a $k-$algebra. Given $n \in \mathbb{N}$, we say that A is *$n-$finitely unobstructed*, if $\mathrm{Ext}_A^2(M, M) = 0$, for every $M \in \mathrm{Rep}_A^n(k)$. We say that A is *finitely unobstructed*, if it is *$n-$finitely unobstructed* for every $n \in \mathbb{N}$.

Corollary 4.2. *The scheme* Rep_A^n *is smooth for all $n-$finitely unobstructed $k-$algebra.*

Proof. It follows by Theorem 3.4. □

We recast here some basic concepts in order to list examples and applications of the results proven in Section 3.

4.2 Hereditary algebras

Recall that the *projective dimension* $\mathrm{pd}(M)$ of an $M \in A\text{-}\mathsf{Mod}$ is the minimum length of a projective resolution of M.

Definition 4.3. The *global dimension* of a ring A, denoted with $gd(A)$, is the supremum of the set of projective dimensions of all (left) $A-$modules. If $gd(A) \leq 1$, then A is called *hereditary*.

It holds that $gd(A) \leq d$ if and only if $\mathrm{Ext}_A^{d+1}(M, N) = 0$, for all $M, N \in A-$mod, see [3, Prop.2.1.]

4.3 Formally smooth algebras

It is the last concept we need to introduce. Formally smooth (or quasi-free) algebras provide a generalization of the notion of free algebra, since they behave like a free algebra with respect to nilpotent extensions. The definition goes back to J. Cuntz and D. Quillen and it was inspired by the Grothendieck's definition of formal smoothness given in the commutative setting, see [13, 19.3.1]. See also [11, 19] and [16, 4.1.]. For further readings on these topics see [11, 12].

Definition 4.4. (Definition 3.3. [6]). An $A \in \mathcal{N}_k$ is said to be *formally smooth* (or *quasi-free*), if it satisfies the equivalent conditions:
i) any homomorphism $\varphi \in \mathrm{Hom}_{\mathcal{N}_k}(A, R/N)$ where N is a nilpotent (two-sided) ideal in an algebra $R \in \mathcal{N}_k$, can be lifted to a homomorphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{N}_k}(A, R)$ that commutes with the projection $R \to R/N$;
ii) $\mathrm{H}^2(A, M) = 0$ for any $M \in A^e$-Mod;
iii) the kernel Ω^1_A of the multiplication $A \otimes A \to A$ is a projective A^e-module.

Remark 4.5. When A commutative Ω^1_A is nothing but the module of the Kähler differentials (see [11, Section 8]).

If we substitute $A \in \mathcal{C}_k$ and $\mathrm{Hom}_{\mathcal{C}_k}(A, -)$ in Definition 4.4, we obtain the classical definition of regularity in the commutative case (see [16, Proposition 4.1.]). On the other hand, if we ask for a commutative algebra A to be formally smooth in the category $\mathcal{N}_k$ we obtain regular algebras of dimension ≤ 1 only ([6, Proposition 5.1.]). Thus, If $X = \mathrm{Spec}A$ is an affine smooth scheme, then A is not formally smooth unless $\dim X \leq 1$.

4.4 Implications and equivalences

We have the following.

Theorem 4.6. *Let A be a finite-dimensional algebra over k. The following assertions are equivalent:*

(1) A is formally smooth;

(2) $\mathrm{H}^2(A, N) = 0$ for every $N \in A^e$-Mod_f;

(3) A is finitely unobstructed;

(4) Rep^n_A is smooth for every $n \in \mathbb{N}$;

(5) A is hereditary.

Proof. (1) $\Rightarrow$ (2) is trivial. Implication (2) $\Rightarrow$ (3) follows by Theorem 3.1, while (3) $\Rightarrow$ (4) follows by Corollary 4.2. Finally (4) $\Rightarrow$ (5) is [2, Proposition 1] and (5) $\Rightarrow$ (1) follows from [4, Proposition 0.6.]. □

More generally we have the following result.

Theorem 4.7. *Let A be an infinite-dimensional finitely generated algebra over k. Then there is a chain of implications*

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \qquad \textit{but} \qquad (1) \not\Leftarrow (2) \not\Leftarrow (3) \not\Leftarrow (4)$$

where:

(1) A is formally smooth;

(2) A is hereditary;

(3) A is finitely unobstructed;

(4) Rep_A^n is smooth for every $n \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2). This is [6, Proposition 5.1].

(1) $\not\Leftarrow$ (2). Consider $A^1 = \mathbb{C}[x,\delta]/ < x\delta - \delta x = 1 >$ the first Weyl algebra. It is an example of a hereditary but not formally smooth algebra, since it can be proved that $\mathrm{H}^2(A, A^e) \neq 0$ (see [22, Proposition 3]). This is due to William Crawley-Boevey (personal communication).

(2) $\Rightarrow$ (3). This is trivial.

(2) $\not\Leftarrow$ (3). Let $U := U(\mathfrak{g})$ be the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$. *Whitehead's second lemma* (see e.g. [23, Corollary 7.8.12, page 246]) asserts that, in characteristic zero, $\mathrm{H}^2_{Lie}(\mathfrak{g}, N) = 0$ for every $\mathfrak{g}-$module N of finite-dimension over k. In particular, for every $M \in A\text{-}\mathsf{Mod}_f$, we obtain $\mathrm{H}^2_{Lie}(\mathfrak{g}, \mathrm{End}_k(M)) = 0$. By [23, Exercise 7.3.5, page 226], we have that

$$\mathrm{H}^*_{Lie}(\mathfrak{g}, \mathrm{End}_k(M)) \cong \mathrm{Ext}^*_U(M, M).$$

Therefore, $\mathrm{Ext}^2_U(M, M) = 0$. If $\mathfrak{g}$ is finite-dimensional, then U is finitely generated and, thus, U is finitely unobstructed. Now, U has global dimension $\dim_k(\mathfrak{g})$, see [23, Exercise 7.7.2, page 241], and we are done.

(3) $\Rightarrow$ (4). This is Corollary 4.2.

(3) $\not\Leftarrow$ (4). Remark 3.5 shows that there might exist regular points M in Rep_A^n with $\mathrm{Ext}^2_A(M, M) \neq 0$. $\square$

Remark 4.8. The implication (1) $\Rightarrow$ (4) was already known, see [11, Proposition 19.1.4.], [16, Prop.6.3.].

Theorem 4.7 and the argument on $U = U(\mathfrak{g})$, contained in the proof thereby, together imply smoothness of Rep_U^n. This result was known, see e.g. the comment by Le Bruyn in [17].

4.5 Unobstructed Algebras

We now list some examples and results in case A is finitely generated but not necessarily finite-dimensional.

In the remaining part of the section k can be any field.

Example 4.9. We have seen in the proof of Theorem 4.7 that $U(\mathfrak{g})$ is finitely unobstructed for a semisimple Lie algebra $\mathfrak{g}$.

More generally, in [24, Theorem 0.2], there is a characterization of all finite-dimensional Lie algebras $\mathfrak{g}$ over a field k of characteristic zero such that their second cohomology with coefficients in any finite-dimensional module vanishes. Such a Lie algebra is one of the following: (i) a one-dimensional Lie algebra; (ii) a semisimple Lie algebra; (iii) the direct sum of a semisimple Lie algebra and a one-dimensional Lie algebra. Note that a one-dimensional Lie algebra $\mathfrak{g}$ is not semisimple as $[\mathfrak{g}, \mathfrak{g}] = 0 \neq \mathfrak{g}$ (cf. [15, Corollary at page 23]). The same argument as above shows that the universal enveloping algebras of all of these Lie algebras are finitely unobstructed.

The proof of the following result is analogous to [6, Proposition 5.3(4)].

Proposition 4.10. *Let A and S be finitely unobstructed algebras over a field k. If $\mathrm{Ext}^1_A(M, M) = 0$ for every $M \in S\text{-}\mathsf{Mod}_f$, then $S \otimes A$ is finitely unobstructed.*

Remark 4.11. Since $gd\, k[x_1, \ldots, x_n] = n$, from Theorem 4.7 it follows that the algebra $k[x_1, \ldots, x_n]$ is not formally smooth for $n > 1$.
In general, the tensor product of two formally smooth algebras is not formally smooth. Indeed, in the setting of Proposition 4.10, if both A and S are finitely generated algebras over k, then, by [3, Proposition 7.4], we have $\mathrm{pd}(S \otimes A) = \mathrm{pd}(S) + \mathrm{pd}(A)$, where $\mathrm{pd}(\Lambda)$ denotes the projective dimension of a $k-$algebra Λ regarded as a bimodule over itself. Since $\mathrm{pd}(\Lambda) \leq n$ if and only if $\mathrm{H}^{n+1}(A, N) = 0$ for every $N \in A^e\text{-}\mathsf{Mod}$, we get that the algebra $S \otimes A$ is not formally smooth unless $\mathrm{pd}(S) + \mathrm{pd}(A) \leq 1$ i.e. unless S and A are both formally smooth and at least one of them is separable.

By using Proposition 4.10, we can give new examples of algebras whose associated representation scheme is smooth.

Example 4.12. 1) Let A be a finitely unobstructed algebra and let S be a separable algebra (see [6, above Proposition 3.2]), that is $\mathrm{H}^i(S, N) = 0$ for every $i > 0$ and for every $S-$bimodule N. By Theorem 3.1, we get $\mathrm{Ext}^i_S(M, M) = 0$ for every $i > 0$ and for every $M \in S\text{-}\mathsf{Mod}$. By Proposition 4.10, we get that $S \otimes A$ is finitely unobstructed. As a particular case, when $\mathrm{char}(k) = 0$, we have that $M_n(A) \cong M_n(k) \otimes A$ and the group $A-$ring $A[G] \cong k[G] \otimes A$, for every finite group G, are finitely unobstructed as the matrix ring $M_n(k)$ and the group algebra $k[G]$ are separable in characteristic zero (see [6, Example of page 271]).

2) Let A be finitely unobstructed algebra and S a separable algebra, then $\mathrm{Rep}^n_{A \otimes S}$ is smooth. This follows from Proposition 4.10 and example 1).

3) Let $\mathfrak{g}$ be a semisimple Lie algebra and assume $\mathrm{char}(k) = 0$. As observed in Example 4.9, $U := U(\mathfrak{g})$ is finitely unobstructed. Moreover Whitehead's first lemma [23, Corollary 7.8.10] ensures that $\mathrm{H}^1_{Lie}(\mathfrak{g}, N) = 0$ for every $\mathfrak{g}-$module N of finite dimension over k so that, by the same argument used in Example 4.9 for the second group of cohomology, we obtain $\mathrm{Ext}^1_U(M, M) = 0$, for every $M \in U\text{-}\mathsf{Mod}_f$. Thus, by Proposition 4.10, we get that $U \otimes A$ is finitely unobstructed if A is.

4) In analogy with [6, Proposition 5.3(5)], we have that the direct sum of finitely unobstructed algebras is finitely unobstructed too.

Lemma 4.13. *Assume that $\mathrm{H}^2(A, N) = 0$ for some $N \in A^e\text{-}\mathsf{Mod}$ and let Ω^1_A be as in Definition 4.4. Then Ω^1_A is projective with respect to any surjective morphism of A^e-modules with kernel N.*

Proof. One gets that $\mathrm{Ext}^1_{A^e}(\Omega^1_A, N) \cong \mathrm{Ext}^2_{A^e}(A, N) = \mathrm{H}^2(A, N) = 0$ analogously to [6, Proposition 3.3]. The conclusion follows by applying the long exact sequence of $\mathrm{Ext}^*_{A^e}(\Omega^1_A, -)$ to the exact sequence formed by any surjective morphism of A^e-modules and its kernel N. $\square$

The proof of the following result is similar to [6, Proposition 5.3(3)].

Proposition 4.14. *Let A be a finitely unobstructed algebra over a field k. Then the tensor algebra $T_A(P)$ is finitely unobstructed for every $P \in A^e\text{-}\mathsf{Mod}$ which is projective with respect to any surjective morphism in $A^e\text{-}\mathsf{Mod}$ with kernel $\mathrm{End}_k(M)$ for every $M \in A\text{-}\mathsf{Mod}_f$.*

Example 4.15. Let A be a finitely unobstructed algebra over a field k. By Theorem 3.1, Lemma 4.13 and Proposition 4.14, we have that $T_A(\Omega^1_A)$ is finitely unobstructed. The latter, by [6, Proposition 2.3], identifies with ΩA, the DG-algebra of noncommutative differential forms on A.

5 Deformations

The aim of this section is twofold: first, we would like to analyze the relationships between the results of Section 3 and the theory of deformations of module structures as developed in [8, 9, 10]; second, we give a different proof of Corollary 4.2 for finitely unobstructed algebras which are finitely presented or module bicoherent.

5.1 Deformations of modules

Definition 5.1. Let $M \in \mathrm{Rep}^n_A(k)$ and let $\mu : A \to \mathrm{End}_k(M)$ be the associated linear representation. For $(R, \mathfrak{m})$ a local commutative $k-$algebra, an $R-$*deformation* of M is an element $\widetilde{M} \in \mathrm{Rep}^n_A(R)$ whose associate linear representation $\widetilde{\mu} : A \to \mathrm{End}_R(\widetilde{M})$ verifies $\alpha \circ \widetilde{\mu} = \mu$ where $\alpha : \mathrm{End}_R(\widetilde{M}) \to \mathrm{End}_k(M)$ is the morphism of $k-$algebras induceded by the projection $R \to R/\mathfrak{m} \cong k$.
When $R = k[\epsilon] := k[t]/(t^2)$, the ring of dual numbers or $R = k[[t]]$, the ring of formal power series, then an $R-$deformation will be called *infinitesimal* or *formal*, respectively.

For the general theory on deformations of finite-dimensional modules see [8] and [10].

Remark 5.2. It is well-known that the obstructions in extending the infinitesimal deformations of M to formal deformations are in $\mathrm{Ext}^2_A(M, M)$ (see for example [8, 3.6. and 3.6.1.]).

The adjunction in Lemma 2.2 gives the dictionary to describe deformations of $A-$modules in terms of deformations at points of Rep^n_A.

5.2 Deformations of schemes

The theory of local and global deformations of algebraic schemes is an ample and well-established domain of modern algebraic geometry. Sernesi wrote an excellent treatise on this topic [20], and we address the interested reader to it.

We just recall some facts we need to develop our analysis .
Let X be a scheme over k, let $x \in X(k)$ be a $k-$point of X and let $(R, \mathfrak{m})$ be a local commutative $k-$algebra.

Definition 5.3. An $R-$deformation of X at x is an $R-$point x_R of X such that the restriction $\operatorname{Spec} k \to \operatorname{Spec} R$ maps x to x_R. When $R = k[\epsilon]$ or $R = k[[t]]$, then an $R-$deformation will be called *infinitesimal* or *formal*, respectively.

Lemma 5.4. *Let $R \in \mathcal{C}_k$ and let $x : R \to k$ be a rational point of $X = \operatorname{Spec} R$. Then, for all local $S \in \mathcal{C}_k$, there is a bijection*

$$\{S - \textit{deformations of } X \textit{ at } x\} \cong \operatorname{Hom}_{\mathcal{C}_k}(R_x, S)$$

where R_x denotes the localization of R at $\mathfrak{m}_x := \ker x$.

Proof. Let $\alpha : R \to S$ be such that $x = \pi_S\, \alpha$, where $\pi_S : S \to S/\mathfrak{m}_S \cong k$ is the canonical projection with $\mathfrak{m}_S$ the maximal ideal of S. Then if $a \in R - \ker x$, it follows that $\alpha(a)$ is invertible in S and, therefore, by universality, there is a unique morphism $\alpha_x : R_x \to S$ such that $\alpha_x\, j_R^x = \alpha$, where $j_R^x : R \to R_x$ is the canonical map, and hence $x = \pi_S\, \alpha_x\, j_R^x$.
On the other hand, given a morphism $\beta : R_x \to S$ one has that $x = \pi_S\, \beta\, j_R^x$ thus giving the unique $S-$deformation $\beta\, j_R^x$ of X at x. It is, indeed, trivial that $\ker x \subset \ker(\pi_S\, \beta\, j_R^x)$. If $r \in \ker(\pi_S\, \beta\, j_R^x)$ then $\beta(j_R^x(a)) \in \mathfrak{m}_S$ and, therefore, $j_R^x(a) \in \mathfrak{m}_{R_x}$. Thus $a \in \ker x$. □

There is the following criteria.

Theorem 5.5. *[18, Proposition, pag 151] Let R be a noetherian commutative local ring with $k \subset R$ isomorphic to the residue field of R. Then R is regular if and only if, for all finite-dimensional local commutative $k-$algebra S, T and surjective homomorphism of $k-$algebras $S \to T$ the map $\operatorname{Hom}_{\mathcal{C}_k}(R, S) \to \operatorname{Hom}_{\mathcal{C}_k}(R, T)$ is surjective.*

Corollary 5.6. *Let $R \in \mathcal{C}_k$ and let $x : R \to k$ be a rational point of $X := \operatorname{Spec} R$. Then R is regular if and only if, for all finite-dimensional local commutative $k-$algebra S, T a surjective homomorphism of $k-$algebras $S \to T$ induces a surjection*

$$\{S - \textit{deformations of } X \textit{ at } x\} \to \{T - \textit{deformations of } X \textit{ at } x\}$$

5.3 Smoothness of Rep_A^n

Combining the adjunction of Lemma 2.2 with Corollary 5.6 and Theorem 3.4 we have the following criteria.

Theorem 5.7. *Let $A \in \mathcal{N}_k$ be finitely generated. Let $M \in \mathrm{Rep}_A^n(k)$ and let $f : V_n(A) \to k$ be the associated point. Then the following are equivalent:*

(1) M is a regular point;

(2) for all finite-dimensional local commutative $k-$algebras S, T a surjective homomorphism of $k-$algebras $S \to T$ induces a surjection;

$$\{S - \textit{deformations of } M\} \to \{T - \textit{deformations of } M\}$$

(3) $\mathrm{Harr}^2(A, {}_f k) = 0$.

5.4 On the work of Geiss and de la Peña

The following result is well-known.

Lemma 5.8. *[8, 4.4.3.] Let X be a scheme over k and let $x \in X(k)$. Suppose that there exists an open neighbourhood U of x such that for all $y \in U$ the following conditions hold:*

1. *For each $y' \in T_{U,y}$ there exists a map η : $\mathrm{Spec}\ k[[t]] \to U$ such that $\eta \circ p_{t,\epsilon}^* = y'$, where $p_{t,\epsilon} : k[[t]] \to k[\epsilon]$ is the canonical projection;*
2. $\dim T_y U = \dim T_x U$.

Then x is a regular point of X.

Geiss and de la Peña proved that, if A is finite-dimensional, then $M \in \mathrm{Rep}_A^n(k)$ is regular if $\mathrm{Ext}_A^2(M, M) = 0$. Their proof is based on Lemma 5.8.

The assumption that $\mathrm{Ext}_A^2(M, M) = 0$ ensures that the first hypothesis of Lemma 5.8 is satisfied, since the obstructions to integrate the infinitesimal deformation are in $\mathrm{Ext}_A^2(M, M)$.

The second step is to control the dimension of the tangent space at a point of Rep_A^n. We write $M \in \mathrm{Rep}_A^n(k)$ and $T_M \mathrm{Rep}_A^n$ to stress the dependence on M. In this case the tangent space admits a nice description in terms of derivations. Recall that if $N \in A^e\text{-}\mathsf{Mod}$, then a derivation $d : A \to N$ is a linear map such that $d(ab) = ad(b) + d(a)b$ for all $a, b \in A$. We denote by $\mathrm{Der}(A, N)$ the vector space of all derivations $A \to N$.

Proposition 5.9. *[11, 12.4.] For all $M \in \mathrm{Rep}_A^n(k)$, it holds*

$$T_M \mathrm{Rep}_A^n \cong \mathrm{Der}(A, \mathrm{End}_k(M))$$

Let now $N = \mathrm{End}_k(M)$ with $M \in \mathrm{Rep}_A^n(k)$. It is easy to check (see [11, 5.4.]) that we have the following long exact sequence

$$0 \to \mathrm{Ext}_A^0(M, M) \to N \to \mathrm{Der}_M(A, N) \to \mathrm{Ext}_A^1(M, M) \to 0$$

and therefore

$$\dim_k T_M \mathrm{Rep}_A^n = n^2 - \dim_k(\mathrm{Ext}_A^0(M, M)) + \dim_k(\mathrm{Ext}_A^1(M, M)). \tag{5.1}$$

Thus, one can control the dimension of the tangent space if one can control the dimension of the first two Ext's. But, Theorem 3.1, implies that this is equivalent to control the dimension of $\mathrm{H}^i(A,N) \cong \mathrm{Ext}^i_A(M,M)$.
If A is finite-dimensional, then the bar resolution of A is made of finite-dimensional linear spaces which are free A^e-modules. In this case the differentials d^i are regular map between affine spaces and, by using a subdeterminantal argument, it is easy to see that the functions $z^i : \mathrm{Rep}^n_A \to \mathbb{R} : M \mapsto \dim_k(\ker d^i)$ are upper semicontinuos for all i. Furthermore the rank-nullity theorem is valid, and, by using it backward with respect to the indexes i, one finds, under the assumption $\mathrm{Ext}^2(M,M) = 0$ that $0 = z^2(M') + z^1(M') - n^2 \dim_k(A)$ in an open neighborhood of M in Rep^n_A. From this it follows, by upper semicontinuity, that both z^1 and z^2 are constant in that neighborhood of M. It is then again easy to see that $\dim_k T_M \mathrm{Rep}^n_A = z^1(M)$ and, by Lemma 5.8, it follows that M is regular.

We show now that this approach remains valid for a quite broad class of algebras.

Definition 5.10. ([12, 3.5.1]) An algebra $A \in \mathcal{N}_k$ is called *bimodule coherent* if the kernel of any homomorphism between finitely generated free $A-$bimodule is a finitely generated $A-$bimodule.

Consider the kernel Ω^1_A of the multiplication $A \otimes A \to A$, viewed as a map of A^e-modules: if A is generated as a $k-$algebra by m elements, then Ω^1_A is generated, as A^e-module, by m elements. Therefore there exists a surjective homomorphism

$$\omega : F_0 \twoheadrightarrow \Omega^1_A \tag{5.2}$$

with $F_0 \cong (A^e)^m$. Analogously, we can consider the kernel K of (5.2) and construct a surjective homomorphism

$$F_1 \twoheadrightarrow K \hookrightarrow F_0$$

If A is bimodule coherent, then, continuing in this way, we obtain a free resolution of A made by finitely generated $A-$bimodules

$$\ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0\ . \tag{5.3}$$

By applying $\mathrm{Hom}_{A^e}(-,N)$ to (5.3) we get a cochain complex

$$\begin{aligned} 0 \quad &\longrightarrow \mathrm{Hom}_{A^e}(A,N) \longrightarrow \mathrm{Hom}_{A^e}(A^e,N) \xrightarrow{d^0_M} \\ &\xrightarrow{d^0_M} \mathrm{Hom}_{A^e}(F_0,N) \xrightarrow{d^1_M} \mathrm{Hom}_{A^e}(F_1,N) \xrightarrow{d^2_M} \ldots \end{aligned} \tag{5.4}$$

By Theorem 3.1 the cohomology groups of the sequence (5.4) are the Ext groups. By hypothesis $\mathrm{Ext}^2_A(M,M) = 0$ whence

$$z^1(M) + z^2(M) = \dim \mathrm{Hom}_{A^e}(F_0,N) = mn^2. \tag{5.5}$$

In this case everything works in the same way as A it was finite-dimensional and we can carry over verbatim the proof given in [8, 9] to deduce that $\mathrm{Ext}^2_A(M,M) =$

0 implies that M is regular since $z^1 = \dim_k T_M \mathrm{Rep}_A^n$ is locally constant.

Suppose now that A finitely presented i.e. $A \cong F/J$ with J finitely generated (see 2.1). We can repeat the same steps as above to obtain the resolution (5.3), but now, we only know F_0 and F_1 to be surely finitely generated. Indeed F_0 as before and the free module F_1 is finitely generated because, if A is finitely presented, then also Ω_A^1 is finitely presented as $A-$bimodule by [6, Cor.2.11].

For $i = 0, 1$ set $f_i := \mathrm{rank}\, F_i$. One has $V_i := \mathrm{Hom}_{A^e}(F_i, N) \cong M_n(k)^{f_i}$, so these spaces are finite-dimensional over k and their dimensions do not depend on M. Consider the sets

$$C_i := \{(M, A_i, \ldots, A_{f_i}) \ : \ d_M^i(A_1, \ldots, A_{f_i}) = \mathbf{0}_{V_i}\} \subset \mathrm{Rep}_A^n(k) \times V_{i-1}.$$

They are closed in $\mathrm{Rep}_A^n \times V_{i-1}$. The set

$$C_{i_M} := \{(A_i, \ldots, A_{f_i}) \in V_{i-1} \ : \ (M, A_i, \ldots, A_{f_i}) \in C_i\}$$

is a cone in $V_{i-1.}$ Thus, for $i = 1, 2,$ the functions z^i, defined before, are upper semicontinuous (this argument is similar to [5, Lemma 4.3.]). Then the result on smoothness follows as for finite-dimensional algebras.

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