The Nori-Hilbert scheme is not smooth for $2-$Calabi Yau algebras

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Abstract

Let k be an algebraically closed field of characteristic zero and let A be a finitely generated $k-$algebra. The Nori - Hilbert scheme of A, Hilb_A^n, parameterizes left ideals of codimension n in A, and it is well known that Hilb_A^n is smooth when A is formally smooth. In this paper we will study Hilb_A^n for $2-$Calabi Yau algebras. The main examples of these are surface group algebras and preprojective algebras. For the former we show that the Nori-Hilbert scheme is smooth for $n = 1$ only, while for the latter we show that the smooth components of Hilb_A^n that contain simple representations are precisely those that only contain simple representation. Under certain conditions we can generalize this last statement to arbitrary $2-$Calabi Yau algebras.

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1 Introduction

Let A be a finitely generated associative $k-$algebra with k an algebraically closed field of characteristic zero. In this paper we study the Nori - Hilbert scheme Hilb_A^n whose $k-$points parameterize left ideals of codimension n of A.

When A is commutative, this is nothing but the classical Hilbert scheme Hilb_X^n of the $n-$points on $X = \mathrm{Spec}\, A$. It is well-known that Hilb_X^n is smooth when X is a quasi-projective irreducible and smooth curve or surface. The

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scheme Hilb_A^n is smooth when A is formally smooth, hence of global dimension one. This has been proved by L.Le Bruyn (see [30, Prop.6.3.]). The same holds when A is finitely unobstructed [3].

The main result of this paper is to show that the above results do not extend in dimension two in the non commutative case.

The smoothness results on Hilb_X^n are heavily based on the use of Serre Duality, so it seems natural to investigate the geometry of Hilb_A^n when A is a Calabi Yau algebra of global dimension two. These are algebras for which $\mathrm{Ext}^\bullet_{A^e}(A, A) \cong A[2]$, which implies that the double shift is a Serre functor for their derived category.

The main examples of 2-dimensional Calabi Yau algebras are tame and wild preprojective algebras (see Bocklandt [6]) and group algebras of fundamental groups of compact orientable surfaces with nonzero genus (see Kontsevich [20, Corollary 6.1.4.]). In this paper we will investigate the smoothness of the Nori-Hilbert scheme for these two types of algebras. The main results are the following:

Theorem 1.1. *Let $A_g = k[\pi_1(S)]$ be the group algebra of the fundamental group of a compact orientable surface S of genus $g > 1$. The scheme $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{A_g}}$ is irreducible of dimension $(2g-2)n^2 + n + 1$ and it is smooth if and only if $n = 1$.*

Theorem 1.2. *Let $\Pi(Q)$ be the preprojective algebra and let α be a dimension vector for which there exist simple representations. The component of $\mathrm{Hilb}^{\mathrm{n}}_{\Pi(\mathrm{Q})}$ containing the α-dimensional representations is irreducible of dimension $1 + 2\sum_{a\in Q_1} \alpha_{h(a)}\alpha_{t(a)} + \sum_{v\in Q_0}(\alpha_v - 2\alpha_v^2)$ and it is smooth if and only if Q has one vertex and $\alpha = 1$ (or equivalently all α-dimensional representations are simple).*

After these two results we look into the case of more general $2-$CY algebras. Using results by Van den Bergh [41], we show that locally the representation space of any finitely generated $2-$CY algebra can be seen as the representation space of a preprojective algebra. This fact will allow us to generalize the main result to all finitely generated $2-$CY algebras.

Theorem 1.3. *Let A be a finitely generated $2-CY$ algebra and let S be a simple representation such that the component of S in $\mathrm{Rep}_A^n /\!/ \mathrm{GL}_n$ is at least 2-dimensional. The component of $\mathrm{Hilb}^{\mathrm{n}}_{\Pi(\mathrm{Q})}$ containing S is smooth if and only if all representations in this component are simple.*

The paper goes as follows. In section 3 we recall the definition and the principal known results on the smoothness of Hilb_A^n. We also introduce quivers and generalize the notion Hilb_A^α to arbitrary dimension vectors α. We consider the representation scheme Rep_A^α of an associative algebra A and the open subscheme U_A^α whose points correspond to $\alpha-$dimensional cyclic $A-$modules. The general linear group GL_α acts naturally on U_A^α. We show that $\mathrm{U}_A^\alpha/\mathrm{GL}_\alpha$ represents Hilb_A^α and that $\mathrm{U}_A^\alpha \to \mathrm{Hilb}_A^\alpha$ is an universal categorical quotient and a $\mathrm{GL}_\alpha-$principal bundle.

After introducing Calabi Yau algebras in section 4, we carefully analyze the tangent space of Rep_A^α and of Rep_A^n, representation scheme of the n-dimensional representations of A, in section 5. If A is a $2-$CY algebra having a suitable resolution, we find a sharp upper bound for the dimension of the tangent space of a point in U_A^α corresponding to an $A-$module M. In Theorem 5.2 we prove that this dimension is completely controlled by $\dim_k(\mathrm{End}_A(M))$. This is achieved by using Hochschild cohomology and the equality $\dim_k(\mathrm{End}_A(M)) = \dim_k(\mathrm{Ext}_A^2(M, M))$ given by the Calabi Yau condition. This method was inspired by a similar one use by Geiss and de la Peña (see [18]), which works for finite-dimensional $k-$algebras only.

We then prove our first two main theorems 1.1 and 1.2 by combining our results on the tangent spaces of $\mathrm{Rep}_{A_g}^n$ and $\mathrm{U}_{A_g}^n$ to the description of $\mathrm{Rep}_{A_g}^n$ and $\mathrm{Rep}_{\Pi(Q)}^\alpha$ given in [6, 7, 14, 35] and in [12, 7].

In section 6 we show that locally the representation space of a $2-$CY algebra is the representation space of a preprojective algebra and we deduce from this that the smooth semisimple locus equals the simple locus. Finally, we combine the results from 6 and 5.2 to prove Theorem 1.3.

2 Notations

Unless otherwise stated we adopt the following notations:

- k is an algebraically closed field of characteristic zero.
- B is a commutative $k-$algebra.
- $F = k\{x_1, \ldots, x_m\}$ denotes the associative free $k-$algebra on m letters.
- $A \cong F/J$ is a finitely generated associative $k-$algebra.
- $\mathcal{N}_-$, $\mathcal{C}_-$ and Set denote the categories of $-$algebras, commutative $-$algebras and sets, respectively.
- The term "$A-$module" indicates a left $A-$module.

- -mod and -bimod denote the categories of left $-$modules and $-$bimodules, respectively.
- we write $\mathrm{Hom}_{\mathcal{A}}(B, C)$ in a category $\mathcal{A}$ with B, C objects in $\mathcal{A}$. If $\mathcal{A} = A\text{-}\mathsf{mod}$, then we will write $\mathrm{End}_A(-)$.
- A^{op} denotes the opposite algebra of A.
- $A^e := A \otimes A^{\mathrm{op}}$ denotes the envelope of A. It is an $A-$bimodule and a $k-$algebra. One can identify A-bimod with A^e-mod and we will do it thoroughly this paper.
- $\mathrm{Ext}^i_{-}(\ ,\ )$ denotes the Ext groups on the category -mod.
- Q will denote a quiver, Q_0 its vertices and Q_1 its arrows. The maps $h, t : Q_0 \to Q_1$ assign to each arrow its head and tail.
- kQ will be the path algebra of Q.
- $\alpha : Q_0 \to \mathbb{N}$ will denote a dimension vector and its size is $n = |\alpha| = \sum_{v \in Q_0} \alpha_v$.
- If R is a ring $\mathrm{Mat}_n(R)$ denotes the ring of $n \times n$ matrices with elements in R.
- $\mathrm{Mat}_\alpha(R) := \prod_{v \in Q_0} \mathrm{Mat}_{\alpha_v \times \alpha_v}(R)$ and its group of invertible elements is GL_α.
- The standard module of $\mathrm{Mat}_\alpha(R)$ will be denoted by $R^\alpha = \oplus_{v \in Q_0} R^{\alpha_v}$ and $\mathrm{Mat}_\alpha(R)$ sits inside $\mathrm{Mat}_n(R) = \mathrm{End}_R(R^\alpha)$.

3 Nori - Hilbert schemes

3.1 Definitions

The Nori - Hilbert scheme is the representing scheme of a functor of points $\mathcal{C}_k \to \mathsf{Set}$, which is given on objects by

$$\mathrm{Hilb}^n_A(B) := \{ \begin{array}{l} \text{left ideals } I \subset A \otimes_k B \text{ such that } M = (A \otimes_k B)/I \\ \text{is a projective } B - \text{module of rank } n \}. \end{array} \tag{3.1}$$

where $A \in \mathcal{N}_k$, $B \in \mathcal{C}_k$. Hilb^n_A is a closed subfunctor of the Grassmannian functor, so it is representable by a scheme (see [40, Proposition 2]). Nori introduced it for $A = \mathbb{Z}\{x_1, \ldots, x_m\}$ in [33]. It was then defined in a more

general setting in [40] and in [36]. Van den Bergh showed that for $A = F$ the scheme Hilb^n_F is smooth of dimension $n^2(m-1)+n$, (see [40]).
It is also called *the non commutative Hilbert scheme* (see [17, 36]) or the *Brauer-Severi scheme* of A (see [30, 29, 40]), in analogy with the classical Brauer-Severi varieties parameterizing left ideals of codimension n of central simple algebras (see [4]).

Let now A be commutative and $X = \operatorname{Spec} A$. The $k-$points of Hilb^n_A parameterize zero-dimensional subschemes $Y \subset X$ of length n. It is the simplest case of Hilbert scheme parameterizing closed subschemes of X with fixed Hilbert polynomial P, in this case P is the constant polynomial n. The scheme Hilb^n_A is usually called the *Hilbert scheme of $n-$points on X* (see for example Chapter 7 in [10, 21] and Chapter 1 in [32]).

There is the following fundamental result.

Theorem 3.1. *(see [15, 22, 16]) If X is an irreducible smooth quasi projective variety of dimension d, $(d = 1, 2)$ then the Hilbert scheme of the $n-$points over X is a smooth irreducible scheme of dimension dn.*

This theorem can be partially extended to the Nori - Hilbert scheme. If A is an associative finitely generated algebra, then Hilb^n_A is smooth if A is finitely unobstructed i.e. if $\mathrm{Ext}^2_A(M, M) \cong 0$ for all finite dimensional $A-$modules M. This follows by [3, Corollary 4.2.] and Theorem 3.13.

Remark 3.2. If A is hereditary then it is finitely unobstructed and it was well known that Hilb^n_A is smooth for hereditary algebras which are finite dimensional (see [9, Proposition 1]).

If $A = kQ/J$ is the path algebra of a quiver with relations then to every left ideal $I \in \mathrm{Hilb}^n_A(B)$ we can assign a dimension vector $\alpha : v \mapsto \operatorname{rank}(vA \otimes_k B)/I$. So we can define the subset $\mathrm{Hilb}^\alpha_A(B) \subset \mathrm{Hilb}^n_A(B)$ containing all ideals with dimension vector α. We denote its representing scheme by Hilb^α_A. Hilb^n_A decomposes as a disjoint union of all Hilb^α_A with $|\alpha| = n$.

3.2 Representation schemes

Let $A \in \mathcal{N}_k$, $B \in \mathcal{C}_k$ and $\rho : A \to \mathrm{Mat}_n(B)$ an $n-$dimensional representation of A over B. The covariant functor

$$\mathcal{C}_k \longrightarrow \mathsf{Set}$$

given by the assignment $B \mapsto \mathrm{Hom}_{\mathcal{C}_k}(A, \mathrm{Mat}_n(B))$ is represented by a commutative algebra $V_n(A)$ (see [34, Ch.4, §1]). We write Rep^n_A to denote $\operatorname{Spec} V_n(A)$. It is considered as a $k-$scheme.

Let now $A = kQ/J$ be a path algebra of a quiver with relations. We set Q_0 for the set of vertices, Q_1 for the set of arrows and $h, t : Q_1 \to Q_0$ for the head and tail maps. A and kQ are ℓ-algebras with $\ell = k^{Q_0}$ and we chose a set of relations $\{r_i | i \in J\}$ such that each r_i sits in $vkQw$ for some $v, w \in Q_0$, which we also will denote by $h(r_i), t(r_i)$.

Fix a dimension vector $\alpha : Q_0 \to \mathbb{N}$ and set $n = |\alpha| = \sum_{v \in Q_0} \alpha_v$. Let k^α be the ℓ-module consisting of the direct sum of α_v copies of the simple module corresponding to each vertex v. The space $\mathrm{Mat}_n(k)$ can be given the structure of a ℓ-bimodule/ℓ-algebra by identifying it with $\mathrm{Hom}_k(k^\alpha, k^\alpha)$. An α-dimensional representation ρ is a ℓ-algebra morphism from A to $\mathrm{Mat}_n(k)$, this morphism extends the ℓ-module structure on k^α to an A-module structure.

For any commutative k-algebra B we set $B^\alpha = k^\alpha \otimes B$ and $\mathrm{Mat}_n(B) = \mathrm{Mat}_n(k) \otimes B$.

Definition 3.3. Let $A = kQ/J$ and $B \in \mathcal{C}_k$. By an *α-dimensional representation of* A over B we mean a homomorphism of $\ell-$algebras $\rho : A \to \mathrm{Mat}_n(B)$.

It is clear that this is equivalent to giving an $A-$module structure on B^α. The assignment $B \to \mathrm{Hom}_{\mathcal{N}_\ell}(A, \mathrm{Mat}_n(B))$ defines a covariant functor

$$\mathcal{C}_k \longrightarrow \mathsf{Set}. \tag{3.2}$$

This functor is represented by a commutative $k-$algebra. More precisely, there is the following

Lemma 3.4. *[34, Ch.4, §1 extended to quivers] For all $A \in \mathcal{N}_\ell$ and each dimension vector α there exists a commutative $k-$algebra $V_\alpha(A)$ and a representation $\pi_A : A \to \mathrm{Mat}_n(V_\alpha(A))$ such that $\rho \mapsto \mathrm{Mat}_n(\rho) \cdot \pi_A$ gives an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}_k}(V_n(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{N}_\ell}(A, \mathrm{Mat}_n(B)) \tag{3.3}$$

for all $B \in \mathcal{C}_k$.

Definition 3.5. We denote $\mathrm{Rep}^\alpha_A := \mathrm{Spec}\, V_\alpha(A)$. It is considered as a $k-$scheme.

Remark 3.6. Any path algebra with relations $A = kQ/I$ can also be seen as the quotient of a free algebra $A \cong F/J$, so it makes sense to define both

Rep_A^n and Rep_A^α. It is known that there is the following relation between the two

$$\mathrm{Rep}_A^n = \coprod_{|\alpha|=n} \mathrm{Rep}_A^\alpha \times_{\mathrm{GL}_\alpha} \mathrm{GL}_n$$

where the action of GL_α on GL_n is by multiplication.

Examples 3.7.

1. If $A = F$, then $\mathrm{Rep}_F^n(k) \cong \mathrm{Mat}_n(k)^m$ (because a free algebra is the path algebra of a quiver with one vertex, a dimension vector in this case is just a number n).
2. If $A = F/J$, the $B-$points of Rep_A^n can be described in the following way:

$$\mathrm{Rep}_A^n(B) = \{(X_1, \ldots, X_m) \in \mathrm{Mat}_n(B)^m \,:\, f(X_1, \ldots, X_m) = 0 \text{ for all } f \in J\}.$$

The scheme Rep_A^n is a closed subscheme of Rep_F^n.

3. If $A = \mathbb{C}[x, y]$, then

$$\mathrm{Rep}_A^n(\mathbb{C}) \;=\; \{(M_1,\, M_2) \;:\; M_1,\; M_2 \in \mathrm{Mat}_2(\mathbb{C}) \text{ and } M_1M_2 = M_2M_1\}$$

is the *commuting scheme*, see [38].

4. If $A = kQ$, then $\mathrm{Rep}_A^\alpha(k) \cong \bigoplus_{a\in Q_0} \mathrm{Mat}_{\alpha_{h(a)}\times\alpha_{t(a)}}(k)$. (for each arrow a, $\rho(a)$ will be an $n \times n$ matrix with zeros everywhere except on a block of size $\alpha_{h(a)} \times \alpha_{t(a)}$.)

If A is finitely generated, Rep_A^n is of finite type. Note that Rep_A^n may be quite complicated. It is not reduced in general and it seems to be hopeless to describe the coordinate ring of its reduced structure. The scheme Rep_A^α is also known as *the scheme of α-dimensional A-modules.*

3.3 Hilb_A^α as a principal bundle

For any dimension vector α with $|\alpha| = n$ we can identify the B-points of the $n-$dimensional affine scheme $\mathbb{A}_k^n$ with the elements of the module B^α.

Definition 3.8. For each $B \in \mathcal{C}_k$, consider the set

$$\mathrm{U}_A^\alpha(B) \;=\; \{(\rho, v) \in \mathrm{Rep}_A^\alpha(B) \times \mathbb{A}_k^n(B) \;:\; \rho(A)(Bv) = B^\alpha\}. \tag{3.4}$$

The assignment $B \mapsto \mathrm{U}_A^\alpha(B)$ is functorial in B, so that we get a subfunctor U_A^α of $\mathrm{Rep}_A^\alpha \times \mathbb{A}_k^n$ which is clearly open. We denote by U_A^α the corresponding open subscheme.

Remark 3.9. Note that points $\rho \in \mathrm{Rep}_A^\alpha$ such that there is $v \in \mathbb{A}_k^n$ with $(\rho, v) \in \mathrm{U}_A^\alpha$ correspond to $\alpha-$dimensional cyclic $A-$modules.

Let GL_α be the general linear group scheme over k whose $B-$points form the group $\mathrm{GL}_\alpha(B)$ of invertible matrices in $\mathrm{Mat}_\alpha(B) = \mathrm{End}_\ell(B^\alpha)$.

Definition 3.10. Given $B \in \mathcal{C}_k$, $\mathrm{GL}_\alpha(B)$ acts on $\mathrm{Rep}_A^\alpha(B)$:

$$\begin{array}{ccc} \mathrm{GL}_\alpha(B) \times \mathrm{Rep}_A^\alpha(B) & \longrightarrow & \mathrm{Rep}_A^\alpha(B) \\ (g, \rho) & \longrightarrow & \rho^g \; : \; \rho^g(a) = g\rho(a)g^{-1}, \end{array} \tag{3.5}$$

and on $\mathrm{Rep}_A^\alpha(B) \times \mathbb{A}_k^n(B)$:

$$\begin{array}{ccc} \mathrm{GL}_\alpha(B) \times \mathrm{Rep}_A^\alpha(B) \times \mathbb{A}_k^n(B) & \longrightarrow & \mathrm{Rep}_A^\alpha(B) \times_k \mathbb{A}_k^n(B) \\ (g, \rho, v) & \longrightarrow & (\rho^g, gv). \end{array} \tag{3.6}$$

The open subscheme U_A^α is clearly closed under the action above.

Remark 3.11. The $A-$module structures induced on B^α by two representations ρ and ρ' are isomorphic if and only if there exists $g \in \mathrm{GL}_\alpha(B)$ such that $\rho' = \rho^g$.

Definition 3.12. We denote by $\mathrm{Rep}_A^\alpha//\mathrm{GL}_\alpha = \mathrm{Spec}\, V_\alpha(A)^{\mathrm{GL}_\alpha(k)}$ the categorical quotient (in the category of $k-$schemes) of Rep_A^α by GL_α. It is the *(coarse) moduli space of α-dimensional representations* of A.

We have the following result

Theorem 3.13. *The scheme $\mathrm{U}_A^\alpha/\mathrm{GL}_\alpha$ represents Hilb_A^α and $\mathrm{U}_A^\alpha \to \mathrm{Hilb}_A^\alpha$ is an universal categorical quotient and a $\mathrm{GL}_\alpha-$principal bundle. Therefore the scheme Hilb_A^α is smooth iff U_A^α is smooth.*

Proof. The statement is proved in [39] for Hilb_A^n. The generalization from free algebras to path algebras and arbitrary dimension vectors is straightforward. By Remark 3.6 it follows

$$\mathrm{U}_A^n = \coprod_{|\alpha|=n} \mathrm{U}_A^\alpha \times_{\mathrm{GL}_\alpha} \mathrm{GL}_n$$

and hence

$$\mathrm{Hilb}_A^n \cong \mathrm{U}_A^n/\mathrm{GL}_n = \coprod_{|\alpha|=n} \mathrm{U}_A^\alpha/\mathrm{GL}_\alpha.$$

□

Consider now the forgetful map $\mathrm{Rep}_A^\alpha \times \mathbb{A}_k^\alpha \longrightarrow \mathrm{Rep}_A^\alpha$, which sends (ρ, v) in ρ and set V_A^α for the image of U_A^α. Theorem 3.13 implies that if a point in V_A^α is smooth, so is the corresponding point in Hilb_A^α. The same holds for the image V_A^n of the analogous forgetful map $\mathrm{Rep}_A^n \times \mathbb{A}_k^n \longrightarrow \mathrm{Rep}_A^n$. This result leads us to study the local geometry of Rep_A^α and Rep_A^n. For general algebras this study is quite hard so we will restrict to a special class of algebras: $2-$Calabi Yau algebras.

4 Calabi Yau algebras

Calabi Yau algebras have been defined by V.Ginzburg in [20] and R.Bocklandt in [6] following the notion of Calabi Yau triangulated category introduced by Kontsevich. For alternative approaches and further reading see [1], [23], [24] and [26]. We first recall the following

Definition 4.1. ([19, Definition 20.6.1]) An algebra A is called *homologically smooth* if A has a finite resolution by finitely-generated projective (left) A^e-modules.

Definition 4.2. ([20, Definition 3.2.3]) A homologically smooth algebra A is *$d-$Calabi Yau ($d-CY$* for short) if there are A^e-module isomorphisms

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} A & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases} \tag{4.1}$$

We note some properties of Calabi Yau algebras

Proposition 4.3. *If A is $d-CY$ then*

1. *The global dimension of A is $\leq d$.*
2. *If there exists a nonzero finite-dimensional $A-$module M, then the global dimension of A is exactly d.*
3. *If $M, N \in A$-mod are finite-dimensional, then*
$$\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^{d-i}_A(N, M)^*.$$
4. *For every finite dimensional A-module M there is a trace map $\mathrm{Tr}_M : \mathrm{Ext}^d(M, M) \to k$, compatible with the product of $\mathrm{Ext}'s$: $\mathrm{Tr}_N(fg) = (-1)^{i(d-i)}\mathrm{Tr}_M(gf)$ for $f \in \mathrm{Ext}^i(M, N)$ and $g \in \mathrm{Ext}^{d-i}(N, M)$.*

Proof. These are standard results, see for example [2, Proposition 2.4.], [5, Section 2] or [6, Prop.2.2] for proofs. $\square$

Examples 4.4.

1. The polynomial algebra $k[x_1, ..., x_n]$ is $n-$CY.
2. Let X be an affine smooth Calabi Yau variety (i.e. the canonical sheaf is trivial) of dimension n, then $\mathbb{C}[X]$ is $n-$CY.
3. If Q is a quiver, denote by $\overline{Q}$ the *double* quiver of Q obtained by adjoining an arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q. The *preprojective* algebra is the associative algebra

$$\pi(Q) := k(\overline{Q})/ < \sum_{a \in Q_1} [a, a^*] >$$

where $[x, y] = xy - yx$ denotes the commutator. If A is a positively graded $2-$CY, then it is the preprojective algebra of a non-Dynkin quiver (see [6, Theorem 3.2.]).

4. Let $k[\pi_1(M)]$ be the group algebra of the fundamental group of a compact orientable manifold M of dimension n. Kontsevich proves that $k[\pi_1(M)]$ is $n-$CY (see [20, Corollary 6.1.4.]). This algebra is not positively graded. Thus, if S is a surface of genus g, the algebra $A_g := k[\pi_1(S)]$ is $2-$Calabi Yau. The fundamental group $\pi_1(S)$ has the presentation

$$< X_1, Y_1, \dots, X_g, Y_g | X_1 Y_1 X_1^{-1} Y_1^{-1} \dots X_g Y_g X_g^{-1} Y_g^{-1} = 1 > . \quad (4.2)$$

5 Local geometry of Rep_A^α.

A point $x \in \mathrm{Rep}_A^\alpha(k)$ corresponds to a pair (M, μ) where $M \cong k^\alpha$ has an $A-$module structure given by the $\ell-$algebra morphism $\mu : A \to \mathrm{End}_k(M) \cong \mathrm{Mat}_n(k)$. The linear representation μ makes $\mathrm{End}_k(M)$ an A^e-module.

We write M for a point x in $\mathrm{Rep}_A^\alpha(k)$ and $T_M \mathrm{Rep}_A^\alpha$ to denote the tangent space to Rep_A^α at x and to stress the dependence on M.

Proposition 5.1. *[19, 12.4.] For $M \in \mathrm{Rep}_A^\alpha(k)$, $T_M \mathrm{Rep}_A^\alpha \cong \mathrm{Der}_\ell(A, \mathrm{End}_k(M))$.*

Proof. An element $p \in T_M \mathrm{Rep}_A^\alpha$ corresponds to a morphism of $\ell-$algebras $q : A \to \mathrm{Mat}_n(k[\epsilon])$ such that $q(a) = \theta(a)\epsilon + \mu(a)$ for all $a \in A$, where $\mu : A \to \mathrm{End}_k(M)$ is the $\ell-$algebra morphism associated to M. By using

$q(ab) = q(a)q(b)$ one can easily see that $\theta \in \mathrm{Der}_\ell(A, \mathrm{End}_k(M))$ and $\theta(\ell) = 0$. On the other hand, for all $\theta \in \mathrm{Der}_\ell(A, \mathrm{End}_k(M))$, the pair (θ, μ) gives a point of $T_M \mathrm{Rep}_A^\alpha$ in the obvious way. $\square$

Let now $M \in \mathrm{Rep}_A^\alpha(k)$. It is easy to check (see [19, 5.4.]) that we have the following exact sequence

$$0 \to \mathrm{Ext}_A^0(M, M) \to \mathrm{End}_\ell(M) \to \mathrm{Der}_\ell(A, \mathrm{End}_k(M)) \to \mathrm{Ext}_A^1(M, M) \to 0$$

and therefore

$$\begin{aligned} \dim_k T_M \mathrm{Rep}_A^\alpha &= \dim_k \mathrm{Der}_\ell(A, \mathrm{End}_k(M)) && (5.1)\\ &= \alpha^2 + \dim_k(\mathrm{Ext}_A^1(M, M)) - \dim_k(\mathrm{Ext}_A^0(M, M)) && (5.2) \end{aligned}$$

where α^2 stands for the inner product of α with itself: $\alpha^2 = \sum_{v \in Q_0} \alpha_v^2 = \dim_k \mathrm{End}_\ell(M) = \dim_k(\mathrm{Mat}_\alpha(k))$.
In an analogous way one can see that

$$\dim_k T_M \mathrm{Rep}_A^n = n^2 - \dim_k(\mathrm{Ext}_A^0(M, M)) + \dim_k(\mathrm{Ext}_A^1(M, M)). \quad (5.3)$$

Hence the local dimension of Rep_A^α (and of Rep_A^n) is controlled by the dimensions of Ext_A^0 and Ext_A^1.
If A is a $2-$CY admitting a suitable resolution one can actually say more.

Theorem 5.2. *Let A be a $2-CY$ and $F_\bullet$ be a resolution by finitely-generated projective (left) A^e-modules. Suppose that the functions*

$$c^i \ : \ \mathrm{Rep}_A^n(k) \longrightarrow \mathbb{N}, \qquad M \longmapsto c_M^i := \dim_k(\mathrm{Hom}_{A^e}(F_i, \mathrm{End}_k(M))$$

are locally constant for $i = 0, 1, 2$. Then the dimension of the tangent space $T_M \mathrm{Rep}_A^\alpha$ is an increasing function of $\dim_k(\mathrm{End}_A(M))$ on the irreducible components of $\mathrm{Rep}_A^\alpha(k)$.

Proof. To compute the dimension of the tangent space $T_M \mathrm{Rep}_A^\alpha$ at $M \in \mathrm{Rep}_A^\alpha(k)$ we need to compute the groups $\mathrm{Ext}_A^i(M, M)$, $i = 0, 1$. We use the isomorphisms

$$\mathrm{Ext}_A^i(M, M) \cong \mathrm{H}^i(A, \mathrm{End}_k(M)) \quad (5.4)$$

(see [11, Corollary 4.4.]) where $\mathrm{H}^i(A, \mathrm{End}_k(M))$ denotes the Hochschild cohomology with coefficients in $\mathrm{End}_k(M)$. Note also that

$$\mathrm{End}_A(M) \cong \mathrm{Ext}_A^0(M, M) \cong \mathrm{Hom}_{A^e}(A, \mathrm{End}_k(M)).$$

Take the resolution $F_\bullet$ of A and consider the associate complex

$$\begin{aligned}
0 \quad &\longrightarrow \mathrm{Hom}_{A^e}(A, \mathrm{End}_k(M)) \longrightarrow \mathrm{Hom}_{A^e}(F_0, \mathrm{End}_k(M)) \xrightarrow{d_M^0} \\
&\longrightarrow \mathrm{Hom}_{A^e}(F_1, \mathrm{End}_k(M)) \xrightarrow{d_M^1} \mathrm{Hom}_{A^e}(F_2, \mathrm{End}_k(M)) \xrightarrow{d_M^2} \ldots
\end{aligned} \tag{5.5}$$

Set

$$\begin{array}{lll}
k^i & : \ \mathrm{Rep}_A^n(k) \longrightarrow \mathbb{N}, & M \longmapsto k_M^i := \dim_k \ker d_M^i \\
h^i & : \ \mathrm{Rep}_A^n(k) \longrightarrow \mathbb{N}, & M \longmapsto h_M^i := \dim_k \mathrm{Ext}_A^i(M, M).
\end{array} \tag{5.6}$$

The following relations hold by the rank-nullity theorem

$$\begin{cases} h_M^i = k_M^i + k_M^{i-1} - c_M^{i-1} \\ h_M^0 = k_M^0 \end{cases} \tag{5.7}$$

Recall that $\dim T_M \mathrm{Rep}_A^\alpha = \alpha^2 + h_M^1 - h_M^0$, but, since A is $2-$CY, we have $h_M^0 = h_M^2$, so that

$$\begin{aligned}
\dim T_M \mathrm{Rep}_A^\alpha &= \alpha^2 + h_M^1 - h_M^2 \\
&= \alpha^2 + (k_M^1 + k_M^0 - c_M^0) - (k_M^2 + k_M^1 - c_M^1) \\
&= \alpha^2 - c_M^0 + c_M^1 - k_M^2 + h_M^0.
\end{aligned} \tag{5.8}$$

The algebra A has global dimension 2, therefore $h^3 \equiv 0$ on $\mathrm{Rep}_A^n(k)$. From (5.7) it follows then that $k_M^3 + k_M^2 = c_M^2$. The function c^2 is locally constant, so by observing that the functions k^i are (locally) upper semicontinuous, it follows that the functions k^3 and k^2 are locally constant as well. Therefore by (5.8) one has that $\dim T_M \mathrm{Rep}_A^\alpha = N + h_M^0$ where N is locally constant. □

The two main examples of Calabi Yau algebras under consideration fit into this picture.

Proposition 5.3. *Let S be a compact orientable surface S of genus g. The algebra $A_g = k[\pi_1(S)]$ admits a finite free resolution.*

Proof. A resolution $F_\bullet$ is provided by Davison in the proof of [14, Theorem 5.2.2.]:

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

where $F_i = A \otimes k^{d_i} \otimes A$ and d_i is the number of non-degenerated $i-$th dimensional simplices in a simplicial complex Δ homeomorphic to S. □

Since the F_i's are finitely generated and free, the functions $c^i = n^2 \mathrm{rank} F_i$ are constant.

Proposition 5.4. *The preprojective algebra of a non Dynkin quiver $A = \pi(Q)$ admits a resolution by finitely-generated projective A^e-modules $F_\bullet$ such that the functions c^i are constant.*

Proof. Here we follow [6]. Consider the standard projective resolution given in [6, Remark 4.5.]

$$\bigoplus_{i\in Q_0} F_{ii} \longrightarrow \bigoplus_{(a,a^*)} F_{t(a)h(a)} \oplus F_{t(a^*)h(a^*)} \longrightarrow \bigoplus_{i\in Q_0} F_{ii} \xrightarrow{m} A$$

where $F_{ij} := Ai \otimes jA$ and $i, j \in Q_0$. The crucial observation now is that, if $M \in \mathrm{Rep}^n_A(k)$, then

$$\dim_k(\mathrm{Hom}_{A^e}(F_{ij}, \mathrm{End}_k(M))) = \dim_k(i\mathrm{End}_k(M)j) = \alpha_i\alpha_j.$$

This means that the dimensions $\dim_k(\mathrm{Hom}_{A^e}(F_{ij}, \mathrm{End}_k(M))$ are constant. □

5.1 $\mathrm{Hilb}^n_{A_g}$

In this section we prove Theorem 1.1. We start with two preliminary lemmas.

Lemma 5.5. *Let A be an associative $k-$algebra. A codimension n ideal $I \subset A$ is two-sided if and only if $h^0_{A/I} = n$.*

Proof. If I is two-sided, then $h^0_{A/I} = n$. Let now I be such that $h^0_{A/I} = n$. We have $End_A(A/I) = \mathcal{I}/I$ where $\mathcal{I}$ is the idealizer of I, that is the subalgebra of A which is maximal among those algebras where I is two-sided. Therefore, $I \subset \mathcal{I} \subset A$ and $\mathcal{I}/I \cong A/I$. This implies $\mathcal{I} = A$ and I two-sided. □

Now we fix $g > 1$ and $A = A_g$.

Lemma 5.6. *For all $n \geq 1$ there is $I \in \mathrm{Hilb}^n_A(\mathbb{C})$ which is a two-sided ideal.*

Proof. Recall that

$$A = \mathbb{C}[< X_1, Y_1, \ldots, X_g, Y_g | X_1Y_1X_1^{-1}Y_1^{-1} \ldots X_gY_gX_g^{-1}Y_g^{-1} = 1 >] \quad (5.9)$$

so that $A_1 \cong \mathbb{C}[x, y]$. Let J be a $\mathbb{C}-$point in $\mathrm{Hilb}^n_{A_1}$. Consider the following composition

$$A \xrightarrow{\alpha} A_1 \xrightarrow{\pi} A_1/J$$

where α maps X_1 to x, Y_1 to y and the all the others X_i and Y_i to 1. The map π is the quotient map. The kernel I of this composition gives the required two-sided ideal of A. □

Proof of Theorem 1.1 In [35] it is shown that Rep_A^n is irreducible for every n. and its dimension is

$$\dim \mathrm{Rep}_A^n = \begin{cases} (2g-1)n^2+1 & if \;\; g>1 \\ n^2+n & if \;\; g=1 \end{cases}$$

This result and Theorem 3.13 imply that $\mathrm{Hilb}_\mathrm{A}^\mathrm{n}$ is irreducible of dimension $(2g-2)n^2+n+1$, if $g>1$.

In the case $g>1$ it is wellknown that there exist simple representations of A for any dimension. For those points $h^0=1$ is minimal. By Lemma 5.5 we know that Hilb_A^n contain $k-$points corresponding to two-sided ideals and for those points $h^0=n$. Thus V_A^n contains $k-$points where the dimension of the tangent space is different by Theorem 5.2 and Proposition 5.3. This means that it is not smooth, or equivalently, $\mathrm{Hilb}_{A_g}^n$ is not smooth. □

5.2 $\mathrm{Hilb}_{\Pi(Q)}^\alpha$

The situation for $\mathrm{Hilb}_{\Pi(Q)}^\alpha$ is a bit more complicated.

First of all $\mathrm{Hilb}_{\Pi(Q)}^\alpha$ might not be irreducible. Take for instance $Q = \circ \to \circ$ with dimension vector $(1,1)$. In this case $\mathrm{Rep}_{\Pi(Q)}^\alpha$ is the union of 2 intersecting lines and all representations are cyclic, so $\mathrm{Hilb}_{\Pi(Q)}^\alpha$ is not smooth. If we take the dimension vector $(1,2)$ then $\mathrm{Rep}_{\Pi(Q)}^\alpha$ is the union of two planes intersecting in the zero representation, so it is still irreducible. The zero representation is not cyclic so $U_{\Pi(Q)}^\alpha$ so is smooth an hence so is $\mathrm{Hilb}_{\Pi(Q)}^\alpha$. If we take the dimension vector $(1,3)$ then $\mathrm{Rep}_{\Pi(Q)}^\alpha$ is the union of two 3-dimensional space intersecting in the zero representation, but now no representation is cyclic so $\mathrm{Hilb}_{\Pi(Q)}^\alpha$ is empty.

To avoid these pathologies we will restrict to the case where $\mathrm{Rep}_{\Pi(Q)}^\alpha$ contains simple representations. The quivers and dimension vectors for which there exist simple representations have been characterized by Crawley-Boevey in the same paper. To state the result we define a quadratic form on the space of dimension vectors:

$$p(\alpha) = 1 - \alpha\cdot\alpha + \sum_{a\in Q_1} \alpha_{h(a)}\alpha_{t(a)}$$

Theorem 5.7 (Crawley-Boevey)**.** *[12]*

- $\mathrm{Rep}_{\Pi(Q)}^\alpha$ *contains simple representations if and only if* α *is a positive root,* $p(\alpha) > \sum_1^r p(\beta^i)$ *for each decomposition of* $\alpha = \beta^1+\cdots+\beta^r$ *into* $r \geq 2$ *positive roots.*

- *If* $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *contains simple representations then* $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *is an irreducible variety of dimension* $1+2\sum_{a\in Q_1}\alpha_{h(a)}\alpha_{t(a)}+\sum_{v\in Q_0}(\alpha_v-2\alpha_v^2)$ *and the quotient variety* $\mathrm{Rep}^{\alpha}_{\Pi(Q)}/\!\!/\mathrm{GL}_\alpha$ *has dimension* $p(\alpha)$.

In [31] Le Bruyn observes the following interesting property of dimension vectors of simples.

Lemma 5.8. *If* α *is the dimension vector of a simple representation then there is an extended Dynkin subquiver of* Q *with imaginary root* δ *such that* $\alpha \geq \delta$

Remark 5.9. Note that combined with Crawley-Boevey's result this implies the quotient variety has dimension at least 2 unless Q is extended Dynkin. We will call these cases wild.

We will also need a local description of the quotient space of representations.

Theorem 5.10 (Crawley-Boevey)**.** *[13] If* ξ *is a point in* $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *corresponding to a semisimple representation with decomposition* $S_1^{e_1}\oplus\cdots\oplus S_k^{e_k}$ *then there is a quiver* Q_L *and a* $\mathrm{Stab}_\xi=\mathrm{GL}_\beta$*-equivariant morphism* $\kappa:\mathrm{Rep}^{\beta}_{\Pi(Q_L)}\to\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *which maps* 0 *to* ξ*. The corresponding quotient map*

$$\mathrm{Rep}^{\beta}_{\Pi(Q_L)}/\!\!/\mathrm{GL}_\beta\to\mathrm{Rep}^{\alpha}_{\Pi(Q)}/\!\!/\mathrm{GL}_\alpha$$

is étale at 0*. The vertices of* Q_L *correspond to the simple factors in* ξ *and the dimension vector* β *assigns to each vertex the mulitplicity of the corresponding simple.*

Remark 5.11. This means that if ζ' is a β-dimensional semisimple representation of $\Pi(Q')$ that is 'close enough' to the 0, the corresponding representation $\zeta'=\kappa(\zeta')\in\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ is semisimple. The stabilizers of these two points are the same so the decomposition in simples has the same structure. To determine the dimensions of the simples of ζ one can look at the centralizer of Stab_ζ in GL_α:

$$C_{\mathrm{GL}_\alpha}\mathrm{Stab}_\zeta=\prod_i\mathrm{GL}_{\dim S_i}.$$

The dimension of each simple in ζ must be at least the dimension of the corresponding simple in ζ', because $\mathrm{Stab}_{\zeta'}=\mathrm{Stab}_\zeta$ and $\mathrm{GL}_\beta\subset\mathrm{GL}_\alpha$ so $C_{\mathrm{GL}_\beta}\mathrm{Stab}_{\zeta'}\subset C_{\mathrm{GL}_\alpha}\mathrm{Stab}_\zeta$.

Lemma 5.12. *If* M *is a semisimple representation with decomposition* $S_1^{e_1}\oplus\cdots\oplus S_k^{e_k}$ *then* M *is cyclic if and only if* $e_i\leq\dim S_i\forall i$.

Proof. Because M is semisimple, the map $\rho_M : A \to \mathrm{End}_k(M)$ factorizes through $\oplus_i \mathrm{Mat}_{\dim S_i}(k)$. Using the idempotents $\mathbb{1}_{e_i}$ we can split up every cyclic vector v into cyclic vectors $\mathbb{1}_{e_i}$ for $\mathbb{1}_{e_i} M$ and vice versa. This reduces the problem to showing that the $\mathrm{Mat}_d(k)$-representation $(k^d)^{\oplus e}$ is cyclic if and only if $e \leq d$. This condition is clearly necessary as otherwise $\dim_k \mathrm{Mat}_d(k) < \dim_k (k^d)^{\oplus e}$. It is also sufficient because we can take $b_1 \oplus \cdots \oplus b_e$ where (b_i) is the standard basis. $\square$

Lemma 5.13. *If* $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *contains simple representations and* $\alpha \neq (1)$ *then there exists a cyclic* α*-dimensional representation* M *with* $\mathrm{End}_A(M) \neq k$*.*

If, after deleting the zero vertices, Q *is not extended Dynkin then one can choose this representation to be semisimple.*

Proof. We assume that α is sincere: $\forall v \in Q_0 : \alpha_v \neq 0$, if not we can delete the vertices with $\alpha_v = 0$.

We first do the one vertex case. If Q has 0 or 1 loop then the only dimension vector with simples is (1). If Q has more than 1 loop, then Crawley-Boevey's criterion implies that there are simple representations in every dimension vector. For $\alpha = (n)$ we can take the direct sum of n different 1-dimensional simple representations, which is cyclic by lemma 5.12.

If Q has more than one vertex we have to distinguish between three cases.

1. Q is Dynkin. In this case there are no dimension vectors with simple representations except the elementary ones.

2. Q is extended Dynkin. In this case the only sincere dimension vector which has simple representations is the imaginary root. In each of these cases we can find a cyclic representation which is not indecomposable. If $Q = \tilde{A}_n$ then the zero representation is cyclic because the dimension vector only contains ones. In the other cases assume that Q is oriented with arrows that move away from a chosen vertex with dimension 1 as illustrated below in the case of $\tilde{E}_8$

$$\begin{tikzcd} & & 3 & & & & & \\ 2 & 4 \arrow[l] & 6 \arrow[l] \arrow[u] & 5 \arrow[l] & 4 \arrow[l] & 3 \arrow[l] & 2 \arrow[l] & 1 \arrow[l] \end{tikzcd}$$

We pick a representation which assigns to each arrow in Q a map of maximal rank except for terminal arrows, for which we take a matrix with rank equal to the terminal dimension -1. To the starred arrows we assign zero maps. This is a cyclic representation of $\Pi(Q)$: because the dimension jumps by at most one between two adjacent vertices, we can find a vector $\vec{w}$ such that in each vertex v, $v\vec{w}$ and $v\Pi(Q)_{\geq 1}\vec{w}$ generate vk^{α}. The endomorphism ring of this representation contains $k \oplus k$ because there is a direct summand in the terminal vertex.

3. Q is wild, so by remark 5.9 $\dim \mathrm{Rep}^{\alpha}_{\Pi(Q)} /\!\!/ \mathrm{GL}_{\alpha} \geq 2$. We work by induction on the pair $|\alpha| = \sum_v \alpha_v$. If α only consists of ones then by lemma 5.12 the zero representation is semisimple and cyclic so we are done.

   If $\alpha_v > 1$ for some v, 5.8 shows that we can always find a subquiver of extended dynkin type (or a 1 vertex 1 loop quiver, which is essentially $\tilde{A}_0$) such that α is bigger than the imaginary root δ. We can find a semisimple representation in $\rho \in \mathrm{Rep}_{\alpha} Q$ which is the direct sum of a simple nonzero representation of the extended Dynkin subquiver, together with elementary simples with multiplicity $\alpha_v - \delta_v$ for each vertex v. By theorem 5.10 there is a GL_{β}-equivariant a morphism $\mathrm{Rep}^{\beta}_{\Pi}(Q_L) \to \mathrm{Rep}^{\alpha}_{\Pi(Q)}$ that maps 0 to ρ, which induces a morphism $\mathrm{Rep}^{\beta}_{\Pi(Q_L)} /\!\!/ \mathrm{GL}_{\beta} \to \mathrm{Rep}^{\alpha}_{\Pi(Q)} /\!\!/ \mathrm{GL}_{\alpha}$ that is étale at the zero.

   This implies that the dimension of $\mathrm{Rep}^{\beta}_{\Pi(Q_L)} /\!\!/ \mathrm{GL}_{\beta}$ is the same as the dimension of $\dim \mathrm{Rep}^{\alpha}_{\Pi(Q)} /\!\!/ \mathrm{GL}_{\alpha}$ and also that $\mathrm{Rep}^{\beta}_{\Pi(Q_L)}$ contains simples (just lift a simple 'close enough' to ρ. So (Q_L, β) is again wild and $|\beta| < |\alpha|$.

   By induction there is a semisimple cyclic representation $\xi \in \mathrm{Rep}_{\beta} \pi(Q_L)$, which we can chose in the appropriate neighborhood of the zero representation because $\pi(Q_L)$ is graded and hence $\mathrm{Rep}_{\beta} \pi(Q_L)$ has a k^*-action by scaling. Under the étale morphism, ξ corresponds to a semisimple point $\rho' \in \mathrm{Rep}_{\alpha} \pi(Q)$ which has the same stabilizer.

   By remark 5.11 the dimensions of the simples in the decomposition of ρ' are not smaller than those in the decomposition of ξ. Lemma 5.12 now implies that the representation ρ is also cyclic.

$\square$

Theorem 5.14. *Let $\Pi(Q)$ be the preprojective algebra and let α be a dimension vector for which there exist simple representations. The component of $\mathrm{Hilb}^{\mathrm{n}}_{\Pi(\mathrm{Q})}$ containing the α-dimensional representations is irreducible of dimension $1 + 2\sum_{a \in Q_1} \alpha_{h(a)}\alpha_{t(a)} + \sum_{v \in Q_0}(\alpha_v - 2\alpha_v^2)$ and it is smooth if and only if Q has one vertex and $\alpha = 1$ (or equivalently all α-dimensional representations are simple).*

Proof. In [12] it is shown that in this case $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ is an irreducible variety with dimension $p(\alpha)-1+\alpha\cdot\alpha$. Using the fact that $\mathrm{Hilb}^{\mathrm{n}}_{\Pi(\mathrm{Q})}$ is a quotient of an open subset of $\mathrm{Rep}^{\alpha}_{\Pi(Q)} \times k^{\alpha}$ with fibers of dimension $\alpha\cdot\alpha$. we arrive at the desired formula for the dimension. Unless $\alpha = (1)$ the previous lemma shows that we can always find a cyclic representation ρ with $\mathrm{End}(\rho) \neq k$. By theorem 5.2 this representation will correspond to a singularity in the Hilbert space. ☐

The crucial element in the proof for preprojective algebras rests on the fact that one can describe the representation space around any semisimple point again as the representation space of a preprojective algebra. If we want to generalize our result to other Calabi Yau algebras we need to find a similar description. This will be done in the final part of the paper.

6 The local structure of representations spaces of $2-$Calabi Yau algebras

In this section we explain how the local structure of the representation space of a $2-$Calabi Yau algebra can be seen as the representation space of a preprojective algebra. This result enables us to show that the semisimple representations that correspond to smooth points in the representation space are precisely the simple points. Moreover we show that if a neighborhood of a semisimple contains simples and the dimension of the quotient space is at least 2 then we can also find non-simple semisimple cyclic representations. This implies that there is a singularity in de corresponding component of the Hilbert scheme.

The results described here follow from a combination of results by many authors. First we will explain the A_∞-perspective on deformation theory as developed by Kontsevich and Soibelman [27, 28] and apply it to representation theory. This point of view is also studied by Segal [37]. Then we will use results by Van den Bergh on complete Calabi Yau algebras in [41] to show that locally $2-$CY algebras can be seen as completed preprojective algebras. This observation is a generalization of a result by Crawley-Boevey in [13]. It also allows us to classify the semisimple representations that correspond to smooth points in the representation space of a Calabi Yau algebra.

6.1 Deformation theory

We are going to reformulate some concepts in geometric representation theory to the setting of deformation theory. To do this we need to recall some basics about A_∞-algebras from [25] and [27].

Let ℓ be a finite dimensional semisimple algebra over k. An A_∞-algebra is a graded ℓ-bimodule B equipped with a collection of products $(\mu_i)_{i\geq 1}$, which are ℓ-bimodule morphisms of degree $2-i$

$$\mu_i : \underbrace{B \otimes_\ell \cdots \otimes_\ell B}_{i \text{ factors}} \to B$$

subject to the relations[1]

$$[M_n] \quad \sum_{u+v+j=n} \pm \mu_{u+v+1}(1^{\otimes u} \otimes \mu_j \otimes 1^{\otimes v}) = 0.$$

Note that μ_1 has degree 1 and $[M_1]$ implies $\mu_1^2 = 0$, so B has the structure of a complex. Moreover if $\mu_i = 0$ for $i > 2$ we get a dg-algebra, so A_∞-algebras can be seen as generalizations of dg-algebras. If it is clear which product we are talking about we drop the index i.

Morphisms between 2 A_∞-algebras B and C are defined as collections of ℓ-bimodule morphisms $(F_i)_{i\geq 1}$ of degree $1-i$

$$F_i : B \otimes_\ell \cdots \otimes_\ell B \to C$$

subject to the relations

$$\sum_{u+v+j=n} \pm F_{u+v+1}(1^{\otimes u} \otimes \mu_j \otimes 1^{\otimes v}) + \sum_{i_1+\cdots+i_l=n} \pm \mu_l(F_{i_1} \otimes \cdots \otimes F_{i_k}) = 0.$$

The power of A_∞-structures lies in the fact that they can be transported over quasi-isomorphisms between two complexes. If B is an A_∞-algebra, C a complex of k-bimodules and $\phi : B \to C$ a quasi-isomophism then we can find an A_∞-structure on C and a quasi-A_∞-isomorphism $F_\bullet : B \to C$ with $F_1 = \phi$.

An important result in the theory of A_∞-algebras is the minimal model theorem [27, 28, 25]:

Theorem 6.1. *Every A_∞-algebra is A_∞-isomorphic to the product of a minimal one (i.e. $\mu_1 = 0$) and a contractible one (i.e. $\mu_{>1} = 0$ and zero homology). Two A_∞-algebras are quasi-isomorphic if they have isomorphic minimal factors.*

[1] for the specific sign convention we refer to [25]

Given an A_∞-algebra B we can define the Maurer-Cartan equation

$$\mu(x) + \mu(x,x) + \mu(x,x,x) + \cdots = 0$$

The standard way to make sense of this equation is to demand that $x \in B_1 \otimes \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal in $R = k[t]/(t^n)$ (or some other local artinian commutative ring $R = k \oplus \mathfrak{m}$) and to let R commute with the μ_i. The set of solutions will be denoted by $\mathrm{MC}(B)_\mathfrak{m}$ and as such $\mathrm{MC}(B)$ can be considered as a functor from local artinian rings to sets. [2]

If B_0 and B_1 are finite dimensional we can also make sense of this by looking at the local ring

$$\widehat{\mathrm{MC}}(B) = k[\![B_1^*]\!]/\langle \xi\mu^1 + \xi\mu^2 + \xi\mu^3 + \ldots | \xi \in B_1^* \rangle$$

where $\xi\mu^k$ is interpreted as the homogeneous polynomial function that maps $x \in B_1$ to $\xi(\mu_k(x,\ldots,x))$. B_0 has an infinitesimal action on $\widehat{\mathrm{MC}}$

$$b \cdot \xi := \sum_{i=0}^{\infty} \pm \xi\mu_b^i$$

where $\xi\mu_b^k$ is interpreted as the element in $(B_1^*)^{\otimes k-1}$ that maps x to

$$\xi(\mu_k(b,\ldots,x) \pm \cdots \pm \mu_k(x,\ldots,b)).$$

We denote the ring of invariants of this action by

$$\widehat{\mathrm{MC}}^{\mathrm{inv}}(B) := \{f \in \widehat{\mathrm{MC}}(B) | \forall b \in B_0 : b \cdot f = 0\}$$

If $F_\bullet : B \to C$ is an A_∞-isomorphism then the map

$$\phi_F : \widehat{\mathrm{MC}}(C) \to \widehat{\mathrm{MC}}(B) : \xi \mapsto \sum_{i=0}^{\infty} \xi F^i$$

is an isomophism which maps $\widehat{\mathrm{MC}}^{\mathrm{inv}}(C)$ to $\widehat{\mathrm{MC}}^{\mathrm{inv}}(B)$.

The set of solutions to the Maurer-Cartan equations for an A_∞-algebra is the product of the Maurer-Cartan equations of its 2 factors. Likewise, the corresponding local ring is the completed tensor product of the local rings of the two factors and the invariant ring is the product of the two invariant rings. If B is contractible then as vector spaces $B_0 \cong \mathrm{Ker}\mu_1|_{B_1}$. As the higher products vanish $\widehat{\mathrm{MC}}(B) \cong k[\![B_0^*]\!]$ and the invariant ring is $\widehat{\mathrm{MC}}^{\mathrm{inv}}(B) = k$

[2] In fact it is a functor to groupoids, because one can integrate the $B_0 \otimes \mathfrak{m}$-action on $\mathrm{MC}(B)_\mathfrak{m}$

6.2 Representation spaces

For $A = kQ/J$ a path algebra of a quiver with relations, we can describe the space Rep_A^α as a deformation problem. Fix an α-dimensional representation ρ and construct the following complex $R^\bullet$:

$$R^i = \mathrm{Hom}_{\ell^e}(A \otimes_\ell \cdots \otimes_\ell A, \mathrm{Mat}_n(k))$$

with the following products

$$\begin{aligned}\mu_1 f(a_1, \dots, a_{i+1}) &= \rho(a_1)f(a_2, \dots, a_{i+1}) - f(a_1a_2, \dots, a_{i+1}) + \dots \\ &\quad \pm f(a_1, \dots, a_ia_{i+1}) \mp f(a_1, \dots, a_i)\rho(a_{i+1}) \\ \mu_2(f, g)(a_1, \dots, a_{i+j}) &= f(a_1, \dots, a_i)g(a_{i+1}, \dots, a_{i+j})\end{aligned}$$

The Maurer-Cartan equation for this algebra reduces to finding ℓ-linear maps $f : A \to \mathrm{Mat}_n(k) \otimes \mathfrak{m}$ for which

$$\rho(a)f(b) - f(ab) + f(a)\rho(b) + f(a)f(b) = 0,$$

which is precisely the condition that $\rho + f$ is a α-dimensional representation. So the map $f \mapsto (\rho(a) + f(a))_{a\in Q_1}$ maps $MC(R)_\mathfrak{m}$ bijectively to the $k \oplus \mathfrak{m}$-points that lie over the point $\rho \in \mathrm{Rep}_\alpha A$. In this way $R^\bullet$ captures the local information of the representation scheme Rep_A^α around ρ.

It is well known that $R^\bullet$ is quasi-isomorphic to the complex $\mathrm{Ext}_A^\bullet(\rho, \rho)$ with its corresponding A_∞-structure because we can interprete the complex $R^\bullet$ as

$$\mathrm{Hom}_{A^e}(\mathcal{A}_\bullet, M \otimes M^\vee)$$

where $\mathcal{A}_\bullet$ is the bar resolution of A and M is the A-module corresponding to the representation ρ.

6.3 Koszul Duality

In general if $A = kQ/J$ and none of the relations r_i contains paths of length ≤ 1, we can consider the zero representation corresponding to the module $\underline{\ell} := A/Q_1A$. In this case the Koszul dual of A is defined as the Ext-A_∞-algebra of $\underline{\ell}$:

$$A^! := \mathrm{Ext}_A^\bullet(\underline{\ell}, \underline{\ell}).$$

Note that $\mathrm{Ext}_A^0(\underline{\ell}, \underline{\ell}) = \ell$ and as a ℓ-bimodule $\mathrm{Ext}_A^1(\underline{\ell}, \underline{\ell})$ is spanned by elements $[a]$ corresponding to the arrows while $\mathrm{Ext}^2(\underline{\ell}, \underline{\ell})$ is spanned by elements $[r_i]$ corresponding to a minimal set of relations. The complete structure of

the A_∞-products can become very complicated but one has the following identity [37]
$$\mu([a_1],\dots,[a_k]) = \sum_i c_i[r_i] \quad (*)$$
where c_i is the coefficient of the path $a_1\dots a_k$ in r_i.

For every dimension vector α we also have a zero representation $\rho_0 = \underline{\ell}\otimes_\ell k^\alpha$ and in that case

$$\mathrm{Ext}^\bullet_A(\rho_0,\rho_0) = \mathrm{Ext}^\bullet_A(\underline{\ell}\otimes_\ell k^\alpha, \underline{\ell}\otimes_\ell k^\alpha) = k^\alpha\otimes_\ell \mathrm{Ext}^\bullet_A(\underline{\ell},\underline{\ell})\otimes_\ell k^\alpha.$$

If $\{b_i | i\in\mathcal{I}\}$ is a graded ℓ-basis for $A^!$, then elements in $\mathrm{Ext}^\bullet_A(\rho_0,\rho_0)$ can be seen as linear combinations $\sum B_i b_i$ where B_i is an $\alpha_{h(b_i)}\times\alpha_{t(b_i)}$-matrix. The higher multiplications are matrix-versions of the original ones:

$$\mu(B_1b_1,\dots B_ib_i) = B_1\dots B_i\mu(b_1,\dots,b_i).$$

In combination with $(*)$ it is easy to see that, just as expected, $\sum A_i[a_i]\in \mathrm{Ext}^1_A(\rho_0,\rho_0)\otimes\mathfrak{m}$ is a solution to the Maurer-Cartan equation if and only if the matrices A_i satisfy the relations. From the point of view of local rings we see that
$$\widehat{\mathrm{MC}}(\mathrm{Ext}^\bullet_A(\rho_0,\rho_0)) \cong k\widehat{[\mathrm{Rep}^\alpha_A]}_{\rho_0}.$$
It can also easily be checked that

$$\widehat{\mathrm{MC}}^{\mathrm{inv}}(\mathrm{Ext}^\bullet_A(\rho_0,\rho_0)) \cong k\widehat{[\mathrm{Rep}^\alpha_A]^{\mathrm{GL}_\alpha}}_{\rho_0}.$$

Now we return to the general situation and look at a semisimple ρ with decomposition $\rho = \sigma_1^{\oplus e_1}\oplus\dots\oplus\sigma_m^{e_m}$. We can rewrite

$$\mathrm{Ext}_A(\rho,\rho) = \bigoplus_{i,j}\bigoplus_{r=1}^{e_i}\bigoplus_{s=1}^{e_j}\mathrm{Ext}(\sigma_i,\sigma_j) = k^\epsilon\otimes_l \mathrm{Ext}(\underline{\rho},\underline{\rho})\otimes_l k^\epsilon.$$

In this notation $\underline{\rho}$ is the representation that contains only one copy of each simple, $l = k^m$ is the semisimple algebra $\mathrm{Ext}^0_A(\underline{\rho},\underline{\rho})$ and k^ϵ is the module over this algebra with dimension vector $\epsilon = (e_1,\dots,e_m)$. If we can find an l-algebra B such that $B^! = \mathrm{Ext}^\bullet_A(\underline{\rho},\underline{\rho})$, then we can say that locally (up to a product with an affine space) the space of α-dimensional representations of A around ρ looks like the space of ϵ-dimensional representations of B around the zero representation.

Morally, the algebra B should be the Koszul dual of $E := \mathrm{Ext}^\bullet_A(\underline{\rho}, \underline{\rho})$, so we need to take a look at the construction of the Koszul dual of an A_∞-algebra. We restrict to the relevant case where $E = l \oplus V$ is a finite dimensional augmented l-algebra with an A_∞-structure on E such that $\mu_1(l) = 0$, μ_2 is the ordinary multiplication and $\mu_n(\dots, l, \dots) = 0$ for all $n > 2$.

Using a graded l-basis $\mathcal{B}$ for V we can define a differential d on the completed tensor-algebra $\hat{T}_l V^*$ with ($V^* := \mathrm{Hom}(V, k)$ and $\forall b \in \mathcal{B} : \deg b^* = 1 - \deg b$): For each $b, b_1, \dots, b_k \in \mathcal{B}$ and we set the coeffient $b_1^* \otimes \dots \otimes b_k^*$ in db^* equal the coefficient of b in $\mu(b_1 \otimes \dots \otimes b_k)$. With this identification the graded Leibniz rule for d and the $[M_n]$ combine into the rule $d^2 = 0$. We will call the dg-algebra $E^! := (\widehat{T_l V^*}, d)$ the Koszul dual of (B, μ). One can check that if $E = \mathrm{Ext}^\bullet_A(\underline{\ell}, \underline{\ell}) = A^!$ then $E^!$ is formal and its homology is the completion of A by path-length concentrated in degee 0. In other words $\hat{A}$ is the minimal model of $E^!$.

In general $E^!$ might not be formal, but $H_0(E^!)$ is enough to construct the Maurer-Cartan equation for E. Indeed, the Maurer-Cartan equation for E only depends on $\mu_i|_{E_1^{\otimes i}}$. In $E^!$ these are encoded in the map $d : E^!_{-1} \to E^!_0$.

Note that because all degrees in $E = \mathrm{Ext}^\bullet_A(\underline{\rho}, \underline{\rho})$ are nonnegative, the degrees in $E^!$ are nonpositive. The degree zero part of $E^!$ is the completed tensor algebra $\widehat{T_l E_1^*}$, which can be seen as a completed path algebra of a quiver Q_L with m vertices and $\dim i E_1^* j = \dim \mathrm{Ext}^1_A(\sigma_i, \sigma_j)$ arrows from i to j. This quiver is called the *local quiver of* ρ. $E^!_{-1} = \widehat{kQ_L} \otimes_l E_2^* \otimes_l \widehat{kQ_L}$ and the image of $d|_{E^!_{-1}}$ is the $\widehat{kQ_L}$-ideal generated by the ds_i where the s_i form an l-basis for E_2^* Hence, $H_0(E^!)$ is the completed path algebra of the quiver Q_L with relations ds_i and

$$\mathrm{Ext}^i_{H_0(E^!)}(l, l) = E_i \text{ for } i \leq 2 \text{ and } \mu_n|_{\mathrm{Ext}^1_{H_0(E^!)}(l,l)^{\otimes n}} = \mu_n|_{\mathrm{Ext}^1_{H_0(E^!)}(l,l)^{\otimes n}}$$

This allows us to conclude

Theorem 6.2. *If ρ is an α-dimensional semisimple representation of A with decomposition $\rho = \sigma_1^{\oplus e_1} \oplus \dots \oplus \sigma_m^{e_m}$ then the local structure of the representation space around ρ is the same (up to a product with an affine space) as the local structure of the representation space around the ϵ-dimensional zero representation of $H_0(E^!)$ with $E = \mathrm{Ext}^\bullet_A(\underline{\rho}, \underline{\rho})$ and $\epsilon = (e_1, \dots, e_n)$. If $H(E^!)$ is the completion of a path algebra with relations L we can write*

$$\widehat{k[\mathrm{Rep}^\alpha_A]}_\rho \cong \widehat{k[\mathrm{Rep}^\epsilon_L]}_{\rho_0} \otimes k[\![X_1, \dots, X_k]\!] \text{ and } (\widehat{k[\mathrm{Rep}^\alpha_A]^{\mathrm{GL}_\alpha}})_\rho \cong (\widehat{k[\mathrm{Rep}^\epsilon_L]^{\mathrm{GL}_\epsilon}})_{\rho_0}.$$

Remark 6.3. The number k equals the difference $\dim \mathrm{GL}_\alpha - \dim \mathrm{GL}_\epsilon = \alpha \cdot \alpha - \epsilon \cdot \epsilon$ and we can also identify $\widehat{k[\mathrm{Rep}^\epsilon_L]}_{\rho_0} \otimes k[\![X_1, \dots, X_k]\!]$ with

$$k\widehat{[\mathrm{Rep}^\epsilon_L \times_{\mathrm{GL}_\epsilon} \mathrm{GL}_\alpha]}_{(\rho_0,1)}$$

Remark 6.4. If A is hereditary then $\mathrm{Ext}^{\geq 2}_A(\rho, \rho) = 0$ and the Maurer-Cartan equation becomes trivial. The algebra $H_0(E^!)$ is equal to $E^!$ and is just the completed path algebra of the local quiver without any relations. Hence, locally the representation space of an hereditary algebra looks like the representation space of a quiver without relations. This result is an analog of the local quiver theorem by Le Bruyn in [30].

We will now have a look at generalizations of this result to $2-$CY.

6.4 Generalizations to $2-$Calabi Yau algebras

Suppose for now that A is $2-$CY and M is a semisimple module with $\mathrm{End}_A(M) = l = k^m$. In this case $\mathrm{Ext}^1_A(M, M)$ has a nondegenerate antisymmetric l-bilinear form $\langle f, g\rangle := \mathrm{Tr}_M(fg)$ and hence we can find a symplectic l-basis of the form $\{[a_i], [a_i]^* | i \in \mathcal{I}\}$ such that $\langle [a_i], [a_j]\rangle = 0$, $\langle [a_i]^*, [a_j]^*\rangle = 0$ and $\langle [a_i], [a_j]^*\rangle = \delta_{ij}$. Similarly $\mathrm{Ext}^0(M, M)$ is dual to $\mathrm{Ext}^2(M, M)$ so each 'vertex' $[v] \in l$ has a dual element $[v^*]$ and we have $[a_i][a_i^*] = [v^*]$ and $[a_i^*][a_i] = -[w^*]$ for some v and w which we can consider as the head and tail of a_i in the local quiver.

If we take the Koszul dual of $\mathrm{Ext}^\bullet_A(M, M)$, it is the completed path algebra of the local quiver Q with an extra loop v^* in every vertex v. If we put $z = \sum_{v \in Q_0} v^*$ then we get

$$dz = \sum_{a \in Q_1} aa^* - a^*a + h.o.t.$$

Following the same reasoning as in the proof of Theorem 11.2.1 in [41] one can show that up to a change of variables these higher order terms vanish. This implies that

$$H_0(\mathrm{Ext}^\bullet_A(M, M)^!) \cong \widehat{kQ}/\langle dz\rangle \cong \widehat{kQ}/\langle \sum_{a \in Q_1} aa^* - a^*a\rangle.$$

This last algebra is the completed preprojective algebra so in this case $L = \Pi(Q')$ for some quiver Q'. Locally representation spaces of $2-$CY algebras look like preprojective algebras around the zero representation. This result can be seen as a generalization of theorem 5.10.

To solve the question which semisimple representations are smooth, we need to classify the local quivers and dimension vectors for which the zero representation of the preprojective algebra is smooth. Note that by construction such a dimension vector is sincere, i.e. $\forall v \in Q_0 : \alpha_v \neq 0$.

Theorem 6.5. *The only quivers and sincere dimension vectors for which* $\mathrm{Rep}_{\Pi}^{\alpha}$ *is smooth in the zero representation are disjoint unions of quivers with one vertex and an arbitrary number of loops and dimension vector* 1*, or quivers with one vertex and no loops and arbitrary dimension vector.*

Proof. First note that if the quiver is a disjoint union of two subquivers, the preprojective algebra is the direct sum of two smaller preprojective algebras and the representation space is the product of the corresponding representation spaces of these smaller algebras. So we can assume that Q is connected.

The tangent space to the zero ρ_0 representation in $\mathrm{Rep}_\alpha \Pi$ is equal to $\mathrm{Rep}_\alpha Q$ because the derivative

$$\sum [\rho(a), \rho_0(a^*)] + [\rho_0(a), \rho(a^*)] = \sum [\rho(a), 0] + [0, \rho(a)]$$

is identical to zero. Therefore the zero representation is smooth if and only if $\mathrm{Rep}_\alpha \Pi = \mathrm{Rep}_\alpha Q$. This means that the relation $\sum [\rho(a), \rho(a^*)] = 0$ must be identical to zero. This only happens when all arrows are loops and the dimension in the vertex is 1 or there are no arrows. $\square$

Corollary 6.6. *If ρ is a semisimple representation of a $2-CY$ algebra then ρ is a smooth point in the representation space if it is a direct sum of simples without extensions between them, if a simple has no self-extensions it can occur with higher multiplicity. If ρ has simple representations in its neighborhood then ρ itself must be simple.*

Finally we need to look at cyclic representations.

Lemma 6.7. *If A is a $2-CY$ algebra and ρ is a non-simple semisimple representation such that the component of* $\mathrm{Rep}_A^\alpha /\!\!/ \mathrm{GL}_\alpha$ *containing ρ contains simples and has dimension at least* 2*, then this component contains a cyclic non-simple representation.*

Proof. By the conditions on ρ and lemma 5.13 we can find such a representation σ, corresponding to a point in $\mathrm{Rep}^\epsilon L /\!\!/ \mathrm{GL}_\epsilon$, which can chose in any neighborhood of ρ_0 by rescaling. Artin's aproximation applied to the isomorphism $\widehat{k[\mathrm{Rep}_A^\alpha]^{\mathrm{GL}_\alpha}}_\rho \cong \widehat{k[\mathrm{Rep}_L^\epsilon]^{\mathrm{GL}_\epsilon}}_{\rho_0}$ implies that there is a diagram of étale covers $\mathrm{Rep}_A^\alpha /\!\!/ \mathrm{GL}_\alpha \leftarrow U \to \mathrm{Rep}_L^\epsilon /\!\!/ \mathrm{GL}_\epsilon$. Pulling back and pushing forward we can find a semisimple representation $\tilde{\sigma}$ of A. Again by remark 5.11 the dimensions of the simple factors of $\tilde{\sigma}$ are at least those of σ, so lemma 5.12 impies that $\tilde{\sigma}$ is cyclic. $\square$

This lemma and the corollary before it immediately imply Theorem 1.3.

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