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# The Nori-Hilbert scheme is not smooth for 2–Calabi Yau algebras

Raf Bocklandt, Federica Galluzzi\*, Francesco Vaccarino†

## Abstract

Let  $k$  be an algebraically closed field of characteristic zero and let  $A$  be a finitely generated  $k$ –algebra. The Nori - Hilbert scheme of  $A$ ,  $\text{Hilb}_A^n$ , parameterizes left ideals of codimension  $n$  in  $A$ , and it is well known that  $\text{Hilb}_A^n$  is smooth when  $A$  is formally smooth. In this paper we will study  $\text{Hilb}_A^n$  for 2–Calabi Yau algebras. The main examples of these are surface group algebras and preprojective algebras. For the former we show that the Nori-Hilbert scheme is smooth for  $n = 1$  only, while for the latter we show that the smooth components of  $\text{Hilb}_A^n$  that contain simple representations are precisely those that only contain simple representation. Under certain conditions we can generalize this last statement to arbitrary 2–Calabi Yau algebras.

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## 1 Introduction

Let  $A$  be a finitely generated associative  $k$ –algebra with  $k$  an algebraically closed field of characteristic zero. In this paper we study the Nori - Hilbert scheme  $\text{Hilb}_A^n$  whose  $k$ –points parameterize left ideals of codimension  $n$  of  $A$ .

When  $A$  is commutative, this is nothing but the classical Hilbert scheme  $\text{Hilb}_X^n$  of the  $n$ –points on  $X = \text{Spec } A$ . It is well-known that  $\text{Hilb}_X^n$  is smooth when  $X$  is a quasi-projective irreducible and smooth curve or surface. The

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scheme  $\text{Hilb}_A^n$  is smooth when  $A$  is formally smooth, hence of global dimension one. This has been proved by L. Le Bruyn (see [30, Prop.6.3.]). The same holds when  $A$  is finitely unobstructed [3].

The main result of this paper is to show that the above results do not extend in dimension two in the non commutative case.

The smoothness results on  $\text{Hilb}_X^n$  are heavily based on the use of Serre Duality, so it seems natural to investigate the geometry of  $\text{Hilb}_A^n$  when  $A$  is a Calabi Yau algebra of global dimension two. These are algebras for which  $\text{Ext}_{A^e}^\bullet(A, A) \cong A[2]$ , which implies that the double shift is a Serre functor for their derived category.

The main examples of 2-dimensional Calabi Yau algebras are tame and wild preprojective algebras (see Bocklandt [6]) and group algebras of fundamental groups of compact orientable surfaces with nonzero genus (see Kontsevich [20, Corollary 6.1.4.]). In this paper we will investigate the smoothness of the Nori-Hilbert scheme for these two types of algebras. The main results are the following:

**Theorem 1.1.** *Let  $A_g = k[\pi_1(S)]$  be the group algebra of the fundamental group of a compact orientable surface  $S$  of genus  $g > 1$ . The scheme  $\text{Hilb}_{A_g}^n$  is irreducible of dimension  $(2g - 2)n^2 + n + 1$  and it is smooth if and only if  $n = 1$ .*

**Theorem 1.2.** *Let  $\Pi(Q)$  be the preprojective algebra and let  $\alpha$  be a dimension vector for which there exist simple representations. The component of  $\text{Hilb}_{\Pi(Q)}^n$  containing the  $\alpha$ -dimensional representations is irreducible of dimension  $1 + 2 \sum_{a \in Q_1} \alpha_{h(a)} \alpha_{t(a)} + \sum_{v \in Q_0} (\alpha_v - 2\alpha_v^2)$  and it is smooth if and only if  $Q$  has one vertex and  $\alpha = 1$  (or equivalently all  $\alpha$ -dimensional representations are simple).*

After these two results we look into the case of more general 2-CY algebras. Using results by Van den Bergh [41], we show that locally the representation space of any finitely generated 2-CY algebra can be seen as the representation space of a preprojective algebra. This fact will allow us to generalize the main result to all finitely generated 2-CY algebras.

**Theorem 1.3.** *Let  $A$  be a finitely generated 2-CY algebra and let  $S$  be a simple representation such that the component of  $S$  in  $\text{Rep}_A^n // \text{GL}_n$  is at least 2-dimensional. The component of  $\text{Hilb}_{\Pi(Q)}^n$  containing  $S$  is smooth if and only if all representations in this component are simple.*

The paper goes as follows. In section 3 we recall the definition and the principal known results on the smoothness of  $\text{Hilb}_A^n$ . We also introduce

quivers and generalize the notion  $\text{Hilb}_A^\alpha$  to arbitrary dimension vectors  $\alpha$ . We consider the representation scheme  $\text{Rep}_A^\alpha$  of an associative algebra  $A$  and the open subscheme  $U_A^\alpha$  whose points correspond to  $\alpha$ -dimensional cyclic  $A$ -modules. The general linear group  $\text{GL}_\alpha$  acts naturally on  $U_A^\alpha$ . We show that  $U_A^\alpha/\text{GL}_\alpha$  represents  $\text{Hilb}_A^\alpha$  and that  $U_A^\alpha \rightarrow \text{Hilb}_A^\alpha$  is an universal categorical quotient and a  $\text{GL}_\alpha$ -principal bundle.

After introducing Calabi Yau algebras in section 4, we carefully analyze the tangent space of  $\text{Rep}_A^\alpha$  and of  $\text{Rep}_A^n$ , representation scheme of the  $n$ -dimensional representations of  $A$ , in section 5. If  $A$  is a 2-CY algebra having a suitable resolution, we find a sharp upper bound for the dimension of the tangent space of a point in  $U_A^\alpha$  corresponding to an  $A$ -module  $M$ . In Theorem 5.2 we prove that this dimension is completely controlled by  $\dim_k(\text{End}_A(M))$ . This is achieved by using Hochschild cohomology and the equality  $\dim_k(\text{End}_A(M)) = \dim_k(\text{Ext}_A^2(M, M))$  given by the Calabi Yau condition. This method was inspired by a similar one use by Geiss and de la Peña (see [18]), which works for finite-dimensional  $k$ -algebras only.

We then prove our first two main theorems 1.1 and 1.2 by combining our results on the tangent spaces of  $\text{Rep}_{A_g}^n$  and  $U_{A_g}^n$  to the description of  $\text{Rep}_{A_g}^n$  and  $\text{Rep}_{\Pi(Q)}^\alpha$  given in [6, 7, 14, 35] and in [12, 7].

In section 6 we show that locally the representation space of a 2-CY algebra is the representation space of a preprojective algebra and we deduce from this that the smooth semisimple locus equals the simple locus. Finally, we combine the results from 6 and 5.2 to prove Theorem 1.3.

## 2 Notations

Unless otherwise stated we adopt the following notations:

- $k$  is an algebraically closed field of characteristic zero.
- $B$  is a commutative  $k$ -algebra.
- $F = k\{x_1, \dots, x_m\}$  denotes the associative free  $k$ -algebra on  $m$  letters.
- $A \cong F/J$  is a finitely generated associative  $k$ -algebra.
- $\mathcal{N}_-, \mathcal{C}_-$  and  $\text{Set}$  denote the categories of  $-$ algebras, commutative  $-$ algebras and sets, respectively.
- The term " $A$ -module" indicates a left  $A$ -module.

- $\text{-mod}$  and  $\text{-bimod}$  denote the categories of left  $\text{-modules}$  and  $\text{-bimodules}$ , respectively.
- we write  $\text{Hom}_{\mathcal{A}}(B, C)$  in a category  $\mathcal{A}$  with  $B, C$  objects in  $\mathcal{A}$ . If  $\mathcal{A} = A\text{-mod}$ , then we will write  $\text{End}_A(-)$ .
- $A^{\text{op}}$  denotes the opposite algebra of  $A$ .
- $A^e := A \otimes A^{\text{op}}$  denotes the envelope of  $A$ . It is an  $A$ -bimodule and a  $k$ -algebra. One can identify  $A\text{-bimod}$  with  $A^e\text{-mod}$  and we will do it thoroughly this paper.
- $\text{Ext}_-^i(, )$  denotes the Ext groups on the category  $\text{-mod}$ .
- $Q$  will denote a quiver,  $Q_0$  its vertices and  $Q_1$  its arrows. The maps  $h, t : Q_0 \rightarrow Q_1$  assign to each arrow its head and tail.
- $kQ$  will be the path algebra of  $Q$ .
- $\alpha : Q_0 \rightarrow \mathbb{N}$  will denote a dimension vector and its size is  $n = |\alpha| = \sum_{v \in Q_0} \alpha_v$ .
- If  $R$  is a ring  $\text{Mat}_n(R)$  denotes the ring of  $n \times n$  matrices with elements in  $R$ .
- $\text{Mat}_{\alpha}(R) := \prod_{v \in Q_0} \text{Mat}_{\alpha_v \times \alpha_v}(R)$  and its group of invertible elements is  $\text{GL}_{\alpha}$ .
- The standard module of  $\text{Mat}_{\alpha}(R)$  will be denoted by  $R^{\alpha} = \bigoplus_{v \in Q_0} R^{\alpha_v}$  and  $\text{Mat}_{\alpha}(R)$  sits inside  $\text{Mat}_n(R) = \text{End}_R(R^{\alpha})$ .

### 3 Nori - Hilbert schemes

#### 3.1 Definitions

The Nori - Hilbert scheme is the representing scheme of a functor of points  $\mathcal{C}_k \rightarrow \text{Set}$ , which is given on objects by

$$\text{Hilb}_A^n(B) := \left\{ \begin{array}{l} \text{left ideals } I \subset A \otimes_k B \text{ such that } M = (A \otimes_k B)/I \\ \text{is a projective } B\text{-module of rank } n \end{array} \right\}. \quad (3.1)$$

where  $A \in \mathcal{N}_k$ ,  $B \in \mathcal{C}_k$ .  $\text{Hilb}_A^n$  is a closed subfunctor of the Grassmannian functor, so it is representable by a scheme (see [40, Proposition 2]). Nori introduced it for  $A = \mathbb{Z}\{x_1, \dots, x_m\}$  in [33]. It was then defined in a more

general setting in [40] and in [36]. Van den Bergh showed that for  $A = F$  the scheme  $\text{Hilb}_F^n$  is smooth of dimension  $n^2(m-1) + n$ , (see [40]).

It is also called *the non commutative Hilbert scheme* (see [17, 36]) or the *Brauer-Severi scheme* of  $A$  (see [30, 29, 40]), in analogy with the classical Brauer-Severi varieties parameterizing left ideals of codimension  $n$  of central simple algebras (see [4]).

Let now  $A$  be commutative and  $X = \text{Spec } A$ . The  $k$ -points of  $\text{Hilb}_A^n$  parameterize zero-dimensional subschemes  $Y \subset X$  of length  $n$ . It is the simplest case of Hilbert scheme parameterizing closed subschemes of  $X$  with fixed Hilbert polynomial  $P$ , in this case  $P$  is the constant polynomial  $n$ . The scheme  $\text{Hilb}_A^n$  is usually called the *Hilbert scheme of  $n$ -points on  $X$*  (see for example Chapter 7 in [10, 21] and Chapter 1 in [32]).

There is the following fundamental result.

**Theorem 3.1.** (see [15, 22, 16]) *If  $X$  is an irreducible smooth quasi projective variety of dimension  $d$ , ( $d = 1, 2$ ) then the Hilbert scheme of the  $n$ -points over  $X$  is a smooth irreducible scheme of dimension  $dn$ .*

This theorem can be partially extended to the Nori - Hilbert scheme. If  $A$  is an associative finitely generated algebra, then  $\text{Hilb}_A^n$  is smooth if  $A$  is finitely unobstructed i.e. if  $\text{Ext}_A^2(M, M) \cong 0$  for all finite dimensional  $A$ -modules  $M$ . This follows by [3, Corollary 4.2.] and Theorem 3.13.

*Remark 3.2.* If  $A$  is hereditary then it is finitely unobstructed and it was well known that  $\text{Hilb}_A^n$  is smooth for hereditary algebras which are finite dimensional (see [9, Proposition 1]).

If  $A = kQ/J$  is the path algebra of a quiver with relations then to every left ideal  $I \in \text{Hilb}_A^n(B)$  we can assign a dimension vector  $\alpha : v \mapsto \text{rank}(vA \otimes_k B)/I$ . So we can define the subset  $\text{Hilb}_A^\alpha(B) \subset \text{Hilb}_A^n(B)$  containing all ideals with dimension vector  $\alpha$ . We denote its representing scheme by  $\text{Hilb}_A^\alpha$ .  $\text{Hilb}_A^n$  decomposes as a disjoint union of all  $\text{Hilb}_A^\alpha$  with  $|\alpha| = n$ .

### 3.2 Representation schemes

Let  $A \in \mathcal{N}_k$ ,  $B \in \mathcal{C}_k$  and  $\rho : A \rightarrow \text{Mat}_n(B)$  an  $n$ -dimensional representation of  $A$  over  $B$ . The covariant functor

$$\mathcal{C}_k \longrightarrow \text{Set}$$

given by the assignment  $B \mapsto \text{Hom}_{\mathcal{C}_k}(A, \text{Mat}_n(B))$  is represented by a commutative algebra  $V_n(A)$  (see [34, Ch.4, §1]). We write  $\text{Rep}_A^n$  to denote  $\text{Spec } V_n(A)$ . It is considered as a  $k$ -scheme.

Let now  $A = kQ/J$  be a path algebra of a quiver with relations. We set  $Q_0$  for the set of vertices,  $Q_1$  for the set of arrows and  $h, t : Q_1 \rightarrow Q_0$  for the head and tail maps.  $A$  and  $kQ$  are  $\ell$ -algebras with  $\ell = k^{Q_0}$  and we chose a set of relations  $\{r_i | i \in J\}$  such that each  $r_i$  sits in  $vkQw$  for some  $v, w \in Q_0$ , which we also will denote by  $h(r_i), t(r_i)$ .

Fix a dimension vector  $\alpha : Q_0 \rightarrow \mathbb{N}$  and set  $n = |\alpha| = \sum_{v \in Q_0} \alpha_v$ . Let  $k^\alpha$  be the  $\ell$ -module consisting of the direct sum of  $\alpha_v$  copies of the simple module corresponding to each vertex  $v$ . The space  $\text{Mat}_n(k)$  can be given the structure of a  $\ell$ -bimodule/ $\ell$ -algebra by identifying it with  $\text{Hom}_k(k^\alpha, k^\alpha)$ . An  $\alpha$ -dimensional representation  $\rho$  is a  $\ell$ -algebra morphism from  $A$  to  $\text{Mat}_n(k)$ , this morphism extends the  $\ell$ -module structure on  $k^\alpha$  to an  $A$ -module structure.

For any commutative  $k$ -algebra  $B$  we set  $B^\alpha = k^\alpha \otimes B$  and  $\text{Mat}_n(B) = \text{Mat}_n(k) \otimes B$ .

**Definition 3.3.** Let  $A = kQ/J$  and  $B \in \mathcal{C}_k$ . By an  $\alpha$ -dimensional representation of  $A$  over  $B$  we mean a homomorphism of  $\ell$ -algebras  $\rho : A \rightarrow \text{Mat}_n(B)$ .

It is clear that this is equivalent to giving an  $A$ -module structure on  $B^\alpha$ . The assignment  $B \rightarrow \text{Hom}_{\mathcal{N}_\ell}(A, \text{Mat}_n(B))$  defines a covariant functor

$$\mathcal{C}_k \longrightarrow \text{Set}. \quad (3.2)$$

This functor is represented by a commutative  $k$ -algebra. More precisely, there is the following

**Lemma 3.4.** [34, Ch.4, §1 extended to quivers] For all  $A \in \mathcal{N}_\ell$  and each dimension vector  $\alpha$  there exists a commutative  $k$ -algebra  $V_\alpha(A)$  and a representation  $\pi_A : A \rightarrow \text{Mat}_n(V_\alpha(A))$  such that  $\rho \mapsto \text{Mat}_n(\rho) \cdot \pi_A$  gives an isomorphism

$$\text{Hom}_{\mathcal{C}_k}(V_n(A), B) \xrightarrow{\cong} \text{Hom}_{\mathcal{N}_\ell}(A, \text{Mat}_n(B)) \quad (3.3)$$

for all  $B \in \mathcal{C}_k$ .

**Definition 3.5.** We denote  $\text{Rep}_A^\alpha := \text{Spec } V_\alpha(A)$ . It is considered as a  $k$ -scheme.

*Remark 3.6.* Any path algebra with relations  $A = kQ/I$  can also be seen as the quotient of a free algebra  $A \cong F/J$ , so it makes sense to define both

$\text{Rep}_A^n$  and  $\text{Rep}_A^\alpha$ . It is known that there is the following relation between the two

$$\text{Rep}_A^n = \coprod_{|\alpha|=n} \text{Rep}_A^\alpha \times_{\text{GL}_\alpha} \text{GL}_n$$

where the action of  $\text{GL}_\alpha$  on  $\text{GL}_n$  is by multiplication.

**Examples 3.7.**

1. If  $A = F$ , then  $\text{Rep}_F^n(k) \cong \text{Mat}_n(k)^m$  (because a free algebra is the path algebra of a quiver with one vertex, a dimension vector in this case is just a number  $n$ ).
2. If  $A = F/J$ , the  $B$ -points of  $\text{Rep}_A^n$  can be described in the following way:

$$\text{Rep}_A^n(B) = \{(X_1, \dots, X_m) \in \text{Mat}_n(B)^m : f(X_1, \dots, X_m) = 0 \text{ for all } f \in J\}.$$

The scheme  $\text{Rep}_A^n$  is a closed subscheme of  $\text{Rep}_F^n$ .

3. If  $A = \mathbb{C}[x, y]$ , then

$$\text{Rep}_A^n(\mathbb{C}) = \{(M_1, M_2) : M_1, M_2 \in \text{Mat}_2(\mathbb{C}) \text{ and } M_1 M_2 = M_2 M_1\}$$

is the *commuting scheme*, see [38].

4. If  $A = kQ$ , then  $\text{Rep}_A^\alpha(k) \cong \bigoplus_{a \in Q_0} \text{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(k)$ . (for each arrow  $a$ ,  $\rho(a)$  will be an  $n \times n$  matrix with zeros everywhere except on a block of size  $\alpha_{h(a)} \times \alpha_{t(a)}$ .)

If  $A$  is finitely generated,  $\text{Rep}_A^n$  is of finite type. Note that  $\text{Rep}_A^n$  may be quite complicated. It is not reduced in general and it seems to be hopeless to describe the coordinate ring of its reduced structure. The scheme  $\text{Rep}_A^\alpha$  is also known as *the scheme of  $\alpha$ -dimensional  $A$ -modules*.

**3.3  $\text{Hilb}_A^\alpha$  as a principal bundle**

For any dimension vector  $\alpha$  with  $|\alpha| = n$  we can identify the  $B$ -points of the  $n$ -dimensional affine scheme  $\mathbb{A}_k^n$  with the elements of the module  $B^\alpha$ .

**Definition 3.8.** For each  $B \in \mathcal{C}_k$ , consider the set

$$\text{U}_A^\alpha(B) = \{(\rho, v) \in \text{Rep}_A^\alpha(B) \times \mathbb{A}_k^n(B) : \rho(A)(Bv) = B^\alpha\}. \quad (3.4)$$

The assignment  $B \mapsto \text{U}_A^\alpha(B)$  is functorial in  $B$ , so that we get a subfunctor  $\text{U}_A^\alpha$  of  $\text{Rep}_A^\alpha \times \mathbb{A}_k^n$  which is clearly open. We denote by  $\text{U}_A^\alpha$  the corresponding open subscheme.

*Remark 3.9.* Note that points  $\rho \in \text{Rep}_A^\alpha$  such that there is  $v \in \mathbb{A}_k^n$  with  $(\rho, v) \in U_A^\alpha$  correspond to  $\alpha$ -dimensional cyclic  $A$ -modules.

Let  $\text{GL}_\alpha$  be the general linear group scheme over  $k$  whose  $B$ -points form the group  $\text{GL}_\alpha(B)$  of invertible matrices in  $\text{Mat}_\alpha(B) = \text{End}_\ell(B^\alpha)$ .

**Definition 3.10.** Given  $B \in \mathcal{C}_k$ ,  $\text{GL}_\alpha(B)$  acts on  $\text{Rep}_A^\alpha(B)$ :

$$\begin{aligned} \text{GL}_\alpha(B) \times \text{Rep}_A^\alpha(B) &\longrightarrow \text{Rep}_A^\alpha(B) \\ (g, \rho) &\longrightarrow \rho^g : \rho^g(a) = g\rho(a)g^{-1}, \end{aligned} \quad (3.5)$$

and on  $\text{Rep}_A^\alpha(B) \times \mathbb{A}_k^n(B)$ :

$$\begin{aligned} \text{GL}_\alpha(B) \times \text{Rep}_A^\alpha(B) \times \mathbb{A}_k^n(B) &\longrightarrow \text{Rep}_A^\alpha(B) \times_k \mathbb{A}_k^n(B) \\ (g, \rho, v) &\longrightarrow (\rho^g, gv). \end{aligned} \quad (3.6)$$

The open subscheme  $U_A^\alpha$  is clearly closed under the action above.

*Remark 3.11.* The  $A$ -module structures induced on  $B^\alpha$  by two representations  $\rho$  and  $\rho'$  are isomorphic if and only if there exists  $g \in \text{GL}_\alpha(B)$  such that  $\rho' = \rho^g$ .

**Definition 3.12.** We denote by  $\text{Rep}_A^\alpha // \text{GL}_\alpha = \text{Spec } V_\alpha(A)^{\text{GL}_\alpha(k)}$  the categorical quotient (in the category of  $k$ -schemes) of  $\text{Rep}_A^\alpha$  by  $\text{GL}_\alpha$ . It is the (coarse) moduli space of  $\alpha$ -dimensional representations of  $A$ .

We have the following result

**Theorem 3.13.** *The scheme  $U_A^\alpha / \text{GL}_\alpha$  represents  $\text{Hilb}_A^\alpha$  and  $U_A^\alpha \rightarrow \text{Hilb}_A^\alpha$  is an universal categorical quotient and a  $\text{GL}_\alpha$ -principal bundle. Therefore the scheme  $\text{Hilb}_A^\alpha$  is smooth iff  $U_A^\alpha$  is smooth.*

*Proof.* The statement is proved in [39] for  $\text{Hilb}_A^n$ . The generalization from free algebras to path algebras and arbitrary dimension vectors is straightforward. By Remark 3.6 it follows

$$U_A^n = \coprod_{|\alpha|=n} U_A^\alpha \times_{\text{GL}_\alpha} \text{GL}_n$$

and hence

$$\text{Hilb}_A^n \cong U_A^n / \text{GL}_n = \coprod_{|\alpha|=n} U_A^\alpha / \text{GL}_\alpha.$$

□

Consider now the forgetful map  $\text{Rep}_A^\alpha \times \mathbb{A}_k^\alpha \longrightarrow \text{Rep}_A^\alpha$ , which sends  $(\rho, v)$  in  $\rho$  and set  $V_A^\alpha$  for the image of  $U_A^\alpha$ . Theorem 3.13 implies that if a point in  $V_A^\alpha$  is smooth, so is the corresponding point in  $\text{Hilb}_A^\alpha$ . The same holds for the image  $V_A^n$  of the analogous forgetful map  $\text{Rep}_A^n \times \mathbb{A}_k^n \longrightarrow \text{Rep}_A^n$ . This result leads us to study the local geometry of  $\text{Rep}_A^\alpha$  and  $\text{Rep}_A^n$ . For general algebras this study is quite hard so we will restrict to a special class of algebras: 2-Calabi Yau algebras.

## 4 Calabi Yau algebras

Calabi Yau algebras have been defined by V.Ginzburg in [20] and R.Bocklandt in [6] following the notion of Calabi Yau triangulated category introduced by Kontsevich. For alternative approaches and further reading see [1], [23], [24] and [26]. We first recall the following

**Definition 4.1.** ([19, Definition 20.6.1]) An algebra  $A$  is called *homologically smooth* if  $A$  has a finite resolution by finitely-generated projective (left)  $A^e$ -modules.

**Definition 4.2.** ([20, Definition 3.2.3]) A homologically smooth algebra  $A$  is  *$d$ -Calabi Yau* ( *$d$ -CY* for short) if there are  $A^e$ -module isomorphisms

$$\text{Ext}_{A^e}^i(A, A^e) \cong \begin{cases} A & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases} \quad (4.1)$$

We note some properties of Calabi Yau algebras

**Proposition 4.3.** *If  $A$  is  $d$ -CY then*

1. *The global dimension of  $A$  is  $\leq d$ .*
2. *If there exists a nonzero finite-dimensional  $A$ -module  $M$ , then the global dimension of  $A$  is exactly  $d$ .*
3. *If  $M, N \in A\text{-mod}$  are finite-dimensional, then*

$$\text{Ext}_A^i(M, N) \cong \text{Ext}_A^{d-i}(N, M)^*.$$

4. *For every finite dimensional  $A$ -module  $M$  there is a trace map  $\text{Tr}_M : \text{Ext}^d(M, M) \rightarrow k$ , compatible with the product of Ext's:  $\text{Tr}_N(fg) = (-1)^{i(d-i)} \text{Tr}_M(gf)$  for  $f \in \text{Ext}^i(M, N)$  and  $g \in \text{Ext}^{d-i}(N, M)$ .*

*Proof.* These are standard results, see for example [2, Proposition 2.4.], [5, Section 2] or [6, Prop.2.2] for proofs.  $\square$

#### Examples 4.4.

1. The polynomial algebra  $k[x_1, \dots, x_n]$  is  $n$ -CY.
2. Let  $X$  be an affine smooth Calabi Yau variety (i.e. the canonical sheaf is trivial) of dimension  $n$ , then  $\mathbb{C}[X]$  is  $n$ -CY.
3. If  $Q$  is a quiver, denote by  $\overline{Q}$  the *double* quiver of  $Q$  obtained by adjoining an arrow  $a^* : j \rightarrow i$  for each arrow  $a : i \rightarrow j$  in  $Q$ . The *preprojective* algebra is the associative algebra

$$\pi(Q) := k(\overline{Q}) / \langle \sum_{a \in Q_1} [a, a^*] \rangle$$

where  $[x, y] = xy - yx$  denotes the commutator. If  $A$  is a positively graded 2-CY, then it is the preprojective algebra of a non-Dynkin quiver (see [6, Theorem 3.2.]).

4. Let  $k[\pi_1(M)]$  be the group algebra of the fundamental group of a compact orientable manifold  $M$  of dimension  $n$ . Kontsevich proves that  $k[\pi_1(M)]$  is  $n$ -CY (see [20, Corollary 6.1.4.]). This algebra is not positively graded. Thus, if  $S$  is a surface of genus  $g$ , the algebra  $A_g := k[\pi_1(S)]$  is 2-Calabi Yau. The fundamental group  $\pi_1(S)$  has the presentation

$$\langle X_1, Y_1, \dots, X_g, Y_g \mid X_1 Y_1 X_1^{-1} Y_1^{-1} \dots X_g Y_g X_g^{-1} Y_g^{-1} = 1 \rangle. \quad (4.2)$$

## 5 Local geometry of $\text{Rep}_A^\alpha$ .

A point  $x \in \text{Rep}_A^\alpha(k)$  corresponds to a pair  $(M, \mu)$  where  $M \cong k^\alpha$  has an  $A$ -module structure given by the  $\ell$ -algebra morphism  $\mu : A \rightarrow \text{End}_k(M) \cong \text{Mat}_n(k)$ . The linear representation  $\mu$  makes  $\text{End}_k(M)$  an  $A^e$ -module.

We write  $M$  for a point  $x$  in  $\text{Rep}_A^\alpha(k)$  and  $T_M \text{Rep}_A^\alpha$  to denote the tangent space to  $\text{Rep}_A^\alpha$  at  $x$  and to stress the dependence on  $M$ .

**Proposition 5.1.** [19, 12.4.] For  $M \in \text{Rep}_A^\alpha(k)$ ,  $T_M \text{Rep}_A^\alpha \cong \text{Der}_\ell(A, \text{End}_k(M))$ .

*Proof.* An element  $p \in T_M \text{Rep}_A^\alpha$  corresponds to a morphism of  $\ell$ -algebras  $q : A \rightarrow \text{Mat}_n(k[\epsilon])$  such that  $q(a) = \theta(a)\epsilon + \mu(a)$  for all  $a \in A$ , where  $\mu : A \rightarrow \text{End}_k(M)$  is the  $\ell$ -algebra morphism associated to  $M$ . By using

$q(ab) = q(a)q(b)$  one can easily see that  $\theta \in \text{Der}_\ell(A, \text{End}_k(M))$  and  $\theta(\ell) = 0$ . On the other hand, for all  $\theta \in \text{Der}_\ell(A, \text{End}_k(M))$ , the pair  $(\theta, \mu)$  gives a point of  $T_M \text{Rep}_A^\alpha$  in the obvious way.  $\square$

Let now  $M \in \text{Rep}_A^\alpha(k)$ . It is easy to check (see [19, 5.4.]) that we have the following exact sequence

$$0 \rightarrow \text{Ext}_A^0(M, M) \rightarrow \text{End}_\ell(M) \rightarrow \text{Der}_\ell(A, \text{End}_k(M)) \rightarrow \text{Ext}_A^1(M, M) \rightarrow 0$$

and therefore

$$\begin{aligned} \dim_k T_M \text{Rep}_A^\alpha &= \dim_k \text{Der}_\ell(A, \text{End}_k(M)) & (5.1) \\ &= \alpha^2 + \dim_k(\text{Ext}_A^1(M, M)) - \dim_k(\text{Ext}_A^0(M, M)) & (5.2) \end{aligned}$$

where  $\alpha^2$  stands for the inner product of  $\alpha$  with itself:  $\alpha^2 = \sum_{v \in Q_0} \alpha_v^2 = \dim_k \text{End}_\ell(M) = \dim_k(\text{Mat}_\alpha(k))$ .

In an analogous way one can see that

$$\dim_k T_M \text{Rep}_A^n = n^2 - \dim_k(\text{Ext}_A^0(M, M)) + \dim_k(\text{Ext}_A^1(M, M)). \quad (5.3)$$

Hence the local dimension of  $\text{Rep}_A^\alpha$  (and of  $\text{Rep}_A^n$ ) is controlled by the dimensions of  $\text{Ext}_A^0$  and  $\text{Ext}_A^1$ .

If  $A$  is a 2-CY admitting a suitable resolution one can actually say more.

**Theorem 5.2.** *Let  $A$  be a 2-CY and  $F_\bullet$  be a resolution by finitely-generated projective (left)  $A^e$ -modules. Suppose that the functions*

$$c^i : \text{Rep}_A^n(k) \longrightarrow \mathbb{N}, \quad M \longmapsto c_M^i := \dim_k(\text{Hom}_{A^e}(F_i, \text{End}_k(M)))$$

*are locally constant for  $i = 0, 1, 2$ . Then the dimension of the tangent space  $T_M \text{Rep}_A^\alpha$  is an increasing function of  $\dim_k(\text{End}_A(M))$  on the irreducible components of  $\text{Rep}_A^\alpha(k)$ .*

*Proof.* To compute the dimension of the tangent space  $T_M \text{Rep}_A^\alpha$  at  $M \in \text{Rep}_A^\alpha(k)$  we need to compute the groups  $\text{Ext}_A^i(M, M)$ ,  $i = 0, 1$ . We use the isomorphisms

$$\text{Ext}_A^i(M, M) \cong H^i(A, \text{End}_k(M)) \quad (5.4)$$

(see [11, Corollary 4.4.]) where  $H^i(A, \text{End}_k(M))$  denotes the Hochschild cohomology with coefficients in  $\text{End}_k(M)$ . Note also that

$$\text{End}_A(M) \cong \text{Ext}_A^0(M, M) \cong \text{Hom}_{A^e}(A, \text{End}_k(M)).$$

Take the resolution  $F_\bullet$  of  $A$  and consider the associate complex

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{A^e}(A, \mathrm{End}_k(M)) \longrightarrow \mathrm{Hom}_{A^e}(F_0, \mathrm{End}_k(M)) \xrightarrow{d_M^0} \\ &\longrightarrow \mathrm{Hom}_{A^e}(F_1, \mathrm{End}_k(M)) \xrightarrow{d_M^1} \mathrm{Hom}_{A^e}(F_2, \mathrm{End}_k(M)) \xrightarrow{d_M^2} \dots \end{aligned} \quad (5.5)$$

Set

$$\begin{aligned} k^i &: \mathrm{Rep}_A^n(k) \longrightarrow \mathbb{N}, & M &\longmapsto k_M^i := \dim_k \ker d_M^i \\ h^i &: \mathrm{Rep}_A^n(k) \longrightarrow \mathbb{N}, & M &\longmapsto h_M^i := \dim_k \mathrm{Ext}_A^i(M, M). \end{aligned} \quad (5.6)$$

The following relations hold by the rank-nullity theorem

$$\begin{cases} h_M^i = k_M^i + k_M^{i-1} - c_M^{i-1} \\ h_M^0 = k_M^0 \end{cases} \quad (5.7)$$

Recall that  $\dim T_M \mathrm{Rep}_A^\alpha = \alpha^2 + h_M^1 - h_M^0$ , but, since  $A$  is 2-CY, we have  $h_M^0 = h_M^2$ , so that

$$\begin{aligned} \dim T_M \mathrm{Rep}_A^\alpha &= \alpha^2 + h_M^1 - h_M^2 \\ &= \alpha^2 + (k_M^1 + k_M^0 - c_M^0) - (k_M^2 + k_M^1 - c_M^1) \\ &= \alpha^2 - c_M^0 + c_M^1 - k_M^2 + h_M^0. \end{aligned} \quad (5.8)$$

The algebra  $A$  has global dimension 2, therefore  $h^3 \equiv 0$  on  $\mathrm{Rep}_A^n(k)$ . From (5.7) it follows then that  $k_M^3 + k_M^2 = c_M^2$ . The function  $c^2$  is locally constant, so by observing that the functions  $k^i$  are (locally) upper semicontinuous, it follows that the functions  $k^3$  and  $k^2$  are locally constant as well. Therefore by (5.8) one has that  $\dim T_M \mathrm{Rep}_A^\alpha = N + h_M^0$  where  $N$  is locally constant.  $\square$

The two main examples of Calabi Yau algebras under consideration fit into this picture.

**Proposition 5.3.** *Let  $S$  be a compact orientable surface  $S$  of genus  $g$ . The algebra  $A_g = k[\pi_1(S)]$  admits a finite free resolution.*

*Proof.* A resolution  $F_\bullet$  is provided by Davison in the proof of [14, Theorem 5.2.2.]:

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

where  $F_i = A \otimes k^{d_i} \otimes A$  and  $d_i$  is the number of non-degenerated  $i$ -th dimensional simplices in a simplicial complex  $\Delta$  homeomorphic to  $S$ .  $\square$

Since the  $F_i$ 's are finitely generated and free, the functions  $c^i = n^2 \mathrm{rank} F_i$  are constant.

**Proposition 5.4.** *The preprojective algebra of a non Dynkin quiver  $A = \pi(Q)$  admits a resolution by finitely-generated projective  $A^e$ -modules  $F_\bullet$  such that the functions  $c^i$  are constant.*

*Proof.* Here we follow [6]. Consider the standard projective resolution given in [6, Remark 4.5.]

$$\bigoplus_{i \in Q_0} F_{ii} \longrightarrow \bigoplus_{(a, a^*)} F_{t(a)h(a)} \oplus F_{t(a^*)h(a^*)} \longrightarrow \bigoplus_{i \in Q_0} F_{ii} \xrightarrow{m} A$$

where  $F_{ij} := Ai \otimes jA$  and  $i, j \in Q_0$ . The crucial observation now is that, if  $M \in \text{Rep}_A^n(k)$ , then

$$\dim_k(\text{Hom}_{A^e}(F_{ij}, \text{End}_k(M))) = \dim_k(i\text{End}_k(M)j) = \alpha_i \alpha_j.$$

This means that the dimensions  $\dim_k(\text{Hom}_{A^e}(F_{ij}, \text{End}_k(M)))$  are constant.  $\square$

## 5.1 $\text{Hilb}_{A_g}^n$

In this section we prove Theorem 1.1. We start with two preliminary lemmas.

**Lemma 5.5.** *Let  $A$  be an associative  $k$ -algebra. A codimension  $n$  ideal  $I \subset A$  is two-sided if and only if  $h_{A/I}^0 = n$ .*

*Proof.* If  $I$  is two-sided, then  $h_{A/I}^0 = n$ . Let now  $I$  be such that  $h_{A/I}^0 = n$ . We have  $\text{End}_A(A/I) = \mathcal{I}/I$  where  $\mathcal{I}$  is the idealizer of  $I$ , that is the subalgebra of  $A$  which is maximal among those algebras where  $I$  is two-sided. Therefore,  $I \subset \mathcal{I} \subset A$  and  $\mathcal{I}/I \cong A/I$ . This implies  $\mathcal{I} = A$  and  $I$  two-sided.  $\square$

Now we fix  $g > 1$  and  $A = A_g$ .

**Lemma 5.6.** *For all  $n \geq 1$  there is  $I \in \text{Hilb}_A^n(\mathbb{C})$  which is a two-sided ideal.*

*Proof.* Recall that

$$A = \mathbb{C}[\langle X_1, Y_1, \dots, X_g, Y_g | X_1 Y_1 X_1^{-1} Y_1^{-1} \dots X_g Y_g X_g^{-1} Y_g^{-1} = 1 \rangle] \quad (5.9)$$

so that  $A_1 \cong \mathbb{C}[x, y]$ . Let  $J$  be a  $\mathbb{C}$ -point in  $\text{Hilb}_{A_1}^n$ . Consider the following composition

$$A \xrightarrow{\alpha} A_1 \xrightarrow{\pi} A_1/J$$

where  $\alpha$  maps  $X_1$  to  $x$ ,  $Y_1$  to  $y$  and the all the others  $X_i$  and  $Y_i$  to 1. The map  $\pi$  is the quotient map. The kernel  $I$  of this composition gives the required two-sided ideal of  $A$ .  $\square$

*Proof of Theorem 1.1* In [35] it is shown that  $\text{Rep}_A^n$  is irreducible for every  $n$ . and its dimension is

$$\dim \text{Rep}_A^n = \begin{cases} (2g-1)n^2 + 1 & \text{if } g > 1 \\ n^2 + n & \text{if } g = 1 \end{cases}$$

This result and Theorem 3.13 imply that  $\text{Hilb}_A^n$  is irreducible of dimension  $(2g-2)n^2 + n + 1$ , if  $g > 1$ .

In the case  $g > 1$  it is wellknown that there exist simple representations of  $A$  for any dimension. For those points  $h^0 = 1$  is minimal. By Lemma 5.5 we know that  $\text{Hilb}_A^n$  contain  $k$ -points corresponding to two-sided ideals and for those points  $h^0 = n$ . Thus  $V_A^n$  contains  $k$ -points where the dimension of the tangent space is different by Theorem 5.2 and Proposition 5.3. This means that it is not smooth, or equivalently,  $\text{Hilb}_{A_g}^n$  is not smooth.  $\square$

## 5.2 $\text{Hilb}_{\Pi(Q)}^\alpha$

The situation for  $\text{Hilb}_{\Pi(Q)}^\alpha$  is a bit more complicated.

First of all  $\text{Hilb}_{\Pi(Q)}^\alpha$  might not be irreducible. Take for instance  $Q = \circ \rightarrow \circ$  with dimension vector  $(1, 1)$ . In this case  $\text{Rep}_{\Pi(Q)}^\alpha$  is the union of 2 intersecting lines and all representations are cyclic, so  $\text{Hilb}_{\Pi(Q)}^\alpha$  is not smooth. If we take the dimension vector  $(1, 2)$  then  $\text{Rep}_{\Pi(Q)}^\alpha$  is the union of two planes intersecting in the zero representation, so it is still irreducible. The zero representation is not cyclic so  $U_{\Pi(Q)}^\alpha$  so is smooth and hence so is  $\text{Hilb}_{\Pi(Q)}^\alpha$ . If we take the dimension vector  $(1, 3)$  then  $\text{Rep}_{\Pi(Q)}^\alpha$  is the union of two 3-dimensional space intersecting in the zero representation, but now no representation is cyclic so  $\text{Hilb}_{\Pi(Q)}^\alpha$  is empty.

To avoid these pathologies we will restrict to the case where  $\text{Rep}_{\Pi(Q)}^\alpha$  contains simple representations. The quivers and dimension vectors for which there exist simple representations have been characterized by Crawley-Boevey in the same paper. To state the result we define a quadratic form on the space of dimension vectors:

$$p(\alpha) = 1 - \alpha \cdot \alpha + \sum_{a \in Q_1} \alpha_{h(a)} \alpha_{t(a)}$$

**Theorem 5.7** (Crawley-Boevey). [12]

- $\text{Rep}_{\Pi(Q)}^\alpha$  contains simple representations if and only if  $\alpha$  is a positive root,  $p(\alpha) > \sum_1^r p(\beta^i)$  for each decomposition of  $\alpha = \beta^1 + \dots + \beta^r$  into  $r \geq 2$  positive roots.

- If  $\text{Rep}_{\Pi(Q)}^\alpha$  contains simple representations then  $\text{Rep}_{\Pi(Q)}^\alpha$  is an irreducible variety of dimension  $1 + 2 \sum_{a \in Q_1} \alpha_{h(a)} \alpha_{t(a)} + \sum_{v \in Q_0} (\alpha_v - 2\alpha_v^2)$  and the quotient variety  $\text{Rep}_{\Pi(Q)}^\alpha // \text{GL}_\alpha$  has dimension  $p(\alpha)$ .

In [31] Le Bruyn observes the following interesting property of dimension vectors of simples.

**Lemma 5.8.** *If  $\alpha$  is the dimension vector of a simple representation then there is an extended Dynkin subquiver of  $Q$  with imaginary root  $\delta$  such that  $\alpha \geq \delta$*

*Remark 5.9.* Note that combined with Crawley-Boevey's result this implies the quotient variety has dimension at least 2 unless  $Q$  is extended Dynkin. We will call these cases wild.

We will also need a local description of the quotient space of representations.

**Theorem 5.10** (Crawley-Boevey). *[13] If  $\xi$  is a point in  $\text{Rep}_{\Pi(Q)}^\alpha$  corresponding to a semisimple representation with decomposition  $S_1^{e_1} \oplus \dots \oplus S_k^{e_k}$  then there is a quiver  $Q_L$  and a  $\text{Stab}_\xi = \text{GL}_\beta$ -equivariant morphism  $\kappa : \text{Rep}_{\Pi(Q_L)}^\beta \rightarrow \text{Rep}_{\Pi(Q)}^\alpha$  which maps 0 to  $\xi$ . The corresponding quotient map*

$$\text{Rep}_{\Pi(Q_L)}^\beta // \text{GL}_\beta \rightarrow \text{Rep}_{\Pi(Q)}^\alpha // \text{GL}_\alpha$$

*is étale at 0. The vertices of  $Q_L$  correspond to the simple factors in  $\xi$  and the dimension vector  $\beta$  assigns to each vertex the multiplicity of the corresponding simple.*

*Remark 5.11.* This means that if  $\zeta'$  is a  $\beta$ -dimensional semisimple representation of  $\Pi(Q')$  that is 'close enough' to the 0, the corresponding representation  $\zeta' = \kappa(\zeta') \in \text{Rep}_{\Pi(Q)}^\alpha$  is semisimple. The stabilizers of these two points are the same so the decomposition in simples has the same structure. To determine the dimensions of the simples of  $\zeta$  one can look at the centralizer of  $\text{Stab}_\zeta$  in  $\text{GL}_\alpha$ :

$$C_{\text{GL}_\alpha} \text{Stab}_\zeta = \prod_i \text{GL}_{\dim S_i}.$$

The dimension of each simple in  $\zeta$  must be at least the dimension of the corresponding simple in  $\zeta'$ , because  $\text{Stab}_{\zeta'} = \text{Stab}_\zeta$  and  $\text{GL}_\beta \subset \text{GL}_\alpha$  so  $C_{\text{GL}_\beta} \text{Stab}_{\zeta'} \subset C_{\text{GL}_\alpha} \text{Stab}_\zeta$ .

**Lemma 5.12.** *If  $M$  is a semisimple representation with decomposition  $S_1^{e_1} \oplus \dots \oplus S_k^{e_k}$  then  $M$  is cyclic if and only if  $e_i \leq \dim S_i \forall i$ .*



we can find a vector  $\vec{w}$  such that in each vertex  $v$ ,  $v\vec{w}$  and  $v\Pi(Q)_{\geq 1}\vec{w}$  generate  $vk^\alpha$ . The endomorphism ring of this representation contains  $k \oplus k$  because there is a direct summand in the terminal vertex.

3.  $Q$  is wild, so by remark 5.9  $\dim \text{Rep}_{\Pi(Q)}^\alpha // \text{GL}_\alpha \geq 2$ . We work by induction on the pair  $|\alpha| = \sum_v \alpha_v$ . If  $\alpha$  only consists of ones then by lemma 5.12 the zero representation is semisimple and cyclic so we are done.

If  $\alpha_v > 1$  for some  $v$ , 5.8 shows that we can always find a subquiver of extended dynkin type (or a 1 vertex 1 loop quiver, which is essentially  $\tilde{A}_0$ ) such that  $\alpha$  is bigger than the imaginary root  $\delta$ . We can find a semisimple representation in  $\rho \in \text{Rep}_\alpha Q$  which is the direct sum of a simple nonzero representation of the extended Dynkin subquiver, together with elementary simples with multiplicity  $\alpha_v - \delta_v$  for each vertex  $v$ . By theorem 5.10 there is a  $\text{GL}_\beta$ -equivariant a morphism  $\text{Rep}_{\Pi(Q_L)}^\beta \rightarrow \text{Rep}_{\Pi(Q)}^\alpha$  that maps 0 to  $\rho$ , which induces a morphism  $\text{Rep}_{\Pi(Q_L)}^\beta // \text{GL}_\beta \rightarrow \text{Rep}_{\Pi(Q)}^\alpha // \text{GL}_\alpha$  that is étale at the zero.

This implies that the dimension of  $\text{Rep}_{\Pi(Q_L)}^\beta // \text{GL}_\beta$  is the same as the dimension of  $\dim \text{Rep}_{\Pi(Q)}^\alpha // \text{GL}_\alpha$  and also that  $\text{Rep}_{\Pi(Q_L)}^\beta$  contains simples (just lift a simple 'close enough' to  $\rho$ . So  $(Q_L, \beta)$  is again wild and  $|\beta| < |\alpha|$ .

By induction there is a semisimple cyclic representation  $\xi \in \text{Rep}_\beta \pi(Q_L)$ , which we can chose in the appropriate neighborhood of the zero representation because  $\pi(Q_L)$  is graded and hence  $\text{Rep}_\beta \pi(Q_L)$  has a  $k^*$ -action by scaling. Under the étale morphism,  $\xi$  corresponds to a semisimple point  $\rho' \in \text{Rep}_\alpha \pi(Q)$  which has the same stabilizer.

By remark 5.11 the dimensions of the simples in the decomposition of  $\rho'$  are not smaller than those in the decomposition of  $\xi$ . Lemma 5.12 now implies that the representation  $\rho$  is also cyclic.

□

**Theorem 5.14.** *Let  $\Pi(Q)$  be the preprojective algebra and let  $\alpha$  be a dimension vector for which there exist simple representations. The component of  $\text{Hilb}_{\Pi(Q)}^\alpha$  containing the  $\alpha$ -dimensional representations is irreducible of dimension  $1 + 2 \sum_{a \in Q_1} \alpha_{h(a)} \alpha_{t(a)} + \sum_{v \in Q_0} (\alpha_v - 2\alpha_v^2)$  and it is smooth if and only if  $Q$  has one vertex and  $\alpha = 1$  (or equivalently all  $\alpha$ -dimensional representations are simple).*

*Proof.* In [12] it is shown that in this case  $\text{Rep}_{\Pi(Q)}^\alpha$  is an irreducible variety with dimension  $p(\alpha) - 1 + \alpha \cdot \alpha$ . Using the fact that  $\text{Hilb}_{\Pi(Q)}^\alpha$  is a quotient of an open subset of  $\text{Rep}_{\Pi(Q)}^\alpha \times k^\alpha$  with fibers of dimension  $\alpha \cdot \alpha$ , we arrive at the desired formula for the dimension. Unless  $\alpha = (1)$  the previous lemma shows that we can always find a cyclic representation  $\rho$  with  $\text{End}(\rho) \neq k$ . By theorem 5.2 this representation will correspond to a singularity in the Hilbert space.  $\square$

The crucial element in the proof for preprojective algebras rests on the fact that one can describe the representation space around any semisimple point again as the representation space of a preprojective algebra. If we want to generalize our result to other Calabi Yau algebras we need to find a similar description. This will be done in the final part of the paper.

## 6 The local structure of representations spaces of 2–Calabi Yau algebras

In this section we explain how the local structure of the representation space of a 2–Calabi Yau algebra can be seen as the representation space of a preprojective algebra. This result enables us to show that the semisimple representations that correspond to smooth points in the representation space are precisely the simple points. Moreover we show that if a neighborhood of a semisimple contains simples and the dimension of the quotient space is at least 2 then we can also find non-simple semisimple cyclic representations. This implies that there is a singularity in the corresponding component of the Hilbert scheme.

The results described here follow from a combination of results by many authors. First we will explain the  $A_\infty$ -perspective on deformation theory as developed by Kontsevich and Soibelman [27, 28] and apply it to representation theory. This point of view is also studied by Segal [37]. Then we will use results by Van den Bergh on complete Calabi Yau algebras in [41] to show that locally 2–CY algebras can be seen as completed preprojective algebras. This observation is a generalization of a result by Crawley-Boevey in [13]. It also allows us to classify the semisimple representations that correspond to smooth points in the representation space of a Calabi Yau algebra.

### 6.1 Deformation theory

We are going to reformulate some concepts in geometric representation theory to the setting of deformation theory. To do this we need to recall some

basics about  $A_\infty$ -algebras from [25] and [27].

Let  $\ell$  be a finite dimensional semisimple algebra over  $k$ . An  $A_\infty$ -algebra is a graded  $\ell$ -bimodule  $B$  equipped with a collection of products  $(\mu_i)_{i \geq 1}$ , which are  $\ell$ -bimodule morphisms of degree  $2 - i$

$$\mu_i : \underbrace{B \otimes_\ell \cdots \otimes_\ell B}_{i \text{ factors}} \rightarrow B$$

subject to the relations<sup>1</sup>

$$[M_n] \quad \sum_{u+v+j=n} \pm \mu_{u+v+1}(1^{\otimes u} \otimes \mu_j \otimes 1^{\otimes v}) = 0.$$

Note that  $\mu_1$  has degree 1 and  $[M_1]$  implies  $\mu_1^2 = 0$ , so  $B$  has the structure of a complex. Moreover if  $\mu_i = 0$  for  $i > 2$  we get a dg-algebra, so  $A_\infty$ -algebras can be seen as generalizations of dg-algebras. If it is clear which product we are talking about we drop the index  $i$ .

Morphisms between 2  $A_\infty$ -algebras  $B$  and  $C$  are defined as collections of  $\ell$ -bimodule morphisms  $(F_i)_{i \geq 1}$  of degree  $1 - i$

$$F_i : B \otimes_\ell \cdots \otimes_\ell B \rightarrow C$$

subject to the relations

$$\sum_{u+v+j=n} \pm F_{u+v+1}(1^{\otimes u} \otimes \mu_j \otimes 1^{\otimes v}) + \sum_{i_1 + \cdots + i_l = n} \pm \mu_l(F_{i_1} \otimes \cdots \otimes F_{i_l}) = 0.$$

The power of  $A_\infty$ -structures lies in the fact that they can be transported over quasi-isomorphisms between two complexes. If  $B$  is an  $A_\infty$ -algebra,  $C$  a complex of  $k$ -bimodules and  $\phi : B \rightarrow C$  a quasi-isomorphism then we can find an  $A_\infty$ -structure on  $C$  and a quasi- $A_\infty$ -isomorphism  $F_\bullet : B \rightarrow C$  with  $F_1 = \phi$ .

An important result in the theory of  $A_\infty$ -algebras is the minimal model theorem [27, 28, 25]:

**Theorem 6.1.** *Every  $A_\infty$ -algebra is  $A_\infty$ -isomorphic to the product of a minimal one (i.e.  $\mu_1 = 0$ ) and a contractible one (i.e.  $\mu_{>1} = 0$  and zero homology). Two  $A_\infty$ -algebras are quasi-isomorphic if they have isomorphic minimal factors.*

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<sup>1</sup>for the specific sign convention we refer to [25]

Given an  $A_\infty$ -algebra  $B$  we can define the Maurer-Cartan equation

$$\mu(x) + \mu(x, x) + \mu(x, x, x) + \cdots = 0$$

The standard way to make sense of this equation is to demand that  $x \in B_1 \otimes \mathfrak{m}$ , where  $\mathfrak{m}$  is the maximal ideal in  $R = k[t]/(t^n)$  (or some other local artinian commutative ring  $R = k \oplus \mathfrak{m}$ ) and to let  $R$  commute with the  $\mu_i$ . The set of solutions will be denoted by  $\text{MC}(B)_\mathfrak{m}$  and as such  $\text{MC}(B)$  can be considered as a functor from local artinian rings to sets. <sup>2</sup>

If  $B_0$  and  $B_1$  are finite dimensional we can also make sense of this by looking at the local ring

$$\widehat{\text{MC}}(B) = k[[B_1^*]] / \langle \xi\mu^1 + \xi\mu^2 + \xi\mu^3 + \dots \mid \xi \in B_1^* \rangle$$

where  $\xi\mu^k$  is interpreted as the homogeneous polynomial function that maps  $x \in B_1$  to  $\xi(\mu_k(x, \dots, x))$ .  $B_0$  has an infinitesimal action on  $\widehat{\text{MC}}$

$$b \cdot \xi := \sum_{i=0}^{\infty} \pm \xi \mu_b^i$$

where  $\xi\mu_b^k$  is interpreted as the element in  $(B_1^*)^{\otimes k-1}$  that maps  $x$  to

$$\xi(\mu_k(b, \dots, x) \pm \cdots \pm \mu_k(x, \dots, b)).$$

We denote the ring of invariants of this action by

$$\widehat{\text{MC}}^{\text{inv}}(B) := \{f \in \widehat{\text{MC}}(B) \mid \forall b \in B_0 : b \cdot f = 0\}$$

If  $F_\bullet : B \rightarrow C$  is an  $A_\infty$ -isomorphism then the map

$$\phi_F : \widehat{\text{MC}}(C) \rightarrow \widehat{\text{MC}}(B) : \xi \mapsto \sum_{i=0}^{\infty} \xi F^i$$

is an isomorphism which maps  $\widehat{\text{MC}}^{\text{inv}}(C)$  to  $\widehat{\text{MC}}^{\text{inv}}(B)$ .

The set of solutions to the Maurer-Cartan equations for an  $A_\infty$ -algebra is the product of the Maurer-Cartan equations of its 2 factors. Likewise, the corresponding local ring is the completed tensor product of the local rings of the two factors and the invariant ring is the product of the two invariant rings. If  $B$  is contractible then as vector spaces  $B_0 \cong \text{Ker}\mu_1|_{B_1}$ . As the higher products vanish  $\widehat{\text{MC}}(B) \cong k[[B_0^*]]$  and the invariant ring is  $\widehat{\text{MC}}^{\text{inv}}(B) = k$

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<sup>2</sup>In fact it is a functor to groupoids, because one can integrate the  $B_0 \otimes \mathfrak{m}$ -action on  $\text{MC}(B)_\mathfrak{m}$

## 6.2 Representation spaces

For  $A = kQ/J$  a path algebra of a quiver with relations, we can describe the space  $\text{Rep}_A^\alpha$  as a deformation problem. Fix an  $\alpha$ -dimensional representation  $\rho$  and construct the following complex  $R^\bullet$ :

$$R^i = \text{Hom}_{\ell^e}(A \otimes_\ell \cdots \otimes_\ell A, \text{Mat}_n(k))$$

with the following products

$$\begin{aligned} \mu_1 f(a_1, \dots, a_{i+1}) &= \rho(a_1)f(a_2, \dots, a_{i+1}) - f(a_1 a_2, \dots, a_{i+1}) + \dots \\ &\quad \pm f(a_1, \dots, a_i a_{i+1}) \mp f(a_1, \dots, a_i)\rho(a_{i+1}) \\ \mu_2(f, g)(a_1, \dots, a_{i+j}) &= f(a_1, \dots, a_i)g(a_{i+1}, \dots, a_{i+j}) \end{aligned}$$

The Maurer-Cartan equation for this algebra reduces to finding  $\ell$ -linear maps  $f : A \rightarrow \text{Mat}_n(k) \otimes \mathfrak{m}$  for which

$$\rho(a)f(b) - f(ab) + f(a)\rho(b) + f(a)f(b) = 0,$$

which is precisely the condition that  $\rho + f$  is a  $\alpha$ -dimensional representation. So the map  $f \mapsto (\rho(a) + f(a))_{a \in Q_1}$  maps  $MC(R)_\mathfrak{m}$  bijectively to the  $k \oplus \mathfrak{m}$ -points that lie over the point  $\rho \in \text{Rep}_\alpha A$ . In this way  $R^\bullet$  captures the local information of the representation scheme  $\text{Rep}_A^\alpha$  around  $\rho$ .

It is well known that  $R^\bullet$  is quasi-isomorphic to the complex  $\text{Ext}_A^\bullet(\rho, \rho)$  with its corresponding  $A_\infty$ -structure because we can interpret the complex  $R^\bullet$  as

$$\text{Hom}_{A^e}(\mathcal{A}_\bullet, M \otimes M^\vee)$$

where  $\mathcal{A}_\bullet$  is the bar resolution of  $A$  and  $M$  is the  $A$ -module corresponding to the representation  $\rho$ .

## 6.3 Koszul Duality

In general if  $A = kQ/J$  and none of the relations  $r_i$  contains paths of length  $\leq 1$ , we can consider the zero representation corresponding to the module  $\underline{\ell} := A/Q_1 A$ . In this case the Koszul dual of  $A$  is defined as the Ext- $A_\infty$ -algebra of  $\underline{\ell}$ :

$$A^! := \text{Ext}_A^\bullet(\underline{\ell}, \underline{\ell}).$$

Note that  $\text{Ext}_A^0(\underline{\ell}, \underline{\ell}) = \ell$  and as a  $\ell$ -bimodule  $\text{Ext}_A^1(\underline{\ell}, \underline{\ell})$  is spanned by elements  $[a]$  corresponding to the arrows while  $\text{Ext}_A^2(\underline{\ell}, \underline{\ell})$  is spanned by elements  $[r_i]$  corresponding to a minimal set of relations. The complete structure of

the  $A_\infty$ -products can become very complicated but one has the following identity [37]

$$\mu([a_1], \dots, [a_k]) = \sum_i c_i[r_i] \quad (*)$$

where  $c_i$  is the coefficient of the path  $a_1 \dots a_k$  in  $r_i$ .

For every dimension vector  $\alpha$  we also have a zero representation  $\rho_0 = \underline{\ell} \otimes_\ell k^\alpha$  and in that case

$$\text{Ext}_A^\bullet(\rho_0, \rho_0) = \text{Ext}_A^\bullet(\underline{\ell} \otimes_\ell k^\alpha, \underline{\ell} \otimes_\ell k^\alpha) = k^\alpha \otimes_\ell \text{Ext}_A^\bullet(\underline{\ell}, \underline{\ell}) \otimes_\ell k^\alpha.$$

If  $\{b_i | i \in \mathcal{I}\}$  is a graded  $\ell$ -basis for  $A^!$ , then elements in  $\text{Ext}_A^\bullet(\rho_0, \rho_0)$  can be seen as linear combinations  $\sum B_i b_i$  where  $B_i$  is an  $\alpha_{h(b_i)} \times \alpha_{t(b_i)}$ -matrix. The higher multiplications are matrix-versions of the original ones:

$$\mu(B_1 b_1, \dots, B_i b_i) = B_1 \dots B_i \mu(b_1, \dots, b_i).$$

In combination with  $(*)$  it is easy to see that, just as expected,  $\sum A_i [a_i] \in \text{Ext}_A^1(\rho_0, \rho_0) \otimes \mathfrak{m}$  is a solution to the Maurer-Cartan equation if and only if the matrices  $A_i$  satisfy the relations. From the point of view of local rings we see that

$$\widehat{\text{MC}}(\text{Ext}_A^\bullet(\rho_0, \rho_0)) \cong k[\widehat{\text{Rep}}_A^\alpha]_{\rho_0}.$$

It can also easily be checked that

$$\widehat{\text{MC}}^{\text{inv}}(\text{Ext}_A^\bullet(\rho_0, \rho_0)) \cong k[\widehat{\text{Rep}}_A^\alpha]^{\text{GL}_\alpha}_{\rho_0}.$$

Now we return to the general situation and look at a semisimple  $\rho$  with decomposition  $\rho = \sigma_1^{\oplus e_1} \oplus \dots \oplus \sigma_m^{\oplus e_m}$ . We can rewrite

$$\text{Ext}_A(\rho, \rho) = \bigoplus_{i,j} \bigoplus_{r=1}^{e_i} \bigoplus_{s=1}^{e_j} \text{Ext}(\sigma_i, \sigma_j) = k^\epsilon \otimes_l \text{Ext}(\underline{\rho}, \underline{\rho}) \otimes_l k^\epsilon.$$

In this notation  $\underline{\rho}$  is the representation that contains only one copy of each simple,  $l = k^m$  is the semisimple algebra  $\text{Ext}_A^0(\underline{\rho}, \underline{\rho})$  and  $k^\epsilon$  is the module over this algebra with dimension vector  $\epsilon = (e_1, \dots, e_m)$ . If we can find an  $l$ -algebra  $B$  such that  $B^! = \text{Ext}_A^\bullet(\underline{\rho}, \underline{\rho})$ , then we can say that locally (up to a product with an affine space) the space of  $\alpha$ -dimensional representations of  $A$  around  $\rho$  looks like the space of  $\epsilon$ -dimensional representations of  $B$  around the zero representation.

Morally, the algebra  $B$  should be the Koszul dual of  $E := \text{Ext}_A^\bullet(\underline{\rho}, \underline{\rho})$ , so we need to take a look at the construction of the Koszul dual of an  $A_\infty$ -algebra. We restrict to the relevant case where  $E = l \oplus V$  is a finite dimensional augmented  $l$ -algebra with an  $A_\infty$ -structure on  $E$  such that  $\mu_1(l) = 0$ ,  $\mu_2$  is the ordinary multiplication and  $\mu_n(\dots, l, \dots) = 0$  for all  $n > 2$ .

Using a graded  $l$ -basis  $\mathcal{B}$  for  $V$  we can define a differential  $d$  on the completed tensor-algebra  $\widehat{T_l V^*}$  with  $(V^* := \text{Hom}(V, k)$  and  $\forall b \in \mathcal{B} : \deg b^* = 1 - \deg b$ ): For each  $b, b_1, \dots, b_k \in \mathcal{B}$  and we set the coefficient  $b_1^* \otimes \dots \otimes b_k^*$  in  $db^*$  equal the coefficient of  $b$  in  $\mu(b_1 \otimes \dots \otimes b_k)$ . With this identification the graded Leibniz rule for  $d$  and the  $[M_n]$  combine into the rule  $d^2 = 0$ . We will call the dg-algebra  $E^! := (\widehat{T_l V^*}, d)$  the Koszul dual of  $(B, \mu)$ . One can check that if  $E = \text{Ext}_A^\bullet(\underline{\ell}, \underline{\ell}) = A^!$  then  $E^!$  is formal and its homology is the completion of  $A$  by path-length concentrated in degree 0. In other words  $\hat{A}$  is the minimal model of  $E^!$ .

In general  $E^!$  might not be formal, but  $H_0(E^!)$  is enough to construct the Maurer-Cartan equation for  $E$ . Indeed, the Maurer-Cartan equation for  $E$  only depends on  $\mu_i|_{E_1^{\otimes i}}$ . In  $E^!$  these are encoded in the map  $d : E_{-1}^! \rightarrow E_0^!$ .

Note that because all degrees in  $E = \text{Ext}_A^\bullet(\underline{\rho}, \underline{\rho})$  are nonnegative, the degrees in  $E^!$  are nonpositive. The degree zero part of  $E^!$  is the completed tensor algebra  $\widehat{T_l E_1^*}$ , which can be seen as a completed path algebra of a quiver  $Q_L$  with  $m$  vertices and  $\dim iE_1^*j = \dim \text{Ext}_A^1(\sigma_i, \sigma_j)$  arrows from  $i$  to  $j$ . This quiver is called the *local quiver of  $\rho$* .  $E_{-1}^! = \widehat{kQ_L} \otimes_l E_2^* \otimes_l \widehat{kQ_L}$  and the image of  $d|_{E_{-1}^!}$  is the  $\widehat{kQ_L}$ -ideal generated by the  $ds_i$  where the  $s_i$  form an  $l$ -basis for  $E_2^*$ . Hence,  $H_0(E^!)$  is the completed path algebra of the quiver  $Q_L$  with relations  $ds_i$  and

$$\text{Ext}_{H_0(E^!)}^i(l, l) = E_i \text{ for } i \leq 2 \text{ and } \mu_n|_{\text{Ext}_{H_0(E^!)}^1(l, l)^{\otimes n}} = \mu_n|_{\text{Ext}_{H_0(E^!)}^1(l, l)^{\otimes n}}$$

This allows us to conclude

**Theorem 6.2.** *If  $\rho$  is an  $\alpha$ -dimensional semisimple representation of  $A$  with decomposition  $\rho = \sigma_1^{\oplus e_1} \oplus \dots \oplus \sigma_m^{\oplus e_m}$  then the local structure of the representation space around  $\rho$  is the same (up to a product with an affine space) as the local structure of the representation space around the  $\epsilon$ -dimensional zero representation of  $H_0(E^!)$  with  $E = \text{Ext}_A^\bullet(\underline{\rho}, \underline{\rho})$  and  $\epsilon = (e_1, \dots, e_n)$ . If  $H(E^!)$  is the completion of a path algebra with relations  $L$  we can write*

$$k[\widehat{\text{Rep}}_A^\alpha]_\rho \cong k[\widehat{\text{Rep}}_L^\epsilon]_{\rho_0} \otimes k[[X_1, \dots, X_k]] \text{ and } (k[\widehat{\text{Rep}}_A^\alpha]^{\text{GL}_\alpha})_\rho \cong (k[\widehat{\text{Rep}}_L^\epsilon]^{\text{GL}_\epsilon})_{\rho_0}.$$

*Remark 6.3.* The number  $k$  equals the difference  $\dim \mathrm{GL}_\alpha - \dim \mathrm{GL}_\epsilon = \alpha \cdot \alpha - \epsilon \cdot \epsilon$  and we can also identify  $k[\widehat{\mathrm{Rep}}_L^\epsilon]_{\rho_0} \otimes k[[X_1, \dots, X_k]]$  with

$$k[\widehat{\mathrm{Rep}}_L^\epsilon \times_{\mathrm{GL}_\epsilon} \mathrm{GL}_\alpha]_{(\rho_0, 1)}$$

*Remark 6.4.* If  $A$  is hereditary then  $\mathrm{Ext}_A^{\geq 2}(\rho, \rho) = 0$  and the Maurer-Cartan equation becomes trivial. The algebra  $H_0(E^!)$  is equal to  $E^!$  and is just the completed path algebra of the local quiver without any relations. Hence, locally the representation space of an hereditary algebra looks like the representation space of a quiver without relations. This result is an analog of the local quiver theorem by Le Bruyn in [30].

We will now have a look at generalizations of this result to 2-CY.

#### 6.4 Generalizations to 2-Calabi Yau algebras

Suppose for now that  $A$  is 2-CY and  $M$  is a semisimple module with  $\mathrm{End}_A(M) = l = k^m$ . In this case  $\mathrm{Ext}_A^1(M, M)$  has a nondegenerate antisymmetric  $l$ -bilinear form  $\langle f, g \rangle := \mathrm{Tr}_M(fg)$  and hence we can find a symplectic  $l$ -basis of the form  $\{[a_i], [a_i]^* | i \in \mathcal{I}\}$  such that  $\langle [a_i], [a_j] \rangle = 0$ ,  $\langle [a_i]^*, [a_j]^* \rangle = 0$  and  $\langle [a_i], [a_j]^* \rangle = \delta_{ij}$ . Similarly  $\mathrm{Ext}^0(M, M)$  is dual to  $\mathrm{Ext}^2(M, M)$  so each 'vertex'  $[v] \in l$  has a dual element  $[v^*]$  and we have  $[a_i][a_i^*] = [v^*]$  and  $[a_i^*][a_i] = -[w^*]$  for some  $v$  and  $w$  which we can consider as the head and tail of  $a_i$  in the local quiver.

If we take the Koszul dual of  $\mathrm{Ext}_A^\bullet(M, M)$ , it is the completed path algebra of the local quiver  $Q$  with an extra loop  $v^*$  in every vertex  $v$ . If we put  $z = \sum_{v \in Q_0} v^*$  then we get

$$dz = \sum_{a \in Q_1} aa^* - a^*a + h.o.t.$$

Following the same reasoning as in the proof of Theorem 11.2.1 in [41] one can show that up to a change of variables these higher order terms vanish. This implies that

$$H_0(\mathrm{Ext}_A^\bullet(M, M)^!) \cong \widehat{kQ} / \langle dz \rangle \cong \widehat{kQ} / \langle \sum_{a \in Q_1} aa^* - a^*a \rangle.$$

This last algebra is the completed preprojective algebra so in this case  $L = \Pi(Q')$  for some quiver  $Q'$ . Locally representation spaces of 2-CY algebras look like preprojective algebras around the zero representation. This result can be seen as a generalization of theorem 5.10.

To solve the question which semisimple representations are smooth, we need to classify the local quivers and dimension vectors for which the zero representation of the preprojective algebra is smooth. Note that by construction such a dimension vector is sincere, i.e.  $\forall v \in Q_0 : \alpha_v \neq 0$ .

**Theorem 6.5.** *The only quivers and sincere dimension vectors for which  $\text{Rep}_\Pi^\alpha$  is smooth in the zero representation are disjoint unions of quivers with one vertex and an arbitrary number of loops and dimension vector 1, or quivers with one vertex and no loops and arbitrary dimension vector.*

*Proof.* First note that if the quiver is a disjoint union of two subquivers, the preprojective algebra is the direct sum of two smaller preprojective algebras and the representation space is the product of the corresponding representation spaces of these smaller algebras. So we can assume that  $Q$  is connected.

The tangent space to the zero  $\rho_0$  representation in  $\text{Rep}_\alpha \Pi$  is equal to  $\text{Rep}_\alpha Q$  because the derivative

$$\sum [\rho(a), \rho_0(a^*)] + [\rho_0(a), \rho(a^*)] = \sum [\rho(a), 0] + [0, \rho(a)]$$

is identical to zero. Therefore the zero representation is smooth if and only if  $\text{Rep}_\alpha \Pi = \text{Rep}_\alpha Q$ . This means that the relation  $\sum [\rho(a), \rho(a^*)] = 0$  must be identical to zero. This only happens when all arrows are loops and the dimension in the vertex is 1 or there are no arrows.  $\square$

**Corollary 6.6.** *If  $\rho$  is a semisimple representation of a 2-CY algebra then  $\rho$  is a smooth point in the representation space if it is a direct sum of simples without extensions between them, if a simple has no self-extensions it can occur with higher multiplicity. If  $\rho$  has simple representations in its neighborhood then  $\rho$  itself must be simple.*

Finally we need to look at cyclic representations.

**Lemma 6.7.** *If  $A$  is a 2-CY algebra and  $\rho$  is a non-simple semisimple representation such that the component of  $\text{Rep}_A^\alpha // \text{GL}_\alpha$  containing  $\rho$  contains simples and has dimension at least 2, then this component contains a cyclic non-simple representation.*

*Proof.* By the conditions on  $\rho$  and lemma 5.13 we can find such a representation  $\sigma$ , corresponding to a point in  $\text{Rep}^\epsilon L // \text{GL}_\epsilon$ , which can chose in any neighborhood of  $\rho_0$  by rescaling. Artin's approximation applied to the isomorphism  $k[\widehat{\text{Rep}}_A^\alpha]_{\rho}^{\text{GL}_\alpha} \cong k[\widehat{\text{Rep}}_L^\epsilon]_{\rho_0}^{\text{GL}_\epsilon}$  implies that there is a diagram

of étale covers  $\text{Rep}_A^\alpha // \text{GL}_\alpha \leftarrow U \rightarrow \text{Rep}_L^\epsilon // \text{GL}_\epsilon$ . Pulling back and pushing forward we can find a semisimple representation  $\tilde{\sigma}$  of  $A$ . Again by remark 5.11 the dimensions of the simple factors of  $\tilde{\sigma}$  are at least those of  $\sigma$ , so lemma 5.12 implies that  $\tilde{\sigma}$  is cyclic.  $\square$

This lemma and the corollary before it immediately imply Theorem 1.3.

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Raf Bocklandt  
 Korteweg de Vries institute  
 University of Amsterdam (UvA)  
 Science Park 904  
 1098 XH Amsterdam  
 The Netherlands  
 e-mail: raf.bocklandt@gmail.com

Federica Galluzzi  
 Dipartimento di Matematica  
 Università di Torino  
 Via Carlo Alberto n.10 ,Torino  
 10123, ITALIA  
 e-mail: federica.galluzzi@unito.it

Francesco Vaccarino  
 Dipartimento di Scienze Matematiche  
 Politecnico di Torino  
 C.so Duca degli Abruzzi n.24, Torino  
 10129, ITALIA  
 e-mail: francesco.vaccarino@polito.it

and  
ISI Foundation  
Via Alassio 11/c  
10126 Torino - Italy  
e-mail: [vaccarino@isi.it](mailto:vaccarino@isi.it)