Boundary value problems for PDEs and some classes of L^p-bounded pseudodifferential operators

Dissertation

Dottorato di Ricerca in Matematica XXVI ciclo

Politecnico di Torino

presented by

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Torino 2014

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Date of examination: March, 2014

Acknowledgements

I would like to express my gratitude to my supervisors, Professors Luigi Rodino and Lamberto Rondoni, who have made available his support in a number of ways and in particular gave me the freedom to follow my own mathematical ways. Moreover, I am grateful to Prof. Hynek Kovarik (Università degli Studi di Brescia) and Prof. Fabio Nicola (Politecnico di Torino) for their helpful discussions on the subject of this thesis, and also for accepting to collaborate with me when I was still studying the basic facts of the theory. I gratefully acknowledge the financial support of the research training of Turin Polytechnic University in Tashkent " Italy-Uzbekistan project". Moreover, I would like to thank the DEPARTMENT OF MATHEMATICAL SCIENCES "GIUSEPPE LUIGI LAGRANGE" for the kind hospitality. During the tenure of my Ph.D , I spent three years in Torino which was mainly possible due to the financial support provided by my family for which I am greatly indebted.
I would also like to thank my family for the support they provided me through my entire life and in particular, I must acknowledge my wife and children, without whose help, encouragement and editing assistance, I would not have finished this thesis. Thank you for the last three years!

Contents

Part 1

Introduction

Notation and Symbols

We briefly introduce some basic notations that we will use throughout the thesis without further reference. More advanced notations can be found in the appendix.

First of all, $\mathbb{N} = \{1, 2, \ldots\}$ denotes the natural numbers and the symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are used as common. I denotes either the unit interval $[0, 1]$ or the identity operator on a normed linear space. In the latter case we sometimes add the space E on which the identity acts by writing I_E. Unless otherwise stated we assume all linear spaces that we consider to be defined over the complex numbers. For two normed linear spaces E and F we denote by $\mathcal{L}(E, F)$ the space of bounded linear operators with the usual operator norm. For the sake of simplicity **we assume all topological spaces to be Hausdorff**. Hence a space is paracompact if and only if it admits partitions of unity subordinated to any open cover. For a locally compact space X we denote by X^+ its one point compactification and set $X^+ = X \cup \{*\}$ if X is compact, where $*$ is a disjoint point.

$C_c(\Omega)$: The space of continuous functions with compact support in Ω.

$C_c^\infty(\Omega)$: The space of C^∞- functions with compact support in Ω.

$(C_c^\infty(\Omega))'$: The space of distributions on Ω.

$supp(u)$: The support of the function u.

$\sum(\psi) := \{z \in C, z \neq 0, |argz| < \psi\}$, $C^+ := \sum(\frac{\pi}{2})$.

$u^+ := \sup(u, 0)$ the positive part of u, $u^- := \sup(-u, 0)$ the negative part.

$f \wedge g := \inf(f, g)$, $f \vee g := \sup(f, g)$.

$$sign\ u(x) = \begin{cases} \frac{u(x)}{|u(x)|} & if\ u(x) \neq 0, \\ 0 & if\ u(x) = 0. \end{cases}$$

$\mathfrak{R}$: Real part, $\mathfrak{I}$: Imaginary part.

χ_Ω: Characteristic function of Ω.

$L^p(X, \mu, K)$: The classical Lebesgue spaces of functions with values in K.

$||\cdot||_p$: The norm of $L^p(X, \mu, K)$.

dx: Lebesgue measure.

$W^{s,p}$: Sobolev spaces.

$H^1(\Omega) := W^{1,2}(\Omega)$, $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

$D_i = \frac{\partial}{\partial x_i}$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ is the Laplacian.

$\mathcal{L}(E, F)$: The space of bounded linear operators from E into F. $\mathcal{L}(E) := \mathcal{L}(E, E)$.

$||T||_{\mathcal{L}(E,F)}$: The operator norm of T in $\mathcal{L}(E, F)$.

$\rho(A)$: Resolvent set of the operator A. $\sigma(A)$: Spectrum of A.

$C^\alpha(R^n)$: where $0 < \alpha < 1$, Hölder space in R^n.

$\mathfrak{D}'(\Omega)$: The space is the dual of $C_0^\infty(\Omega)$ and $\mathfrak{E}'(\Omega)$ is the dual of $C^\infty(\Omega)$.

$\mathfrak{F}$: Fourier transform, we also denote $\hat{u}$.

$H^s(R^n) = (1 - \Delta)^{-s/2} L^2(R^n)$: Sobolev space.

$\mathfrak{L}_p^s(R^n) = (1-\Delta)^{-s/2}L^p(R^n)$, $s \in R$ and $W_p^k(R^n) = \mathfrak{L}_p^k(R^n)$, $1 < p < \infty$: L^p Style Sobolev spaces.

$OPS^m_{\rho,\delta}$: The operator $p(x, D)$ classes with symbol $p(x, \xi) \in S^m_{\rho,\delta}$.

$p(x, D) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha$ is a differential operator.

$q_{jk}(u) = \sup_{x \in R^n}\{(1 + |x|^2)^{j/2} |D^\alpha u(x)| : |\alpha| \leq k\}$: The seminorms on functions on R^n.

$\mathfrak{S}(R^n)$: The space consists of smooth functions u on R^n for which each $q_{jk}(u)$ is finite, with the Frechet space topology determined by these seminorms and $\mathfrak{S}'(\mathbb{R}^n)$ is dual.

$S^m_{\rho,\delta}(\Omega)$: The symbol class on Ω be an open subset of R^n, $m, \rho, \delta \in R$ and $0 \leq \rho, \delta \leq 1$.

$S^m(\Omega)$: The symbol $p(x, \xi)$ classes, if $p \in S^m_{1,0}(\Omega)$ and there are smooth $p_{m-j}(x, \xi)$, homogeneous of degree $m - j$ in ξ for $|\xi| \geq 1$.

$S^m_{\rho,\delta_1,\delta_2}(\Omega \times \Omega \times R^n)$: The symbol classes, where $0 \leq \rho, \delta_1, \delta_2$.

Introduction

In recent years much attention has been extended in the study of differential equations of non-classical types. These articles need, on one hand, fluid mechanics, hydro-and gas dynamics and other applied disciplines, and on the other hand, the actual needs of the mathematical sciences. One of the most important classes of equations of non-classical type is the third-order equation with multiple characteristics

$$u_{xxx} - \alpha u_y = \Phi(x, y, u, u_x, u_{xx}),$$

which is a generalization of linear Korteweg-de Vries-Burgers equation (see [121])

$$\beta u_{xxx} + u_y + \alpha u_x - \mu u_{xx} = 0,$$

special cases which occur in the dissemination of waves in weakly dispersive media (see [67]), the propagation of waves in a cold plasma, magneto-hydrodynamics (see [15]), problems of nonlinear acoustics (see [102]), the hydrodynamic theory of space plasma (see [15]).

A pioneering work in the theory of odd order partial differential equations with multiple characteristics was done by E.Del Vecchio [115-117], H.Block [19], in which they studied the technique of constructing fundamental solutions of these equations.

Consequently, the theory of equations with multiple characteristics has been greatly developed by the Italian mathematician L.Cattabriga [26-27]. In his works, he built the potentials for partial differential equations with multiple characteristics and investigated various properties of these potentials, when the transition lines are straight.

Following the results of L.Cattabriga [26], T.D.Dzhuraev and his research group [30-32], [1-6] developed the theory further, where they proposed new boundary problems and worked on new approaches to the solution of equations. In these works, the technique of constructing the Green's function for the solution of boundary value problems were developed along with fundamental solutions of odd-order equations with multiple characteristics and with many variables, and a study of their asymptotic properties. In the work of E.L.Roetman [100] , the author has identified the largest class of functions in which there exists a unique solution of the Cauchy problem. Other Russian mathematicians with notable contributions in this field are S.N.Kruzhkov and A.V.Fominskogo [80], N.N.Shopolov [105], A.Eleev [33], A.I.Kozhanov [77-79], and others, whose results are taken for the third order equations.

In the first part of Ph.D thesis we develop and study boundary value problems for third-order equations with multiple characteristics in areas with curved boundaries, as well as some properties of the fundamental solutions of the equations, when the transition line is a curve. In addition, we construct a solution of the Cauchy problem in the classes of functions growing at infinity, depending on the behavior of the right-hand side of the equation.

The theory of the equations of even order with multiple characteristics are developed relatively complete. The presentation of fundamental propositions of the theory with a detailed overview of the main results can be found in the works of V.P.Mikhailov [93] E.A.Baderko [14], L.I.Kamynin [63-66], V.A.Solonnikov, C.D. Eydelman, etc.

Theory of nonlinear problems is an important and relevant section of the modern theory of partial differential equations. In spite of the interesting facts and variety of the original research techniques and the analytic solutions of nonlinear problems, this area of mathematics does not yet have a thorough theoretical foundation methods. Boundary problems with nonlinear boundary conditions for the equations of odd order is a relatively new trend. In this regard, work of S.Abdinazarov and A.R.Khashimov [6] may be noted, where the equations of the third order were delivered to boundary value problems with nonlinear boundary conditions along with a study of their existence and uniqueness solution.

In the work by S.N.Kruzhkov and A.V.Fominskogo, the authors studied a generalized solution of the Cauchy problem for the nonlinear Korteweg-de Vries equation, depending on the nature of the nonlinearity.

Our thesis explores both linear and nonlinear boundary value problems for linear and non-linear third-order equation with multiple characteristics in the domain with curved boundaries.

Throughout this thesis under the regular solution of the problem is the function that has continuous derivatives and satisfies the equations inside the domain. The boundary conditions are satisfied by continuity from inside the domain.

The main result of the first chapter is to prove the unique solvability of the general boundary value problem for the third-order equation with multiple characteristics in curved domains.

We consider the following equation

$$L^i(u_i) \equiv \frac{\partial^3 u_i}{\partial x^3} + a_{i1}(x,y)\frac{\partial u_i}{\partial x} + a_{i0}(x,y)u_i - \frac{\partial u_i}{\partial y} = f_i(x,y), \ i = 1,2, \quad (1)$$

in the domains $D_i = \{(x,y) : h_i(y) < x < h_{i+1}(y), \quad 0 < y \leq Y\}$, $i = 1,2$, where $a_{i1}(x,y)$, $a_{i0}(x,y)$ has discontinuous first type in the curve $x = h_2(y)$.

The functions $h_j(y), j = 1,2,3$ are bounded in the domains D_i, and satisfy the Lipschitz conditions:

$$|h_j(y) - h_j(\eta)| \leq C|y - \eta|,$$

where C is a constant.

Definition. The class of functions $C^{i,j}_{x,y}(D)$ said the class of of continuously differentiable functions, if the derivatives of the orders i and j with respective x and y of the functions exist and are continuous.

We will discuss the following problem

Problem. To find the solution of equation (1) in the domain $D_i (i = 1, 2,)$ which is $u_i(x, y) \in C^{3,1}_{x,y}(D_i) \bigcap C^{2,0}_{x,y}(\bar{D}_i)$, that satisfies the following boundary conditions

$$u_i(x, 0) = F_i(x), \quad h_i(0) \leq x \leq h_{i+1}(0), i = 1, 2,$$

$$u_{1x}(h_1(y), y) = \varphi_1(y), \ 0 \leq y \leq Y,$$

$$\alpha_1(y)u_{1xx}(h_1(y), y) + \alpha_2(y)u_1(h_1(y), y) = \varphi_2(y), \ 0 \leq y \leq Y,$$

$$\beta_1(y)u_{2xx}(h_3(y), y) + \beta_2(y)u_{2x}(h_3(y), y) + \beta_3(y)u_2(h_3(y), y) = \varphi_3(y), \ 0 \leq y \leq Y,$$

and the conditions of discontinuous coefficients in the line $x = h_2(y)$

$$l_k(u_1, u_2) \equiv \frac{\partial^k u_1(h_2(y), y)}{\partial x^k} - \frac{\partial^k u_2(h_2(y), y)}{\partial x^k} = r_k(y), \ 0 \leq y \leq Y, \ k = \overline{0, 2},$$

and also the compatibility conditions

$$\alpha_1(0) \cdot F_1''(h_1(0)) + \alpha_2(0) \cdot F_1(h_1(0)) = \varphi_2(0),$$

$$\beta_1(0)F_2''(h_3(0)) + \beta_2(0) \cdot F_2'(h_3(0)) + \beta_3(0) \cdot F_2(h_3(0)) = \varphi_3(0),$$

$$F_1^{(k)}(h_2(0)) - F_2^{(k)}(h_2(0)) = r_k(0), \ k = \overline{0, 2},$$

$$\varphi'(0) = h_1'(0)F_1''(h_1(0)), \ r_0'(0) = h_2'(0)(F_1'(h_2(0)) - F_2'(h_2(0))).$$

We introduce the following notations

$$P_1(y) \equiv \frac{\beta_2^2(y)}{\beta_1^2(y)} - 2\frac{\alpha_2(y)}{\alpha_1(y)} - h_1'(y) + a_{11}(h_1(y), y),$$

$$\tilde{P}_1(y) \equiv K^2 - 2\frac{\alpha_2(y)}{\alpha_1(y)} - h_1'(y) + a_{11}(h_1(y), y),$$

$$P_2(y) \equiv h_3'(y) + 2\frac{\beta_3(y)}{\beta_1(y)} - \frac{\beta_2^2(y)}{\beta_1^2(y)} + \alpha_{21}(h_3(y), y),$$

where K is a positive number.

Theorem 2.1. Let $a_{i0}(x, y) \in C(\bar{D}_i)$, $a_{i1} \in C^{1,0}_{x,y}(\bar{D}_i)$, $i = 1, 2$, $a_{21}(h_2(y), y) \geq a_{11}(h_2(y), y)$ and satisfy one of the following conditions:

$$\text{if } \alpha_1(y) \neq 0, \ \beta_1(y) \neq 0, \text{then let } \beta_2(y)\alpha_1(y) \geq 0, \ P_1(y) \geq 0, \ P_2(y) \geq 0,$$

$$\text{if } \alpha_1(y) \neq 0, \ \beta_1(y) = 0, \text{then let } \beta_2(y) = 0, \ \beta_3(y) \neq 0, \ \tilde{P}_1(y) \geq 0,$$

$$\text{if } \alpha_1(y) = 0, \ \beta_1(y) = 0, \text{then let } \alpha_2(y) \neq 0, \ P_2(y) \geq 0,$$

$$\text{if } \alpha_1(y) = 0, \ \beta_1(y) = 0, \text{then let } \beta_2(y) = 0, \ \alpha_2(y) \neq 0.$$

Then the solution of the problem is unique.

Theorem 2.2. Let the conditions of Theorem 2.1 be satisfied along with the following conditions: $a_{ij} \in C^{1,2}_{x,y}(\bar{D}_i)$, $(i = 1,2,\ j = 0,1)$; $h_i(y) \in C^2[0,Y]$, $i = 1,2$; $h_3(y) \in C^1[0,Y]$; $F_1(x) \in C^4[c_1,c_2]$; $F_2(x) \in C^4[c_3,c_4]$; $\varphi_1(y), \varphi_2(y), \alpha_1(y), \alpha_2(y), \beta_1(y), \beta_2(y), r_0(y) \in C^2[0,Y]$; $r_2(y) \in C^1[0,Y]$; $\varphi_3(y), \beta_3(y), r_1(y) \in C[0,Y]$; $f_i(x,y) \in C^{0,2}_{0,Y}(\bar{D}_i)$; $f_i(x,0) = f_{iy}(x,0) = 0,\ i = 1,2,$

where $c_1 \le h_1(y) < h_2(y) \le c_2$, $c_3 \le h_2(y) < h_3(y) \le c_4$, $c_l = constant$, $l = \overline{1,4}$.

Then the solution of the problem exists.

To prove the uniqueness theorem of the solution, we use the method of energy integrals. For the existence theorem, we find equivalent systems of Volterra second type integral equations.

In the next chapter, we will study boundary value problem and Cauchy problem for model third order equation.

First we consider the following boundary value problem for the equation

$$\frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial y} = f(x,y) \tag{2}$$

in the domain $D = \{(x,y) : h_1(y) < x < h_2(y),\ 0 < y \le Y\}$, where $h_i(y)(i = 1,2)$ are the curves, and the intersection point of the two curves doesn't exist.

Problem. Find the function $u(x,y) \in C^{3,1}_{x,y} \cap C^{3,1}_{x,y}(D) \cap C^{2,0}_{x,y}(\bar{D})$, which is a regular solution of equation (2) in the domain D and satisfies following boundary conditions

$$u(x,0) = F(x),\ h_1(0) \le x \le h_1(0), \tag{3}$$

$$u_x(h_1(y,y)) = \varphi_1(y),\ 0 \le y \le Y, \tag{4}$$

$$u_x(h_2(y),y) = \varphi_2(y),\ 0 \le y \le Y, \tag{5}$$

$$u_{xx}(h_1(y),y) = \varphi_3(y),\ 0 \le y \le Y, \tag{6}$$

and the compatibility conditions

$$F'(h_1(0)) = \varphi_1(0),\ F''(h_1(0)) = \varphi_3(0),\ F'(h_2(0)) = \varphi_2(0).$$

Where $F(x)$, $\varphi_i(x), i \in \{1,3\}$, $f(x,y)$ are the given bounded smooth functions.

Theorem 3.1. If $h_i(y) \in C^1[0,Y], i = 1,2$, then the solution of problem (2)-(6)is unique.

Theorem 3.2. Let $F(x) \in C^3[c_1,c_2]$, $(c_1 \le h_1(y) < h_2(y) \le c_2)$; $x^{\frac{3}{4}+\delta}F_x(x,y)\varphi_2(y) \in C^1[0,Y]$; $f(x,y) \in C^{0,1}_{x,y}(\bar{D})$; $f(x,0) = 0$ and $h_1(y) \in C^1[0,Y]$. Then there exists $u(x,y) \in C^{3,1}_{x,y}(D) \cap C^{2,0}_{x,y}(\bar{D})$ which is the solution of the problem (2)-(6).

The proof of the theorem will be shown in the next paragraph.

Next we concentrate on the Cauchy problem.

We study the model equation

$$u_{xxx} - u_y = F(x, y) \tag{7}$$

in the domain $D = \{(x, y) : -\infty < x < +\infty,\ 0 < y \le Y\}$ with initial conditions

$$u(x, 0) = 0. \tag{8}$$

Moreover, the problem (7)-(8) has been studied by E.L.Roetman (see [100]), but the behavior of its solutions depending on the behavior of the right-hand side of the equation has not been studied. Our purpose of the study is to construct the solutions of the problem (7)-(8) in the classes of functions that are growing at infinity.

Theorem 3.3. Let the function of bounded variation $F(x, y)$ belongs to any bounded subdomain $D_{[a,b]} = \{(x, y) : a < x < b,\ 0 < y \le Y\}$ of D. Suppose, the variation functions $x^{\frac{3}{4}+\delta} F(x, y)$ and $x^{\frac{3}{4}+\delta} F_x(x, y)$ are bounded for any $x < a$, but at large x we can get the following upper bound

$$F(x, y) < c_1 exp\{c_2 |x|^{\frac{3}{2}-\eta}\},$$

where δ, η are sufficiently small positive number, c_1, c_2 are some constant.

Then the function

$$u(x, y) = -\frac{1}{\pi} \int_0^t \int_{-\infty}^{+\infty} U(x - \xi; y - \eta) F(\xi, \tau) d\xi d\tau$$

satisfies in the domain D the equation (7) and the condition (8).

$U(x - \xi; y - \eta)$ is a fundamental solution of the equation $u_{xxx} - u_y = 0$ and has the form

$$U(x, y; \xi, \eta) = \frac{1}{(y - \eta)^{1/3}} f\left(\frac{x - \xi}{(y - \tau)^{1/3}}\right) \equiv U(x - \xi; y - \tau),$$

where $f(t) = \int_0^\infty \cos(\lambda^3 - \lambda t) d\lambda$ is the Airy function which satisfies the following equation

$$f''(t) + \frac{t}{3} f(t) = 0,$$

and is true for the following asymptotes:

$$f^{(n)}(t) \sim C_n^+ t^{\frac{n}{2}-\frac{1}{4}} \sin(\frac{2}{3} t^{\frac{3}{2}}),\ at\ t \to +\infty,$$

$$f^{(n)}(t) \sim C_n^+ t^{\frac{n}{2}-\frac{1}{4}} exp(-\frac{2}{3} |t|^{\frac{3}{2}}),\ at\ t \to -\infty,$$

$$\int_{-\infty}^0 f(t) dt = \frac{2\pi}{3},\quad \int_0^{+\infty} f(t) dt = \frac{\pi}{3},$$

where C_n^+, C_n^- are positive constants.

The next chapter consists of three sections and it investigates the problem with nonlinear boundary conditions for linear and non-linear equations of the third order with multiple characteristics.

In the first section of the this chapter, we consider the equation (2) in the domain $D = \{(x, y) : h_1(y) < x < h_2(y), 0 < y \leq Y\}$. The curves $x = h_i(y) \in C^1[0, Y], (i = 1, 2)$ define the lateral boundaries of D and the intersection point of the two curves doesn't exist.

Problem. Find in the domain D the regular solution of equation (2), which is continuous together with its derivatives u_x, u_{xx} in the domain $\bar{D}$ and satisfy the boundary conditions

$$u(x, 0) = F(x),\ h_1(0) \leq x \leq h_2(0), \tag{9}$$

$$u_x(h_1(y), y) = g(u(h_1(y), y), y),\ 0 \leq y \leq Y, \tag{10}$$

$$u_{xx}(h_1(y), y) = \varphi_1(y),\ 0 \leq y \leq Y, \tag{11}$$

$$u(h_2(y), y) = \varphi_2(y),\ 0 \leq y \leq Y \tag{12}$$

and the compatibility conditions

$$F'(h_1(0)) = g(u(h_1(0), 0),\ F(h_2(0)) = \varphi_2(0),\ F''(h_1(0)) = \varphi_1(0).$$

The given functions $F(x)$, $g(u, y)$, $\varphi_i(y), (i = 1, 2)$, $f(x, y)$ are bounded, sufficiently smooth functions and the function $g(\xi, y)$ satisfies Lipschitz condition on ξ

$$|g(\xi_1, y) - g(\xi_2, y)| < l(y)|\xi_1 - \xi_2|, \tag{13}$$

where

$$0 < l(y) \leq -k + \sqrt{k^2 + \frac{3k \exp\{-k(h_2(y) - h_1(y))\}}{h_2(y) - h_1(y)}},\ k = \text{const} > 0. \tag{14}$$

Theorem 4.1. If the conditions (13)-(14) are satisfied, the solution of the problem (2), (9)-(12) is unique.

The energy integrals are used to prove the uniqueness of the solutions of (2), (9) - (12) .

Theorem 4.2. Let $F(x) \in C^3[c_1, c_2]$, $(c_1 \leq h_1(y) < h_2(y) \leq c_2)$, $\varphi_i(y) \in C^{3-j}[0, Y]$, $(i, j = 1, 2)$, $|g(u, y)| < M$ for any fixed $|u| < \infty$ and satisfy the condition of Theorem 3.1.1. Then the solution of problem (2), (9)-(12) exists.

In the proof of the existence of solutions of (2), (9) - (12), we constructed the Green's function for the auxiliary problem and used the method of potentials so that our problem became a nonlinear integral equation of Hammerstein type. The existence and uniqueness of solutions of nonlinear integral equation were proved by the method of successive approximations. In the second section we investigate the nonlinear boundary value problem for the nonlinear equations of odd order with multiple characteristics.

Linear boundary value problem for nonlinear equations with multiple characteristics of the third order was considered by T.D.Dzhuraeva (see [31]), and non-linear boundary value problems for linear equations with multiple characteristics in the works of Abdinazarov and Khashimov (see [6]).

Problem. Determine the function $u(x,y)$ in the domain $D = \{(x,y) : h_1(y) < x < h_2(y),\ 0 < y \leq 1\}$ where $u(x,y)$ has the following properties:

1) $u(x,y)$ is a regular solution of equation

$$Lu \equiv u_{xxx} - u_y = f(x,y,u(x,y)) \tag{15}$$

2) $u(x,y) \in C^{3,1}_{x,y}(D) \cap C^{1,0}_{x,y}(\bar{D}) \cap C^{2,0}_{x,y}(\bar{D} \setminus (x = h_1(y))) \cap C(\bar{D})$;

3) the solution satisfies the boundary conditions:

$$u(x,0) = u_0(x),\ h_1(0) \leq x \leq h_2(0), \tag{16}$$

$$u_x(h_1(y),y) = g(u(h_1(y),y),y),\ 0 \leq y \leq 1, \tag{17}$$

$$u_{xx}(h_1(y),y) = \sigma(u(h_1(y),y),y),\ 0 \leq y \leq 1, \tag{18}$$

$$u(h_2(y),y) = \varphi(y),\ 0 \leq y \leq 1. \tag{19}$$

The given functions $u_0(x)$, $g(\xi,y)$, $\sigma(\eta,y)$, $\varphi(y)$, $f(x,y)$ are bounded and smooth in the domain, which satisfy the compatibility conditions at the end points in suitable domain

$$u_0'(h_1(0)) = g(u(h_1(0),0),0),\ u_0''(h_1(0)) = \sigma(u(h_1(0),0),0),\ u_0(h_2(0)) = \varphi(0).$$

We will prove the theorem, which is unique solvability of the problem (15)-(19).

Theorem 4.3. Let $h_i(y) \in C^1([0,1])\ i = 1,2$ and $g(u(h_1(y),y),y) \in C([0,1])$; $\sigma(u(h_(y),y),y) \in C([0,1])$; $f(x,y,u(x,y)) \in C(\bar{D})$; $|g(u_1,y) - g(u_2,y)| \leq l(y)|u_1 - u_2|$; $|\sigma(u_1,y) - \sigma(u_2,y)| \leq k(y)|u_1 - u_2|$, $|f(x,y,u_1) - f(x,y,u_2)| \leq p(x,y)|u_1 - u_2|$.

Then the solution of (15)-(19) is unique.

Theorem 4.4. Let the conditions of Theorem 4.3 be satisfied and let the following conditions hold

$$\varphi(y) \in C^1[0,1], u_0(x) \in C^3[c_1,c_2]\ (c_1 \leq h_1(y) < h_2(y) \leq c_2).$$

Moreover, there exist constants M,N_1,N_2,M_i $(i \in \{1,7\})$, such that for a fixed $y \in [0,1]$ and $|u| < \infty$, the inequalities given below are true $|g(u,y)| < N_1$, $|\sigma(u,y)| < N_2$, $|g_u(u,y)| < M_1$, $|g_y(u,y)| < M_2$, $|\sigma_u(u,y)| < M_3$, $|\sigma_y(u,y)| < M_4$,

for $(x,y) \in D$ and any fixed $|u| < \infty$

$$|f(x,y,u(x,y))| < M,\ |f_x(x,y,u(x,y))| < M_5,$$

$$|f_y(x,y,u(x,y))| < M_6,\quad |f_u(x,y,u(x,y))| < M_7$$

Then the solution of (15)-(19) exists.

To prove the existence and uniqueness theorems, we will use methods of integral energy and theory of integral equations.

In the last part of the thesis we analyze basic properties of pseudodifferential operators, such as the behavior of products and adjoints of such operators, their continuity on L^2, L^p and Sobolev spaces (see Appendix A.2). There are numerous excellent books giving more leisurely and complete treatments et al. Taylor's Pseudodifferential Operators [109], Boggiatto, Buzano an Rodino's Global hypoellipticity and spectral theory [20], Nicola and Rodino's Global Pseudo-Differential Calculus on Euclidean Spaces [99], Schulze's Boundary value problems and singular pseudo-differential operators, Shubin's Pseudodifferential operators in R^n and Pseudodifferential operators and spectral theory and Wong's An introduction to pseudo-differential operators [119]. M.E.Taylor studies the pseudodifferential operators and some of their basic properties , such as the behavior of products and adjoints of such operators, their continuity on L^2 and Sobolev spaces, the fact that they do not increase the singular support of distributions to which they are applied, and the Garding inequality, generalizing following inequality

$$Re(Pu,u) \geq c_1||u||^2_{H^m} - c_2||u||^2_{L_2},\ u \in C_0^\infty(\Omega)$$

for a partial differential operator

$$P = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha,$$

assuming

$$ReP_{2m}(x,\xi) = Re\left(\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha\right) \geq c|\xi|^{2m}.$$

Here

$$||u||^2_{H^m} = \sum_{|\alpha|\leq m} ||D^\alpha u||^2_{L^2}$$

defines a norm on a space H^m known as a Sobolev space, when P is a second order scalar differential operator, Garding's inequality is proved simply by integration by parts. They applied this calculus of pseudodifferential operators to some basic questions of existence and regularity of solutions to elliptic, hyperbolic, and parabolic equations, and elliptic boundary value problems. They also study the behavior of various classes of pseudodifferential operators on L^p and Hölder spaces and include a treatment of estimates for solutions to regular elliptic boundary value problems within these categories. In this work they make use of results of Marcinkiewicz, Mikhlin, and Hörmander on continuity of certain Fourier multipliers on $L^p(R^n)$. Other results are devoted to the Calderon-Vaillancourt theorem on L^2 boundedness of pseudodifferential operators in a borderline case, and to Hörmander-Melin inequalities, on the semiboundedness of second order pseudodifferential operators. The continuity on L^p and Hölder space theory of pseudodifferential operators have been studied by multiple authors like Marcinkiewicz [87], Mikhlin [94], Hörmander [49-55], Stein [106], and Taibleson [109], and we discuss some of the results that were obtained by them.

$$P(D)u = \int e^{ix\xi} p(\xi)\hat{u}(\xi)d\xi$$

on $L^p(R^n)$ and $C^\alpha(R^n)$. $P(D)$ simply multiplies the Fourier transform of u by $p(\xi)$, hence $P(D)$ is called a Fourier multiplier. It can also be written as a convolution operator

$$P(D)u = \hat{p} * u.$$

The basic results on continuity of such an operator on L^p and C^α are merely stated here, and the reader is referred to various places in the literature for proofs. We can see Taibleson's theorem and show it is equivalent to a condition which is somewhat parallel to Hörmander's version of the Marcinkiewicz multiplier theorem. Marcinkiewicz [87] studied the L^p continuity of convolution operators on the torus T^n and Mikhlin [94] translated some of these results to the R^n setting.

In the thesis we study the L^p - boundedness of vector weighted pseudodifferential operators with symbols which have derivatives with respect to x only up to order k, in the Hölder continuous sense, where $k > n/2$ (the case $1 < p \leq 2$) and $k > n/p$ (the case $2 < p < \infty$). First, set $m(\xi)$ bounded continuous function, $||a|| = ||a||_{m,k}$ if $a \in \Lambda_k^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $||a|| = ||a||_{m,k,k'}$ if $a \in \Lambda_{k,k'}^m(\mathbb{R}^n \times \mathbb{R}^n)$. Then we have the following theorems.

Theorem 5.1.17. Let $1 < p \leq 2$, $k > n/2$, $k \notin \mathbb{N}$, E a compact subset of $\mathbb{R}^n$ and $\Omega_1 = \{x \in \mathbb{R}^n | d(x, E) \leq 1\}$. If $a \in \Lambda_k^m(\mathbb{R}^n \times \mathbb{R}^n)$ and *supp* $a \subseteq E \times \mathbb{R}^n$, then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(E)$ with its norm bounded by $C_{E,n,p,k}|\Omega_1|^{1/p}|m(\xi)|||a||$, where $||$ denotes the Lebesgue measure.

Theorem 5.1.18. Let $2 < p < \infty$, $k > n/p$, $k \notin \mathbb{N}$ and E a compact subset of $\mathbb{R}^n$. If $a \in \Lambda_k^m(\mathbb{R}^n \times \mathbb{R}^n)$ and *supp* $a \subseteq E \times \mathbb{R}^n$, then $a(x, D)$ is continuous from $L_{loc}^p(\mathbb{R}^n)$ to $L^p(E)$.

Theorem 5.1.19. let $1 < p \leq 2$, $k > n/2$, $k' > n/p$ and $k, k' \notin \mathbb{N}$. If $a \in \Lambda_{k,k'}^m(\mathbb{R}^n \times \mathbb{R}^n)$, then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with its norm bounded by $C_{n,p,k,k'}|m(\xi)|||a||$.

Theorem 5.1.20. Let $2 < p < \infty$, $k > n/p$, $k' > n/2$ and $k, k' \notin \mathbb{N}$. If $a \in \Lambda_{k,k'}^n(R^n \times \mathbb{R}^n)$, then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with its norm bounded by $C_{n,p,k,k'}|m(\xi)|||a||$. This results also are given by Hwang [57], we only analyze our approach some other Symbol classes and we use to proof our results the same technical methods.

The main purposes objectives of the thesis are:

- The study of general boundary value problems for third-order equation with multiple characteristics and discontinuous coefficients in curved domains.

- The study of the properties of potentials for the third-order equation, when the transition line is a curve.

- The study of problems with nonlinear boundary conditions for linear and non-linear equation of the third order with multiple characteristics in curved domains.

- The construction of the solution of the Cauchy problem in classes of functions growing at infinity, depending on the behavior of the right-hand side of the equation.

- To get the L^p- boundedness for the Pseudodifferential operators with some Symbol classes.

The general procedure for the study: To apply methods of energy integrals, Green's functions, potential theory and integral equations, Pseudodifferential operators, Symbol classes, Fourier transform.

Scientific novelty of dissertational research.

- We prove the unique solvability of the general linear boundary value problems for third-order equation with multiple characteristics and discontinuous coefficients in curved domains.

- The problem of a nonlinear boundary conditions for linear and non-linear third-order equation with multiple characteristics in curved domains.

- The solution of the Cauchy problem in classes of functions growing at infinity, depending on the behavior of the right-hand side of the equations.

- The result of L^p- boundedness for some Symbol classes for the Pseudodifferential operators.

The theoretical and practical value. The results of the work are primarily of theoretical interest. They can be applied in the study of linear and non-linear problems for a wide class of partial differential equations, and can also be used to study specific applications leading to such equations. We also studied the boundedness and continuity problems for Pseudodifferential operators are very important to study the various classes of symbols.

Thesis results were regularly discussed at the seminar on "Modern problems of the theory of partial differential equations" (Institute of Mathematics, Academy of Sciences of Uzbekistan, (Heads of seminar are professors T.D.Dzhuraev and M.S.Salakhitdinov, both members of Uzbekistan Academy of Sciences). The main results were also discussed at the seminar on "Modern Problems of Computational Mathematics and Mathematical Physics" (Head of seminar professor Sh.Alimov member of Uzbekistan Academy of Sciences). Several time are given the talks in the Universita Degli Studi di Torino (professors L.Rodino and J.Seiler). Some parts of the thesis were also presented at various international conferences.

Structure of the thesis. The thesis consists of an introduction, six chapters and a list of bibliography. The equation numbering is twofold: first number indicates the number of the chapter, while the second number indicates the number of the formula in it. The numbering of the allegations is twofold: first number indicates the number of the chapter and the second represents the approval number in it.

We shall review the content of the Ph.D thesis.

The first chapter we give introduction, second chapter we study the boundary value problem for third order linear equation with multiple characteristics and discontinues coefficients and this consists of three sections. The first section contains the formulation a problem, the second section the general properties of the potentials, finally last section shown the solvability of the classical initial-boundary value problems. In the third chapter we give Cauchy and linear boundary value problems, there are two sections, which are first section a boundary value problems, second section the Cauchy problem in the class increasing functions at infinity. Next chapter we introduce the nonlinear boundary value problems. In the chapter V we give some Symbol classes of Pseudodifferential operators and finally last chapter VI is Appendix.

Part 2

Boundary value problem for third order linear equation with multiple characteristics and discontinues coefficients

2.1 The formulation a problem

We will study the domain $D_i = \{(x,y) : h_i(y) < x < h_{i+1}, 0 \le y \le Y, \}\ i = 1,2$. The intersection point of the curves $h_j(y),\ (j = 1,2,3)$ don't exist on the bound of domain D_i. Next, the functions satisfy Lipschiz condition

$$|h_j(y) - h_j(\eta)| \le C|y - \eta| \tag{2.1}$$

where C - positive constant.

In the domain D_i we consider following equation

$$L^i(u_i) \equiv \frac{\partial^3 u_i}{\partial x^3} + a_{i1}(x,y)\frac{\partial u_i}{\partial x} + a_{i0}(x,y)u_i - \frac{\partial u_i}{\partial y} = f_i(x,y),\ (i = 1,2) \tag{2.2}$$

and study the following problem: to find regular solutions of equation (2.2) in the domain D_i, such that $u_i(x,y) \in C^{3,1}_{x,y}(D_i) \cap C^{2,0}_{x,y}(\bar{D}_i)$ and satisfy the boundary conditions

$$u_i(x,0) = F_i(x),\ h_i(0) \le x \le h_{i+1}(0),\ i = 1,2, \tag{2.3}$$

$$u_{1x}(h_1(y),y) = \varphi_1(y),\ 0 \le y \le Y, \tag{2.4}$$

$$\alpha_1(y)u_{1xx}(h_1(y),y) + \alpha_2(y)u_1(h_1(y),y) = \varphi_2(y),\quad 0 \le y \le Y, \tag{2.5}$$

$$\beta_1(y)u_{2xx}(h_3(y),y) + \beta_2(y)u_{2x}(h_3(y),y) + \beta_3(y)u_2(h_3(y),y) = \varphi_3(y),\ 0 \le y \le Y, \tag{2.6}$$

and the coupling conditions on the coefficients are discontinues of the line $x = h_2(y)$

$$l_k(u_1,u_2) \equiv \frac{\partial^k u_1(h_2(y),y)}{\partial x^k} - \frac{\partial^k u_2(h_2(y),y)}{\partial x^k} = r_k(y),\ 0 \le y \le Y,\ k \in \{0,2\} \tag{2.7}$$

as well as the appropriate compatibility conditions

$$\left.\begin{array}{c} F_1'(h_1(0)) = \varphi_1(0), \\ \alpha_1(0)\cdot F_2''(h_1(0)) + \alpha_2(0)\cdot F_1(h_1(0)) = \varphi_2(0), \\ \beta_1(0)\cdot F_2''(h_3(0)) + \beta_2(0)\cdot F_2'(h_3(0)) + \beta_3(0)\cdot F_2(h_3(0)) = \varphi_3(0), \\ F_1^{(k)}(h_2(0)) - F_2^{(k)}(h_2(0)) = r_k(0),\ k = \overline{0,2}, \\ \varphi_1'(0) = h_1'(0)F_1''(h_1(0)),\ r_0'(0) = h_2'(0)(F_1'(h_2(0)) - F_2'(h_2(0))). \end{array}\right\} \tag{2.8}$$

Note that the problem (2.2)-(2.7) at $\alpha_1(y) = \beta_1(y) = \beta_2(y) = 0,\ \beta_3(y) \equiv 1$ was studied in [26] in the rectangular region. In the work [1] , was considered the equation (2.2) with the boundary conditions (2.3) - (2.6) in the domain $\Omega = \{(x,y) :\ 0 < x < 1,\ 0 < y \le 1\}$.

Always assume that

$$\left.\begin{array}{c} h_i(y) \in C^2[0,Y],\ i = 1,2;\ h_3(y) \in C^1[0,Y];\ F_1(x) \in C^4[c_1,c_2]; \\ F_2(x) \in C^4[c_3,c_4];\ \varphi_1(y),\ \varphi_2(y),\ \alpha_1(y),\ \alpha_2(y),\ \beta_1(y),\ \beta_2(y),\ r_0(y) \in C^2[0,Y]; \\ r_2(y) \in C^1[0,Y];\ \varphi_3(y),\ r_1(y),\ \beta_3(y) \in C[0,Y]; \\ f_i(x,y) \in C^{0,2}_{x,y}(\bar{D}_i);\ f_i(x,0) = f_{iy}(x,0) = 0,\ i = 1,2, \end{array}\right\} \tag{2.9}$$

where $c_1 \le h_1(y) < h_2(y) \le c_2$, $c_3 \le h_2(y) < h_3(y) \le c_4$, c_l-constant, $l \in \{1,4\}$

We introduce the following notation

$$P_1(y) \equiv \frac{\beta_2^2(y)}{\beta_1^2(y)} - 2\frac{\alpha_2(y)}{\alpha_1(y)} - h_1'(y) + a_{11}(h_1(y), y),$$
$$\tilde{P}_1(y) \equiv K^2 - 2\frac{\alpha_2(y)}{\alpha_1(y)} - h_1'(y) + a_{11}(h_1(y), y),$$
$$P_2(y) \equiv h_3'(y) + 2\frac{\beta_3(y)}{\beta_1(y)} - \frac{\beta_2^2(y)}{\beta_1^2(y)} + \alpha_{21}(h_3(y), y),$$

where K- is a sufficiently large positive number. We assume that one of the conditions is satisfied:

$$\text{If } \alpha_1(y) \neq 0, \beta_1(y) \neq 0, \text{ then } \beta_2(y)\alpha_1(y) \geq 0, \; P_1(y) \geq 0, \; P_2(y) \geq 0, \tag{2.10}$$

$$\text{If } \alpha_1(y) \neq 0, \; \beta_1(y) = 0, \text{then } \beta_2(y) = 0, \; \beta_3(y) \neq 0, \; \tilde{P}_1(y) \geq 0, \tag{2.11}$$

$$\text{If } \alpha_1(y), \; \beta_1(y) = 0, \text{then } \alpha_2(y) \neq 0, \; P_2(y) \geq 0, \tag{2.12}$$

$$\text{If } \alpha_1(y) = 0, \; \beta_1(y) = 0, \text{then } \beta_2(y) = 0, \; \alpha_2(y) \neq 0. \tag{2.13}$$

Theorem 2.1. If one of the conditions (2.10) - (2.13) is satisfied and also $a_{i0}(x,y) \in C(\bar{D}_i)$, $a_{i1}(x,y) \in C^{1,0}_{x,y}(\bar{D}_i)$, $i = 1,2$, $a_{21}(h_2(y),y) \geq a_{11}(h_2(y),y)$, then the solution (2.2)-(2.7) is unique.

Proof. We consider the case of (2.10). Suppose that there are two solutions of the problem, which are $u_{i1}(x,y)$, $u_{i2}(x,y)$, and consider their difference $\bar{v}_i(x,y) = u_{i1}(x,y) - u_{i2}(x,y)$. The function $\bar{v}_i(x,y)$ satisfies the homogenous equation $L^i(\bar{v}_i) = 0$ and the homogeneous boundary conditions

$$\left.\begin{array}{c} \bar{v}_i(x,0) = 0, \; h_i(0) \leq x \leq h_{i+1}(0), \; i = 1,2, \\ \bar{v}_{1x}(h_1(y),y) = 0, \; 0 \leq y \leq Y, \\ \alpha_1(y)\bar{v}_{1xx}(h_1(y),y) + \alpha_2(y)\bar{v}_1(h_1(y),y) = 0, \; 0 \leq y \leq Y \\ \beta_1(y)\bar{v}_{2xx}(h_3(y),y) + \beta_2(y)\bar{v}_{2x}(h_3(y),y) + \beta_3(y)\bar{v}_2(h_3(y),y) = 0, \; 0 \leq y \leq Y, \\ l_k(\bar{v}_1,\bar{v}_2) = 0, \; 0 \leq y \leq Y, \; k \in \{0,2\}. \end{array}\right\} \tag{2.14}$$

We prove that the function $\bar{v}_i(x,y)$ is identically equal to zero. We set

$$\bar{v}_i(x,y) = v_i(x,y) \cdot \exp(M_i y), \; i = 1,2, \tag{2.15}$$

where $M_i = const > 0$,

Then the functions $v_i(x,y)$ have solutions of the equations

$$M^i(v_i) \equiv v_{ixxx} + a_{i1}(x,y)v_{ix} + (a_{i0} - M_i)v_i - v_{iy} = 0, \; (i = 1,2) \tag{2.16}$$

with boundary conditions

$$\left.\begin{array}{c} v_i(x,0) = 0, \; h_i(0) \leq x \leq h_{i+1}(0), \; i = 1,2 \\ v_{1x}(h_1(y),y) = 0, \; 0 \leq y \leq Y, \\ \alpha_1(y)v_{1xx}(h_1(y),y) + \alpha_2(y)v_1(h_1(y),y) = 0, \; 0 \leq y \leq Y, \\ \beta_1(y)v_{2xx}(h_3(y),y) + \beta_2(y)v_{2x}(h_3(y),y) + \beta_3(y)v_2(h_3(y),y) = 0, \; 0 \leq y \leq Y, \\ l_k(v_1,v_2) = 0, \; 0 \leq y \leq Y, \; k \in \{0,2\}. \end{array}\right\} \tag{2.17}$$

We have the identity

$$\int\int_{D_i} C(x,y)v_i(x,y)M^i(v_i)dxdy = 0,\ i = 1,2, \tag{2.18}$$

where

$$C(x,y) = \exp\left\{-\frac{\beta_2(y)}{\beta_1(y)}(x - h_2(y))\right\}. \tag{2.19}$$

We integrate the (2.18) by parts and using the corresponding homogeneous boundary conditions (2.17), we obtain

$$\sum_{i=1}^{2}\int\int_{D_i}\left(-\frac{1}{2}C_{xxx} - \frac{1}{2}\frac{\partial(Ca_{i1})}{\partial x} + (a_{i0}(x,y) - M_i)C + \frac{1}{2}\frac{\partial C}{\partial y}\right)v_i^2 dxdy+$$

$$+\frac{3}{2}\int\int_{D_i} C_x v_{ix}^2 dxdy - \frac{1}{2}\sum_{i=1}^{2}\int_{h_i(Y)}^{h_{i+1}(Y)} Cv_i^2|_{y=Y}dx-$$

$$-\frac{1}{2}\int_0^Y CP_1(y)v_1^2|_{x=h_1(y)}dy - \frac{1}{2}\int_0^Y CP_2(y)v_2^2|_{x=h_3(y)}dy-$$

$$-\frac{1}{2}\int_0^Y C\left(a_{21}(x,y) - a_{11}(x,y)\right)v_2^2|_{x=h_2(y)}dy - \frac{1}{2}\int_0^Y Cv_{2x}^2|_{x=h_3(y)}dy = 0. \tag{2.20}$$

To get (2.20), we used the fact that the function $C(x,y)$ at $x = h_3(y)$ satisfies equation $C_x + \frac{\beta_2(y)}{\beta_1(y)}C = 0$.

From (2.19) and conditions (2.10), we get $C(x,y) > 0,\ C_x(x,y) \le 0,\ P_1(y) \ge 0,\ P_2(y) \ge 0$. We will choose the numbers M_i $(i = 1,2)$, which satisfy the inequality

$$M_i > \frac{1}{\min_{D_i} C(x,y)}\max\left(-\frac{1}{2}C_{xxx} - \frac{1}{2}\frac{\partial(a_{i1}C)}{\partial x} + a_{i0}(x,y)C + \frac{1}{2}\frac{\partial C}{\partial y}\right)$$

It is always possible in the view of the assumptions of Theorem 2.1.

Then, given the conditions of Theorem 2.1, we can conclude that (2.20) is possible only if $v_1(h_1(y),y) = v_2(h_3(y),y) = v_{2x}(h_3(y),y) = v_i(x,y) = v_i(x,Y) = 0,\ i = 1,2$. Hence, $\bar{v}_i(x,y) \equiv 0$ in the domain $D_i (i = 1,2)$ and the above problem is proved. The other cases are treated similarly.

Theorem 2.2. Let the condition of Theorem 2.1 be satisfied by the conditions below $a_{ij} \in C^{0,2}_{x,y}(\bar{D}_i)$ $(i = 1,2,\ j = 0,1)$; $h_i(y) \in C^2[0,Y],\ i = 1,2$; $h_3(y) \in C^1[0,Y]$; $F_1 \in C^4[c_1,c_2]$;$F_2(x) \in C^4[c_3,c_4]$, $\varphi_1(y),\varphi_2(y),\alpha_1(y),\alpha_2(y),\beta_1(y),\beta_2(y),r_0(y) \in C^2[0,Y]$; $r_2(y) \in C^1[0,Y]$; $\varphi_3(y),r_1(y),\beta_3(y) \in C[0,Y]$; $f_i(x,y) \in C^{0,2}_{x,y}(\bar{D}_i)$; $f_i(x,0) = f_{iy}(x,0) = 0,\ i = 1,2,$

where $c_1 \le h_1(y) < h_2(y) \le c_2,\ c_3 \le h_2(y) < h_3(y) \le c_4$.

Then $u_i(x,y)$- the solutions of problem (2.2,(2.3)-(2.7) exist which are continuous with first and second derivatives u_{ix}, u_{ixx} in closed domain $\bar{D}_i, (i = 1,2)$.

Proof. Let $F_1(x) \in C^4[c_1,c_2]$, $F_2(x) \in C^4[c_3,c_4]$, $(c_1 \le h_1(y) < h_2(y) \le c_2,\ c_3 \le h_2(y) < h_3(y) \le c_4)$, and

$$\left.\begin{aligned} f_1(x,y) \in C^{0;2}_{x,y}(\bar{D}_1),\ f_1(x,0) = f_{1y}(x,0) = 0,\\ f_2(x,y) \in C^{0;2}_{x,y}(\bar{D}_2),\ f_2(x,0) = 0. \end{aligned}\right\} \tag{2.21}$$

Then we can write $F_i(x) \equiv 0,\ (i = 1, 2)$.

We construct solution

$$u_i(x, y) = v_i(x, y) + F_i(x),\ i = 1, 2, \tag{2.22}$$

we get by $v_i(x, y)$ the following problems

$$L^{(i)}(v_i) = \bar{f}_i(x, y),\ i = 1, 2, \tag{2.23}$$

$$\left.\begin{array}{c} v_i(x, 0) = 0,\ h_i(0) \le x \le h_{i+1},\ i = 1, 2, \\ v_{1x}(h_1(y), y) = \bar{\varphi}_1(y),\ 0 \le y \le Y, \\ \alpha_1(y)v_{1xx}(h_1(y), y) + \alpha_2(y)v_1(h_1(y), y) = \bar{\varphi}_2(y),\ 0 \le y \le Y, \\ \beta_1(y)v_{2xx}(h_3(y), y) + \beta_2(y)v_{2x}(h_3(y), y) + \beta_3(y)v_2(h_3(y), y) = \bar{\varphi}_3(y),\ 0 \le y \le Y, \end{array}\right\} \tag{2.24}$$

with coupling conditions of coefficients on the line $x = h_2(y)$

$$l_k(v_1, v_2) \equiv \frac{\partial^k v_1(h_2(y), y)}{\partial x^k} - \frac{\partial^k v_2(h_2(y), y)}{\partial x^k} = \bar{r}_k(y),\ 0 \le y \le Y,\ k \in \{0, 2\} \tag{2.25}$$

where

$$\bar{f}_i(x, y) = f_i(x, y) - F_i''(x) - a_{i1}(x, y)F_i'(x) - a_{i0}(x, y)F_i(x),\ i = 1, 2$$

$$\bar{\varphi}_1(y) = \varphi_1(y) - F_1'(h_1(y)),$$

$$\bar{\varphi}_2(y) = \varphi_2(y) - \alpha_1(y)F_1''(h_1(y)) - \alpha_2(y)F_1(h_1(y)),$$

$$\bar{\varphi}_3(y) = \varphi_3(y) - \beta_1(y)F_2''(h_3(y)) - \beta_2(y)F_2'(h_3(y)) - \beta_3(y)F_2(h_3(y)),$$

$$\bar{r}_k(y) = r_k(y) - \sum_{i=1}^{2} \frac{\partial^k F_i(h_2(y))}{\partial x^k},\quad k \in \{0, 2\}.$$

The functions $\bar{\varphi}_1(y)$, $\bar{\varphi}_2(y)$, $\bar{\varphi}_3(y)$, $\bar{r}_k(y)$ satisfy the coupling conditions $\bar{\varphi}_1(0) = \bar{\varphi}_2(0) = \bar{\varphi}_3(0) = \bar{r}_k(0) = 0,\ k =\in \{0, 2\}$.

Firstly, we consider the inhomogeneous model equations

$$\frac{\partial^3 u_i}{\partial x^3} - \frac{\partial u_i}{\partial y} = g_i(x, y),\ i = 1, 2. \tag{2.26}$$

We can show that functions

$$W_i(x, y) = \frac{1}{\pi} \int_0^Y \int_{h_i(\eta)}^{h_{i+1}(\eta)} U(x, y, \xi, \eta) g_i(\xi, \eta) d\xi d\eta,\ i = 1, 2, \tag{2.27}$$

satisfy the equations (2.26) and with initial condition $W_i(x, 0) = 0$, if $g_i(\xi, \eta) \in C^{0,1}_{x,y}(\bar{D})$, $g_i(x, 0) = 0$, $i = 1, 2$, where $U(x, y; \xi, \eta)$- fundamental solution of (2.26)(see [26]), which is

$$U(x, y; \xi, \eta) = \begin{cases} \frac{1}{(y-\eta)^{1/3}} f(\frac{x-\xi}{(y-\eta)^{1/3}}), & y > \eta,\ x \neq \xi \\ 0 & y \le \eta \end{cases} \tag{2.28}$$

where

$$f(t) = \int_0^\infty \cos(\lambda^3 - \lambda t) d\lambda,\ t = (x - \xi)/(y - \eta)^{1/3}.$$

The function $f(t)$- Airy function which satisfies the equation below

$$f''(t) + \frac{t}{3} f(t) = 0. \quad (2.29)$$

The $f(t)$ function has the asymptotic (see [35])

$$f^n(t) \sim C_n^+ t^{\frac{n}{2}-\frac{1}{4}} \sin(\frac{2}{3} t^{3/2}) \ at \ t \to +\infty, \quad (2.30)$$

$$f^n(t) \sim C_n^+ t^{\frac{n}{2}-\frac{1}{4}} \exp(-\frac{2}{3} |t|^{3/2}) \ at \ t \to -\infty, \quad (2.31)$$

C_n^+, C_n^- - const.

We find the solutions of the problems (2.26), (2.24) of the form

$$u_i(x,y) = \omega_i(x,y) + W_i(x,y), \ i = 1,2. \quad (2.32)$$

Then by function $\omega(x,y), \ i = 1,2$ we get the following problem

$$\tilde{L}^i \omega_i \equiv \frac{\partial^3 \omega_i}{\partial x^3} - \frac{\partial \omega_i}{\partial y} = 0, \ i = 1,2 \quad (2.33)$$

$$\left.\begin{array}{c} \omega_i(x,0) = 0, \ h_i(0) \le x \le h_{i+1}(0), \ i = 1,2, \\ \omega_{1x}(h_1(y),y) = \tilde{\varphi}_1(y), \ 0 \le y \le Y, \\ \alpha_1(y)\omega_{1xx}(h_1(y),y) + \alpha_2(y)\omega_1(h_1(y),y) = \tilde{\varphi}_2(y), \ 0 \le y \le Y, \\ \beta_1(y)\omega_{2xx}(h_3(y),y) + \beta_2(y)\omega_{2x}(h_3(y),y) + \beta_3(y)\omega_2(h_3(y),y) = \tilde{\varphi}_3(y), \ 0 \le y \le Y, \\ l_k(\omega_1,\omega_2) = \tilde{r}_k(y), \ k \in \{0,2\}, \ 0 \le y \le Y, \end{array}\right\} \quad (2.34)$$

where

$$\left.\begin{array}{c} \tilde{\varphi}_1(y) = \varphi_1(y) - W_{1x}(h_1(y),y) - F_1'(h_1(y)), \\ \tilde{\varphi}_2(y) = \varphi_2(y) - \alpha_1(y)W_{1xx}(h_1(y),y) \\ -\alpha_2(y)W_1(h_1(y),y) - \alpha_1(y)F_1''(h_1(y)) - \alpha_2(y)F_1(h_1(y)), \\ \tilde{\varphi}_3(y) = \varphi_3(y) - \beta_1(y)W_{2xx}(h_3(y),y) - \beta_2(y)W_{2x}(h_3(y),y) \\ -\beta_3(y)W_2(h_3(y),y) - \beta_1(y)F_2''(h_3(y)) - \beta_2(y)F_2'(h_3(y)) - \beta_3(y)F_2(h_3(y)), \\ \tilde{r}_k = r_k(y) - \sum_{i=1}^{2}(\frac{\partial^k W_i(h_2(y),y)}{\partial x^k} + \frac{\partial^k F_i(h_2(y))}{\partial x^k}), \ k \in \{0,2\}. \end{array}\right\} \quad (2.35)$$

The functions $\tilde{\varphi}_j(y)$, $\tilde{r}_k(y)$, $(j \in \{1,3\}, k \in \{0,2\})$ satisfy the coupling conditions $\tilde{\varphi}_j(0) = \tilde{r}_k(0) = 0$, $(j \in \{1,3\}, \ k \in \{0,2\})$. Therefore the function $\bar{\varphi}_j(y)$, $\bar{r}_k(y)$ $(j \in \{1,3\})$, $k \in \{0,2\}$ satisfy the condition (2.9), we suppose the functions $g_i(x,y)$ satisfy the condition (2.21). We differentiate (2.27) and get

$$W_{ix}(x,y) = -\frac{1}{\pi} \int \int_{D_i} U_x(x,y;\xi,\eta) g_i(\xi,\eta) d\xi d\eta, \ i = 1,2, \quad (2.36)$$

Let $i = 1$. Suppose

$$\omega_1(x,y) = 3(y-\eta)^{1/3} f' \left(\frac{x-\xi}{(y-\eta)^{1/3}} \right) + (x-\xi) \int_{-\infty}^{(x-\xi)/(y-\eta)^{1/3}} f(t_1) dt_1.$$

We differentiate above function several time and get the following

$$\frac{\partial \omega_1}{\partial x} = \int_{-\infty}^{(x-\xi)/(y-\eta)^{1/3}} f(t_1)dt_1,\ \omega_{1xx} = U,\ \omega_{1xxx} = U_x = \omega_{1y} = -\omega_{1\eta}.$$

From (2.36) we can get

$$W_{1x}(x,y) = -\frac{1}{\pi}\int\int_{D_1} \omega_1(x,y;\xi,\eta)g_{1\eta}(\xi,\eta)d\xi d\eta + \frac{1}{\pi}\int_{h_1(y)}^{x} \omega_1(x,y;\xi,y)g_1(\xi,y)d\xi.$$

Let $x = h_2(y)$. Then

$$W_{1x}(h_2(y),y) = -\frac{1}{\pi}\int\int_{D_1} \omega_1(h_2(y),y;\xi,\eta)g_{1\eta}(\xi,\eta)d\xi d\eta + \frac{1}{\pi}\int_{h_1(y)}^{h_2(y)} [h_2(y)-\xi)]g_1(\xi,y)d\xi.$$

Hence, in view of (2.21) it is easy to show that

$$W_{1x}[h_2(y),y] \in C^1[0,1].$$

The solution of (2.33), (2.34) is given in the form

$$\omega_i(x,y) = \int_0^y U(x,y;h_i(\eta),\eta)\rho_{2i-1}(\eta)d\eta + \int_0^y \frac{\partial^{(2-i)}U(x,y;h_{i+1}(\eta),\eta)}{\partial x^{2-i}}\rho_{2i}(\eta)d\eta +$$

$$\int_0^y V(x,y;h_i(\eta),\eta)\delta_i(\eta)d\eta,\ i = 1,2. \tag{2.37}$$

Hence

$$V(x,y;\xi,\eta) = \begin{cases} \frac{1}{(y-\eta)^{1/3}}\varphi(\frac{x-\xi}{(y-\eta)^{1/3}}), & y > \eta,\ x \neq \xi \\ 0 & y \leq \eta \end{cases} \tag{2.38}$$

where

$$\varphi(t) = \int_0^\infty \exp(-\lambda^3 - \lambda t) + \sin(\lambda^3 - \lambda t)d\lambda,\ t = (x-\xi)/(y-\eta)^{1/3}.$$

The function $\varphi(t)$ is the Airy function that satisfies

$$\varphi''(t) + \frac{t}{3}\varphi(t) = 0. \tag{2.39}$$

The function $\varphi(t)$ has the asymptotes (see [35])

$$\varphi^{(n)}(t) \sim C_n^+ t^{\frac{n}{2}-\frac{1}{4}} \sin\left(\frac{2}{3}t^{3/2}\right)\ at\ t \to +\infty, \tag{2.40}$$

$$\varphi^{(n)}(t) \sim C_n^+ t^{\frac{n}{2}-\frac{1}{4}} \exp\left(-\frac{2}{3}|t|^{3/2}\right)\ at\ t \to -\infty, \tag{2.41}$$

C_n^+, C_n^- - const.

Before proceeding to the proof of the existence of solutions of (2.33)-(2.34), we will present the following lemmas, which we will need in the future.

2.2 General properties of potentials

Lemma 2.3. Let the functions $h(y) \in C^{\alpha}(0 \le y \le Y)$, $\alpha > \frac{3}{4}$ and $\rho(y)$- be continuous in the interval [0,Y]. Then

$$\lim_{(x,y)\to(h(y),y)} \int_0^y \frac{\partial^2 U(x,y;h(\eta),\eta)}{\partial\xi^2}\rho(\eta)d\eta = \frac{\pi}{3}\rho(y)+$$

$$\int_0^y \frac{\partial^2 U(h(y),y;h(\eta),\eta)}{\partial\xi^2}\rho(\eta)d\eta, \ at\ x < h(y), \tag{2.42}$$

$$\left|\frac{\partial^2 U(h(y),y;h(\eta),\eta)}{\partial\xi^2}\right| \le \frac{C}{|y-\eta|^{\frac{5}{4}-\alpha}}, \tag{2.43}$$

Proof. Due to the fact that the function $f(t)$ value is true (2.29), we obtain the following expression

$$\frac{\partial^2 U(x,y;h(\eta),\eta)}{\partial\xi^2} = -\frac{x-h(\eta)}{3(y-\eta)^{4/3}} f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right).$$

Here we assume that $y > \eta$. Then

$$\int_0^y \frac{\partial^2 U(x,y;h(\eta),\eta)}{\partial\xi^2}\rho(\eta)d\eta = -\int_0^y \frac{x-h(\eta)}{3(y-\eta)^{4/3}} f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right)\rho(\eta)d\eta.$$

We transform this expression as follows

$$-\int_0^y \frac{x-h(\eta)}{3(y-\eta)^{4/3}} f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right)\rho(\eta)d\eta =$$

$$-\int_0^y \frac{x-h(y)}{3(y-\eta)^{4/3}} f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right)\rho(\eta)d\eta-$$

$$\int_0^y \frac{h(y)-h(\eta)}{3(y-\eta)^{4/3}} f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right)\rho(\eta)d\eta-$$

$$\int_0^y \frac{x-h(y)}{3(y-\eta)^{4/3}} \left\{ f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right) - f\left(\frac{x-h(y)}{(y-\eta)^{1/3}}\right)\right\}\rho(\eta)d\eta =$$

$$J_1(x,y) + J_2(x,y) + J_3(x,y). \tag{2.44}$$

Let us consider each integral on the right side of (2.44) separately. We have

$$J_1(x,y) = \int_0^y \frac{\partial^2 U(x,y;h(y),\eta)}{\partial\xi^2}(\rho(\eta)-\rho(y))d\eta+$$

$$\int_0^y \frac{\partial^2 U(x,y;h(y),y)}{\partial\xi^2}\rho(y)d\eta = J_{11}(x,y) + \rho(y)J_{12}(x,y).$$

Then

$$\lim_{(x,y)\to(h(y),y)} J_{12}(x,y) = \lim_{(x,y)\to(h(y),y)} -\int_0^y \frac{x-h(y)}{3(y-\eta)^{4/3}} f\left(\frac{x-h(y)}{(y-\eta)^{1/3}}\right)d\eta =$$

$$\lim_{(x,y)\to(h(y),y)} \int_{-\infty}^{\frac{x-h(y)}{y^{1/3}}} f(t)dt = \int_{-\infty}^{0} f(t)dt = \frac{\pi}{3};$$

where $t = \frac{x-h(y)}{(y-\eta)^{1/3}}$.

Now, assume $t = \frac{h(y)-x}{(y-\eta)^{1/3}}$, we have

$$J_{11}(x,y) = \int_{\frac{h(y)-x}{y^{1/3}}}^{+\infty} \left\{ f(-t) \left(\rho(y - \frac{(h(y)-x)^3}{t^3}) - \rho(y) \right) \right\} dt.$$

Further, the continuity $\rho(\eta)$ at the point y for any $\varepsilon > 0$ for available that $\delta(\varepsilon)$, such that

$$|\rho(y) - \rho(y-\eta)| < \varepsilon \tag{2.45}$$

at $h < \delta(\varepsilon)$.

For any fixed $y > 0$ we can always assume that $0 < \delta(\varepsilon) < y^{1/3}$.

Then

$$\frac{h(y)-x}{y^{1/3}} < \frac{h(y)-x}{\delta(\varepsilon)} < +\infty$$

and it makes sense to view

$$J_{11}(x,y) = \int_{\frac{h(y)-x}{y^{1/3}}}^{\frac{h(y)-x}{\delta(\varepsilon)}} \{\cdot\}dt + \int_{\frac{h(y)-x}{\delta(\varepsilon)}}^{+\infty} \{\cdot\}dt \equiv J_{111}(x,y) + J_{112}(x,y).$$

Obviously, we get

$$|J_{111}(x,y)| \leq 2 \max_{0\leq\eta\leq 1} |\rho(\eta)| \int_{\frac{h(y)-x}{y^{1/3}}}^{\frac{h(y)-x}{\delta(\varepsilon)}} |f(-t)|dt,$$

for fixed $\delta(\varepsilon)$ and $y > 0$ due to the fact that the function $f(-t)$ value is true (2.31), we have

$$\lim_{(x,y)\to(h(y),y)} |J_{111}(x,y)| = 0.$$

Further, we are noting that

$$\frac{h(y)-x}{\delta(\varepsilon)} \leq z < +\infty$$

we have the inequality

$$0 \leq \frac{h(y)-x}{z} \leq \delta(\varepsilon),$$

we see that

$$|J_{112}(x,y)| \leq \sup_{|h|\leq\delta(\varepsilon)} |\rho(y) - \rho(y-h)| \int_{0}^{+\infty} |f(-t)|dt.$$

By (2.31) we have

$$|J_{112}(x,y)| \leq c \max_{|h|\leq\delta(\varepsilon)} |\rho(y) - \rho(y-h)| \int_{0}^{+\infty} |t|^{-\frac{1}{4}} e^{-\frac{1}{2}|t|^{3/2}} dt =$$

$$c \max_{|h|\leq\delta(\varepsilon)} |\rho(y) - \rho(y-\eta)| \int_{0}^{+\infty} |z|^{-\frac{1}{2}} e^{-|z|} dz.$$

Hence, by (2.45) and any ε it follows that

$$\lim_{(x,y)\to(h(y),y)} |J_{112}(x,y)| = 0.$$

Finally, we have

$$\lim_{(x,y)\to(h(y),y)} J_1(x,y) = \frac{\pi}{3}\rho(y).$$

Now we prove that

$$\lim_{(x,y)\to(h(y),y)} J_2(x,y) = \int_0^y \frac{h(y)-h(\eta)}{3(y-\eta)^{4/3}} f\left(\frac{h(y)-h(\eta)}{(y-\eta)^{1/3}}\right)\rho(\eta)d\eta.$$

To this end, we estimate the following difference

$$|J_2(x,y) - J_2(h(y),y)| \le \int_0^y \frac{|h(y)-h(\eta)|}{(y-\eta)^{4/3}} \left| f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right) - f\left(\frac{h(y)-h(\eta)}{(y-\eta)^{1/3}}\right)\right| \rho(\eta)d\eta.$$

It is known that $f(t) \in C^\infty(\mathbb{R}^1)$ (see [35]). Then

$$\left| f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right) - f\left(\frac{h(y)-h(\eta)}{(y-\eta)^{1/3}}\right)\right| \le K_1 \frac{|x-h(y)|}{(y-\eta)^{5/12}}.$$

Therefore

$$|J_2(x,y) - J_2(h(y),y)| \le K_2 |x-h(y)| y^{\alpha - \frac{3}{4}}.$$

Accordingly

$$\lim_{(x,y)\to(h(y),y)} |J_2(x,y) - J_2(h(y),y)| = 0.$$

Note that, by the mean value theorem

$$\left| f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right) - f\left(\frac{x-h(\eta)}{(y-\eta)^{1/3}}\right)\right| = f'\left(\frac{x-x(\lambda,\eta)}{(y-\eta)^{1/3}}\right) \frac{h(y)-h(\eta)}{(y-\eta)^{1/3}};$$

where $0<\lambda<1$, $x(\lambda,\eta) = h(y) + \lambda(h(\eta)-h(y))$.

In view of this, we have

$$|J_3(x,y)| \le K_1|x-h(y)| \int_0^y \frac{|h(y)-h(\eta)|}{(y-\eta)^{5/3}} |f'\left(\frac{x-x(\lambda,\eta)}{(y-\eta)^{1/3}}\right)|\rho(\eta)d\eta.$$

This integral is evaluated depending on the location of the point (x,y) and the nature of the curve $h(y)$.

1) If the function $h(y)$ is monotonically decreasing, then $x < x(\lambda,\eta)$.

Therefore, in this case, we use the asymptotic expansion (2.31).

Then

$$|J_3(x,y)| = K_1|x-h(y)| \int_0^y \frac{h(y)-h(\eta)}{(y-\eta)^{5/3}} \left(\frac{|x-x(\lambda,\eta)|^{1/4}}{(y-\eta)^{1/12}}\right) \times$$
$$\exp\left(-\frac{2}{3}\frac{|x-x(\lambda,\eta)|^{3/2}}{(y-\eta)^{1/2}}\right)\rho(\eta)d\eta \le K_2|x-h(y)|y^{\alpha-\frac{2}{3}}. \tag{2.46}$$

2) If the function $h(y)$ is monotonically increasing, then the expression $x - x(\lambda,\eta)$ will be alternating. The function $J_3(x,y)$ is estimated as follows

$$|J_3(x,y)| \le K_1|x-h(y)| \int_0^y \frac{|h(y)-h(\eta)|}{(y-\eta)^{5/3}} \frac{|x-x(\lambda,\eta)|^{1/4}}{(y-\eta)^{1/12}} \rho(\eta)d\eta \le$$

$$K_2|x-h(y)|y^{\alpha-\frac{3}{4}}. \tag{2.47}$$

3) If $h(y) \equiv const$, then $J_3(x,y) \equiv 0$.

In the investigated cases the nature of the curves $h(y)$ in different segments contained in the definition may be different.

Therefore, the division of produce D follows. The point (x,y) can be in the domain D_1, D_2, D_3.

If the point (x,y) is in the domain D_1 the integral $J_3(x,y)$ to be the estimate (2.46).

If the point (x,y) is in the domains D_2, D_3 then the integral $J_3(x,y)$ will be true to the estimate (2.47).

Hence

$$\lim_{(x,y)\to(h(y),y)} J_3(x,y) \equiv 0.$$

A similar argument proves the validity of estimate (2.43).

Lemma 2.4. Let functions $h(y) \in C^\alpha[0,Y]$, $\alpha > \frac{3}{4}$, and $\rho(y)$ be continuous and bounded variation in [0,Y]. Then

$$\lim_{(x,y)\to(h(y),y)} \int_0^y \frac{\partial^2 U(x,y;h(\eta),\eta)}{\partial\xi^2}\rho(\eta)d\eta =$$

$$-\frac{2\pi}{3}\rho(y) + \int_0^y \frac{\partial^2 U(h(y),y;h(\eta),\eta)}{\partial\xi^2}\rho(\eta)d\eta \tag{2.48}$$

at $x > h(y)$,

$$\lim_{(x,y)\to(h(y),y)} \int_0^y \frac{\partial^2 V(x,y;h(\eta),\eta)}{\partial\xi^2}\rho(\eta)d\eta =$$

$$\int_0^y \frac{\partial^2 V(h(y),y;h(\eta),\eta)}{\partial\xi^2}\rho(\eta)d\eta \tag{2.49}$$

Proof. Similarly as in the proof of lemma 2.3 we obtain expression (2.44) and consider integral $J_1(x,y)$

$$J_1(x,y) = \int_0^y \frac{\partial^2 U(x,y;h(y),\eta)}{\partial\xi^2}(\rho(\eta)-\rho(y))d\eta +$$

$$\int_0^y \frac{\partial^2 U(h(y),y;h(\eta),\eta)}{\partial\xi^2}\rho(y)d\eta = J_{11}(x,y) + \rho(y)J_{12}(x,y).$$

Then

$$\lim_{(x,y)\to(h(y),y)} J_{12}(x,y) = \lim_{(x,y)\to(h(y),y)} -\int_0^y \frac{x-h(y)}{3(y-\eta)^{4/3}} f\left(\frac{x-h(y)}{(y-\eta)^{1/3}}\right)d\eta =$$

$$= -\lim_{(x,y)\to(h(y),y)} \int_{\frac{x-h(y)}{y^{1/3}}}^{+\infty} f(t)dt = -\int_0^{+\infty} f(t)dt = -\frac{2\pi}{3};$$

where

$$t = \frac{x-h(y)}{(y-\eta)^{1/3}}.$$

The expression $J_{11}(x,y)$ is of the form below

$$J_{11}(x,y) = \left(\int_0^{y-\delta} + \int_{y-\delta}^{y}\right) \frac{x-h(y)}{3(y-\eta)^{4/3}} f\left(\frac{x-h(y)}{(y-\eta)^{1/3}}\right)(\rho(\eta)-\rho(y))d\eta =$$

$$J_{111}(x,y) + J_{112}(x,y).$$

Any $\delta > 0$, then $\lim_{(x,y)\to(h(y),y)} J_{111}(x,y) = 0$.

We leave the proof of integral $J_{112}(x,y)$ to the reader and we give without proof the following lemma in (see [35]).

Lemma 2.5. Let the variation of function $P(x)$ be bounded on interval $[a,b]$ and let

$$\max\left|\int_\alpha^\beta Q(x)dx\right| < M, \text{ where } (\alpha,\beta)\subset(a,b).$$

Then

$$\left|\int_a^b P(x)Q(x)dx\right| < M\{|P(a)| + V_a^b(P(x))\},$$

where $V_a^b(P(x))$- full variation function $P(x)$ on the interval $[a,b]$.

We use above inequality and get

$$|J_{112}(x,y)| < K\{\max_{y-\delta\le\eta\le y} |\rho(\eta)-\rho(y)| + V_{y-\delta}^{y}(\rho(y))\},$$

where

$$\max\left|\int_{y_1}^{y_2} \frac{x-h(y)}{3(y-\eta)^{4/3}} f\left(\frac{x-h(y)}{(y-\eta)^{1/3}}\right)d\eta\right| < K,\ (y_1,y_2)\subset(y-\delta,y).$$

Hence

$$\lim_{(x,y)\to(h(y),y)} |J_{111}(x,y)| = 0.$$

Finally, we have

$$\lim_{(x,y)\to(h(y),y)} J_1(x,y) = \frac{2\pi}{3}\rho(y).$$

Similar to the proof obtained in above results, we get

$$\lim_{(x,y)\to(h(y),y)} J_2(x,y) = \int_0^y \frac{h(y)-h(\eta)}{3(y-\eta)^{4/3}} f\left(\frac{h(y)-h(\eta)}{(y-\eta)^{1/3}}\right)\rho(\eta)d\eta,$$

$$\lim_{(x,y)\to(h(y),y)} J_3(x,y) = 0.$$

Similarly we can prove the validity of (2.49). Only in this case we use the fact that

$$\int_0^{+\infty} \varphi(t)dt = 0.$$

Lemma 2.6. If the function $g(x) \in C^\gamma$ $(0 < \gamma < 1)$, then

$$G(x) = \int_a^x \frac{g(s)}{(x-s)^\gamma} ds, \quad 0 < \gamma < 1, \ (g(a) = 0),$$

this can be differentiated by x, and we can get

$$G'(x) = \frac{g(x)}{(x-a)^\gamma} - \gamma \int_a^x \frac{g(s) - g(x)}{(x-s)^{\gamma+1}} ds.$$

The proof is elementary (see [47]).

Lemma 2.7. If $h(y) \in C^1(0 \le y \le Y)$, then

$$\lim_{\eta \to z} \int_\eta^z \left(\frac{\partial}{\partial z} \int_t^z \frac{1}{(z-y)^{(2-k)/3}} \frac{\partial^k U(h(y), y; h(t), t)}{\partial x^k} dy \right) dt = 0, \tag{2.50}$$

$$\lim_{\eta \to z} \int_\eta^z \left(\frac{\partial}{\partial z} \int_t^z \frac{1}{(z-y)^{(2-k)/3}} \frac{\partial^k V(h(y), y; h(t), t)}{\partial x^k} dy \right) dt = 0, \tag{2.51}$$

where $k \in \{0, 1\}$.

Proof. Let $k = 0$. We set

$$J(z, \eta) = \int_\eta^z \left(\frac{\partial}{\partial z} \int_t^z \frac{1}{(z-y)^{(2-k)/3}} f\left(\frac{h(y) - h(t)}{(y-t)^{1/3}} \right) dy \right) dt = 0$$

We perform the integration in the following way

$$dv = \frac{dy}{(z-y)^{2/3}(y-t)^{1/3}}$$

$$G(z, y, t) \equiv v = \int_y^z \frac{du}{(z-u)^{2/3}(u-t)^{1/3}},$$

$$u = f\left(\frac{h(y) - h(t)}{(y-t)^{1/3}} \right),$$

$$du = \left\{ \frac{h'(y)}{(y-t)^{1/3}} - \frac{(h(y) - h(t))}{3(y-t)^{4/3}} \right\} f'\left(\frac{h(y) - h(t)}{(y-t)^{1/3}} \right) dy$$

where $G(z, z, t) = 0$, $G(z, t, t) = \frac{2\pi}{3}$. Then

$$J(z, \eta) = \int_\eta^z \left(\frac{\partial}{\partial z} \left\{ f\left(\frac{h(y) - h(t)}{(y-t)^{1/3}} \right) G(z, y, t)|_{y=z}^z + \right. \right.$$

$$\int_t^z G(z, y, t) \left(\frac{h'(y)}{(y-t)^{1/3}} - \frac{h(y) - h(t)}{3(y-t)^{4/3}} \right) f'\left(\frac{h(y) - h(t)}{(y-t)^{1/3}} \right) dydt. (*)$$

Hence, using the properties of the function $G(z, y, t)$, and given the fact that

$$\frac{\partial}{\partial z} G(z, y, t) = \frac{\partial}{\partial z} \int_y^z (z-u)^{2/3}(u-t)^{1/3} du =$$

$$\frac{\partial}{\partial z} \int_0^{\frac{z-y}{z-t}} s^{-2/3}(1-s)^{-1/3} ds = \frac{1}{z-t} \frac{(y-t)^{2/3}}{(z-y)^{2/3}},$$

from (*), performing the differentiation, we obtain

$$J(z,\eta) = \int_\eta^z \frac{dt}{z-t} \int_t^z \frac{(y-t)^{2/3}}{(z-y)^{2/3}} \left\{ \frac{h'(y)}{(y-t)^{1/3}} - \frac{h(y)-h(t)}{3(y-t)^{4/3}} \right\} \times$$
$$f'\left(\frac{h(y)-h(t)}{(y-t)^{1/3}}\right) dy.$$

Using the asymptotic $f(t)$ have

$$|J(z,\eta)| \le c_2(z-\eta).$$

Then

$$\lim_{\eta\to z} |J(z,\eta)| = 0.$$

Let $k=1$, then

$$J(z,\eta) = \int_\eta^z \left(\frac{\partial}{\partial z} \int_t^z \frac{1}{(z-y)^{1/3}(y-t)^{2/3}} f'\left(\frac{h(y)-h(t)}{(y-t)^{1/3}}\right) dy\right) dt.$$

By making arguments similar to the above, we obtain the following expression

$$J(z,\eta) = \int_\eta^z \frac{dt}{z-t} \int_t^z \frac{(y-t)^{1/3}}{(z-y)^{1/3}} \left\{ \frac{h(y)-h(t)}{3(y-t)^{2/3}} f\left(\frac{h(y)-h(t)}{(y-t)^{1/3}}\right) \times\right.$$
$$\left(h'(y) - \frac{h(y)-h(t)}{3(y-t)}\right) dy.$$

Hence

$$|J(z,\eta)| \le c_4(z-\eta).$$

Then

$$\lim_{\eta\to z} |J(z,\eta)| = 0.$$

Similarly we can prove the validity of (2.51).

Lemma 2.8. If $h_i(y)$, $h_j(y) \in C^1[0,Y]$ and $h_i(y) < h_j(y)$, then

$$\lim_{\eta\to z} \int_\eta^z \frac{1}{(z-y)^{(2-k)/3}} \frac{\partial^{k+2} U(h_i(y),y;h_j(\eta),\eta)}{\partial x^{k+2}} dy = 0, \tag{2.52}$$

$$\lim_{\eta\to z} \int_\eta^z \frac{1}{(z-y)^{(2-k)/3}} \frac{\partial^{k+1} U(h_i(y),y;h_j(\eta),\eta)}{\partial x^k \partial y} dy = 0, \tag{2.53}$$

$$\lim_{\eta\to z} \int_\eta^z \frac{1}{(z-y)^{(2-k)/3}} \left(\int_\eta^y \frac{\partial^k U(h_j(y),y;h_i(\eta),\eta)}{\partial x^k} dt\right) dy = 0, \tag{2.54}$$

$$\lim_{\eta\to z} \int_\eta^z \frac{1}{(z-y)^{(2-k)/3}} \left(\int_\eta^y \frac{\partial^k V(h_j(y),y;h_i(\eta),\eta)}{\partial x^k} dt\right) dy = 0, \tag{2.54}$$

where $k \in \{0,1\}$, $\eta \le y \le z$.

Proof. Let $k=1$. We set

$$J(z,\eta) = \int_\eta^z \frac{1}{(z-y)^{1/3}} \frac{\partial^3 U(h_i(y),y;h_j(\eta),\eta)}{\partial x^3} dy.$$

Transform the integral as follows

$$J(z,\eta)=\int_\eta^z \frac{1}{(z-y)^{1/3}}\frac{\partial^3 U(h_i(y),y;h_j(\eta),\eta)}{\partial x^3}dy+$$

$$\int_\eta^z \frac{1}{(z-y)^{1/3}}\left\{\frac{\partial^3 U(h_i(y),y;h_j(\eta),\eta)}{\partial x^3}-\frac{\partial^3 U(h_i(y),y;h_j(\eta),\eta)}{\partial x^3}\right\}dy=$$

$$J_1(z,\eta)+J_2(z,\eta).$$

By condition of lemma $h_i(y)<h_j(y)$. Therefore, using the asymptotic expansion (2.31), we have

$$|J_1(z,\eta)|\le K_1\int_\eta^z \frac{1}{(z-y)^{1/3}}\frac{|h_i(y)-h_j(\eta)|^{5/4}}{|y-\eta|^{21/12}}\exp\left(-\frac{3|h_i(y)-h_j(\eta)|^{3/2}}{2|y-\eta|^{1/2}}\right)dy\le$$

$$\le K_2(z-\eta)^{2/3}$$

It follows that

$$\lim_{\eta\to z}|J_1(z,\eta)|=0.$$

We are applying the mean value theorem to the expression $J_2(z,\eta)$ and transform it as follows

$$J_2(z,\eta)=\int_\eta^z \frac{1}{(z-y)^{1/3}}[h_i(y)-h_j(\eta)]\frac{\partial^4 U(h_i(y),y;h_j(y)+\lambda(h_i(y)-h_j(y)),\eta)}{\partial x^4}d\eta=$$

$$\int_\eta^z \frac{h_i(y)-h_j(y)}{(z-\eta)^{1/3}(y-\eta)^{5/3}}f^{(IV)}\left(\frac{h_i(y)-h_j(y)-\lambda(h_j(\eta)-h_j(y))}{(y-\eta)^{1/3}}\right)d\eta.$$

So as we explore this integral at $\eta\to z$, we should take advantage of the asymptotic behavior (1.31). Indeed, if $\eta\to z$ that $\eta\to y$. Consequently $\lambda(h_j(\eta)-h_j(y))\to 0$. Since by assumption $h_j(\eta)-h_j(y)\ge c_0>0$, the argument functions $f^{(IV)}(t)$ are committed to the $-\infty$. This makes it possible to use the asymptotic expansion (2.31). Hence at $z-\eta_1<\varepsilon_1$, $y-\eta<\varepsilon_2$ the integral is estimated as follows

$$|J_2(z,\eta)|\le K_2(z-\eta)^{25/36},$$

where ε_1 and ε_2 are sufficiently small positive number. Hence

$$\lim_{\eta\to z}|J_2(z,\eta)|=0.$$

the validity of the remaining relations can be proved similarly.

Lemma 2.9. If $h(y)\in C^2[0,Y]$, then

$$\left|\frac{\partial}{\partial z}\int_\eta^z\left(\frac{\partial}{\partial z}\int_t^z \frac{1}{(z-y)^{(2-k)/3}}\frac{\partial^k U(h(y),y;h(t),t)}{\partial x^k}dy\right)dt\right|<c_4 \tag{2.56}$$

$$\left|\frac{\partial}{\partial z}\int_\eta^z\left(\frac{\partial}{\partial z}\int_t^z \frac{1}{(z-y)^{(2-k)/3}}\frac{\partial^k V(h(y),y;h(t),t)}{\partial x^k}dy\right)dt\right|<c_5 \tag{2.57}$$

where $k\in\{0,1\}$, c_4,c_5 - const >0.

Proof. Let $k=1$.

We set
$$I=\frac{\partial}{\partial z}\int_\eta^z\left[\frac{\partial}{\partial z}\int_t^z\frac{1}{(z-y)^{1/3}(y-t)^{2/3}}f'\left(\frac{h(y)-h(t)}{(y-t)^{1/3}}\right)dy\right]dt.$$
Using the properties of the function
$$G(z,y,t)\equiv v=\int_y^z\frac{du}{(z-u)^{1/3}(u-t)^{2/3}},$$
$$G(z,z,t)=0,\ G(z,t,t)=\frac{2\pi}{3},$$
we take differentiation, as well as taking into account Lemma 2.6, we have
$$I=\frac{\partial}{\partial z}\int_\eta^z\frac{t}{(z-t)}\int_t^z\frac{1}{(z-y)^{1/3}}f''\left(\frac{h(y)-h(t)}{(y-\eta)^{1/3}}\right)\left(h'(y)-\frac{h(y)-h(t)}{3(y-\eta)}\right)dy,$$
where
$$\frac{\partial}{\partial z}\left(f'\left(\frac{h(y)-h(t)}{(y-t)^{1/3}}\right)G(z,y,t)|_{y=t}^z\right)=0.$$
Integrating by parts, we have
$$I=\frac{\partial}{\partial z}\int_\eta^z\frac{t}{(z-t)}\int_t^z\frac{(z-y)^{2/3}}{(y-t)^{1/3}}\left\{f\left(\frac{h(y)-h(t)}{(y-\eta)^{1/3}}\right)\{(h'(y)-\right.$$
$$\frac{2h(y)-h(t)}{3(y-\eta)}\left[h'(y)-\frac{h(y)-h(t)}{3(y-\eta)}\right]+\left(h''(y)(y-t)+\frac{2(h(y)-h(t))}{9(y-\eta)}\right)\left[h'(y)-\frac{h(y)-h(t)}{(y-\eta)^{2/3}}\right]+$$
$$f\left(\frac{h(y)-h(t)}{(y-t)^{1/3}}\right)[h(y)-h(t)]\left[h'(y)-\frac{h(y)-h(t)}{3(y-\eta)}\right]^2dy,$$
where
$$f\left(\frac{h(y)-h(t)}{(y-t)^{1/3}}\right)\left[h'(y)-\frac{h(y)-h(t)}{3(y-\eta)}\right]\int_y^z\frac{du}{(z-u)^{2/3}}|_{y=t}^z dy=0.$$
Differentiating and taking into account that
$$\lim_{\eta\to z}\int_t^z\frac{(z-y)^{2/3}}{(y-t)^{1/3}}\left\{f\left(\frac{h(y)-h(t)}{(y-\eta)^{1/3}}\right)\left[h'(y)-\frac{2h(y)-h(t)}{3(y-\eta)}\right]\times\right.$$
$$\left[h'(y)-\frac{h(y)-h(t)}{3(y-\eta)}\right]+\frac{h(y)-h(t)}{(y-\eta)^{2/3}}\left[(y-t)h''(y)+\frac{2(h(y)-h(t))}{9(y-t)}\right]+$$
$$f'\left(\frac{h(y)-h(t)}{(y-t)^{1/3}}\right)(h(y)-h(t))\left[\frac{h(y)-h(t)}{3(y-t)}+h'(y)\right]^2dy=0,$$
We get
$$I=\int_\eta^z\frac{dt}{2(z-t)}\int_t^z\frac{1}{(z-y)^{1/3}(y-t)^{1/3}}\left\{f\left(\frac{h(y)-h(t)}{(y-\eta)^{1/3}}\right)\{(h'(y)-\right.$$
$$\frac{2(h(y)-h(t))}{3(y-t)}+\frac{h(y)-h(t)}{(y-t)^{2/3}}\left[(y-t)h''(y)+\frac{2(h(y)-h(t))}{9(y-t)}\right]+$$

$$f'\left(\frac{h(y)-h(t)}{(y-t)^{1/3}}\right)(h(y)-h(t))\left[\frac{h(y)-h(t)}{3(y-t)}+h'(y)\right]^2 dy.$$

In view of the condition of the lemma we have

$$|I| < c_4.$$

Other cases can be proved similarly

Lemma 2.10. If $h_i(y) \in C^1[0,Y]$, then

$$\left|\frac{\partial}{\partial z}\int_\eta^z (\frac{1}{(z-y)^{(2-k)/3}}\left[\int_\eta^y \frac{\partial^k U(h_2(y),y;h_1(t),t)}{\partial x^k}dt\right]dy\right| < c_6 \tag{2.58}$$

$$\left|\frac{\partial}{\partial z}\int_\eta^z (\frac{1}{(z-y)^{(2-k)/3}}\left[\int_\eta^y \frac{\partial^k V(h_2(y),y;h_1(t),t)}{\partial x^k}dt\right]dy\right| < c_7 \tag{2.59}$$

where $k \in \{0,1\}$, c_6, c_7 - const > 0.

Proof Let $k=0$. We consider

$$I = \frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{1/3}}\left[\int_\eta^y \frac{1}{(z-y)^{1/3}} f(\frac{h_2(y)-h_1(t)}{(y-\eta)^{1/3}})dt\right]dy =$$

$$\frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{1/3}}\left\{\int_\eta^y \frac{1}{(z-y)^{1/3}} f\left(\frac{h_2(y)-h_1(t)}{(y-\eta)^{1/3}}\right)dt+\right.$$

$$\left.\int_\eta^y \frac{1}{(y-t)^{1/3}}\left[f\left(\frac{h_2(y)-h_1(t)}{(y-t)^{1/3}}\right)-f\left(\frac{h_2(y)-h_1(y)}{(y-t)^{1/3}}\right)\right]dtdy =\right.$$

$$I_1 + I_2.$$

If we put

$$s = \frac{h_2(y)-h_1(y)}{(y-t)^{1/3}},$$

then

$$I_1 = \frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{2/3}}\left[3(h_2(y)-h_1(y))^2\int_\mu^\infty \frac{f(s)}{s^3}ds\right]dy,$$

where

$$\mu = \frac{h_2(y)-h_1(y)}{(y-t)^{1/3}}.$$

Differentiating, we get

$$I_1 = \int_\eta^z \frac{1}{(z-y)^{2/3}}\left\{6[h_2(y)-h_1(y)][h_2'(y)-h_1(y)]\int_\mu^\infty \frac{f(s)}{s^3}ds-\right.$$

$$\left.\frac{3}{(y-\eta)^{1/3}}\left[\frac{h_2'(y)-h_1'(y)}{h_2(y)-h_1(y)}(y-\eta)^{-1/3}\right]f\left(\frac{h_2(y)-h_1(y)}{(y-t)^{1/3}}\right)dy.\right.$$

It follows that $|I_1| < c_6$, where $c_6 - const > 0$. In respect that

$$\lim_{\eta\to z}\int_\eta^z \frac{1}{(y-t)^{1/3}}\left[f\left(\frac{h_2(y)-h_1(t)}{(y-t)^{1/3}}\right)-f\left(\frac{h_2(y)-h_1(y)}{(y-t)^{1/3}}\right)\right]dt = 0,$$

We have

$$I_2 = \int_\eta^z \frac{1}{(z-y)^{2/3}} \left\{ \frac{\partial}{\partial y} \int_\eta^y \frac{1}{(y-t)^{1/3}} \left[f\left(\frac{h_2(y)-h_1(t)}{(y-t)^{1/3}}\right) - f\left(\frac{h_2(y)-h_1(t)}{(y-t)^{1/3}}\right)\right] dt \right\} dy.$$

Differentiating, we get

$$I_2 = \int_\eta^z \frac{1}{(z-y)^{2/3}} \left\{ \frac{1}{(y-t)^{1/3}} \left[f\left(\frac{h_2(y)-h_1(y)}{(y-t)^{1/3}}\right) + f\left(\frac{h_2(y)-h_1(t)}{(y-t)^{1/3}}\right)\right] \right\} dy+$$

$$+\int_\eta^z \frac{1}{(z-y)^{2/3}} \left\{ \frac{\partial}{\partial y} \int_\eta^y \frac{1}{(y-t)^{1/3}} \left[f\left(\frac{h_2(y)-h_1(y)}{(y-t)^{1/3}}\right) - f\left(\frac{h_2(y)-h_1(t)}{(y-t)^{1/3}}\right)\right] dt \right\} dy =$$

$$I_{21} + I_{22}.$$

It is easy to show that $|I_{21}| < c_{62}$, $c_{62} = const > 0$. We get $s = y - t$.

Then

$$I_{22} = \int_\eta^z \frac{1}{(z-y)^{2/3}} \left\{ \frac{\partial}{\partial y} \int_\eta^{y-\eta} \frac{1}{(s)^{1/3}} \left[f\left(\frac{h_2(y)-h_1(y-s)}{(s)^{1/3}}\right) - \right.\right.$$

$$\left.\left. f\left(\frac{h_2(y)-h_1(y)}{(s)^{1/3}}\right) dsdy = \int_\eta^z \frac{1}{(z-y)^{2/3}} \left\{ \int_0^y \frac{1}{(z-y)^{1/3}} \times \right.\right.$$

$$\left[\frac{h_2'(y)-h_1'(y)}{(y-t)^{1/3}} f'\left(\frac{h_2(y)-h_1(t)}{(y-t)^{1/3}}\right) - \frac{h_2'(y)-h_1'(y)}{(y-t)^{1/3}} f'\left(\frac{h_2(y)-h_1(y)}{(y-t)^{1/3}}\right)\right] dtdy+$$

$$\int_\eta^z \frac{1}{(z-y)^{2/3}} \frac{1}{(y-\eta)^{1/3}} \left[f\left(\frac{h_2(y)-h_1(\eta)}{(y-\eta)^{1/3}}\right) - f\left(\frac{h_2(y)-h_1(y)}{(y-\eta)^{1/3}}\right)\right] dy$$

Hence

$$|I_{22}| < c_{63},\ c_{63} - \text{ const} > 0.$$

Finally we can take the following bound

$$|I| < c_6,\ c_6 - \text{const} > 0.$$

Similarly we can prove the other case.

2.3 Solvability of the classical initial-boundary value problems

Now we will analyze the solution of problems (2.33)-(2.34).

The solution of problems (2.33)-(2.34) is given by (2.37) with boundary conditions and matching condition, and we use lemma 2.3-2.10 to get

$$\frac{d}{dz} \int_0^z \frac{\tilde{\varphi}(y)}{(z-y)^{1/3}} dy = \frac{2\pi}{\sqrt{3}} f'(0)\rho_1(z) + \frac{2\pi}{\sqrt{3}} \varphi'(0)\delta_1(z)+$$

$$\int_0^z \rho_1'(\eta) d\eta \frac{\partial}{\partial z} \int_\eta^z \left[\frac{\partial}{\partial z} \int_t^z \frac{1}{(z-y)^{1/3}} \frac{\partial U(h_1(y), y; h_1(t), t)}{\partial x} dy \right] dt+$$

$$\int_0^z \delta_1'(\eta)d\eta \frac{\partial}{\partial z}\int_\eta^z \left[\frac{\partial}{\partial z}\int_t^z \frac{1}{(z-y)^{1/3}}\frac{\partial V(h_1(y),y;h_1(t),t)}{\partial x}dy\right]dt+$$

$$\int_\eta^z \rho_2(\eta)d\eta \frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{1/3}}\frac{\partial^3 U(h_1(y),y;h_2(t),t)}{\partial x^3}dy,$$

$$\bar{\varphi}_2(z) = -\frac{2\pi}{3}\alpha_1(z)\rho_1(z) + \alpha_1(z)\int_0^z \frac{\partial^2 U(h_1(z),y;h_1(\eta),\eta)}{\partial x^2}\rho_1(\eta)d\eta+$$

$$\alpha_1(z)\int_0^z \frac{\partial^3 U(h_1(z),y;h_2(\eta),\eta)}{\partial x^3}\rho_2(\eta)d\eta+$$

$$\alpha_1(z)\int_0^z \frac{\partial^2 V(h_1(z),z;h_1(\eta),\eta)}{\partial x^2}\delta_1(\eta)d\eta+$$

$$\alpha_2(z)\left\{\int_0^z U(h_1(z),z;h_1(\eta),\eta)\rho_1(\eta)d\eta+\right.$$

$$\int_0^z \frac{\partial U(h_1(z),z;h_2(\eta),\eta)}{\partial x}\rho_2(\eta)d\eta + \int_0^z V(h_1(z),z;h_1(\eta),\eta)\delta_1(\eta)d\eta; \tag{2.61}$$

$$\tilde{\varphi}_3(z) = \beta_1(z)\left(\int_0^z \left(\int_\mu^{+\infty} f(t_1)dt_1\right)\rho_3'(z)d\eta\right) + \frac{\pi}{3}\rho_4(z)+$$

$$+\beta_1(z)\int_0^z U_{xx}(h_3(z),z;h_3(\eta),\eta)\rho_4(\eta)d\eta + \beta_1(z)\left(\int_0^z\left(\int_\mu^{+\infty}\varphi(t_1)dt_1\right)\delta_2'(\eta)d\eta\right)+$$

$$\beta_2(z)\int_0^z U_x(h_3(z),z;h_2(\eta),\eta)\rho_3(\eta)d\eta + \beta_2(z)\int_0^z U_x(h_3(z),z;h_3(\eta),\eta)\rho_4(\eta)d\eta+$$

$$\beta_2(z)\int_0^z V_x(h_3(z),z;h_2(\eta),\eta)\delta_2(\eta)d\eta+$$

$$\beta_3(z)\int_0^z U(h_3(z),z;h_2(\eta),\eta)\rho_3(\eta)d\eta + \beta_3(z)\int_0^z U(h_3(z),z;h_3(\eta),\eta)\rho_4(\eta)d\eta)+$$

$$\beta_3(z)\int_0^z V(h_3(z),z;h_2(\eta),\eta)\delta_2(\eta)d\eta, \tag{2.62}$$

$$\int_0^z \frac{\bar{r}_0'(y)dy}{(z-y)^{2/3}} = \int_0^z \rho_1(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{2/3}}[\int_\eta^y U(h_2(y),y;h_1(t),t)dt]dy+$$

$$\int_0^z \rho_2(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{2/3}(y-\eta)^{1/3}} f\left(\frac{h_2(y)-h_1(\eta)}{(y-\eta)^{1/3}}\right)dy$$

$$\int_0^z \delta_1(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{2/3}}[\int_\eta^y V(h_2(y),y;h_1(t),t)dt]dy-$$

$$\frac{2\pi}{\sqrt{3}}f(0)\rho_3(z) - \frac{2\pi}{\sqrt{3}}\varphi(0)\delta_2(z)-$$

$$\int_0^z \rho_3(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{2/3}}U(h_2(y),y;h_2(\eta),\eta)dy-$$

$$\int_0^z \delta_2(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{2/3}}V(h_2(y),y;h_2(\eta),\eta)dy-$$

$$\int_0^z \rho_4(\eta)d\eta \frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{2/3}}U(h_2(y),y;h_3(\eta),\eta)dy, \tag{2.63}$$

$$\tilde{r}_1(z)=\frac{\pi}{3}\rho_2(z)+\int_0^z \frac{\partial^2 U(h_2(z),z;h_2(\eta),\eta)}{\partial x^2}\rho_2(\eta)d\eta+$$

$$\int_0^z U_x(h_2(z),z;h_1(\eta),\eta)\rho_1(\eta)d\eta+\int_0^z V_x(h_2(z),z;h_1(\eta),\eta)\delta_1(\eta)d\eta-$$

$$\int_0^z U_x(h_2(z),z;h_2(\eta),\eta)\rho_3(\eta)d\eta-\int_0^z U_x(h_2(z),z;h_3(\eta),\eta)\rho_4(\eta)d\eta-$$

$$\int_0^z V_x(h_2(z),z;h_2(\eta),\eta)\delta_2(\eta)d\eta. \tag{2.64}$$

$$\tilde{r}_2(z)=\int_0^z U_{xx}(h_2(z),z;h_1(\eta),\eta)\rho_1(\eta)d\eta+$$

$$\int_0^z U_{xx}(h_2(z),z;h_2(\eta),\eta)\rho_2(\eta)d\eta+\int_0^z V_{xx}(h_2(z),z;h_1(\eta),\eta)\delta_1(\eta)d\eta+$$

$$\frac{2\pi}{3}\rho_3(z)-\int_0^z U_{xx}(h_2(z),z;h_2(\eta),\eta)\rho_3(\eta)d\eta-\int_0^z U_{xx}(h_2(z),z;h_2(\eta),\eta)\rho_3(\eta)d\eta-$$

$$\int_0^z V_{xx}(h_2(z),z;h_2(\eta),\eta)\delta_2(\eta)d\eta, \tag{2.65}$$

where $\mu=(h_2(z)-h_1(\eta))/(y-\eta)^{-1/3}$.

The system of integral equations (2.60)-(2.65) is equivalent to the system of Volterra integral equations of the second kind

$$\Psi_l(z)=R_l(z)+\sum_{s=1}^{6}\int_0^z N_{sl}(z,\eta)\Psi_s(\eta)d\eta,\ l\in\{1,6\}, \tag{2.66}$$

where $\Psi_l(z)$- unknown functions which match the densities $\rho_{2i-1}^{(3-i)}(z)$, $\rho_{2i}(z)$, $\delta_i^{(3-i)}$, $(i=1,2)$, accordingly $R_l(z)$ are known functions, which are expressed in terms of the given functions $\tilde{\varphi}_{k+1}(z)$ and $\tilde{r}_k(z)$, $(k\in\{0,2\})$; $N_{sl}(z,\eta)$ - matrix whose elements are expressed in terms of the fundamental solutions of the equation (2.33).

It is easy to show that the kernel $N_{sl}(z,\eta)$ has a weak singularity of the form

$$|N_{sl}(z,\eta)|<\frac{c}{|z-\eta|^{2/3}},\ c=\text{const}>0, \tag{2.67}$$

Then, from the general theory [113] that the system (2.66) is uniquely solvable in the class of continuous functions which can be represented in the form

$$\Psi_l(z)=R_l(z)+\sum_{s=1}^{6}\int_0^z H_{sl}(z,\eta)R_s(\eta)d\eta,\ l\in\{1,6\}. \tag{2.68}$$

$H_{sl}(z,\eta)$ - resolution has a weak singularity of the form (2.67).

According to the representations (see [1]), (2.35) and (2.26) the solution of (2.25), (2.14) is given by following formula

$$u_i(x,y) = -\frac{1}{\pi}\int\int_{D_{yi}} G_i(x,y;\xi,\eta)g_i(\xi,\eta)d\xi d\eta, \quad i=1,2, \tag{2.69}$$

where $G_i(x,y;\xi,\eta)$- known functions, which are expressed fundamental solutions of the equation (2.33).

Now we are turn to the solution of the problem (2.2)-(2.7). As we have seen, the function (2.69) for the given $g_i(x,y)$ of the relevant class satisfies the equation (2.25) and the homogeneous boundary conditions (2.14). The solution of (2.2) and (2.14) that we are looking is in the form (2.69), where $g_i(x,y)$ is to be determined, i.e, now we chose $g_i(x,y)$, so that the function (2.69) satisfies the equation (2.2). We substitute (2.69) to the equation (2.2), and obtain

$$g_i(x,y) = f_i(x,y) + \frac{1}{\pi}\int\int_{D_{yi}} K_i(x,y;\xi,\eta)g_i(\xi,\eta)d\xi d\eta, \quad i=1,2, \tag{2.70}$$

where

$$K_i(x,y;\xi,\eta) = a_{1i}(x,y)G_{ix}(x,y;\xi,\eta) + a_{0i}(x,y)G_i(x,y;\xi,\eta), \quad i=1,2.$$

Thus, we determine the functions $g_i(x,y)$ obtained from the integral equation (2.70). If we take into account, $G_i(x,y;\xi,\eta)$ which is expressed in terms of functions $U(x,y;\xi,\eta)$, it is easy to see that for the function $G_i(x,y;\xi,\eta)$ same estimates hold as those for $U(x,y;\xi,\eta)$. Consequently, the kernels $K_i(x,y;\xi,\eta)$ have a weak singularity. Hence, by the uniqueness theorem it is implied that the integral equation (2.70) is uniquely solvable. It's enough that the functions $f_i(x,y)$ satisfy the condition (2.9). According to the results presented above, the solutions of problems (2.2)-(2.7) have the form:

$$u_i(x,y) = \omega_i(x,y) + Z_i(x,y) + H_i(x,y), \; i=1,2,$$

where the functions $\omega_i(x,y)$- the solution of the problems (2.33)-(2.34), $Z_i(x,y)$- the solutions of the problems (2.2)and (2.14), $H_i(x,y)$- the solutions of the following equation

$$H_{ixxx} - H_{iy} + a_{1i}(x,y)H_{ix} + a_{0i}(x,y)H_i = (-a_{1i}\omega_{ix} + a_{0i}(x,y)\omega_i), \; i=1,2,$$

satisfy the homogeneous boundary conditions (2.14).

Part 3

Linear boundary value problems and Cauchy problems for third-order equations with multiple characteristics

3.1 A problem for the third-order equation with multiple characteristics

In this section we consider the following problem.

Problem We consider the equation

$$\tilde{L}(u) \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial y} = f(x, y) \tag{3.1}$$

in the domain $D = \{(x, y) : h_1(y) < x < h_2(y),\ 0 < y \leq 1\}$ with boundary conditions

$$u(x, 0) = F(x),\ h_1(0) \leq x \leq h_2(0), \tag{3.2}$$

$$u_x(h_1(y), y) = \varphi_1(y),\quad 0 \leq y \leq 1, \tag{3.3}$$

$$u_{xx}(h_1(y), y) = \varphi_2(y),\ 0 \leq y \leq 1, \tag{3.4}$$

$$u_x(h_2(y), y) = \varphi_3(y),\ 0 \leq y \leq 1, \tag{3.5}$$

and a condition of compatibility

$$F'(h_1(0)) = \varphi_1(0),\ F''(h_1(0)) = \varphi_2(0),\ F'(h_2(0)) = \varphi_3(0).$$

Where $F(x)$, $\varphi_i(x)$, $i \in \{1, 3\}$, $f(x, y)$- are given, which are bounded as well as sufficiently smooth functions; the curves $x = h_i(y) \in C^1[0, 1]$, $(i = 1, 2)$ are defined on the lateral boundaries and don't have intersection points.

We note that a similar study for the equation (3.1) with other boundary conditions was carried out in [27], [30-32], [1-6].

Uniqueness of solutions of the problem

Theorem 3.1. If $h_i(y) \in C^1[0, 1]$, $i = 1, 2$, then the solution $u(x, y) \in C^{3,1}_{x,y}(D) \cap C^{2,0}_{x,y}(\bar{D})$ of the problems (3.1)-(3.5) is unique.

Proof. Suppose there are two solutions of the problem $u_1(x, y)$ and $u_2(x, y)$. We set $v(x, y) = u_1(x, y) - u_2(x, y)$. Then for the function $v(x, y)$ we get the following problem

$$\tilde{L}(v) = 0,$$

$$\left.\begin{aligned} v(x, 0) = 0,\ h_1(0) \leq x \leq h_2(0), \\ v_x(h_1(y), y) = 0,\ 0 \leq y \leq 1, \\ v_{xx}(h_1(y), y) = 0,\ 0 \leq y \leq 1, \\ v_x(h_2(y), y) = 0,\ 0 \leq y \leq 1. \end{aligned}\right\} \tag{3.6}$$

We consider the identity

$$\int\int_D v_{xx}(v_{xxx} - v_y) dx dy = 0. \tag{3.7}$$

Integrating by parts and using the homogeneous boundary conditions (3.6) to (3.7) , we obtain

$$\frac{1}{2}\int_0^1 v_{xx}^2(h_2(y),y)dy + \frac{1}{2}\int_{h_1(1)}^{h_2(1)} v_x^2(x,1)dx = 0.$$

Hence $v_{xx}(h_2(y),y) = 0$, $v_x(x,1) = 0$. Therefore $v(x,1) = \text{const}$.

Now let $v(x,y) = \omega(x,y)e^{My}$, $M = \text{const} \neq 0$.

Then

$$\int\int_D \omega_{xx} L_1(\omega) \cdot e^{My} dxdy = \frac{M}{2}\int\int_D \omega_x^2 \cdot e^{My} dxdy = 0,$$

where

$$L_1(\omega) = \omega_{xxx} - \omega_y - M\omega. \tag{3.8}$$

Hence $\omega_x(x,y) = 0$. Therefore $\omega(x,y) = w(y)$. Substituting the function $w(y)$ on (3.8) and (3.2) we get

$$\begin{aligned} w'(y) + Mw(y) = 0, \\ w(0) = 0. \end{aligned} \tag{3.9}$$

It is known that the solution of (3.9) is trivial. Which means that $v(x,y) = 0$ in the closed domain $\bar{D}$.

Existence of the solution to the above problem.

Theorem 3.2. Let $F(x) \in C^3[c_1,c_2]$ $c_1 \leq h_1(0) < h_2(0) \leq c_2$; $\varphi_1(y)$, $\varphi_3(y) \in C^2[0,Y]$, $\varphi_2 \in C^1[0,Y]$, $f(x,y) \in C^{0,1}_{x,y}(\bar{D})$, $f(x,0) = 0$ and $h_i(y) \in C^1[0,Y]$.

Then the solution $u(x,y) \in C^{3,1}_{x,y}(D) \cap C^{2,0}_{x,y}(\bar{D})$ of the problems (3.1)-(3.5) exists.

Proof. Let $F(x) \in C^3[c_1,c_2]$ $(c_1 \leq h_1(y) < h_2(y) \leq c_2)$. Then, without loss of generality, we can put $F(x) = 0$.

Indeed, if we put

$$u(x,y) = v(x,y) + F(x),$$

then for the function $v(x,y)$ we get the following problem:

$$\left.\begin{aligned} \tilde{L}(v) = \bar{f}(x,y), \\ v(x,0) = 0,\ h_1(0) \leq x \leq h_2(0), \\ v_x(h_1(y),y) = \bar{\varphi}_1(y),\ 0 \leq y \leq 1, \\ v_{xx}(h_1(y),y) = \bar{\varphi}_2(y),\ 0 \leq y \leq 1, \\ v_x(h_2(y),y) = \bar{\varphi}_3(y),\ 0 \leq y \leq 1, \end{aligned}\right\} \tag{3.10}$$

where

$$\bar{\varphi}_1(y) = \varphi_1(y) - F'(h_1(y)), \quad \bar{\varphi}_2(y) = \varphi_2(y) - F''(h_1(y)),$$

$$\bar{\varphi}_3(y) = \varphi_3(y) - F'(h_2(y)), \quad \bar{f}(x,y) = f(x,y) - F'''(x).$$

As shown in [26], the function

$$W(x,y) = \frac{1}{\pi}\int\int_D U(x,y;\xi,\eta)\bar{f}(\xi,\eta)d\xi d\eta \tag{3.11}$$

satisfies (3.1) and the condition $W(x,y)=0$, if $\bar f(\xi,\eta)\in C^{0,1}_{x,y}(\bar D)$, $\bar f(x,0)=0$. Here $U(x,y;\xi,\eta)$-fundamental solution of equation (3.1) (see [26]), is defined by (2.28). With this in mind, the solution of (3.10)is in the form

$$v(x,y)=\omega(x,y)+W(x,y).$$

Then for the function $\omega(x,y)$ we get the problems

$$\tilde L\omega=0, \tag{3.12}$$

$$\left.\begin{array}{c}\omega(x,0),\ h_1(0)\le x\le h_2(0),\\ \omega_x(h_1(y),y)=\tilde\varphi_1(y),\ 0\le y\le 1,\\ \omega_{xx}(h_1(y),y)=\tilde\varphi_2(y),\ 0\le y\le 1,\\ \omega_x(h_2(y),y)=\tilde\varphi_3(y),\ 0\le y\le 1,\end{array}\right\} \tag{3.13}$$

where

$$\tilde\varphi_1(y)=\varphi_1(y)-F'(h_1(y))-W_x(h_1(y),y),$$

$$\tilde\varphi_2(y)=\varphi_2(y)-F''(h_1(y))-W_{xx}(h_1(y),y),$$

$$\tilde\varphi_3(y)=\varphi_3(y)-F'(h_2(y))-W_x(h_2(y),y),$$

The solution of (3.12)-(3.13) that we are looking for is in the form

$$\omega(x,y)=\int_0^y U(x,y;0,\eta)\alpha_1(\eta)d\eta+\int_0^y U(x,y;1,\eta)\alpha_2(\eta)d\eta+$$

$$\int_0^y V(x,y;0,\eta)\alpha_3(\eta)d\eta. \tag{3.14}$$

Here $U(x,y;\xi,\eta)$, $V(x,y;\xi,\eta)$ are defined respectively by the formulas (2.28) and (2.38). Satisfying the boundary conditions (3.13), we obtain

$$\tilde\varphi_1(y)=\int_0^y U_x(h_1(y),y;h_1(\eta),\eta)\alpha_1(\eta)d\eta+\int_0^y U_x(h_1(y),y;h_2(\eta))\alpha_2(\eta)d\eta+$$

$$\int_0^y V_x(h_1(y),y;h_1(\eta),\eta)\alpha_3(\eta)d\eta,$$

$$\tilde\varphi_2(y)=\int_0^y U_{xx}(h_1(y),y;h_1(\eta),\eta)\alpha_1(\eta)d\eta+\int_0^y U_{xx}(h_1(y),y;h_2(\eta),\eta)\alpha_2(\eta)d\eta+$$

$$\int_0^y V_{xx}(h_1(y),y;h_1(\eta),\eta)\alpha_3(\eta)d\eta,$$

$$\tilde\varphi_3(y)=\int_0^y U_x(h_2(y),y;h_1(\eta),\eta)\alpha_1(\eta)d\eta+\int_0^y U_x(h_2(y),y;h_2(\eta),\eta)\alpha_2(\eta)d\eta+$$

$$\int_0^y V_x(h_2(y),y;h_1(\eta),\eta)\alpha_3(\eta)d\eta,$$

Applying Abel's transformation, given Lemma 2.3-2.10 and after some simple calculations we obtain the system of the form

$$\int_0^z \frac{\tilde{\varphi}_1''(y)}{(z-y)^{1/3}}dy = \frac{2\pi}{\sqrt{3}}f'(0)\alpha_1'(z) + \frac{2\pi}{\sqrt{3}}\varphi'(0)\alpha_3'(z) + \int_0^z \alpha_1'(\eta)d\eta\times$$
$$\frac{\partial}{\partial z}\int_\eta^z \left(\frac{\partial}{\partial z}\int_t^z \frac{1}{(z-y)^{1/3}}\frac{\partial U(h_1(y),y;h_1(t),t)}{\partial x}dy\right)dt + \int_0^z \alpha_3'(\eta)d\eta\times$$
$$\frac{\partial}{\partial z}\int_\eta^z \left(\frac{\partial}{\partial z}\int_t^z \frac{1}{(z-y)^{1/3}}\frac{\partial V(h_1(y),y;h_1(t),t)}{\partial x}dy\right)dt+$$
$$\int_0^z \alpha_2(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{1/3}}\frac{\partial^2 U(h_1(y),y;h_2(t),t)}{\partial x\partial y}dy, \tag{3.15}$$

$$\tilde{\varphi}_2(z) = \frac{\pi}{3}\alpha_1'(z) + \int_0^z \alpha_1'(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \frac{\partial^2 U(h_1(z),z;h_1(t),t)}{\partial z^2}dt+$$
$$\int_0^z \alpha_2(\eta)\frac{\partial}{\partial z}\left(\frac{\partial^2 U(h_1(z),z;h_2(\eta),\eta)}{\partial x^2}\right)d\eta, \tag{3.16}$$

$$\int_0^z \frac{\tilde{\varphi}_3''(y)}{(z-y)^{1/3}}dy = \int_0^z \alpha_1'(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \frac{\partial}{\partial z}\left(\int_t^z \frac{1}{(z-y)^{1/3}}\frac{\partial U(h_2(y),y;h_1(t),t)}{\partial x}dy\right)dt+$$
$$\int_0^z \alpha_2(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \frac{1}{(z-y)^{1/3}}\frac{\partial^2 U(h_2(y),y;h_2(t),t)}{\partial x\partial y}dy+$$
$$\int_0^z \alpha_3'(\eta)d\eta\frac{\partial}{\partial z}\int_\eta^z \left(\frac{\partial}{\partial z}\int_t^z \frac{1}{(z-y)^{1/3}}\frac{\partial^2 V(h_2(y),y;h_1(t),t)}{\partial x\partial y}dy\right)dt. \tag{3.17}$$

The system of integral equations (3.15), (3.16), (3.17) is equivalent to a system of Volterra integral equations of the second kind (see [113])

$$\alpha_l(z) = R_l(z) + \sum_{s=1}^{3}\int_0^z N_{sl}(z,\eta)\alpha_s(\eta)d\eta,\ l\in\{1,3\} \tag{3.18}$$

where $\alpha_l(z)$- unknown functions, $R_l(z)$- known function, which are expressed at given functions $\tilde{\varphi}_i(z)$ $(i\in\{1,3\})$,and $N_{sl}(z,\eta)$- matrix whose elements are expressed in terms of the fundamental solution of equation (3.1).

It is easy to show that the kernel has a weak singularity of the form

$$|N_{sl}(z,\eta)| < \frac{C}{|z-\eta|^{1/2}}, \tag{3.19}$$

where $C = \text{const} > 0$. Then, from the general theory [113] , the system (3.18) is uniquely solvable in the class of continuous functions and can be represented in the form

$$\alpha_l(z) = R_l(z) + \sum_{s=1}^{3}\int_0^z H_{sl}(z,\eta)R_s(\eta)d\eta,\ l\in\{1,3\}.$$

Here the resolution $H_{sl}(z,\eta)$ has a weak singularity of the form (3.19).

3.2 The solution of Cauchy problem for third-order equation with multiple characteristics in the class increasing functions at infinity.

In the section we study the solution of the equation

$$u_{xxx} - u_y = F(x,y) \tag{3.20}$$

in the domain $D = \{(x,y) : -\infty < x < +\infty,\ 0 < y \leq Y\}$, with initial condition

$$u(x,0) = 0. \tag{3.21}$$

Note that the problem (3.20)-(3.21) has been considered in [100], but the behavior of its solutions with $|x| \to \infty$, depending on the behavior of the right-hand side of the equation, has not been studied.

The purpose of the study is to construct solutions of (3.20)-(3.21) in the classes of functions growing at infinity.

It is known (see [26]), that is the fundamental solution of (3.20) has the following form

$$U(x,y;\xi,\eta) = \frac{1}{(y-\eta)^{1/3}} f\left(\frac{x-\xi}{(y-\eta)^{1/3}}\right) \equiv U(x-\xi; y-\eta),$$

where $f(t) = \int_0^\infty \cos(\lambda^3 - \lambda t)d\lambda$, $-\infty < t < +\infty$, is the Airy function, which satisfies the equation (2.29) with relations (2.30), (2.31) and following properties

$$\begin{gathered} \int_{-\infty}^{0} f(t)dt = \tfrac{2\pi}{3}, \quad \int_0^{+\infty} f(t)dt = \tfrac{\pi}{3}, \\ \int_{-\infty}^{+\infty} U(x,y)dx = \int_{-\infty}^{+\infty} f(t)dt = \pi. \end{gathered} \tag{3.22}$$

Theorem 3.3. Let the function of bounded variation $F(x,y)$ belongs to any bounded subdomain $D_{[a,b]} = \{(x,y) : a < x < b,\ 0 < y_0 < Y\}$ of domain D. Suppose that the variation functions $x^{\frac{3}{4}+\delta}F(x,y)$ and $x^{\frac{3}{4}+\delta}F_x(x,y)$ are bounded by any $x < a$ and at high of x

$$F(x,y) < c_1 \exp\left\{c_2|x|^{\frac{3}{2}-\eta}\right\}, \tag{3.23}$$

where δ, η- sufficiently small positive numbers, c_1, c_2 - some constants.

Then the function

$$u(x,y) = -\frac{1}{\pi}\int_0^y \int_{-\infty}^{+\infty} U(x-\xi; y-\tau)F(\xi,\tau)d\xi d\tau \tag{3.24}$$

satisfies the equation (3.20) in the domain D and initial condition (3.21).

Proof. Differentiating formally the expression (3.24) with respective x and y, we obtain

$$\frac{\partial^3 u(x,y)}{\partial x^3} = -\frac{1}{\pi}\int_0^y \int_{-\infty}^{+\infty} \frac{\partial^3 U(x-\xi, y-\tau)}{\partial x^3} F(\xi,\tau)d\xi d\tau =$$

$$\frac{1}{\pi}\int_0^y\int_{-\infty}^{+\infty}\frac{\partial^3 U(x-\xi;y-\tau)}{\partial\xi^3}F(\xi,\tau)d\xi d\tau =$$
$$\frac{1}{\pi}\int_0^y\int_{-\infty}^{+\infty}\frac{\partial^3 U(\xi;y-\tau)}{\partial\xi^3}F(x-\xi,\tau)d\xi d\tau.$$

On the other hand

$$\frac{\partial^3 U(x-\xi;y-\tau)}{\partial x^3} = \frac{1}{3(y-\tau)}(U(x-\xi;y-\tau)+(x-\xi)U_x(x-\xi;y-\tau)) =$$
$$\frac{\partial U(x-\xi;y-\tau)}{\partial y},$$
$$u_y(x,y) = -F(x,y)-\frac{1}{\pi}\int_0^y d\tau\int_{-\infty}^{+\infty}U_y(x-\xi;y-\tau)d\xi.$$

By the change of variables $z=\frac{x-\xi}{(y-\tau)^{1/3}}$ we obtain

$$\frac{\partial^3 u}{\partial x^3} = \frac{1}{\pi}\int_0^y\int_{-\infty}^{+\infty}\frac{1}{3(y-\tau)}(f(z)+zf'(z))F\left(x-z(y-\tau)^{\frac{1}{3}},\tau\right)dzd\tau. \tag{3.25}$$

Furthermore, we have

$$u_{xxx}(x,y) = \frac{1}{3\pi}\int_0^y\frac{d\tau}{y-\tau}\int_{-\infty}^{+\infty}\frac{d}{dz}(zf(z))F\left(x-z(y-\tau)^{\frac{1}{3}},\tau\right)dz =$$
$$\frac{1}{3\pi}\int_0^y\frac{d\tau}{t-\tau}\left(\int_{-\infty}^{-r}+\int_{-r}^{r}+\int_r^{+\infty}\right)\frac{d}{dz}(zf(z))F\left(x-z(y-\tau)^{1/3},\tau\right)dz =$$
$$I_1(x,y)+I_2(x,y)+I_3(x,y), \tag{3.26}$$

where r - a sufficiently large positive number.

Now we study $I_1(x,y)$ in the domain $D_{[a,b]}$ a sufficiently large positive r.

$$I_1(x,y) = \frac{1}{3\pi}\int_0^y\frac{d\tau}{y-\tau}\int_{-\infty}^{+\infty}(f(z)+zf'(z))F(x-z(y-\tau)^{\frac{1}{3}},\tau)dz =$$
$$I_{11}(x,y)+I_{12}(x,y). \tag{3.27}$$

First, we consider the second term on the right hand side of (3.27). By (3.24) and (2.31) we have

$$I_{12}(x,y) = \int_0^y\frac{d\tau}{y-\tau}\int_{-\infty}^{r}zf'(z)F\left(x-z(y-\tau)^{\frac{1}{3}},\tau\right)dz =$$
$$= O\left(\int_0^y\frac{d\tau}{y-\tau}c_2\int_r^{+\infty}z^{\frac{5}{4}}\exp\left\{-z^{\frac{3}{2}}\left(c_1-c_3z^{-\eta}|\frac{x}{z}+(y-\tau)^{1/3}|^{\frac{3}{2}-\tau}\right)\right\}dz\right),$$

where $c_i=\text{const}$, $(i\in\{1,3\})$.

Hence the $I_{12}(x,y)$ uniformly converges to zero at $r\to+\infty$ in the domain $D_{[a,b]}$. Similarly, we can show the uniform convergence of the integral $I_{11}(x,y)$. As a result, we obtain the convergence of the integral $I_1(x,y)$.

Now we study $I_3(x,y)$ in the domain $D_{[a,b]}$ for a sufficiently large positive number r. Integrating by parts in the expression for $I_3(x,y)$, we obtain

$$I_3(x,y) = -\frac{1}{3\pi}\int_0^y \frac{d\tau}{y-\tau}\left(zf(z)F\left(x-z(y-\tau)^{\frac{1}{3}},\tau\right)|_r^{+\infty}+\right.$$

$$\left.\int_r^{+\infty} zf(z)F_\xi\left(x-z(y-\tau)^{1/3},\tau\right)(y-\tau)^{1/3}dz\,\right) = I_{31}(x,y)+I_{32}(x,y).$$

We have to use (2.30) and obtained

$$I_{31}(x,y) \le \left|\frac{1}{3\pi}\int_0^y \frac{d\tau}{y-\tau}\int_r^{+\infty}\left\{z^{-\delta}\sin\left(\frac{2}{3}z^{\frac{2}{3}}\right)\left(\frac{x}{z}-(y-\tau)^{1/3}\right)^{-\frac{3}{4}-\delta}\times\right.\right.$$

$$\left(x-z(y-\tau)^{\frac{1}{3}}\right)^{\frac{3}{4}+\delta}F\left(x-z(y-\tau)^{\frac{1}{3}},\tau\right)dz\,\}\,| \le \frac{1}{3\pi}\int_0^y \frac{d\tau}{y-\tau}M_1M_2\frac{(y-\tau)^{\frac{\delta}{3}}}{(x-\xi)^{\delta}}|_{-\infty}^{-r} =$$

$$\frac{1}{3\pi}\frac{M_1M_2}{(x+r)^\delta}\int_0^y \frac{d\tau}{(y-\tau)^{1-\frac{\delta}{3}}} - \frac{1}{3\pi}\lim_{\xi\to-\infty}\frac{M_1M_2}{(x-\xi)^\delta}\int_0^y\frac{d\tau}{(y-\tau)^{1-\frac{\delta}{3}}} = \frac{1}{3\pi(x+r)^\delta}y^{\frac{\delta}{3}}.$$

Therefore $I_{31}(x,y)$ uniformly converges to zero at $r\to+\infty$ in the domain $D_{[a,b]}$.

Now we consider the expression $I_{32}(x,y)$

$$I_{32}(x,y) = -\frac{1}{\pi}\int_0^y \frac{d\tau}{(y-\tau)^{\frac{2}{3}}}\int_r^{+\infty} z^{-\delta}\sin\left(\frac{2}{3}z^{\frac{3}{2}}\right)\left(\frac{x}{y}-(y-\tau)^{1/3}\right)^{-\frac{3}{4}-\delta}\times$$

$$\left(x-z(y-\tau)^{\frac{1}{3}}\right)^{\frac{3}{4}+\delta}F_\xi\left(x-z(y-\tau)^{\frac{1}{3}},\tau\right)dz = -\frac{1}{\pi}\int_0^y\frac{d\tau}{(y-\tau)^{\frac{2}{3}}}\times$$

$$\int_\rho^{+\infty} v^{-\frac{2}{3}}\sin\left(\frac{2}{3}v\right)\left|\frac{x}{v^{\frac{2}{3}}}-(y-\tau)^{\frac{1}{3}}\right|^{-\frac{3}{4}-\delta}\left|x-v^{\frac{2}{3}}(y-\tau)^{\frac{1}{3}}\right|^{\frac{3}{4}+\delta}\times$$

$$F_\xi\left(x-v^{\frac{2}{3}}(y-\tau)^{\frac{1}{3}},\tau\right)\frac{2}{3}v^{-\frac{1}{3}}dv = -\frac{2}{3\pi}\int_0^y\frac{d\tau}{(y-\tau)^{\frac{2}{3}}}\int_\rho^{+\infty}v^{-\frac{2\delta-1}{3}}\times$$

$$\sin\left(\frac{2}{3}v\right)\mu(v)\left|x-v^{\frac{2}{3}}(y-\tau)^{\frac{1}{3}}\right|^{\frac{3}{4}+\delta}F_\xi\left(x-v^{\frac{2}{3}}(y-\tau)^{\frac{1}{3}},\tau\right)dv,$$

where $\rho = \frac{(x+r)^{\frac{3}{2}}}{(y-\tau)^{\frac{1}{2}}}$, $v = z^{\frac{3}{2}}$.

This integral at sufficiently large positive r is bounded by the following expression

$$-\frac{2}{3\pi}\int_0^y\frac{d\tau}{(y-\tau)^{\frac{2}{3}}}\left\{\left|x-r(y-\tau)^{\frac{1}{3}}\right|^{\frac{3}{4}+\delta}\left|F_\xi(x-r(y-\tau)^{\frac{1}{3}},\tau)\right|+\right.$$

$$\underset{v\ge\frac{(x+r)^{\frac{3}{2}}}{(y-\tau)^{\frac{1}{2}}}}{V}\left[\left(x-v^{\frac{2}{3}}(y-\tau)^{\frac{1}{3}}\right)^{\frac{3}{4}+\delta}F_\xi\left(x-v^{\frac{2}{3}}(y-\tau)^{\frac{1}{3}},\tau\right)\right]\times$$

$$\left.\sup\left\{\left|\int_m^n v^{-\frac{2-\delta}{3}}\mu(v)\sin(\frac{2}{3}v)dv\right|\right\}\right.,$$

where $\rho\le m\le n$.

The existence of the integrals

$$\int_0^{\infty} x^{-p}\sin x dx = \frac{\pi}{2\Gamma(p)}\left(\sin\frac{p\pi}{2}\right)^{-1},\ 0<p<2,$$

it means that the argument of the sup uniformly tends to zero at $r\to\infty$ in the domain $D_{[a,b]}$. From equation (3.24), it is not difficult to show the realization of condition (3.21).

Now select the class of solutions $u(x,y)$ depending on the behavior of the functions $F(x,y)$ at $|x|\to\infty$.

By (3.22) we have

$$u(x,y) = -\frac{1}{\pi}\int_0^y\int_{-\infty}^{+\infty} U(x-\xi;y-\tau)F(\xi,\tau)d\xi d\tau =$$

$$-\frac{1}{\pi}\int_0^y\int_{-\infty}^{+\infty} U(\xi;y-\tau)F(x-\xi,\tau)d\xi d\tau =$$

$$-\frac{1}{\pi}\int_0^y\int_{-\infty}^{+\infty} f(z)F(x-z(y-\tau)^{\frac{1}{3}},\tau)dzd\tau = -\frac{1}{\pi}\int_0^y d\tau\left(\int_{-\infty}^{-r}+\int_{-r}^{r}+\int_r^{+\infty}\right)\times$$

$$f(z)\cdot F(x-z(y-\tau)^{\frac{1}{3}},\tau)dz = u_1(x,y)+u_2(x,y)+u_3(x,y),$$

where r is a sufficiently large positive number.

Now we consider the expression $u_1(x,y)$ in the domain $D_{[a,b]}$.

Then

$$\exp\left(-c_2|x|^{\frac{3}{2}+\eta}\right)\cdot u_1(x,y) = -\frac{1}{\pi}\int_0^y\int_{-\infty}^{-r}\exp\left(-c_2|x|^{\frac{3}{2}+\eta}\right)\times$$

$$f(z)F(x-z(y-\tau)^{\frac{1}{3}},\tau)dzd\tau = u_1'(x,y).$$

According to the conditions of the theorem and the relations (3.22), the integral in the domain $D_{[a,b]}$ is estimated as follows

$$u_1'(x,y) = O\left(-\frac{1}{\pi}\int_0^y\int_r^{+\infty}\exp\left\{-c_2\left||x|^{\frac{3}{2}+\eta}-c_2|z|^{3/2}+c_3|x-z(y-\tau)^{1/3}|\right|^{\frac{3}{2}-\tau}\right\}dzd\tau\right).$$

Hence $u_1'(x,y)$ uniformly converges to zero at $r\to+\infty$ in the domain $D_{[a,b]}$. Furthermore $|u_1'(x,y)|\le K$. Therefore $|u_1(x,y)|\le K\cdot\exp(c_2|x|^{\frac{3}{2}-\eta})$ for sufficiently large x.

Now we study $u_3(x,t)$ for a sufficiently large positive r in the domain $D_{[a,b]}$. According to (2.30) we have

$$\left|x^{\frac{3}{4}+\delta}u_3(x,y)\right| \le |\frac{1}{\pi}\int_0^y d\tau\int_r^{+\infty}|x|^{\frac{3}{4}+\delta}z^{-1-\delta}\sin(\frac{2}{3}z^{3/2})\cdot|\frac{x}{z}-(y-\tau)^{\frac{1}{3}}|^{-\frac{3}{4}-\delta}\times$$

$$|x-z(y-\tau)^{\frac{1}{3}}|^{\frac{3}{4}+\delta}|F(x-z(y-\tau)^{1/3},\tau)dz| \le$$

$$M_3\left|\frac{1}{\pi}\int_0^y d\tau\int_r^{\infty}z^{-1-\delta}\sin\left(\frac{2}{3}z^{\frac{3}{2}}\right)\cdot\left(\frac{\left|x-z(y-\tau)^{\frac{1}{3}}+z(y-\tau)^{\frac{1}{3}}\right|}{\left|x-z(y-\tau)^{\frac{1}{3}}\right|}\right)^{\frac{3}{4}+\delta}dz\right| =$$

$$M_3 \left| \frac{1}{\pi} \int_0^y d\tau \int_r^\infty z^{-1-\delta} \sin\left(\frac{2}{3} z^{\frac{3}{2}}\right) \cdot \left(1 + \frac{z(y-\tau)^{\frac{1}{3}}}{x - z(y-\tau)^{\frac{1}{3}}}\right)^{\frac{3}{4}+\delta} dz \right| \leq$$

$$M_4 \left| \frac{1}{\pi} \int_0^y d\tau \int_\rho^\infty q^{-\frac{1}{3}-\delta} \sin(q) \left| 1 + \frac{(y-\tau)^{\frac{1}{3}}}{\frac{x}{\frac{3}{2} q^{\frac{2}{3}}} - (y-\tau)^{\frac{1}{3}}} \right|^{\frac{3}{4}+\delta} dq \right| \leq M_5,$$

where

$$M_3 = \left\{ \max_{(x,y)\in D_{[a,b]}} \left| P(x - r(y-\tau)^{\frac{1}{3}}) \right| + V_{z \geq r_1(x,y)\in D_{[a,b]}} \left(P\left(x - r(y-\tau)^{\frac{1}{3}}\right)\right) \right\},$$

$$\rho = \frac{2}{3} r^{3/2}.$$

Hence $u_3(x,y)$ uniformly convergence to zero at $r \to +\infty$.

Therefore we have

$$|x|^{\frac{3}{4}+\delta} |u_3(x,y)| \leq M,$$

$$|u_3(x,y)| \leq M|x|^{-\frac{3}{4}-\delta},$$

for $x < a$ and any $y \geq y_0 > 0$.

In our approach, any a, b and y_0 are true in the domain D

Part 4

Nonlinear boundary value problem for linear and nonlinear third-order equation with multiple characteristics.

4.1 A problem for the third-order equation with multiple characteristics and nonlinear boundary conditions

We consider a plane (x,y) domain $D = \{(x,y):\ h_1(y) < x < h_2(y),\ 0 \le y \le Y\}$. The curves $x = h_i(y) \in C^1[0,Y]$, $(i = 1,2)$, are defined on the lateral boundaries of domain D and are without intersection points. In the domain D we study the following problem for the equation (3.1).

Problem Find the solutions of the equation (3.1) in the regular domain D , there exist the derivatives u_x, u_{xx}, which are continuous in $\bar{D}$ and satisfy the boundary conditions

$$u(x,0) = F(x),\ h_1(0) \le x \le h_2(0), \tag{4.1}$$

$$u_x(h_1(y),y) = g(u(h_1(y),y),y),\ 0 \le y \le Y, \tag{4.2}$$

$$u_{xx}(h_1(y),y) = \varphi_1(y),\ 0 \le y \le Y, \tag{4.3.}$$

$$u(h_2(y),y) = \varphi_2(y),\ 0 \le y \le Y \tag{4.4}$$

and the compatibility conditions

$$F'(h_1(0)) = g(u(h_1(0),0)),\ F(h_2(0)) = \varphi_2(0),\ F''(h_1(0)) = \varphi_1(0).$$

Here given functions $F(x)$, $g(u,y)$, $\varphi_i(y)$, $(i = 1,2)$, $f(x,y)$- are bounded and sufficiently smooth, and the function $g(\xi,y)$ satisfies a Lipschitz condition with respective to ξ

$$|g(\xi_1,y) - g(\xi_2,y)| < l(y)|\xi_1 - \xi_2|, \tag{4.5}$$

where

$$0 < l(y) \le -k + \sqrt{k^2 + \frac{3k\exp\{-k(h_2(y) - h_1(y))\}}{h_2(y) - h_1(y)}},\ k = \mathrm{const} > 0. \tag{4.6}$$

Theorem 4.1. If the condition (4.5),(4.6) are true, then a solution of (3.1) (4.1) - (4.4) is unique.

Proof. Suppose that there are two solutions $u_1(x,y)$, $u_2(x,y)$ of the problem. Then $v(x,y) = u_1(x,y) - u_2(x,y)$ satisfies the equation $\tilde{L}(v) = 0$ and conditions

$$\left.\begin{array}{c} v(x,0) = 0,\ h_1(0) \le x \le h_2(0), \\ v_x(h_1(y),y) = g(u_1(h_1(y),y),y) - g(u_2(h_1(y),y),y),\quad 0 \le y \le Y, \\ v_{xx}(h_1(y),y) = 0,\ 0 \le y \le Y, \\ v(h_2(y),y) = 0,\ 0 \le y \le Y. \end{array}\right\} \tag{4.7}$$

We consider the identity

$$\int\int_D C(x,y)vL(v)dxdy = 0, \tag{4.8}$$

where

$$C(x,y) = \exp\{-kx - \beta k^3 y\},\ \beta \ge 1,\ k > 0,\ \beta, k = const. \tag{4.9}$$

Integrating the identity (4.8) by parts and using the suitable boundary conditions (4.7), we get

$$\int_0^Y C_x(x,y)v(g(u_1(h_1(y),y),y)-g(u_2(h_2(y),y),y))|_{x=h_1(y)}dy-$$

$$\frac{1}{2}\int\int_D (C_{xxx}-C_y)v^2dxdy+\frac{3}{2}\int\int_D C_xv_x^2dxdy-$$

$$\frac{1}{2}\int_0^Y \left((C_{xx}+h_1'(y)C)v^2|_{x=h_1(y)}+Cv_x^2|_{x=h_2(y)}\right)dy+$$

$$\frac{1}{2}\int_0^Y Cv_x^2|_{x=h_1(y)}dy-\frac{1}{2}\int_{h_1(0)}^{h_2(0)} Cv^2|_{y=Y}dx=0 \tag{4.10}$$

We set

$$I=\frac{1}{2}\int\int_D (C_{xxx}-C_y)v^2dxdy+\frac{1}{2}\int_{h_1(0)}^{h_2(0)} Cv^2|_{y=Y}dx+$$

$$\frac{1}{2}\int_0^Y \{(k^2+h_1'(y))Cv|_{x=h_1(y)}+Cv_x^2|_{x=h_2(y)}\}dy. \tag{4.11}$$

By choosing a sufficiently large number k, we can always assume that $k_2+h_1'(y)>0$, i.e $I\geq 0$.

By (4.11) and from expression (4.10) we have

$$I\equiv -\frac{3}{2}\int\int_D kCv_x^2dxdy-\int_0^Y kCv(g(u_1(h_1(y),y),y)-g(u_2(h_1(y),y),y))dy+$$

$$\frac{1}{2}\int_0^Y Cv_x^2|_{x=h_1(y)}dy. \tag{4.12}$$

Taking into account the inequality

$$v^2(h_1(y),y)\leq (h_2(y)-h_1(y))\int_{h_1(y)}^{h_2(y)} v_x^2dy$$

and conditions (4.5) from (4.12) we can get

$$I\leq \int\int_D\{\frac{1}{2}l^2(y)C(h_1(y),y)(h_2(y)-h_1(y))+$$

$$kC(h_1(y),y)(h_2(y)-h_1(y))l(y)-\frac{3}{2}kC(h_2(y),y)\}v_x^2dxdy.$$

Which the condition (4.6) is true, we can get the inequality $I\leq 0$. Hence $I=0$. It follows that $v\equiv 0$ in the domain D. Then from (4.11) we get $v(x,y)=0$ in the domain $\bar{D}$, if $\beta>1$. Let $\beta=1$. Then from 3.11 we have the following additional conditions

$$v(h_1(y),y)=0,\ v_x(h_2(y),y)=0,\ v(x,Y)=0,$$

when the problem reduces to Cattabriga problem, and is simultaneously satisfied by the above conditions and (4.7)

The uniqueness of the solution of this problem is proved in the paper (see [26-27]).

Theorem 4.2. Let $F(x) \in C^3[c_1, c_2]$, $(c_1 \le h_1(y) < h_2(y) \le c_2)$; $\varphi_i(y) \in C^{3-i}[0, Y]$, $(i = 1, 2)$; $|g(u, y)| < M$ for any fixed $|u| < \infty$ and satisfies the conditions of Theorem 4.1. Then the solution of the problems (3.1),(4.1)-(4.4) exists.

Proof. First, we consider the auxiliary problem: it is required to define in the domain D the regular solution $u(x, y) \in C^{3,1}_{x,y}(D) \cap C^{1,0}_{x,y}(\bar{D}) \cap C^{2,0}_{x,y}(\bar{D} \setminus (x = h_2(y)))$ of the equation (3.1), satisfying the boundary conditions (4.1),(4.3),(4.4) and

$$u_x(h_1(y), y) = \varphi_3(y), \ 0 \le y \le Y \tag{4.2$'$}$$

We construct the Green's function for the problem (3.1), (4.1), (4.3), (4.4) and (4.2').

We have the identity

$$\varphi \tilde{L}(\psi) - \psi M(\varphi) = \frac{\partial}{\partial \xi}(\varphi \psi_{\xi\xi} - \varphi_\xi \psi_\xi - \varphi_{\xi\xi}\psi) - \frac{\partial}{\partial \eta}(\varphi\psi),$$

where $M \equiv \frac{\partial}{\partial y} - \frac{\partial^3}{\partial x^3}$ - differential operator is adjoint to operator $\tilde{L}$; φ and ψ are sufficiently smooth functions. Integrating the identity of the domain D, we get

$$\int\int_D (\varphi \tilde{L}(\psi) - \psi M(\varphi)) d\xi d\eta = \int_\Gamma (\varphi_{\xi\xi}\psi - \varphi_\xi \psi_\xi + \varphi \psi_{\xi\xi}) d\eta + (\varphi\psi) d\xi, \tag{4.13}$$

where $\Gamma = \partial D$.

Now, in the formula (4.13) for the functions ψ and φ we will take the respective functions (any regular solution of equation (3.1)), and $U(x, y; \xi, \eta)$. We call the function $U(x, y; \xi, \eta)$-fundamental solution of the equation (3.1) and it is defined by (2.28).

Let

$$D^\varepsilon = \{(\xi, \eta) : h_1(\eta) < \xi < h_2(\eta), \ 0 < \eta \le y - \varepsilon\},$$

where $\varepsilon > 0$ is sufficiently small number.

Then the identity (4.13) reduces to the following form

$$\int\int_{D^\varepsilon} U(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta = \int_0^{y-\varepsilon} \big\{(u_\xi U - u_\xi U_\xi + u U_{\xi\xi})|_{\xi = h_2(\eta)} -$$

$$(u_{\xi\xi} U - U_\xi u_\xi + u U_{\xi\xi})|_{\xi = h_1(\eta)} \big\} d\eta + \int_{h_1(0)}^{h_2(0)} uU|_{\eta=0} d\xi -$$

$$\int_{h_1(y-\varepsilon)}^{h_2(y-\varepsilon)} uU|_{\eta = y-\varepsilon} d\xi + \int_0^{y-\varepsilon} (h_2'(\eta) uU|_{\xi=h_2(\eta)} - h_1'(\eta) uU|_{\xi = h_1(\eta)}) d\eta.$$

Sending ε to zero and taking into account the equality

$$\lim_{\varepsilon \to \infty} \int_{h_1(y-\varepsilon)}^{h_2(y-\varepsilon)} U(x, y; \xi, y - \varepsilon) u(\xi, y - \varepsilon) d\xi = \pi u(x, y),$$

we get

$$\pi u(x, y) = \int_0^y (u_{\xi\xi} U - u_\xi U_\xi + u U_{\xi\xi})|_{\xi = h_2(\eta)} d\eta -$$

$$\int_0^y (u_{\xi\xi}U - u_\xi U_\xi + uU_{\xi\xi})|_{\xi=h_1(\eta)}d\eta + \int_{h_1(0)}^{h_2(0)} uU|_{\eta=0}d\xi +$$

$$\int_0^y h_2'(\eta)uU|_{\xi=h_2(\eta)}d\eta - \int_0^y h_1'(\eta)uU|_{\xi=h_1(\eta)}d\eta -$$

$$\int\int_{D_y} U(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta. \tag{4.14}$$

Now we suppose that $W(x,y;\xi,\eta)$ - any regular solution of the equation

$$M(v) \equiv \frac{\partial v}{\partial \eta} - \frac{\partial^3 v}{\partial \xi^3} = 0, \tag{4.15}$$

and $u(x,y)$ is any regular solution of (3.1). Then, assuming formula (4.13), $\varphi = W$, $\psi = u$ we have

$$-\int\int_D W(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta = \int_0^y \{(u_{\xi\xi}W - u_\xi W_\xi + uW_{\xi\xi})|_{\xi=h_2(\eta)} -$$

$$-(u_{\xi\xi} - u_\xi W_\xi + uW_{\xi\xi})|_{\xi=h_1(\eta)}\}d\eta + \int_0^y h_2'(\eta)uW|_{\xi=h_2(\eta)}d\eta +$$

$$\int_{h_1(0)}^{h_2(0)} uW|_{\eta=0}d\xi - \int_{h_1(y)}^{h_2(y)} uW|_{\eta=y}d\xi - \int_0^y h_1'(\eta)uW|_{\xi=h_1(\eta)}d\eta. \tag{4.16}$$

From (4.14) and (4.16) we obtain

$$\pi u(x,y) = \int_0^y (u_{\xi\xi}(U-W) + u_\xi(-U+W)_\xi + u(U-W)_{\xi\xi})|_{\xi=h_2(\eta)}d\eta +$$

$$\int_0^y (u_{\xi\xi}(-U+W) - u_\xi(U-W)_\xi + u(-U+W)_{\xi\xi})|_{\xi=h_1(\eta)}d\eta + \int_{h_1(0)}^{h_(0)} u(U-W)|_{\eta=0}d\xi +$$

$$\int_0^y h_2'(\eta)u(U-W)|_{\xi=h_2(\eta)}d\eta - \int_{h_1(y)}^{h_2(y)} uW|_{\eta=y}d\xi + \int_0^y h_1'(\eta)u(U-W)|_{\xi=h_1(\eta)}d\eta -$$

$$\int\int_{D_y} (U(x,y;\xi,\eta) - W(x,y;\xi,\eta))f(\xi,\eta)d\xi d\eta. \tag{4.17}$$

If the regular solution $W(x,y;\xi,\eta)$ of the equation (4.15) satisfies boundary conditions

$$W_\xi|_{\xi=h_2(\eta)} = U_\xi|_{\xi=h_2(y)};$$

$$(W_{\xi\xi} + h_1'(\eta)W)|_{\xi=h_1(\eta)} = (U_{\xi\xi} + h_1'(\eta)U)|_{\xi=h_1(\eta)};\ W|_{\eta=y} = 0, \tag{4.18}$$

then from (3.17) we have

$$\pi u(x,y) = -\int_0^y G(x,y;h_1(\eta),\eta)u_{\xi\xi}(h_1(\eta),\eta)d\eta +$$

$$\int_0^y G_\xi(x,y;h_\eta,\eta)u_\xi(h_1(\eta),\eta)d\eta + \int_0^y G_{\xi\xi}(x,y;h_2(\eta),\eta)u(h_2(\eta),\eta)d\eta +$$

$$\int_{h_1(0)}^{h_2(0)} G(x,y;\xi,0)u(\xi,0)d\xi - \int\int_D G(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta, \tag{4.19}$$

where $G(x,y;\xi,\eta)=U(x,y;\xi,\tau)-W(x,y,\xi,\eta)$- Green's function for the problems (3.1), (4.1),(4.3),(4.4) and (4.2').

The formula (4.19) gives us the solution of the problems (3.1),(4.1),(4.3),(4.4) and (4.2'). We need to prove the existence of the function $W(x,y;\xi,\eta)$ that satisfies the equation (4.15) and the condition (4.18).

Now we consider the following expression

$$-W(x,y;\xi,\eta)=\int_\eta^y U(h_2(\tau),\tau;\xi,\tau)\alpha_1(x,y;\tau)d\tau+$$

$$\int_\tau^y U(h_1(\tau),\tau;\xi,\tau)\alpha_2(x,y;\tau)d\tau+\int_\tau^y V(h_2(\tau),\tau;\xi,\tau)\alpha_3(x,y;\tau)d\tau, \tag{4.20}$$

where $U(x,y;\xi,\tau)$ and $V(x,y;\xi,\tau)$ are given by the formula (2.28), (2.38) and $\alpha_i(x,y;\tau)(i\in\{1,3\})$ unknown functions.

Satisfying boundary condition (4.18) and by Lemma 2.4 and 2.6 from (4.20) we have

$$U(x,y;h_2(\eta),\eta)=\int_\eta^y U(h_2(\tau),\tau;h_2(\tau),\tau)\alpha_1(x,y;\tau)d\tau+$$

$$\int_\tau^y U(h_1(\tau),\tau;h_2(\tau),\tau)\alpha_2(x,y;\tau)d\tau+\int_\tau^y V(h_2(\tau),\tau;h_2(\tau),\tau)\alpha_3(x,y;\tau)d\tau, \tag{4.21}$$

$$U_\xi(x,y;h_2(\eta),\eta)=\int_\eta^y U_\xi(h_2(\tau),\tau;h_2(\tau),\tau)\alpha_1(x,y;\tau)d\tau+$$

$$\int_\tau^y U_\xi(h_1(\tau),\tau;h_2(\tau),\tau)\alpha_2(x,y;\tau)d\tau+\int_\tau^y V_\xi(h_2(\tau),\tau;h_2(\tau),\tau)\alpha_3(x,y;\tau)d\tau, \tag{4.22}$$

$$U_{\xi\xi}(x,y;h_2(\eta),\eta)+h_1'(\tau)U(x,y;h_1(\tau),\tau)=\int_\eta^y U_{\xi\xi}(h_2(\tau),\tau;h_1(\tau),\tau)\alpha_1(x,y;\tau)d\tau+$$

$$\int_\tau^y U_{\xi\xi}(h_1(\tau),\tau;h_1(\tau),\tau)\alpha_2(x,y;\tau)d\tau+\int_\tau^y V_{\xi\xi}(h_2(\tau),\tau;h_1(\tau),\tau)\alpha_3(x,y;\tau)d\tau+$$

$$h_1'(\tau)\int_\tau^y U(h_2(\tau),\tau;h_1(\tau),\tau)\alpha_1(x,y;\tau)d\tau+$$

$$h_1'(\eta)\int_\eta^y U(h_1(\tau),\tau;h_1(\tau),\tau)\alpha_2(x,y;\tau)d\tau+$$

$$h_1(\eta)\int_\eta^y V(h_2(\tau),\tau;h_1(\tau),\tau)\alpha_3(x,y;\tau)d\tau, \tag{4.23}$$

Using the properties of the functions $U(x,y;\xi,\tau)$ and $V(x,y;\xi,\tau)$ and Lemma 2.3-2.8 we find the solution of the problems (4.21)-(4.23), $\alpha_i(x,y,\tau)\in C(D)(i=1,3)$; $\alpha_2(x,y,\eta)\in L_2(D)$.

It is easy to show that the Green's function $G(x,y;\xi,\eta)$ has the same estimate that holds for $U(x,y;\xi,\eta)$ (see [1,26]).

Now we turn to the solution of the problem (3.1), (4.1)-(4.4). We are looking for it to be in the form (4.19). Then (4.19) takes the form

$$\pi u(x,y)=\int_0^y G_\xi(x,y;h_1(\eta),\eta)g(u(h_1(\eta),\eta),\eta)d\eta+H(x,y) \tag{4.24}$$

where

$$H(x,y)=\int_0^y G_{\xi\xi}(x,y;h_2(\eta),\eta)\varphi_2(\eta)-$$

$$\int_0^y G(x,y;h_1(\eta),\eta)\varphi_1(\eta)d\eta+\int_{h_1(0)}^{h_2(0)} G(x,y;\xi,0)F(\xi)d\xi-$$

$$\int\int_D G(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta.$$

From (4.24) at $x=h_1(y)$ we arrive at the Nonlinear Integral Equations of the Hammerstein type for the function $\tau(y)=u(h_1(y),y)$

$$\tau(y)=\int_0^y G_\xi(h_1(y),y;h_1(\eta),\eta)g(\tau(\eta),\eta)d\eta+H(y), \tag{4.25}$$

where

$$H(y)=\frac{\sqrt{3}}{2\pi\varphi'(0)}\int_0^y G_1(h_1(y),y,\eta)d\eta\int_0^\eta\left(\frac{\varphi'(t)-h_1'(t)F'''(h_1(t))}{(\tau-t)^{1/3}}-\frac{\frac{\partial W_{\xi\xi}(h_1(t),t)}{\partial t}}{(\tau-t)^{1/3}}\right)dt+$$

$$\int_0^y G_3(h_1(y),y,\eta)(\varphi_2(\eta)-F(h_2(\eta))-W(h_2(\eta),\eta))d\eta-\int_0^y G_2(h_1(y),y,\eta)\times$$

$$(F'(h_1(\eta))+W_x(h_1(\eta),\eta))d\eta+F(h_1(y))+W(h_1(y),y).$$

The equation (4.25) will be solved by the method of successive approximations.

Let

$$|H(y)|<N_1,\ |g(u,y)|<M,\ |\tau(y)|<N,\ N=N_1+1 \tag{4.26}$$

We set

$$\tau^0(y)\equiv H(y)\le N_1<N,$$

$$\tau^{(n)}(y)=H(y)+\int_0^y G_\xi(h_1(y),y;h_1(\eta),\eta)g(\tau^{(n-1)}(\eta),\eta)d\eta. \tag{4.27}$$

Hence, by setting $n=1$ and using the estimate (4.27) we have

$$|\tau^{(1)}(y)|\le N_1+C_1M\int_0^y(y-\tau)^{-2/3}d\eta=N_1+3C_1My^{1/3}. \tag{4.28}$$

Because $|\tau^1(y)|<N$, we require the inequality to satisfy

$$N_1+3C_1My^{1/3}<N.$$

By the choice N here we have the inequality $3C_1My^{1/3}<1$ which is true at

$$y<\left(\frac{1}{3C_1M}\right)^3 \tag{4.29}$$

Then from (4.27) at $n=2$ we find that

$$|\tau^{(2)}(y)|\le N_1+3C_1My^{1/3}<N,$$

if $y < (\frac{1}{3C_1M})^3$.

Hence, by induction we conclude that all the successive approximations will be assessed the same amount, if the inequality (4.26) is satisfied.

Now we show that the limit of the sequence exists. It is enough to prove the convergence of the series

$$\tau^0 + (\tau^1 - \tau^0) + (\tau^2 - \tau^1) + ... + (\tau^n - \tau^{(n-1)}) + ... \tag{4.30}$$

We estimate the absolute values of the terms of (4.30). We have

$$|\tau^{(1)} - \tau^{(0)}| \leq C_1 M \int_0^y (y-\tau)^{-1/3} d\tau = C_1 M y^{1/3} B(1, \frac{2}{3}),$$

where $B(\nu, \mu) = \int_0^1 x^{\nu-1}(1-x)^{\mu-1}dx$ - Beta Function.

$$|\tau^2 - \tau^1| \leq C_1 \int_0^y \frac{1}{(y-\eta)^{-2/3}} |g(\tau^{(1)}(\eta)) - g(\tau^0(\eta))| d\tau \leq$$

$$C_1 L \int_0^y \frac{1}{(y-\tau)^{-2/3}} |\tau^{(1)}(\eta) - \tau^{(0)}(\eta)| d\tau \leq C_1^2 ML \int_0^y (y-\eta_1)^{-2/3} d\eta_1 \int_0^{\eta_1} (\eta_1 - \eta_2)^{-2/3} d\eta_2 =$$

$$C_1^2 MLy^{2/3} B\left(1, \frac{2}{3}\right) B\left(\frac{5}{3}, \frac{2}{3}\right),$$

where L-const, which is $L \geq l(y)$.

Using induction, it is easy to show that

$$|\tau^{(n)} - \tau^{(n-1)}| \leq C_1^{n-1} L^{n-1} M y^{\frac{2n}{3}} \prod_{j=1}^{n} B\left(\frac{2(j-1)}{3} + 1, \frac{2}{3}\right).$$

It follows that each term of series (4.30) does not exceed the relevant terms of the power module series

$$\sum_{n=1}^{\infty} C_1^n L^{n-1} M y^{\frac{n}{3}} \prod_{j=1}^{n} B\left(\frac{2(j-1)}{3} + 1, \frac{2}{3}\right).$$

We will show the convergence of (4.30). Applying D'alembert principle we get

$$\lim_{n\to} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} C_1 L M y^{\frac{1}{3}} B\left(\frac{2(n-1)}{3} + 1, \frac{2}{3}\right) = 0. \tag{4.31}$$

Then the series (4.30) converges absolutely and uniformly. Therefore, the sequence $\{\tau^{(n)}(y)\}$ converges uniformly to $\tau(y)$.

We have proved that the solution of (3.1) (4.1)-(4.4) exists in the domain $D_1 = \{(x, y) : h_1(y) \leq x \leq h_2(y), 0 \leq y \leq Y_0\}$ for some Y_0, although the problem was given in the domain $D = \{(x, y) : h_1(y) \leq x \leq h_2(y),\ 0 \leq y \leq Y\}$. If it turns out that $Y_0 \geq Y$, obviously, our problem is completely solved. If $Y_0 < Y$, however, it was found that the solution can be extended in domain $D_1 = \{(x, y) : h_1(y) \leq x \leq h_2(y), 0 \leq y \leq Y_0\}$. This can be done as follows.

We consider the problem (3.1), (4.1)-(4.4) in the domain $D_1 = \{(x, y) : h_1(y) \leq x \leq h_2(y), 0 \leq y \leq Y_0\}$.

To solve the problem in this domain by applying the above scheme, we obtain a nonlinear integral equation of Volterra second form

$$\tau(y) = H(y) + \int_{y_0}^{y} G_\xi(h_1(y), y; h_1(\eta), \eta) g(\tau(\eta), \eta) d\eta,$$

which can be solved by successive approximations in the domain $D_2 = \{(x, y) : h_1(y) \le x \le h_2(y), 0 \le y \le Y_1\}$. If, it turns out that even after that , $Y_1 < Y$ then the above procedure can be repeated and eventually run out $[0, Y]$.

4.2 A nonlinear boundary value problem for nonlinear third-order equation with multiple characteristics

In this section we study a nonlinear boundary value problem for the nonlinear third-order equation with multiple characteristics in the domain having curved boundary.

Problem Is required to determine in the domain $D = \{(x, y) : h_1(y) < x < h_2(y),\ 0 < y \leq 1\}$ the function $u(x, y)$ with the following properties:

1) It is a regular solution of the equation

$$Lu = u_{xxx} - u_y = f(x, y, u(x, y)); \tag{4.32}$$

2) $u(x, y) \in C^{3,1}_{x,y}(D) \cap C^{2,0}_{x,y}(\bar{D} \setminus (x = h_1(y)) \cap C(\bar{D}))$;

3) It satisfies the following conditions:

$$u(x, 0) = u_0(x),\ h_1(0) \leq x \leq h_2(0), \tag{4.33}$$

$$u_x(h_1(y), y) = g(u(h_1(y), y), y),\ 0 \leq y \leq 1, \tag{4.34}$$

$$u_{xx}(h_1(y), y) = \sigma(u(h_1(y), y), y),\ 0 \leq y \leq 1, \tag{4.35}$$

$$u(h_2(y), y) = \varphi(y),\ 0 \leq y \leq 1. \tag{4.36}$$

The given functions $u_0(x)$, $g(\xi, \eta)$, $\sigma(\eta, y)$, $\varphi(y)$, $f(x, y, u(x, y))$ are required to be bounded and smooth in their domain, as well as to satisfy the compatibility conditions at the corner points of consideration domains i.e.

$$u_0'(h_1(0)) = g(u(h_1(0), 0), 0),\ u_0''(h_1(0)) = \sigma(u(h_1(0), 0), 0),\ u_0(h_2(0)) = \varphi(0).$$

The uniqueness of solution of the problem

Theorem 4.3. Let $h_i(y) \in C^1(0 \leq y \leq 1)$, $i = 1, 2$ and $g(u(h_1(y), y), y) \in C(0 \leq y \leq 1)$, $\sigma(u(h_1(y), y), y) \in C(0 \leq y \leq 1)$, $f(x, y, u(x, y)) \in C(\bar{D})$, $|g(u_1, y) - g(u_2, y)| \leq l(y)|u_1 - u_2|$, $|\sigma(u_1, y) - \sigma(u_2, y)| \leq k(y)|u_1 - u_2|$, $f(x, y, u_1) - f(x, y, u_2) \leq p(x, y)|u_1 - u_2|$. Then the solution of the problems (4.32)-(4.36) is unique.

Proof Suppose that, there are two solutions to this problem, which are $u_1(x, y)$, $u_2(x, y)$. We consider the difference between them $\omega(x, y) = u_1(x, y) - u_2(x, y)$. Then we get for $\omega(x, y)$ the following problem:

$$L(\omega) \equiv \omega_{xxx} - \omega_y = f(x, y, u_1(x, y)) - f(x, y, u_2(x, y)), \tag{4.37}$$

$$\left.\begin{array}{c} \omega(x, 0) = 0,\ h_1(0) \leq x \leq h_2(0), \\ \omega_x(h_1(y), y) = g(u(h_1(y), y), y) - g(u_2(h_2(y), y), y),\ 0 \leq y \leq 1, \\ \omega_{xx}(h_1(y), y) = \sigma(u_1(h_1(y), y), y) - \sigma(u_2(h_2(y), y), y),\ 0 \leq y \leq 1, \\ \omega(h_2(y), y) = 0,\ 0 \leq y \leq 1, \end{array}\right\} \tag{4.38}$$

and

$$\left.\begin{aligned}|g(u_1,y)-g(u_2,y)|\le l(y)|u_1-u_2|=l(y)|\omega|,\\ |\sigma(u_1,y)-\sigma(u_2,y)|\le k(y)|u_1-u_2|=k(y)|\omega|,\\ |f(x,y,u_1)-f(x,y,u_2)|\le p|u_1-u_2|=p(\omega),\end{aligned}\right\} \tag{4.39}$$

Integrating identity

$$v\omega\tilde{L}(\omega)\equiv\omega_{xxx}-\omega_y=v\omega\{f(x,y,u_1(x,y))-f(x,y,u_2(x,y))\}$$

in the domain D, where $v=e^{-\alpha x-\beta y}$, $\alpha>(\sqrt{2}-1)k$, $\beta>\alpha^3+p$ we have

$$\int_0^1 v\omega\omega_{xx}\big|_{h_1(y)}^{h_2(y)}dy-\frac{1}{2}\int_0^1 v\omega_x^2\big|_{h_1(y)}^{h_2(y)}dy-\int_0^1 v_x\omega\omega_x\big|_{h_1(y)}^{h_2(y)}dy+$$
$$\frac{1}{2}\int_0^1 v_{xx}\omega^2\big|_{h_1(y)}^{h_2(y)}dy+\frac{3}{2}\int\int_D v_x\omega_x^2dxdy-\frac{1}{2}\int\int_D v_{xxx}\omega^2dxdy-$$
$$\frac{1}{2}\int_{h_1(y)}^{h_2(y)} v\omega^2\big|_0^1dx-\frac{1}{2}\int_0^1 h_1'(y)v\omega^2\big|_{x=h_1(y)}dy+\frac{1}{2}\int_0^1 h_2'(y)v\omega^2\big|_{x=h_2(y)}dx+$$
$$\frac{1}{2}\int\int_D v_y\omega^2dxdy=\int\int_D v\omega(f(x,y,u_1(x,y))-f(x,y,u_2(x,y)))dxdy. \tag{4.40}$$

We have used the boundary conditions (4.38) and introduced the following notation:

$$I=\frac{1}{2}\int_0^1 v\omega_x^2\big|_{x=h_2(y)}dy+\frac{1}{2}\int_{h_1(1)}^{h_2(1)} v\omega^2\big|_{y=1}+\frac{3\alpha}{2}\int\int_D v\omega_x^2dxdy\ge 0. \tag{4.41}$$

According to the conditions (4.38)-(4.39) from (4.40) ,we get

$$I\le\frac{1}{2}\int_0^1(2k(y)-h_1'(y)+l^2(y)-2\alpha l(y)-\alpha^2)v\omega^2\big|_{x=h_1(y)}dy+$$
$$\frac{1}{2}\int_D(\alpha^3-\beta+p(x,y))v\omega^2dxdy. \tag{4.42}$$

The constants α and β can be chosen so that there will be a relationship $I\le 0$. Since by assumption $I\ge 0$ it follows that $I=0$.

Then from (4.41) we obtain the following conditions: if $\omega_x(x,y)=0$ at $x=h_1(y)$; if $\omega(x,y)=0$ at $y=1$; if $\omega_x(x,y)=0$ at $(x,y)\in D$.

Hence we have

$$\omega(x,y)=w(y),\quad (x,y)\in D.$$

As $\omega(h_2(y),y)=0$ then $w(y)=0$. By the continuity $\omega(x,y)$ we get $\omega(x,y)=0$ in the domain $\bar{D}$.

The existence of solutions of the problem

Theorem 4.4. Suppose that along with the terms of the uniqueness theorem, the following conditions are satisfied

$$\varphi(y)\in C^1[0,1];\ u_0(x)\in C^3[c_1,c_2]\ (c_1\le h_1(y)<h_2(y)\le c_2).$$

Moreover, let there exist constants M, N_1, N_2, M_i $(i = \in \{1, 7\})$ such that for $y \in [0, 1]$ any fixed $|u| < \infty$ we get the inequalities

$$|g(u, y)| < N_1, \; |\sigma(u, y)| < N_2, \; |g_u(u, y)| < M_1,$$

$$|g_y(u, y)| < M_2, \; |\sigma_u(u, y)| < M_3 \; |\sigma_y(u, y)| < M_4,$$

for $(x, y) \in D$ and for any fixed $|u| < \infty$

$$|f(x, y, u(x, y))| < M, \; |f_x(x, y, u(x, y))| < M_5,$$

$$|f_y(x, y, u(x, y))| < M_6, \; |f_u(x, y, u(x, y))| < M_7,$$

Then the solution of the problem (4.32)-(4.36) exists.

Proof. The solution of (4.32)-(4.36) has the representation in (see [6])

$$u(x, y) = \frac{1}{\pi} \int_0^y G_\xi(x, y; h_1(\eta), \eta) g(\tau(\eta), \eta) d\eta + \frac{1}{\pi} \int_{h_1(0)}^{h_2(0)} G(x, y; \xi, 0) u_0(\xi) d\xi +$$

$$\frac{1}{\pi} \int_0^y G_{\xi\xi}(x, y; h_2(\eta), \eta) \varphi(\eta) d\eta - \frac{1}{\pi} \int_0^y G(x, y; h_1(\eta), \eta) \sigma(\tau(\eta), \eta) d\eta -$$

$$\frac{1}{\pi} \int \int_D G(x, y; \xi, \eta) f(\xi, \eta, u(\xi, \eta)) d\xi d\eta, \tag{4.43}$$

where

$$u(h_1(y), y) = \tau(y), \tag{4.44}$$

$G(x, y; \xi, \eta) = U(x, y; \xi, \eta) - W(x, y; \xi, \eta)$ - Green's function, $U(x, y; \xi, \eta)$ - fundamental solution of the equation (4.32), $W(x, y; \xi, \eta)$ - a regular solution of the following problems

$$-W_{\xi\xi\xi} + W_\eta = 0,$$

$$W|_{\eta=y} = 0,$$

$$(U_{\xi\xi} + h_1'(y)U)|_{\xi=h_1(\eta)} = (W_{\xi\xi} + h_1'(y)W|_{\xi=h_1(\eta)},$$

$$U|_{\xi=h_2(\eta)} = W|_{\xi=h_2(\eta)}, \quad U_\xi|_{\xi=h_2(\eta)} = W_\xi|_{\xi=h_2(\eta)}.$$

Now we pass to the limit at $x \to h_1(y)$, and according to the notation (4.44) from (4.43) we have

$$\tau(y) = \frac{1}{\pi} \int_0^y G_\xi(h_1(y), y; h_1(\eta), \eta) g(\tau(\eta), \eta) d\eta + \frac{1}{\pi} \int_{h_1(0)}^{h_2(0)} G(h_1(y), y; \xi, 0) u_0(\xi) d\xi +$$

$$\frac{1}{\pi} \int_0^y G_{\xi\xi}(h_1(y), y; h_2(\eta), \eta) \varphi(\eta) d\eta - \frac{1}{\pi} \int_0^y G(h_1(y), y; h_1(\eta), \eta) \sigma(\tau(\eta), \eta) d\eta -$$

$$\frac{1}{\pi} \int \int_D G(h_1(y), y; \xi, \eta) f(\xi, \eta, u(\xi, \eta)) d\xi d\eta. \tag{4.45}$$

The system (4.43)-(4.45)- is a system of nonlinear integral equations of Hammerstein type with respective $u(x, y)$ and $\tau(y)$. The unique solvability of this system will be proven by the contraction mapping principle. Let set G_θ of a pair of continuous functions $F\{u(x,y),\tau(y)\}$ in the domain $D_\theta\{(x,y) : h_1(y) < x < h_2(y),\ 0 \leq y \leq \theta\}$ with bounded norm $||F|| = ||u|| + ||\tau||$ in the interval $0 \leq y \leq \theta$, where

$$||u|| = \max_{(x,y)\in D} |u|, \quad ||\tau|| = \max_{0\leq y\leq\theta} |\tau|.$$

Let $G_{\theta,N} = \{F : F \in G_\theta,\ ||F|| \leq N\}$ be a subset of the G_θ.

We denote the right hand side of (4.43), (4.45) respectively by $A_1(u,\tau)$, $A_2(u,\tau)$ and we define the map $A = (A_1(u,\tau), A_2(u(\tau),\tau))$.

We establish an estimate $u_x(x,y)$ in the domain $\bar{D}$.

$$u_x(x,y) = \frac{1}{\pi}\int_0^y G_{\xi x}(x,y;h_1(\eta),\eta)g(\tau(\eta),\eta)d\eta + \frac{1}{\pi}\int_{h_1(0)}^{h_2(0)} G_x(x,y;\xi,0)u_0(\xi)d\xi +$$

$$\frac{1}{\pi}\int_0^y G_{\xi\xi x}(x,y;h_2(\eta),\eta)\varphi(\eta)d\eta - \frac{1}{\pi}\int_0^y G_x(x,y;h_1(\eta),\eta)\sigma(\tau(\eta,\eta))d\eta -$$

$$\frac{1}{\pi}\int\int_D G_x(x,y;\xi,\eta)f(\xi,\eta,u(\xi,\eta))d\xi d\eta.$$

$$||u_x(x,y)|| \leq |J_1| + |J_2| + |J_3| + |J_4| + |J_5|,$$

where

$$J_1 = \frac{1}{\pi}\int_0^y G_{\xi x}(x,y;h_1(\eta),\eta)g(\tau(\eta),\eta)d\eta,$$

$$J_2 = \frac{1}{\pi}\int_{h_1(0)}^{h_2(y)} G_x(x,y;\xi,0)u_0(\xi)d\xi,$$

$$J_3 = \frac{1}{\pi}\int_0^y G_{\xi\xi x}(x,y;h_2(\eta),\eta)\varphi(\eta)d\eta,$$

$$J_4 = \frac{1}{\pi}\int_0^y G_x(x,y;h_1(\eta),\eta)\sigma(\tau(\eta),\eta)d\eta,$$

$$J_5 = \frac{1}{\pi}\int\int_D G_x(x,y;\xi,\eta)f(\xi,\eta,u(\xi,\eta))d\xi d\eta.$$

According to Lemmas 2.3-2.10, and by the condition of the theorem we can get

$$|J_1| \leq K_1, \quad |J_4| \leq K_4, \quad |J_5| \leq K_5,$$

and at $u_0(x) \in C^3[h_1(0), h_2(0)]$, we have $|J_2| \leq K_2$, at $\varphi(y) \in C^1[0,1]$, we have $|J_3| \leq K_3$, where $K_i = const > 0$, $i = \overline{1,5}$. Hence, we have that

$$||u_x(x,y)|| \leq K,\ K = \max_i\{K_i\},\ i \in \{1,5\}.$$

Then, under the conditions of the theorem for each $N > 0$ for a sufficiently small θ and $0 < y < \theta$ the operator A transforms into itself in $G_{\theta,N}$. Thus the inequalities $||A_i|| \leq N/2$, $i = 1,2$ are true when $(u,\tau) \in G_{\theta,N}$. To do this we assume that $A(u,\tau)$ is identified in the $G_{\theta,N}$ $i = 1,2$. Also, a suitable choice θ can be made for the contracting operator A . Then, by the contraction mapping principle, it has a unique fixed point $(u,\tau) \in G_{\theta,N}$.

Therefore, (u,τ) is a solution of the systems (4.43), (4.45) at $0 < y < \theta$.

Part 5

Some boundedness classes of pseudodifferential operators

5.1 Background materials and basic results continuity and boundedness of pseudodifferential operators. Symbol Classes

In this section we give basic results and background material of global pseudodifferential calculus. These results were developed more systematically by a number of people in long time. There are the results by Hörmander [49-55], Taylor [109], Beal [17] and others. Now we give some important results, which is to help to study our new problems. So we give the some basic results, which is Taylor [109] gets the following results:

Definition 5.1. Let Ω be an open subset of R^n, $m, \rho, \delta \in R$, and suppose $0 \leq \rho, \delta \leq 1$. We define the symbol class $S^m_{\rho,\delta}(\Omega)$ to consist of the set of $p \in C^\infty(\Omega \times R^n)$ with the property that, for any compact $K \subset \Omega$, any multi-indices α, β, there exist a constant $C_{K,\alpha,\beta}$ such that

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{K,\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for all $x \in K$, $\xi \in R^n$. We drop the Ω and use $S^m_{\rho,\delta}$ when the context is clear. The class $S^m_{\rho\delta}$ was introduced by Hörmander in [55]. The subclass $S^m_{1,0}$ defined by Kohn and Nirenberg [106]

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{K,\alpha,\beta}(1+|\xi|)^{m-|\alpha|}.$$

Definition 5.2. The symbol $p(x,y)$ belongs to $S^m(\Omega)$ if $p \in S^m_{1,0}(\Omega)$ and are smooth $p_{m-j}(x, r\xi)$, homogeneous of degree $m-j$ in ξ for $|\xi| \geq 1$, i.e.,

$$p_{m-j}(x, r\xi) = r^{m-j}p_{m-j}(x,\xi), \quad |\xi| \geq 1, \quad r \geq 1$$

such that

$$p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$$

where the asymptotic condition means that

$$p(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S^{m-N-1}_{1,0}(\Omega).$$

If $p(x,\xi)$ is homogeneous of degree m in ξ and if $\varphi(\xi) = 0$ for $|\xi| \leq C_1$, $\varphi(\xi) = 1$ for $|\xi| \geq C_2 > C_1$, $\varphi \in C^\infty$, then $\varphi(\xi)p(x,\xi) \in S^m \subset S^m_{1,0}$.

Proposition 5.3. Let $p \in S^m_{\rho,\delta}(\Omega)$, $q \in S^\mu_{\rho',\delta'}$. Then $p^\alpha_\beta = D^\beta_x D^\alpha_\xi p \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}$, and $p(x,\xi)q(x,\xi) \in S^{m+\mu}_{\rho'',\delta''}$ where $\rho'' = \min(\rho,\rho')$, $\delta'' = \max(\delta,\delta')$.

If $|p(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, then $p(x,\xi)^{-1} \in S^{-m}_{\rho,\delta}$.

Definition 5.4. If $p(x,\xi) \in S^m_{\rho,\delta}$, the operator $p(x,D)$ is said to belong to $OPS^m_{\rho,\delta}$. More generally, if Σ is any symbol class and $p(x,\xi) \in \Sigma$, we say $p(x,D) \in OP\Sigma$.

Theorem 5.5. If $p \in S^m_{\rho,\delta}(\Omega)$, then $p(x,D)$ is continuous operator

$$p(x,D) : C_0^\infty \to C^\infty.$$

If $\delta < 1$, then the map can be extended to a continuous map

$$p(x, D) : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$$

Proof. If $p \in S^m_{\rho,\delta}(\Omega)$, $u \in C_0^\infty(\Omega)$, then the integral

$$p(x, D)u = \int p(x, \xi)\hat{u}(\xi)e^{ix\xi}d\xi$$

is absolutely convergent, and one can differentiate under the integral sign, obtaining always absolutely convergent integrals. To prove above result need a lemma

Lemma 5.6. Let $p \in S^m_{\rho,\delta}(\Omega)$, $v \in C_0^\infty$. Then for all $\xi, \eta \in R^n$,

$$|\int v(x)p(x, \xi)e^{ix\eta}dx| \leq C_N(1 + |\xi|)^{m+\delta N}(1 + |\eta|)^{-N}.$$

Proof. Integration by parts yields

$$|\eta^\alpha \int v(x)p(x, \xi)e^{ix\eta}dx| = |\int D_x^\alpha(v(x)p(x, \xi))e^{ix\eta}dx| \leq C_\alpha(1 + |\xi|)^{m+\delta|\alpha|}.$$

To complete the proof of Theorem 5.5, to show that the functional

$$v \to < p(x, D)u, v >, \quad v \in C_0^\infty(\Omega)$$

is defined $u \in \mathcal{E}'(\Omega)$. There is

$$< p(x, D)u, v >= \int v(x)p(x, \xi)\hat{u}(\xi)e^{ix\xi}d\xi dx = \int p_v(\xi)\hat{u}(\xi)d\xi$$

where $p_v(\xi) = \int v(x)p(x, \xi)e^{ix\xi}dx$. This defined for $\forall u \in \mathcal{E}'(\Omega)$ with $p_v(\xi)$ be rapidly decreasing. But the lemma implies that

$$|p_v(\xi)| \leq C_N(1 + |\xi|)^{m-(1-\delta)N}.$$

5.2 The pseudolocal property.

Theorem 5.7. If $p(x, \xi) \in S^m_{\rho,\delta}(\Omega)$, $\delta < 1$ and if $\rho > 0$, we have for $u \in \mathcal{E}'(\Omega)$,

$$singsupp\ p(x, D)u \subset singsupp\ u.$$

Here the singular support of a distribution u, denoted *singsupp* u is the complement of the open set on which u is smooth.

If $K \in \mathcal{D}'(\Omega \times \Omega)$, then there is associated a map $K : C_0^\infty(\Omega) \to \mathcal{D}'(\Omega)$ defined by $< Ku, v >=< K, u(x)v(y) >$. The converse is also true, and is known as the Schwartz kernel theorem.

Lemma 5.8.(Singular support lemma) Suppose $K \in \mathcal{D}'(\Omega \times \Omega)$ satisfies

$$K : C_0^\infty \to C^\infty(\Omega)$$

and

$$K:\mathcal{E}'(\Omega)\to\mathcal{D}'(\Omega).$$

K is C^∞ off the diagonal in $\Omega\times\Omega$.

Then $singsupp\ Ku\subset singsupp\ u$ for $u\in\mathcal{E}'(\Omega)$.

To prove Theorem 5.7, to study the kernel K of $p(x,D)$ and

$$<K,uv>=<p(x,D)u,v>=\int v(x)p(x,D)u(x)dx=\int\int p(x,\xi)e^{ix\xi}v(x)\hat{u}(\xi)d\xi dx$$

$$=(2\pi)^{-n}\int\int\int p(x,\xi)e^{i(x-y)\xi}v(x)u(y)dyd\xi dx.$$

Thus, with the appropriate interpretation as a distribution integral,

$$K=(2\pi)^{-n}\int p(x,\xi)e^{i(x-y)\xi}d\xi$$

Consequently,

$$(x-y)^\alpha K=\int e^{(x-y)\xi}d\xi.$$

The integral is absolutely convergent for large α that $m-\rho|\alpha|<-n$, generally with j derivatives yields abs. convergent integral provided $m-\rho|\alpha|<-n-j$, so $(x-y)^\alpha K\subset C^j(\Omega\times\Omega)$. K is smooth off the diagonal $x=y$, and end proof.

Remark 5.9. For x, y in compact subset of Ω,

$$|D^\beta_{x,y}K|\leq C|x-y|^{-k}$$

where $k\geq 0$ is any integer strictly greater than $(1/\rho)(m+n+|\beta|)$. This isn't sharp, for example, if $p(x,\xi)\in S^m_{1,0}$ it is true that

$$|K(x,y)|\leq\begin{cases}C|x-y|^{-(m+n)} & \text{if } m>-n\\ C|\log|x-y|| & \text{if } m=-n.\end{cases}$$

5.3 Asymptotic expansions of a symbol

Theorem 5.10. Suppose $p_j\in S^{m_j}_{\rho,\delta}(\Omega)$, $m_j\to-\infty$. Then there exists $p\in S^{m_0}_{\rho,\delta}(\Omega)$ such that, for all $N>0$,

$$p-\sum_{j=0}^{N-1}p_j\in S^{m_N}_{\rho,\delta}(\Omega). \tag{5.1}$$

If (5.1) holds and

$$p\sim\sum_{j\geq 0}p_j.$$

Proof. There are K_j, $K_1\subset K_2\subset\ldots\to\Omega$ compact sets and $\varphi\in C^\infty(R^n)$ with $\varphi(\xi)=0$ for $|\xi|\leq 1/2$, $\varphi(\xi)=1$ for $|\xi|\geq 1$. $p(x,\xi)$ is of the form

$$p(x,\xi)=\sum_{j=0}^{\infty}\varphi(\varepsilon_j\xi)p_j(x,\xi) \tag{5.2}$$

where ε_j are small that

$$|D_x^\beta D_\xi^\alpha \varphi(\varepsilon_j \xi) p_j(x,\xi)| \leq 2^{-j}(1+|\xi|)^{m_j+1-\rho|\alpha|+\delta|\beta|}$$

for $|\alpha|+|\beta|+i \leq j$ and $x \in K_i$. (5.2) is convergent and $p(x,\xi)$ satisfies (5.1).

Theorem 5.11. Let $P_j \in S_{\rho,\delta}^{m_j}(\Omega)$, $m_j \to -\infty$, $j \geq 0$. Let $p \in C^\infty(\Omega \times R^n)$ and assume there are $C_{\alpha,\beta}$, $\mu = \mu(\alpha,\beta)$ such that

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^\mu.$$

If there exist $\mu_k \to \infty$ such that

$$|p(x,\xi) - \sum_{j=0}^{k} p_j(x,\xi)| \leq C_k(1+|\xi|)^{-\mu_k} \tag{5.3}$$

then $p \in S_{\rho,\delta}^{m_0}$ and $p \sim \Sigma p_j$ sense that (5.1) holds.

Proof. By Theorem 5.10 there exist $q \in S_{\rho,\delta}^{m_0}(\Omega)$ such that $q \sim \Sigma p_j$ and remains to show that $p - q \in S^{-\infty}$. (5.3) implies that

$$|p(x,\xi) - q(x,\xi)| \leq C_{K,N}(1+|\xi|)^{-N}, \quad x \in K.$$

This inequality holds for $D_x^\beta D_\xi^\alpha(p-q)$, they use the inequality

$$\sum_{|\alpha|=1} \sup_{K_1} |D^\alpha f|^2 \leq C \sup_{K_2} |f| \sum_{|\alpha|\leq 2} |D^\alpha f|, \tag{5.4}$$

where $K_1 \subset int K_2 \subset K_2$, K_j compact. To proof of (5.4) they apply to the functions

$$F_\xi(x,\xi) = p(x,\xi+\eta) - q(x,\xi+\eta)$$

taking $K_1 = K \times 0$, K_2 a small neighborhood of K_1, they get

$$\sup_{x\in K} |\nabla_{x,\xi}(p-q)(x,\xi)|^2 \leq C \sup_{(x,\eta)\in K_2} |p(x,\xi+\eta) - q(x,\xi+\eta)|$$

$$\times(\sum_{|\alpha|\leq 2} \sup_{(x,\eta)\in K_2} |D_{(x,\eta)}^\alpha (p-q)(x,\xi+\eta)|) \leq C'_\mu(1+|\xi|)^{-\mu},$$

the first factor is rapidly decreasing, second factor has polynomial growth and $D_x^\beta D_\xi^\alpha(p-q)$ is rapidly decreasing, the proof is complete.

Proposition 5.12. Let the closed linear operator A generate a contraction semigroup on a Banach space X. Then, for $u \in \mathcal{D}(A^2)$, there exist

$$||Au||^2 \leq 4||u||||A^2u||.$$

Proof. From the identity

$$-tAu = t(t-A)^{-1}A^2u + t^2u - t^2t(t-A)^{-1}u$$

and

$$||t(t-A)^{-1}|| \leq 1,$$

valid for the generator of a contraction semigroup, they get for $t > 0$

$$t||Au|| \leq ||A^2u|| + 2t^2||u||,$$

and

$$||Au|| \leq \inf_{t>0}((1/t)||A^2u|| + 2t||u||) = 2||A^2u||^{1/2}||u||^{1/2}.$$

Corollary 5.13. For all $u \in C_0^\infty(R^n)$,

$$||\frac{\partial}{\partial x_j}u||_{L^\infty}^2 \leq 4||u||_{L^\infty}||\frac{\partial^2}{\partial x_j^2}u||_{L^\infty}.$$

They consider the operator of the form

$$Au(x) = (2\pi)^{-n}\int\int a(x,y,\xi)u(y)e^{i(x-y)\xi}dyd\xi. \tag{5.5}$$

To study the above pseudodifferential operators they give following definitions:

Definition 5.14. Let $0 \leq \rho,\delta_1,\delta_2$. We say $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}(\Omega\times\Omega\times R^n)$ if, on compact subsets of $\Omega\times\Omega$, we have

$$|D_y^\gamma D_x^\beta D_\xi^\alpha a(x,y,\xi)| \leq C(1+|\xi|)^{m-\rho|\alpha|+\delta_1|\beta|+\delta_2|\gamma|}.$$

This inequality and by above lemma shows, if $u \in C_0^\infty(\Omega)$, then

$$\left|\int u(y)a(x,y,\xi)e^{-iy\xi}dy\right| \leq C_N(1+|\xi|)^{m-(1-\delta_2)N}$$

if $\delta_2 < 1$, for $u \in C_0^\infty$ (5.5) is absolutely integrable and $A : C_0^\infty(\Omega) \to C^\infty(\Omega)$, $\delta_2 < 1$.

Definition 5.15. A distribution $A \in \mathcal{D}'(\Omega\times\Omega)$ is said to be properly supported if *supp* A has compact intersection with $K\times\Omega$ and with $\Omega\times K$ for any compact $K\subset\Omega$.

A is properly supported provided $A : C_0^\infty \to \mathcal{E}'(\Omega)$ and $A^t : C_0^\infty \to \mathcal{E}'(\Omega)$, hence $A : C^\infty \to \mathcal{D}'(\Omega)$, if $b(x,y)$ has proper support, then the operator $\tilde{A}$ given by (5.3) with $a(x,y,\xi)$ replaced by $b(x,y)a(x,y,\xi)$ is properly supported.

Definition 5.16. If A is given (5.5) with $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$ and if A is properly supported, we say $A \in OPS^m_{\rho,\delta_1,\delta_2}$.

If $A \in OPS^m_{\rho,\delta_1,\delta_2}$, $\delta_2 < 1$, then $A : C^\infty(\Omega) \to C^\infty(\Omega)$. We know that $A : C_0^\infty(\Omega) \to C^\infty(\Omega)$, if $u \in C^\infty(\Omega)$ and $K \subset \Omega$ is compact, pick $v \in C_0^\infty(\Omega)$ such that *supp* $A \cap (\Omega\times K)$ is contained in $\hat{K}\times\hat{K}$ with $\hat{K}$ compact and $v = 1$ on a neighborhood of $\hat{K}$. It follows that $Au = A(vu)$ on K, so Au, which a priori belongs to $\mathcal{D}'(\Omega)$, is smooth on the interior of K.

Theorem 5.17. Let $A \in OPS^m_{\rho,\delta_1,\delta_2}$ and $0 \leq \delta_2 < \rho \leq 1$. The there is $p(x,\xi) \in S^m_{\rho,\delta}$ with $\delta = max(\delta_1,\delta_2)$, such that $A = p(x,D)$.

In fact, $p(x,\xi) = e^{-ix\xi}A(e^{ix\xi})$, and there exist the asymptotic expansion

$$p(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x}. \tag{5.6}$$

Proof. $p(x,\xi) = e^{-ix\xi}A(e^{ix\xi})$ is smooth function and they apply the linear operator A to

$$u(x) = \int \hat{u}(\xi) e^{ix\xi} d\xi,$$

they get

$$Au(x) = \int \hat{u}(\xi) p(x,\xi) e^{ix\xi} d\xi.$$

To show that $p \in S^m_{\rho,\delta}$ and (5.6) holds. The general term in the sum in (5.6) belongs $S^{m-(\rho-\delta_2)|\alpha|}_{\rho,\delta}$.

Let $b(x,y,\eta) = a(x,x+y,\eta)$ and $\hat{b}(x,\xi,\eta) = (2\pi)^{-n}\int b(x,y,\eta)e^{-iy\xi}dy$ so

$$p(x,\eta) = \int \hat{b}(x,\xi,\eta+\xi)d\xi.$$

The hypotheses on $a(x,y,\xi)$ imply

$$|D_y^\gamma D_x^\beta D_\eta^\alpha b(x,y,\eta)| \leq C(1+|\eta|)^{m+\delta|\beta|+\delta_2|\gamma|-\rho|\alpha|}, \ \delta = \delta_1 \vee \delta_2.$$

$a(x,y,\xi)$ can be replaced of the form $\hat{a}(x,y)a(x,y,\xi)$, here $\tilde{a}(x,y)$ has proper support in $\Omega \times \Omega$, $\tilde{a} = 1$ on an appropriate neighborhood of the diagonal, $a(x,y,\eta)$ is properly supported. Thus x belong to any compact subset of Ω, $b(x,y,\eta)$ vanishes for y outside some compact set. There exist

$$|D_x^\alpha D_\eta^\alpha \hat{b}(x,\xi,\eta)| \leq C_\nu (1+|\eta|)^{m+\delta|\beta|+\delta_2\nu-\rho|\alpha|}(1+|\xi|)^{-\nu}.$$

If to take Taylor expansion of $\hat{b}(x,\xi,\eta+\xi)$ above inequality yields

$$|\hat{b}(x,\xi,\eta+\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!}(iD_\eta)^\alpha \hat{b}(x,\xi,\eta)\xi^\alpha| \leq$$

$$\leq C_\nu |\xi|^N (1+|\xi|)^{-\nu} \sup_{0\leq t\leq 1} (1+|\eta+t\xi|)^{m+\delta_2\nu-\rho N}$$

where $nu \geq 0$. If $\nu = N$ they obtain a bound

$$C(1+|\eta|)^{m-(p-\delta_2)N} \quad if \quad |\xi| \geq \frac{1}{2}|\eta|,$$

if N is large, they get

$$|p(x,\eta) - \sum_{|\alpha|<N} \frac{1}{\alpha!}(iD_\eta)^\alpha D_y^\alpha b(x,y,\eta)|_{y=0}| \leq C(1+|\eta|)^{m+n-(\rho-\delta_2)N}.$$

5.4 Adjoints and products

In this section we give several result the properties of pseudodifferential operator. We can find this results Taylor book's.

Theorem 5.18. If $p(x, D) \in OPS^m_{\rho,\delta}$ $\delta < 1$ is properly supported, then

$$p(x, D)^* \in OPS^m_{\rho,0,\delta}.$$

Proof. There exists

$$(p(x, D)u, v) = (2\pi)^{-n} \int \bar{v}(y) \int \int e^{i(y-x)\xi} p(y, \xi)u(x)dxd\xi dy$$

$$= (2\pi)^n \left(\int \bar{u}(x) \int \int e^{i(x-y)\xi} p(y, \xi)^* v(y)dyd\xi dx \right)^*,$$

so

$$p(x, D)^* v = (2\pi)^{-n} \int \int p(y, \xi)^* e^{i(x-y)\xi} v(y)dyd\xi$$

which is (5.5) with $a(x, y, \xi) = p(y, \xi)^*$.

Theorem 5.19. If $p(x, D) \in OPS^m_{\rho,\delta}$ is properly supported, $0 \le \delta < \rho \le 1$, then

$$p(x, D)^* \in OPS^m_{\rho,\delta}$$

and indeed $p(x, D)^* = p^*(x, D)$ with

$$p^*(x, \xi) \sim \sum_{\alpha \ge 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_x p(x, \xi)^*.$$

Proof. This immediate yield by (5.6).

Theorem 5.19'. Let $p(x, D) \in OPS^m_{\rho',\delta'}$ and $q(x, D) \in OPS^\mu_{\rho'',\delta''}$ be properly supported, $0 \le \delta'' < \rho'' \le 1$. Then

$$p(x, D)q(x, D) \in OPS^{m+\mu}_{\rho,\delta',\delta''}, \quad \rho = \min(\rho', \rho'').$$

They apply Theorem 5.18-5.19 to the operator $q(x, D)^* = q^*(x, D)$.

$$q(x, D)u(x) = q(x, D)^{**}u = (2\pi)^{-n} \int \int q * (y, \xi)^* u(y)dyd\xi.$$

This implies

$$\widehat{q(x, D)u}(\xi) = (2\pi)^{-n} \int e^{-iy\xi} q^*(y, \xi)^* u(y)dy$$

$0 \le \delta'' < \delta'' \le 1$. Thus

$$p(x, D)q(x, D)u = \int e^{ix\xi} p(x, \xi)\widehat{q(x, D)u}(\xi)d\xi$$

$$= (2\pi)^{-n} \int \int e^{i(x-y)\xi} p(x, \xi)q^*(y, \xi)^* u(y)dyd\xi.$$

Thus $p(x,D)q(x,D)$ is of the form (5.5) with $a(x,y,\xi)=p(x,\xi)q^*(y,\xi)^*$ the proof is complete.

Theorem 5.20. Let $p(x,D)\in OPS^m_{\rho',\delta'}$ and $q(x,D)\in OPS_{\rho'',\delta''}$ be properly supported. Suppose $0\le\delta''<\rho\le 1$ with $\rho=\min(\rho',\rho'')$. Then

$$p(x,D)g(x,D)\in OPS^{m+\mu}_{\rho,\delta},\quad \delta=\max(\delta',\delta'')$$

and $p(x,D)q(x,D)=r(x,D)$ with

$$r(x,\xi)\sim\sum_{\alpha\ge0}\frac{i^{|\alpha|}}{\alpha!}D^\alpha_\xi p(x,\xi)D^\alpha_x q(x,\xi).$$

Proof. From (5.6) there exists

$$r(x,\xi)\sim\sum_{\alpha}\frac{i^{|\alpha|}}{\alpha!}D^\alpha_\xi D^\alpha_y(p(x,\xi)q^*(y,\xi)^*)|_{y=x}. \tag{5.7}$$

Consequently,

$$p^*(x,\xi)-p(x,\xi)^*\in S^{m-(\rho-\delta)}_{\rho,\delta};$$

$$r(x,\xi)-p(x,\xi)q(x,\xi)\in S^{m+\mu-(\rho'-\delta'')}_{\rho,\delta}.$$

If $p(x,\xi)\in S^m_{1,0}$, $q(x,\xi)\in S^\mu_{1,0}$, we have

$$p^*(x,\xi)-p(x,\xi)^*\in S^{m-1}_{1,0},$$

$$r(x,\xi)-p(x,\xi)q(x,\xi)\in S^{m+\mu-1}_{1,0}.$$

Remark 5.20'. The Theorem 5.20 remains the hypothesis $\delta''<\min(\rho',\rho'')$ is relaxed to $\delta''<\rho'$, which the terms in (5.7) still have order tending to $-\infty$.

By Hörmander [55] used $q(x,D)^{**}$, doesn't work in this more general case, and a proof is referred to Hörmander [53].

5.5 L^2 and Sobolev space continuity

In this section we give the continuity results and proves. There are results given by Taylor how to prove that if $A\in OPS^m_{\rho,\delta}(\Omega)$ and $\delta<\rho$, then $A:H_{comp}(\Omega)\to H^{s-m}_{loc}(\Omega)$

Proposition 5.21. If $p(x,\xi)\in S^0_{0,0}(R^n)$ has support in $|x|\le C_0$, then $p(x,D):L^2(R^n)\to L^2(R^n)$, continuously.

Proof. To proof this result they write $p(x,\xi)=\int p_\eta(\xi)e^{ix\eta}d\eta$ where

$$p_\eta=(2\pi)^{-n}\int p(x,\xi)e^{-ix\eta}dx.$$

$p(x,\xi)$ implies that $p_\eta(\xi)\le C_N(1+|\eta|)^{-N}$, since

$$\eta^\alpha p_\eta(\xi)=(2\pi)^{-n}\int D^\alpha_x p(x,\xi)e^{-ix\eta}dx.$$

And

$$||p_\eta(D)u||_{L^2} \leq C_N(1+|\eta|)^{-N}||u||_{L^2}.$$

Since $p(x,D) = \int e^{ix\eta}p_\eta(D)d\eta$ and $e^{ix\eta} = 1$, they get that

$$||p(x,D)u||_{L^2} \leq C_N \int (1+|\eta|)^{-N} d\eta ||u||_{L^2} \leq C_1||u||_{L^2}.$$

where $N > n$. If $\delta > 0$ they only conclude that $|p_\eta(\xi)| \leq C_N(1+|\xi|)^{\delta N}(1+|\eta|)^{-N}$. By Hörmander argument's the positive linear functional $|\lambda|$ on $C(K)$, the space of continuous functions on a compact Hausdorff space K is continuous, with norm $||\lambda|| = \lambda(1)$.

Lemma 5.22. If $p(x,\eta) \in S^0_{\rho,\delta}(\Omega)$, $\delta < \rho$, and if $Re\ p(x,\xi) \geq C > 0$, then there exists a $B \in OPS^0_{\rho,\delta}$ such that, with $Re\ P = (1/2)(P+P^*)$,

$$Re\ p(x,D) - B^*B \in OPS^\infty.$$

Proof. They construct the symbol $b(x,\xi) \sim \Sigma b_j(x,\xi)$ with $b_j \in S^{-j(\rho-\delta)}_{\rho,\delta}$. Firstly, $b_0(x,\xi) = (Re\ p(x,\xi))^{1/2} \in S^0_{\rho,\delta}$. Furthermore,

$$Re\ p(x,D) - b_0(x,D)^* b_0(x,D) = R_1 \in OPS^{-(\rho-\delta)}_{\rho,\delta}.$$

By induction there exist the terms $b_0, ..., b_j$ in the asymptotic expansion. There is $b_{j+1} \in S^{-(j+1)(\rho-\delta)}_{\rho,\delta}$ such that

$$Re\ p(x,D) = ((b_0^* + ... + b_j^*) + b_{j+1}^*)((b_0 + ... + b_j) + b_{j+1}) + R_{j+1}.$$

with $R_{j+1} \in OPS^{-j(\rho-\delta)}_{\rho,\delta}$. The right-hand side is equal to

$$Re\ p(x,D) + R_j + b_{j+1}^*(b_0 + ... + b_{j+1}) + (b_0^* + ... + b_{j+1}^*)b_{j+1} + R_{j+1}$$

$$= Re\ p(x,D) + R_j + b_{j+1}^* b_0^* b_{j+1} \mod OPS_\rho^{-(j+1)(\rho-\delta)},$$

$R_j = R_j^*$ so principal symbol is real or, if a matrix, self adjoint. They require b_{j+1} is

$$b_{j+1}^* b_0 + b_0 b_{j+1} = -R_j. \tag{5.8}$$

They pick $b_{j+1} = -(1/2)b_0^{-1}R_j$ in the scalar case.

$p(x,\xi)$ is a $k \times k$ system, with $Re\ p(x,\xi) = (1/2)(p(x,\xi) + p(x,\xi)^*) \geq C > 0$ and $b_0(x,\xi)$ is a positive self-adjoint matrix. It follows that (5.8) has a unique self-adjoint solution $b_{j+1}(x,\xi) = b_{j+1}(x,\xi)^*$. The map $\Phi(A) = Ab_0 + b_0A$ have eigenvalues $\{\lambda_j + \lambda_i\}$ where $\lambda_j > 0$ are the eigenvalues of b_0.

They obtain the following L^2 estimate.

Theorem 5.23. Let $A \in OPS^0_{\rho,\delta}(\Omega)$, $0 \leq \delta < \rho \leq 1$ assume that

$$\lim_{|\xi|\to\infty} \sup |A(x,\xi)| < M < \infty.$$

If $K \subset\subset \Omega$, there is an $R \in OPS^{-\infty}$ such that

$$||Au||^2_{L^2(K)} \leq M^2||u||^2 + (Ru, u).$$

Proof. The operator $C = M^2 - A^*A$ has principal symbol $C(x, \xi) = M^2 - |A(x, \xi)|^2 > 0$, by lemma 5.22 there is $B \in OPS^0_{\rho,\delta}$ such that

$$C - B^*B = M^2 - A^*A - B^*B = -R \in OPS^{-\infty}.$$

Thus

$$||Au||^2_{L^2} \leq (Au, Au) + (Bu, Bu) \leq M^2||u||^2_{L^2} + (Ru, u).$$

Corollary 5.24. If $\lim_{|\xi|\to\infty} A(x, \xi) = 0$, then $A : L^2(K) \to L^2_{loc}(\Omega)$ is compact. From $OPS^m_{\rho,\delta} : H^s \to H^{s-m}$ if $0 \leq \delta < \rho \leq 1$ follows L^2 continuity result of Theorem 5.23, via use of the operators $\Lambda^\sigma \in OPS^\sigma_{1,0}(R^n)$, where $\Lambda^\sigma u = \int (1+|\xi|^2)^{\sigma/2} e^{ix\xi}\hat{u}(\xi)d\xi$. Clearly $\Lambda^\sigma : H^s \to H^{s-\sigma}$ has isomorphism and properly supported.

Theorem 5.25. If $A \in OPS^m_{\rho,\delta}(\Omega)$ is properly supported, $0 \leq \delta < \rho \leq 1$, then

$$A : H^s_{loc}(\Omega) \to H^{s-m}_{loc}(\Omega).$$

Proof. To proof this theorem they show that $\Lambda^{s-m}A\Lambda^{-s} \in OPS^0_{\rho,\delta}$ takes $L^2_{loc}(\Omega)$ to $L^2_{loc}(\Omega)$ by Theorem 5.23. Calderon and Vaillancourt have shown $A \in OPS^0_{\rho,\rho}$ is continuous on L^2, $0 \leq \rho < 1$. The key in the proof of this the L^2 continuity of $p(x, D)$ on $L^2(R^n)$ we assume

$$|D^\beta_x D^\alpha_\xi p(x, \xi)| \leq C_{\alpha,\beta}, \qquad x,\ \xi \in R^n$$

p doesn't has compact support in x. For $OPS^0_{1,0}$ has continuity on L^p, $1 < p < \infty$ and also on Hölder spaces.

5.6 Families of pseudodifferential operators: Friedrichs' mollifiers

$S^m_{\rho,\delta}(\Omega)$ into a Frechet space has the seminorms

$$|p|_{K,\alpha,\beta} = \sup_{x\in K} |D^\beta_x D^\alpha_\xi p(x, \xi)|(1 + \xi)^{-m+\rho|\alpha|-\delta|\beta|}.$$

The map $p(x, \xi) \to p(x, D)$ is a continuous map from $S^m_{\rho,\delta}(\Omega)$ to $\mathcal{S}(H^s_{comp}(\Omega), H^{s-m}_{loc}(\Omega))$ if $\delta < \rho$. If M is a compact manifold, $0 \leq 1 - \rho \leq \delta < \rho \leq 1$ we give $OPS^m_{\rho,\delta}(M)$ a natural Frechet spase topology, all maps

$$OPS^m_{\rho,\delta}(M) \to \mathcal{S}(H^s(M), H^{s-m}(M))$$

are continuous.

Let $p(\xi) \in S^\sigma_{1,0}(R^n)$, $\sigma \leq 0$, and let $p_\varepsilon(\xi) = p(\varepsilon\xi)$ of the chain rule shows $\{p_\varepsilon : 0 < \varepsilon \leq 1\}$ is bounded in $S^0_{1,0}(R^n)$, if we take $p(\xi) \in S^{-\infty}(R^n)$ and $p(\xi) = e^{-|\xi|^2}$.

Definition 5.26. A Friedrichs' molifier on M is a family J_ε of scalar pseudodifferential operators, $0 < \varepsilon \le 1$, such that a) $J_\varepsilon \in OPS^{-\infty}(M)$ for each $\varepsilon \in (0, 1]$; b) $\{J_\varepsilon : 0 < \varepsilon \le 1\}$ is a bounded subset of $OPS^0_{1,0}(M)$; c) $J_\varepsilon u \to u$ in $L^2(M)$ as $\varepsilon \to 0$, for each $u \in L^2(M)$.

Proposition 5.27. Let $A \in OPS^m_{\rho,\delta}(M)$, $1 - \rho \le \delta < \rho$. If J_ε is a Friedrich's mollifier on M, then $[A, J_\varepsilon] = AJ_\varepsilon - J_\varepsilon A$ has following properties: a) $[A, J_\varepsilon] \in OPS^{-\infty}(M), \quad 0 < \varepsilon \le 1$. b) $\{[A, J_\varepsilon] : 0 < \varepsilon \le 1\}$ is a bounded subset of $OPS^{m-\rho\wedge(1-\delta)}_{\rho,\delta}(M)$. Friedrichs' mollifiers apply to prove the weak and strong solutions to pseudodifferential equations.

Definition 5.28. Let M be a compact manifold, $A : C^\infty \to C^\infty(M)$ and $A : \mathcal{D}'(M) \to \mathcal{D}'(M)$. Take $f \in L^2(M)$. A function $u \in L^2(M)$ is said to be a weak solution of the equation

$$Au = f$$

if this equation holds when A is applied to u in the distribution sense. On the other hand, u is said to be a strong solution of above equation if there exists a sequence $u_j \to u$ in $L^2(M)$, with $u_j \in C^\infty(M)$, such that $Au_j = f_j \to f$ in $L^2(M)$.

Proposition 5.29. If $A \in OPS^1_{1,0}(M)$, then every weak solution to above equation is a strong solution.

Proof. With J_ε a Friedrichs' mollifier, let $\varepsilon_j \to 0$ and set $u_j = J_{\varepsilon_j}u$. To show that $||Au_j - f||_{L^2} \to 0$, write

$$Au_j = J_{\varepsilon_j}Au + [A, J_{\varepsilon_j}]u = J_{\varepsilon_j}f + [A, J_{\varepsilon_j}]u.$$

If $A \in OPS^m_{1,0}(M)$ with $m > 1$, weak and strong solutions need not coincide, though they do if A is elliptic (since by elliptic regularity u must belong to $H^m(M)$)

5.7 Garding's inequality

Theorem 5.30. Let $p(x, D) \in OPS^m_{\rho,\delta}(\Omega)$ and assume $0 \le \delta < \rho \le 1$. Suppose $Re\ p(x, \xi) \ge C|\xi|^m$ for $|\xi|$ large, with $C > 0$. Then, for any $s \in R$, for any compact $K \subset \Omega$, and all $u \in C^\infty_0(K)$, we have

$$Re\ (p(x, D)u, u) \ge C_0||u||^2_{H^{m/2}} - C_1||u||^2_{H^s}.$$

Proof. Replacing $p(x, D)$ by $q(x, D) = \Lambda^{-m/2}p(x, D)\Lambda^{-m/2}$, suppose $Re\ p(x, \xi) \ge C > 0$, $p(x, \xi) \in S^0_{\rho,\delta}$. We have $r(x, \xi) = Re\ p(x, \xi) - (1/2)C$, to yield $B \in OPS^0_{\rho,\delta}$ with $r(x, D) - B^*B = S \in OPS^{-\infty}$, and hence

$$Re\ (p(x, D)u, u) - \frac{1}{2}C(u, u) = (Bu, Bu) + Re\ (Su, u)$$

which implies above inequality in the case $m = 0$. The sharp form of Garding inequality which $Re\ p(x, \xi) \ge 0$ if $p(x, D) \in OPS^m_{1,0}$, it follows that

$$Re\ (p(x, D)u, u) \ge -C_1||u||^2_{H^{(m-1)/2}}, \quad u \in C^\infty_0(K).$$

The operator $p(x,D) \in OPS^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, which satisfies the condition

$$Re\ p(x,\xi) \ge C|\xi|^m \quad \text{for} \quad |\xi| \text{ large}$$

some $C = const > 0$, is called strongly elliptic.

5.8 L^p and Hölder space theory of pseudodifferential operators

Fourier multipliers on L^p and Hölder Spaces.

In this section we give the continuity of pseudodifferential operators on L^p and C^α spaces, which is obtained by various authors, scattered throughout the literature, thought perhaps a few results are stated in sharper form here. We describe some results, due to Marcinkiewicz, Mikhlin, Hörmander, Stein, Taibleson and Taylor, on the behavior of following operator

$$P(D)u = \int e^{ix\xi} p(\xi)\hat{u}(\xi)d\xi$$

on $L^p(R^n)$ and $C^\alpha(R^n)$. $P(D)$ is simply multiplies the Fourier transform of u by $p(\xi)$, hence $P(D)$ is called a Fourier multiplier. It also write as a convolution operator

$$P(D)u = \hat{p} * u.$$

Marcinkiewicz [87] studied the L^p continuity of convolution operators on the torus T^n. Mikhlin translated these result to the R^n.

Theorem 5.31.(Mikhlin)[94] $P(D) : L^p \to L^p(R^n)$, $1 < p < \infty$, provided

$$|\xi|^{|\alpha|} |D^\alpha_\xi p(\xi)| \le C_\alpha, \quad |\alpha| \le [\frac{n}{2}] + 1.$$

Hörmander's theorem is following.

Theorem 5.32.(Hörmander) $P(D) : L^p(R^n) \to L^p(R^n)$, $1 < p < \infty$, provided

$$R^{-n} \int_{R<|\xi|<2R} ||\xi|^\alpha p^\alpha(\xi)|^2 d\xi < C, \quad |\alpha| \le [\frac{n}{2}] + 1$$

with C independent of R, $0 < R < \infty$.

We restate this result, let $\Omega = \{\xi \in R^n : 1 \le |\xi| \le 4\}$, and $p_r(\xi) = p(r\xi)$.

Corollary 5.33. $P(D) : L^p(R^n) \to L^p(R^n)$, $1 < p < \infty$, provided

$$||p_r||_{H^{[n/2]+1}(\Omega)} \le C, \quad 0 < r < \infty$$

where $C < \infty$ is independent of r.

This result sharpened as follows:

Theorem 5.34. $P(D) : L^p(R^n) \to L^p(R^n)$, $1 < p < \infty$, provided

$$||p_r||_{H^{n/2+\varepsilon}(\Omega)} \le C, \quad 0 < r < \infty \tag{5.9}$$

for some $\varepsilon > 0$, where C is independent of r.

Theorem 5.35. $P(D) : L^p(R^n) \to L^p(R^n)$, $1 < p < \infty$, provided

$$|\xi^\alpha p^\alpha(\xi)| \leq C_\alpha, \quad \xi \in R^n$$

for all multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$ with each α_j either 0 or 1.

They consider the be behavior of $P(D)$ on $C^\alpha(R^n)$. For $0 < \alpha < 1$, and define $C^\alpha(R^n)$ to consist of those functions u on R^n such that

$$||u||_{C^\alpha} = ||u||_{L^\infty} + \sup_{x,h \in R^n, |h| \leq 1} |h|^{-\alpha}|u(x+h) - u(x)| < \infty.$$

Taibleson [109] has found a necessary and sufficient condition that $P(D) : C^\alpha(R^n) \to C^\alpha(R^n)$. For $s \in R$

$$L_{1,s} = (1 - \Delta)^{s/2} L^1(R^n).$$

Theorem 5.36. $P(D) : C^\alpha(R^n) \to C^\alpha(R^n)$, $0 < \alpha < 1$, iff $\hat{p} \in L_{1,s}$ for some $s > 0$, and the following holds:

$$\sup_{0<t<1} ||t\Delta e^{t\Delta}\hat{p}||_{L^1} + ||\hat{p}||_{L_{1,s}} < \infty.$$

In fact for $\hat{p} = \frac{k_0(x)}{|x|^n}$, the important term in above inequality,

$$\sup_{0<t<1} ||t\Delta e^{t\Delta}\hat{p}||_{L^1},$$

is finite, $||\hat{p}||_{1,s}$ is infinite in this case. Any $\hat{p} \in \mathcal{E}'(R^n)$ belongs to $L_{1,s}$ for $s > 0$ sufficiently large. If $\hat{p} \in \mathcal{E}'(R^n)$, we have $P(D) : C^\alpha(R^n) \to C^\alpha(R^n)$ iff,

$$\sup_{0<t<1} ||t\Delta e^{t\Delta}\hat{p}||_{L^1} < \infty. \tag{5.10}$$

Theorem 5.37. Suppose $\hat{p} \in \mathcal{E}'(R^n)$. Then $P(D) : C^\alpha(R^n) \to C^\alpha(R^n)$, $0 < \alpha < 1$, iff for some C independent of r,

$$||\psi p_r||_{\mathcal{F}L^1} \leq C, \quad 0 < r < \infty. \tag{5.11}$$

Here the $\mathcal{F}L^1$ norm is $||u||_{\mathcal{F}L^1} = ||\hat{u}||_{L^1}$. $\mathcal{F}L^1$ is a Banach algebra under multiplication, since L^1 is a convolution algebra. The Sobolev theorem that elements of $H^{n/2+\varepsilon}(R^n)$ are continuous i.e., $H^{n/2+\varepsilon}(R^n) \subset \mathcal{F}L^1$. (5.11) is just a little weaker than (5.9).

Proof. By Taibleson's theorem, we show (5.11)and (5.9) are equivalent. So suppose (5.11) holds and $\int_{-\infty}^{\infty} \psi(e^{-y}\xi)dy = 1$. Let $\mu_y(\xi) = \psi(\xi)p(e^y\xi)$, so (5.11) is equivalent to $||\mu_y||_{\mathcal{F}L^1} \leq C < \infty$, $-\infty < y < \infty$. We have

$$p(\xi) = \int_{\infty}^{\infty} \mu_y(e^{-y}\xi)dy.$$

To desire to estimate

$$||t|\xi|^2 e^{-t|\xi|^2} p(\xi)||_{\mathcal{F}L^1},$$

$0 < t < 1$. There exist

$$t\int_{-\infty}^{\infty} ||\mu_y(e^{-y}\xi)|\xi|^2 e^{-t|\xi|^2}||_{\mathcal{F}L^1} dy = t\int_{-\infty}^{\infty} e^{2y}||\mu_y(\xi)|\xi|^2 e^{-te^{2y}|\xi|^2}||_{\mathcal{F}L^1} dy$$

since $||g||_{\mathcal{F}L^1} = ||g_r||_{\mathcal{F}L^1}$, $0 < r < \infty$. We dominate the integrand by

$$Ce^{2y}||\mu_y||_{\mathcal{F}L^1}|||\xi|^2 e^{-te^{2y}|\xi|^2}||_{\mathcal{H}^s(\Omega)} \le C'e^{2y}(1 + t^s e^{2ys})e^{-te^{2y}}$$

where $s > n/2$. Thus

$$||t|\xi|^2 e^{-t|\xi|} p(\xi)||_{\mathcal{F}L^1} \le Ct\int_{-\infty}^{\infty} e^{2y} e^{-te^{2y}} dy + Ct^{1+s}\int_{-\infty}^{\infty} e^{2y(1+s)} e^{-te^{2y}} dy = C'' < \infty.$$

(5.3) implies (5.9).

Conversely, (5.9) is equivalent to

$$|||\xi|^2 e^{-|\xi|^2} p_r||_{\mathcal{F}L^1} \le C < \infty$$

for $r > 1$, $\hat{p} \in \mathcal{E}'$, $p(\xi)$ is smooth, so above inequality is true for $0 < r < \infty$. We have

$$||\psi p_r||_{\mathcal{F}L^1} \le ||\psi|\xi|^{-2} e^{|\xi|^2}||_{\mathcal{F}L^1}|||\xi|^2 e^{-|\xi|^2} p_r||_{\mathcal{F}L^1} = C_0|||\xi|^2 e^{-|\xi|^2} p_r||_{\mathcal{F}L^1}.$$

By Stein [106] take the result of $(1 - \Delta)^{m/2}$ on Hölder spaces.

Theorem 5.38. If $k + \alpha$ and $k + \alpha - m$ are both positive and nonintegral, $m \in R$,

$$(1 - \Delta)^{m/2} : C^{k+\alpha}_{comp}(R^n) \to C^{k+\alpha-m}_{loc}(R^n).$$

Here, for $k = 0, 1, 2, .., \alpha \in (0, 1)$, we set $C^{k+\alpha}(R^n) = \{u \in C^k(R^n) : D^\beta u \in C^\alpha(R^n), |\beta| \le k\}$. Finally, we give the Marcinkiewicz interpolation theorem

Theorem 5.39. Let $T : C_0^\infty(R^n) \to L^\infty(R^n)$ satisfy the conditions

$$meas\{x : |Tu(x)| > \lambda\} \le C_1\lambda^{-p}||u||_{L^p}, \tag{5.11'}$$

$$meas\{x : |Tu(x)| > \lambda\} \le C_2\lambda^{-q}||u||_{L^q}, \tag{5.12}$$

where $1 \le p < q$. Then $T : L^r(R^n) \to L^r(R^n)$, $p < r < q$, with operator norm determined by C_1, C_2, r, n.

$T : L^q(R^n) \to L^q(R^n)$ implies (5.12), T is weak (q, q) if (5.12) satisfied.

L^p and C^α behavior of operators in $OPS^m_{1,0}$

Theorem 5.40. Let $p(x, D) \in OPS^0_{1,0}$. Then

$$p(x, D) : L^p_{comp} \to L^p_{loc}, \quad 1 < p < \infty,$$

and

$$p(x, D) : C^\alpha_{comp} \to C^\alpha_{loc}, \quad 0 < \alpha < 1.$$

Here L^p_{comp} is the space of L^p functions with compact support, L^P_{loc} is the space of locally L^p functions.

Proof. Let $u \in B_R$ ball. $p(x,D)$ is pseudolocal if $u \in L^p$, then $p(x,D)u \in L^p$ on B_{2R}. We have $||||_{\mathcal{S}}$ the operator norm of $T : C_0^\alpha(B_R) \to C^\alpha(B_{2R})$ $0 < \alpha < 1$, or of $T : L^p(B_R) \to L^p(B_{2R})$, $1 < p < \infty$. By theorem 5.35 and 5.36 for Fourier multiplier $p_\eta(D)$

$$||p_\eta(D)||_{\mathcal{S}} \leq C \sup_\xi \sum_{|\beta| \leq [n/2]+1} |\xi|^{|\beta|} |p_\eta^\beta(\xi)|$$

where C depends on α or p.

By the pseudolocal property $p(x,\xi)$ vanishes for $|x| \geq 2R$. We write

$$p(x,\xi) = (2\pi)^{-n} \int_{R^n} e^{-ix\eta} p_\eta(\xi) d\eta$$

where

$$p_\eta(\xi) = \int_{R^n} p(x,\xi) e^{ix\eta} dx.$$

It follows that

$$p(x,D)u = (2\pi)^{-n} \int_{R^n} e^{-ix\eta} p_\eta(D) u d\eta.$$

A multiplication operator on L^p, $e^{-ix\eta}$ has norm 1 and a multiplication operator on C^α, $e^{-ix\eta}$ has norm $< C(1+|\eta|)^\alpha$. We have

$$||p(x,D)||_{\mathcal{S}} \leq \int_{R^n} C(\eta) ||p_\eta(D)||_{\mathcal{S}} d\eta$$

where $C(\eta) = 1$ of L^p and $C(\eta) = C(1+|\eta|)^\alpha$ of C^α.

To show that $||p_\eta(D)||_{\mathcal{S}}$ is rapidly decreasing, as $|\eta| \to \infty$. In fact

$$\eta^\gamma p_\eta^{(\beta)}(\xi) = \int_{R^N} D_\xi^\beta p(x,\xi) D_x^\gamma e^{ix\eta} dx = \int_{R^n} D_x^\gamma D_\xi^\beta p(x,\xi) e^{ix\eta} dx.$$

Therefore,

$$|\eta^\gamma||p_\eta^{(\beta)(\xi)}| \leq C_{\beta\gamma}(1+|\xi|)^{-|\beta|}.$$

We have after summing over $|\gamma| \leq N$

$$||p_\eta(D)||_{\mathcal{S}} \leq C_N (1+|\eta|)^{-N}, \quad C_N = C_N(p) \textit{ or } C_N(\alpha)$$

finally we have $||p(x,D)||_{\mathcal{S}} < \infty$.

Theorem 5.41. Let M be a compact manifold. Let $p(x,D) \in OPS_{1,0}^0$ on M. Then

$$p(x,D) : L^p(M) \to L^p(M), \quad 1 < p < \infty$$

and

$$p(x,D) : C^\alpha(M) \to C^\alpha(M), \quad 0 < \alpha < 1.$$

The operator norms are bounded

$$||p(x,D)||_{\mathcal{S}} \leq C \max_{|\alpha| \leq [n/2]+1, |\beta| \leq n+1} \sup_{(x,\xi) \in T^*(M)} |D_x^\beta D_\xi^\alpha p(x,\xi)||\xi|^\alpha$$

where C depends on p or α.

Corollary 5.42. Let $p_j(x,\xi)$ be a bounded set of symbols on M, in $S^0_{1,0}$. Then $p_j(x,D)$ form a bounded family of operators on $L^p(M)$, $1<p<\infty$, and on $C^\alpha(M)$, $0<\alpha<1$.

They take results on $\mathcal{S}^s_p$ Sobolev and $C^{k+\alpha}$ Holder spaces for $k=1,2,..,$ $0<\alpha<1$, $C^{k+\alpha}=\{u\in C^k : D^\beta u\in C^\alpha : D^\beta u\in C^\alpha, |\beta|=k\}$. If M is compact manifold

$$\mathcal{S}^s_p(M)=\{u\in L^p(M) : Pu\in L^p(M), P\in OPD^k\}.$$

For $1<p<\infty$, $u\in\mathcal{S}^k_p$ iff $p(x,D)u\in L^p(M)$ for all $p(x,D)\in OPS^k_{1,0}$. If $p(x,D)\in OPS^k_{1,0}$ and $p(x,D)=\sum a_j(x,D)q_j(x,D)$ with $q_j(x,D)\in OPD^k$ and $a_j(x,D)\in OPS^0_{1,0}$, since $a_j(x,D): L^p(M)\to L^p(M)$, $1<p<\infty$. For $s\in R$,

$$\mathcal{S}^s_p(M)=\{u\in\mathcal{D}'(M) : p(x,D)u\in L^p(M) \quad for\ all \quad p(x,D)\in OPS^s_{1,0}\}.$$

Proposition 5.43. Let $u\in\mathcal{D}'(M)$ and let $q(x,D)\in OPS^s_{,0}$ be elliptic. Then $u\in\mathcal{S}^s_p(M)$ iff $q(x,D)u\in L^p(M)$, if $1<p<\infty$.

Proof. Let $q(x,D)^{-1}\in OPS^{-s}_{1,0}$ is parametrix of $q(x,D)$. If $p(x,D)\in OPS^s_{1,0}$ is given, then $p(x,D)u=p(x,D)q(x,D)^{-1}q(x,D)u(mod C^\infty)=r(x,D)q(x,D)u\in L^p(M)$ since $r(x,D)\in OPS^0_{1,0}$ leaves L^p invariant.

For $s\in R$, $(1-\Delta)^{s/2}\in OPS^s_{1,0}$ we have

$$\mathcal{S}^s_p(M)=(1-\Delta)^{-s/2}L^p(M).$$

Theorem 5.44. Let $p(x,D)\in OPS^m_{1,0}$ on M. Then

$$p(x,D):\mathcal{S}^s_p\to\mathcal{S}^{s-m}_p,\quad 1<p<\infty \tag{5.13}$$

and

$$p(x,D):C^{k+\alpha}\to C^{k+\alpha-m}, \tag{5.14}$$

provided $k+\alpha$ and $k+\alpha-m$ are both positive and nonintegral.

Proof. $a(x,D)\in OPS^s_{1,0}$, $b(x,D)\in OPS^{s-m}_{1,0}$ is elliptic with A^{-1}, B^{-1} parametrices. Then $b(x,D)p(x,D)u=b(x,D)p(x,D)A^{-1}a(x,D) \mod C^\infty$ belongs to L^p if $u\in\mathcal{S}^s_p$ yields $a(x,D)u\in L^p$ and $b(x,D)p(x,D)A^{-1}\in OPS^0_{1,0}$, this proved (5.13).

Next we take $u\in C^{k+\alpha}(R^n)$ in B_R and $p(x,\xi)=0$ for $|x|\geq 2R$ then $p(x,D)u\in C^{k+\alpha-m}$ is provided $k+\alpha$ and $k+\alpha-m$ are positive and nonintegral. We have $\varphi\in C^\infty_0(B_{2R})$, $\varphi=1$ on B_R,

$$\varphi(x)(1-\Delta)^{m/2}u\in C^{k+\alpha-m}(R^n).$$

But $p(x,D)u=p(x,D)(1-\Delta)^{-m/2}\varphi(x)(1-\Delta)^{m/2}u\ mod C^\infty$, since $p(x,D)(1-\Delta)^{-m/2}\in OPS^0_{1,0}$ follows from Theorem 5.10.

They are consider the Strichartz argument is given the operator

$$Tu(x,y)=\int p(y,\xi)\hat{u}(\xi)e^{ix\xi}d\xi.$$

Therefore

$$||p(x,D)u||_{L^p(R^n)} \leq C||Tu||_{L^p(R^n,H^s)} \leq C'||u||_{L^p}|||p|||_s$$

where

$$|||p|||_s = \sup_{0<r<\infty} ||p_r||_{H^s(\Omega,H^s(R^n))}.$$

Here two spaces are respectively C and $H^s(R^n)$, $s > n/2$ and $p_r(x,\xi) = p(x,r\xi)$.

Theorem 5.45. Let $u \in L^p$ be supported in B_R, and let $(n/2)+\varepsilon = k+\sigma$, k an integer, $0<\sigma<1$. Then

$$||p(x,D)u||_{L^p(B_{2R})} \leq C(p) \sup_{0<r<\infty} ||p_r||_k^{1-\sigma}||p_r||_{k+1}^{\sigma}, \quad 1<p<\infty, \tag{5.15}$$

where

$$||p_r||_k = \max_{|\alpha|,|\beta|\leq k} \sup_{(x,\xi)\in R^n\times\Omega} |D_x^\beta D_\xi^\alpha p_r(x,\xi)|.$$

Proof. To show

$$|||p|||_{n/2+\varepsilon} \leq C \sup_{0<r<\infty} ||p_r||_k^{1-\sigma}||p_r||_{k+1}^{\sigma}$$

in Sobolev space.

Theorem 5.46. Suppose $p(x,D) \in OPS_{\rho,\delta}^{-m}$. Then

$$p(x,D): L^p_{comp} \to L^p_{loc}, \quad 1<p<\infty, \quad \text{and} \quad p(x,D): C^\alpha_{comp} \to C^\alpha_{loc}, \quad 0<\alpha<1,$$

provided $m > (n/2)(1-\rho+\delta)$.

Proof. Given

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{-m-\rho|\alpha|+\delta|\beta|},$$

By using (5.15), we have $D_x^\beta D_\xi^\alpha p_r(x,\xi) = r^{|\alpha|} D_x^\beta D_\xi^\alpha p(x,r\xi)$, so

$$||p_r||_k^{1-\sigma}||p_r||_{k+1}^{\sigma} \leq C[\sum_{l,j=0}^{k} r^j(1+r)^{-m-\rho j+\delta l}]^{1-\sigma}[\sum_{\lambda,\mu=0}^{k+1} r^\mu(1+r)^{-m-\rho\mu+\delta\lambda}]^\sigma$$

$$\leq C(1+r)^{-m}[\sum_{l,j=0}^{k}(1+r)^{\delta l+(1-\rho)j}]^{1-\sigma}[\sum_{\lambda,\mu=0}^{k+1}(1+r)^{\delta\lambda+(1-\rho)\mu}]^\sigma$$

$$\leq C(1+r)^{-m+(1-\rho+\delta)(k+\sigma)},$$

is bounded on $0<r<\infty$ provided

$$m \geq (k+\sigma)(1-\rho+\delta) = (\frac{n}{2}+\varepsilon)(1-\rho+\delta).$$

The C^α has the inclusions

$$C^\alpha \subset \cap_{p<\infty} W_p^{\alpha-\varepsilon/2} \subset C^{\alpha-\varepsilon}$$

last inclusion is Sobolev's imbedding theorems.

Theorem 5.47. Suppose $p(x, D) \in OPS^{-m}_{\rho,\delta}$ with $0 \le \delta \le \rho \le 1 (\delta < 1)$. Then

$$p(x, D) : L^p_{comp} \to L^p_{loc}, \quad 1 < p < \infty, \quad if$$
$$m = \frac{n}{2}(1 - \rho + \delta).$$

The Wainger [118] result is sharp, if $\delta = 0$, symbols which are independent of x. Hörmander [55] shows for $0 \le \delta < \rho < 1$, $OPS^{-(n/2)(1-\rho)}_{\rho,\delta}$

$$L^p_{comp} \to L^p_{loc}, \quad 1 < p < \infty.$$

By E.Stein [106] shows if $p(x, D) \in OPS^m_{\delta,\rho}$ and either $0 \le \delta < \rho = 1$ or $0 < \delta = \rho < 1$, then $p(x, D)$ is weak type $(1, 1)$ if $m = -(1 - \rho)n/2$ and $p(x, D)$ is bounded on L^p $(1 < p < \infty)$ if $(1 - \rho)|(1/2) - (1/p)| \le -(m/n)$. Also $p(x, D)$ is bounded on C^α if $0 < \rho$, $1 \ge \delta$ and $m = -(1-\rho)(n/2)$. Fefferman [34] take $p(x, D) : L^\infty \to BMO$. Stein's results derived by Kagan [62] and others, is that, if $0 \le \delta < 1$, $OPS^0_{1,\delta}$ is bounded on L^p $1 < p < \infty$.

5.9 L^p behavior of $OPS^0_{1,\delta}$

Theorem 5.48. If $p(x, D) \in OPS^0_{1,\delta}$, $0 \le \delta < 1$, then $p(x, D) : L^p_{comp} \to L^p_{loc}$, $1 < p < \infty$.

This contains the Theorem 5.47 with special case. they give following $(1, 1)$ estimate.

Proposition 5.49. Suppose $p(x, \xi) \in S^0_{1,\delta}$, $\delta < 1$ has compact x-support. Then for $u \in L^1(R^n)$, $\lambda \in R^+$,

$$meas\{x : |p(x, D)u(x)| \ge \lambda\} \le \frac{c}{\lambda}||u||_{L^1}.$$

Since $p(x, D) : L^2 \to L^2$, by Marcinkiewicz [87] interpolation theorem implies $p(x, D) : L^p \to L^p$ for $1 < p \le 2$ proving above theorem. The result for $2 \le p < \infty$ by duality, since $p(x, D) \in OPS^0_{1,\delta}$ implies $p(x, D)^* \in OPS^0_{1,\delta}$.

Assume $p(x, \xi) = 0$ for $|\xi| \le 1$, $\psi \in C^\infty_0$ on $\{\xi : 1/2 \le |\xi| \le 2\}$ such that

$$\sum_{j=0}^{\infty} \psi(2^{-j}\xi) = 1$$

for $|\xi| \ge 1$. Set $q_i(x, \xi) = p(x, \xi)\psi(2^{-j}\xi)$,

$$p_N(x, \xi) = \sum_{j=0}^{N} q_j(x, \xi).$$

We take estimates on the kernels $k_j(x, x - y)$ of $q_i(x, D)$ given

$$k_j(x, z) = \int e^{iz\xi} p(x, \xi)\psi(2^{-j}\xi)d\xi,$$

the kernel $K_n(x, x - y)$ of $p_N(x, D)$

$$K_N(x, z) = \sum_{j=0}^{N} k_j(x, z).$$

Lemma 5.50. We have

$$\int_{|x|\geq 2t} |K_N(x+x^0, x-y) - K_N(x+x^0, x)|dx \leq C_0,$$

if $|y| \leq t$.

Proof. For $\kappa = [n/2]+1$, $|\alpha| \leq \kappa$, we have

$$(2^j x)^\alpha k_j(x+x^0, x) = 2^{j|\alpha|} \sum_{\beta\leq\alpha} C_{\alpha\beta} \int e^{ix\xi} D_\xi^\beta p(x+x^0,\xi)|\xi|^{|\beta|} D_\xi^{\alpha-\beta}\psi(2^{-j}\xi)|\xi|^{-|\beta|}d\xi.$$

$D_\xi^\beta p(x+x^o,\xi)|\xi|^{|\beta|}$ is bounded subset of $S^0_{1,\delta}$ with x^0 parameter, so

$$||(2^j x)^\alpha k_j(x+x^0,x)||^2_{L^2} \leq C_\alpha 2^{2j|\alpha|} \sum_{\beta\leq\alpha} ||D_\xi^{\alpha-\beta}\psi(2^{-j}\xi)|\xi|^{-|\beta|}||^2_{L^2}$$

$$\leq C'_\alpha 2^{nj}, \quad |\alpha| \leq [\frac{n}{2}]+1, \tag{5.16}$$

where C'_α is independent of j and x^0. And

$$\int |k_j(x+x^0,x)|dx \leq [\int (1+2^{2j}|x|^2)^\kappa |k_j(x+x^0,x)|^2 dx]^{1/2} [\int (1+2^{2j}|x|^2)^{-\kappa}dx]^{1/2} \leq C_1, \tag{5.17}$$

C_1 is independent of j, x^0. By (5.16),

$$\int_{|x|\geq t} |k_j(x+x^0,x)|dx \leq [\int_{|x|\geq t} (2^{2j}|x|^2)^\kappa |k_j(x+x^0,x)|^2dx]^{1/2}$$

$$\times[\int_{|x|\geq t}(2^{2j}|x|^2)^{-\kappa}dx]^{1/2} \leq C_2(2^jt)^{n/2-\kappa}.$$

For $|y| \leq t$,

$$\int_{|x|\geq 2t} |k_j(x+x^0,x-y) - k_j(x+x^0,x)|dx \leq \int_{|x|\geq 2t} |k_j(x+x^0+y,x)|dx$$

$$+\int_{|x|\geq 2t}|k_j(x+x^0,x)|dx \leq 2C_2(2^jt)^{n/2-\kappa}. \tag{5.18}$$

When $2^jt \leq 1$ and $|y| \leq t$, then

$$|e^{-iy\xi}-1| \leq |y||\xi| \leq 2^jt, \quad \text{for } \xi \in supp\ \psi(2^{-j}\xi),$$

and

$$|D_\xi^\delta(e^{-iy\xi}-1)| \leq t^{|\beta|} \leq 2^jt2^{-j|\beta|}, \quad |\beta| \neq 0.$$

By (5.18),

$$\int |k_j(x+x^0,x-y) - k_j(x+x^0,x)|dx \leq C_3(2^jt) \tag{5.19}$$

for $|y| \leq t$ and $2^jt \leq 1$ and with

$$K_N(x,z) = \sum_{j=0}^N k_j(x,z),$$

it follows from (5.18), (5.19) for $|y| \leq t$,

$$\int_{|x|\geq 2t} |K_N(x+x^0, x-y) - K(x+x^0, x)|dx \leq C_4 \sum_{j=0}^{\infty} \min\{(2^j t)^{n/2-\kappa}, 2^j t\} \leq C_0,$$

proved the lemma.

Lemma 5.51.(Calderon and Zygmund)[55]. Let $u \in L^1(R^n)$ and $\lambda > 0$. There exist v, w_k in $L^1(R^n)$ and disjoint cubes I_k, $1 \leq k < \infty$, with centers $x^{(k)}$ such that

$$i)\ u = v + \sum_{k=1}^{\infty} w_k, \quad ||v||_{L^1} + \sum_{k=1}^{\infty} ||w_k||_{L^1} \leq 3||u||_{L^1};$$

$$ii)\ |v(x)| \leq 2^n \lambda;$$

$$iii)\ \int_{I_k} w_k(x)dx = 0 \ \ \textit{and supp}\ w_k \subset I_k;$$

$$iv)\ \sum_{k=1}^{\infty} meas(I_k) \leq \lambda^{-1} ||u||_{L^1}.$$

If u has compact support, the support of v and w_k are contained in a fixed compact set.

It follows from this decomposition lemma that

$$\hat{u}(\xi) = \hat{v}(\xi) + \sum_k \hat{w}_k(\xi), \quad |\hat{v}(\xi) + \sum_k |\hat{w}_k(\xi)| \leq 3||u||_{L^1},$$

and

$$p_N(x, D)u = p_N(x, D)v + \sum_k p_N(x, D)w_k.$$

Set

$$I_k^* = \{x \in R^n : x - x^{(k)} = 2\sqrt{n}(x' - x^{(k)}),\ \textit{some}\ x' \in I_k\},$$

where $(I^*) = \gamma meas\ (I_k)$, $\gamma = (2\sqrt{n}^n)$. For $t_k > 0$,

$$I_k \subset \{x : |x - x^{(k)}| \leq t_k\},$$

$$Y_k = R^n \backslash I_k^* \subset \{x : |x - x^{(k)}| > 2t_k\}.$$

By $iii)$ we find

$$p_N(x, D)w_k = \int K_N(x, x-y)w_k(y)dy$$

$$= \int_{I_k} \{K_N(x, x-y) - K_N(x, x-x^{(k)})\}w_k(y)dy.$$

By Lemma

$$\int_{Y_k} |p_N(x, Dw_k(x)|dx \int_{|y|\leq t_k} \int_{|x|\geq 2t_k} |K_N(x+x^{(k)}, x-y) - K_N(x+x^{(k)}, x)|$$

$$\times |w_k(y + x^{(k)})|dxdy \leq C_0 ||w_k||_{L^1}.$$

Set

$$\Theta^* = \bigcup_{k=1}^{\infty} I_k^*, \quad w = \sum_{k=1}^{\infty} w_k.$$

By $iv)$

$$meas\ (\Theta^*) \leq \frac{\gamma}{\lambda}||u||_{L^1}.$$

Above inequality become

$$\int_{R^n \setminus \Theta} |p_N(x, D)w(x)|dx \leq 3C_0||u||_{L^1}.$$

Since $p_N(x, \xi)$ is bounded in $S^0_{1,\delta}$ we have

$$||p_N(x, D)v||^2_{L^2} \leq C||v||^2_{L^2} \leq C\lambda||u||_{L^1},$$

We have

$$\frac{\lambda}{2}\ meas\{x : |p_N(x, D)w(x)| > \frac{\lambda}{2}\} \leq 3C_0||u||_{L^1}$$

and

$$(\frac{\lambda}{2})^2 meas\{x : |p_N(x, D)v(x)| > \frac{\lambda}{2}\} \leq C\lambda||u||_{L^1}.$$

5.10 L^2 continuity of operators on the a complex Hilbert space

We give recently result by Boggiatto, Buzano and Rodino [20] for pseudodifferential operator in L^2. Let A is

$$Au(x) = \int e^{i(x-y)\xi} a(x, y, \xi)u(y)dyd\xi,$$

on the a complex Hilbert space H and by domain D_A is linear subspace of H. Assume $D_A = H$, if $\bar{D}_A = H$, we call A is densely.

Theorem 5.52. Consider a pseudo-differential operator $A \in L^m_{P,\rho}$, on $L^2(R^n)$ with domain $D_A = S(R^n)$. Then A is closable in $L^2(R^n)$. $\bar{A}$ is the restriction to $D_{\bar{A}} = \{u \in L^2(R^n) | Au \in L^2(R^n)\}$ of the extension $A : S'(R^n) \to S'(R^n)$.

In particularly

$$(\bar{A}u, v) =< Au, \bar{v} >$$

for all $u \in D_{\bar{A}}$ and $v \in S(R^n)$.

Here $P \subset R^n$ is a convex Newton polyhedron and a finite set of points in R^n.

Proof. They consider the restriction $A|_D$ of $A : S'(R^n) \to S'(R^n)$. If $u_n \in S(R^n) = D_A$ is a sequence such that

$$u_n \to u \ \text{ and } Au_n \to v, \ \text{ in } L^2(R^n),$$

then Au_n tends to Au and also v in $S'(R^n)$. By uniqueness of the limit in $S'(R^n)$ there exist $Au = v$ and $u \in D$. The restriction to D of the extension $A : S'(R^n) \to S'(R^n)$ is closure of A in $L^2(R^n)$.

Following proposition is the connection between adjoint and formal adjoint.

Proposition 5.53. Let A is a pseudo-differential operator. Then the adjoint A^* in $L^2(R^n)$ has domain $D_{A^*} = \{u \in L^2(R^n) | A^+ u \in L^2(R^n)\}$ and coincides with

$i)$ The closure of $A^+ : S(R^n) \to S(R^n)$ in $L^2(R^n)$;

$ii)$ The restriction of $A^+ : S'(R^n) \to S'(R^n)$ to D_{A^*};

$iii)$ The adjoint of $\bar{A}$ in $L^2(R^n)$.

Proof. From

$$(A^+ u, v)_{L^2} = (u, Av)_{L^2} \quad \text{for } u, v \in S(R^n)$$

they obtain

$$(\bar{A}^+ u, v)_{L^2} = (u, \bar{A}v)_{L^2}, \quad \text{for } u \in D_{\bar{A}^+},\ v \in D_{\bar{A}}.$$

There exist $\bar{A^+} = (\bar{A})^*$. On other side

$$(u, A^* v)_{L^2} = (Au, v)_{L^2} = (u, A^+ v)_{L^2}, \quad \text{for } \ u \in D_{\bar{A}} \ \textit{and} \ v \in S(R^n).$$

A^* is an extension of A^+ so it is an extension of $\bar{A}^+$ because A^* is closed, since $A^* = (\bar{A})^*$.

Definition 5.54. An operator A on Hilbert space H is called symmetric if

$$(Au, v)_{L^2} = (u, Av)_{L^2}, \quad for\ u, v \in D_A.$$

Proposition 5.55. A densely defined symmetric operator has symmetric closure.

Proof. Let $u_n \in D_A$, $u_n \to 0$ and $Au_n \to v$. Let $w \in D_A$. There exist

$$(u, w)_H = \lim_{n\to\infty} (Au_n, w)_H = \lim_{n\to\infty} (u_n, Aw)_H = 0.$$

$v = 0$ because D_A is dense in H. And $\bar{A}$ is closable. Let $u, v \in D_{\bar{A}}$. Exist sequences $u_n, v_n \in D_A$ such that

$$(\bar{A}, v)_H = \lim_{n\to\infty} (Au_n, v_n)_H = \lim_{n\to\infty} (u_n, Av_n)_H = (u, \bar{A}v)_H$$

Definition 5.56. A densely defined operator A is self-adjoint if $A = A^*$.

Definition 5.57. A densely defined symmetric operator A is called essentially self-adjoint if its closure $\bar{A}$ is self-adjoint.

From Proposition 5.55 they obtain immediately following theorem.

Theorem 5.58. A formally self-adjoint pseudo-differential operator A is essentially self-adjoint in $L^2(R^n)$.

5.11 On some classes of L^P-bounded pseudodifferential operators

Introduction and discussion of the results

The pseudodifferential operators considered in this work are of the standard quantization:

$$a(x,D)u := \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{ix\xi} a(x,\xi)\hat{u}(\xi)d\xi$$

where $x \cdot \xi = \sum_{j=1}^n x_j\xi_j$, $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi}u(x)dx$ is the Fourier transform of u. The function $a(x,\xi)$ is called the symbol of the operator $a(x,D)$. A symbol $a(x,\xi)$ of weighted pseudodifferential operator satisfies the estimates:

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta} m(\xi)\langle\xi\rangle^{\delta|\alpha|-\rho|\beta|},\ x,\ \xi \in \mathbb{R}^n,$$

for all multi-indices $\alpha,\ \beta \in \mathbb{Z}_+^n$, $m(\xi)$ is a positive continuous weight function and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

There is the general framework given by the symbol classes $S^\lambda(\phi,\varphi)$ and $S(m,g)$, introduced respectively by R.Beals [17] and L.Hörmander [55], [54]. L.Rodino [101] is studied a generalization of the Hörmander smooth wave front set and G.Garello [39] get the extension to the inhomogeneous microlocal analysis for weighted Sobolev singularities of L^2 type is performed. Recently G.Garello and A.Morando [43] introduce a vector weighted pseudodifferential operator is characterized by a smooth symbol which is satisfies the estimates:

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta} m(\xi)\Lambda(\xi)^{-|\alpha|},\ x,\ \xi \in R^n,$$

where $m(\xi)$ is a suitable positive continuous weight function, which indicates the order of the symbol, and $\Lambda(\xi) = (\lambda_1(\xi),...,\lambda_n(\xi))$ is a weight vector that estimates the decay at infinitive of the derivatives and study continuity properties in suitable weighted Sobolev spaces of L^p type and microlocal properties.

I.L.Hwang and R.B.Lee [57] give a new proof of the L^p- boundedness, $1 < p < \infty$ on the $S_{0,0}^m$, where $m = -n|1/p - 1/2|$ and $a \in S_{0,0}^m$ satisfies $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^m$ for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $|\alpha| \le k$ and $|\beta| \le k'$, in the Holder continuous sense, where $k > n/2$, $k' > n/p$ (the case $1 < p \le 2$) and $k > n/p$, $k' > n/2$ (the case $2 < p < \infty$). They also study on the class $S_{\delta,\rho}^m$, which were symbols have derivatives with respect to x only up to order k, in the Holder continuous sense, where $k > n/2$ (the case $1 < p \le 2$) and $k > n/p$ (the case $2 < p < \infty$). A symbol $a(x,\xi)$ is said to be of class $S_{\delta,\rho}^m$, where $m \in R$, $0 \le \delta \le \rho \le 1$ and $\delta < 1$, if it satisfies the inequalities

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|},\ x,\xi \in \mathbb{R}^n,$$

for all multi-indices α and β, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

As to the boundedness of the pseudo-differential operators with symbols belonging to the class $S_{\rho,\delta}^m$ or $\Lambda_{\delta,k,k'}^m$, the following theorems are known.

Theorem 5.1.1. Let $1 < p < \infty$, $\delta = \rho = 0$ and $m = -n|1/p - 1/2|$. If k,k' are sufficiently large real numbers and $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta a$ in the distribution sense satisfy (1.1) with $|\alpha| \leq k$ and $|\beta| \leq k'$, then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

The first result presented by Calderon-Vaillancourt [24] in the case $p = 2$ proved if for $\alpha, \beta \in \{0, 1, 2, 3\}^n$ and Coifman-Meyer [21] proved it the case $(1 < p < \infty)$ for $k, k' \geq 2n$. Cordes [29] proved it $(p = 2)$ for $|\alpha|, |\beta| \leq [n/2] + 1$.

Theorem 5.1.2. Let $1 < p < \infty$, $0 \leq \delta = \rho < 1$ and $m = -n(1 - p)|1/p - 1/2|$. If k, k' are sufficiently large real number and $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta a$ in the distribution sense satisfy (1.1) with $|\alpha| \leq k$ and $|\beta| \leq k'$ then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

This result is due to Calderon-Vaillancourt [25] (the case $p = 2$) and Fefferman [34] (the case $1 < p < \infty$); cf. Wang-Li [57]. Calderon-Vaillancourt proved it for $|\beta| \leq 2[n/2] + n$ and $|\alpha| \leq 2m'$ with $m' \in \mathbb{N}$ and $m'(1 - \rho) \geq 5n/4$. Coifman-Meyer [63] proved it (the case $p = 2$) for $|\alpha|, |\beta| \leq m'$ with $m' \in \mathbb{N}$ and $m' \geq [n/2] + 1$. Kato [68] proved it $(p = 2)$ by using the method of Cordes [29]. Beal [17] proved it ($p = 2$ and $-\infty < \rho < 1$). Nagase [98] proved it (the case $2 \leq p \leq \infty$) for $k, k' = [n/2] + 1$. I.L.Hwang [57] proved it (the case $p = 2$ and $-\infty < \rho < 1$) for $\alpha, \beta \in \{0, 1\}^n$.

In the work Miyachi [95],[96] we show following the sharpest results.

Theorem 5.1.3. Let $0 \leq \delta < 1$ and $m = -n(1 - \delta)|1/p - 1/2|$.

1) If $0 < p \leq 1$, $\delta = 0$, $k > n/2$, $k' > n/p$ and $a \in \Lambda_{\delta,k,k'}(\mathbb{R}^n \times \mathbb{R}^n)$, then $a(x, D)$ is continuous from $H^p(D)$ to $L^p(R^n)$, where H^p are the hardy spaces.

2) If $0 < p < 1$, $k > n/2$, $k' > n/p$ and $a \in \Lambda^m_{\delta,k,k'}(\mathbb{R}^n \times \mathbb{R}^n)$, then $a(x, D)$ is continuous from $h^p(\mathbb{R})$ to $L^p(\mathbb{R})$, where h^p are the local Hardy spaces.

3) If $1 < p \leq 2$, $k > n/2$, $k' > n/p$ and $a \in \Lambda^m_{\delta,k,k'}(\mathbb{R} \times \mathbb{R}^n)$, then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

4) If $2 < p < \infty$, $k > n/p$, $k' > n/2$ and $a \in \Lambda_{\delta,k,k'}(\mathbb{R}^n \times \mathbb{R}^n)$, then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Sugimoto [107,108] proved L^p- boundedness results, $0 < p < \infty$, by means of Besov spaces, which are an improvement of Theorem C with $\delta = 0$. Muramatu [96] also obtained some L^2 - boundedness results by means of Besov spaces, which also are an improvement of Theorem C with $0 \leq \delta < 1$. Bourdaud-Meyer [21] proved Theorem C with $p = 2$ and $\delta = 0$, and obtained the sharpest result.

The following theorem is Sugimoto's result ([107, Theorem 2.2]), which is closely related to our problems.

Theorem 5.1.4. let (1) $p = 2$, $\mathbf{q} = (q, q') \in [2, \infty)^2 \cup \infty \times 2, \infty$ or (2) $p \in [1, 2)$, $\mathbf{q} = (q, q') \in (2, \infty) \times [2, \infty) \cup \infty \times 2, \infty$. Then for $a \in \mathbf{B}^{n/2-n/q',n/p-n/q}_{\mathbf{q},(1,1),(0,n/p-n/2)}$ and $f \in \mathcal{S} \cap \mathbf{H}^p$ we have

$$||a(x, D)f||_{L^p} \leq c||a(x, \xi)||\mathbf{B}^{(n/2-n/q',n/p-n/q)}_{\mathbf{q},(1,1),(0,n/p-n/2)}||f||_{\mathbf{H}^p},$$

where c is a constant independent of a and f, $\mathcal{S}$ is the collection of rapidly decreasing functions, $\mathbf{H}^p$ is the Hardy space, and $\mathbf{B_q}$ is the Hardy space, and $\mathbf{B}^{(n/2-n/q', n/p-n/q)}_{\mathbf{q},(1,1),(0,n/p-n/2)}$ is a Besov space defined in [107].

In this work, we prove the L^p- boundedness, $1 < p < \infty$, of pseudodifferetial operators with the support of their symbols being contained in $E \times \mathbb{R}^n$, where E is a compact subset of $\mathbb{R}^n$.

The contents of this work are as follows. First, we give some lemmas and corollaries and formulate the main results, finally is given the proof of main results.

We require the derivatives $\partial_x^\alpha a$ and $\partial_x^\alpha \partial_\xi^\beta a$ up to finite order for L^p - boundedness of pseudodifferential operators.

Definition 5.1.5. Let $m(\xi)$ is positive continuous weight function, $k > 0$ and $k \notin \mathbb{N}$ and we define $\Lambda_{m,k}(\mathbb{R}^\times\mathbb{R}^n)$ to be collection of continuous functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ whose derivatives $\partial_x^\alpha a$ satisfy the following conditions:

(1.1) $\forall x, \xi, h \in \mathbb{R}^n$, a constant $C > 0$ such that for $\alpha \in \mathbb{N}^n$, $|\alpha| \leq [k]$, we have

1) If $|\alpha| \leq [k]$ then $|\partial_x^\alpha a(x,\xi)| \leq Cm(\xi)$.

2) If $|h| \leq 1$ and $|\alpha| = [k]$ then $|\partial_x^\alpha a(x+h,\xi) - \partial_x^\alpha(x,\xi)| \leq Cm(\xi)|h|^{k-[k]}$.

We denote by $||a||_{m(\xi),k}$ the smallest C such that (1.1) holds. The constant C is depending on n.

Definition 5.1.6. Let $m(\xi)$ is positive continuous weight function, $0 \leq \delta < 1$, $k, k' > 0$, and $k, k' \notin \mathbb{N}$. The collection of continuous functions $a : \mathbb{R}^n \times \mathbb{R}^n$ define by $\Lambda_{\delta,k,k'}$ and the derivatives $\partial_x^\alpha \partial_\xi^\beta a$ satisfy the following conditions:

(1.2) $\forall x, \xi, h, \eta \in \mathbb{R}^n$, a constant $C > 0$ such that for $\alpha, \beta \in \mathbb{N}^n$, $|\alpha| \leq [k]$, $|\beta| \leq [k']$ we have

1) If $|\alpha| \leq [k]$ and $|\beta| \leq [k']$ then

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq Cm(\xi)\langle\xi\rangle^{\delta(|\alpha|-|\beta|)}.$$

2) If $|h| \leq 1$, $|\alpha| = [k]$ and $|\beta| \leq [k']$ then

$$|\partial_x^\alpha \partial_\xi^\beta a(x+h,\xi) - \partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq Cm(\xi)\langle\xi\rangle^{\delta(k-|\beta|)}|h|^{k-[k]}.$$

3) If $|\eta| \leq 1$, $|\alpha| \leq [k]$ and $|\beta| = [k']$ then

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi+\eta) - \partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq Cm(\xi)\langle\xi\rangle^{\delta(|\alpha|-k')}|\eta|^{k'-[k']}.$$

4) If $|h| \leq 1$, $|\eta| \leq 1$, $|\alpha| = [k]$ and $|\beta| = [k']$ then

$$\begin{aligned}|\partial_x^\alpha \partial_\xi^\beta a(x+h,\xi+\eta) - \partial_x^\alpha \partial_\xi^\beta a(x+h,\xi) - \partial_x^\alpha \partial_\xi^\beta a(x,\xi+\eta) + \partial_x^\alpha \partial_\xi^\beta a(x,\xi)|\\ \leq Cm(\xi)\langle\xi\rangle^{\delta(k-k')}|h|^{k-[k]}|\eta|^{k'-[k']}.\end{aligned}$$

We denote by $||a||_{m,k,k'}$ the smallest C such that (1.2) holds.

Now we start to prove the L^p- boundedness , $1 < p < \infty$ by using the method of Hwang [56]. For $u, v \in C_0^\infty(\mathbb{R}^n)$, $a \in \Lambda_k^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $\sup a \subseteq E \times \mathbb{R}^n$, where E is a compact subset of $\mathbb{R}^n$, we can write $(a(x,D)u,v)$ in the following form:

$$(a(x,D)u,v) = \sum_{i=1}^{r_1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} b_i(x,\xi)\hat{u}(\xi)h_i(x,\xi)d\xi dx,$$

where $r_1 \in \mathbb{N}$. Similarly, for $u, v \in C_0^\infty(\mathbb{R}^n)$ and $a \in \Lambda^m_{k,k'}(\mathbb{R}^n \times \mathbb{R}^n)$, we also can write $(a(x,D)u, v)$ in the following form:

$$(a(x,D)u,v) = \sum_{i=1}^{r_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} b_i(x,\xi) g_i(x,\xi) h_i(x,\xi) d\xi dx,$$

where $r_2 \in \mathbb{N}$. Here, $b_i(x,\xi)$ are related to $a(x,\xi)$ and its derivatives, g_i, h_i are Wigner functions which have following form:

$$1) \ \ g_i(x,\xi) = \int_{\mathbb{R}^n} e^{-iy\xi} \varphi_i(x-y) u(y) dy,$$

$$2) \ \ h_i(x,\xi) = \int_{\mathbb{R}^n} e^{-iy\lambda} \psi_i(\xi+\lambda) \bar{\hat{v}}(\lambda) d\lambda,$$

where $x, \xi \in \mathbb{R}^n$ and $\varphi_i \in L^p(\mathbb{R}^n)$, $\psi_i \in L^2(\mathbb{R}^n)$ (the case $1 < p \le 2$), $\varphi_i \in L^2(\mathbb{R}^n)$, $\psi_i \in L^p(\mathbb{R}^n)$ (the case $2 < p < \infty$).

Then, by Paley's inequality, we can get

$$|(a(x,D)u,v)| \le C||u||_{L^p}||v||_{L^q}, \quad 1 < p < \infty, \quad 1/p + 1/q = 1,$$

where $C = C_{E,n,p,k}||a||_{m,k}$ or $C_{n,p,k,k'}||a||_{m,k,k'}$.

Lemmas and Corollaries

First, we have the following lemma. Its proof can be found in [17].

Lemma 5.1.7. Let $\varphi_s(\lambda) = (1+|\lambda|^2)^{s/2}$ with $\lambda \in \mathbb{R}^n$ and $0 < s < 1$. Then the Fourier transform of φ_s has the following properties:

$$\hat{\varphi}_s \in C^\infty(\mathbb{R}^n \setminus 0). \tag{5.1.1}$$

There are constants $C_{n,s}$ and $C_{n,s,t}$ such that

$$|\hat{\varphi}_s(x)| \le C_{n,s,t}|x|^{-t-1} \text{ for } |x| > 1 \ \ \textit{and } t \in \mathbb{N}, \tag{5.1.2}$$

and

$$|\hat{\varphi}_s(x)| \le C_{n,s}|x|^{-n-s} \text{ for } 0 < |x| \le 1. \tag{5.1.3}$$

Remark 5.1.8. In fact, if we define $\varphi_{s,\varepsilon}(\lambda) = \varphi_s(\lambda) e^{-\varepsilon|\lambda|^2}$, $\lambda \in \mathbb{R}^n$ and $0 < s, \varepsilon < 1$, then $\hat{\varphi}_{s,\varepsilon}$ satisfies (5.1.1)-(5.1.3) with $C_{n,s}$ and $C_{n,s,t}$ independent of ε.

For $a \in \Lambda^m_{k,k'}(\mathbb{R}^n \times \mathbb{R}^n)$, we define $\hat{a}^1$, $\hat{a}^2$ as follows:

$$1) \ \hat{a}^1(\lambda,\xi) = \int_{\mathbb{R}^n} e^{-ix\lambda} a(x,\xi) dx, \quad \lambda, \xi \in \mathbb{R}^n.$$

$$2) \ \hat{a}^2(x,y) = \int_{\mathbb{R}^n} e^{-i\xi y} a(x,\xi) d\xi, \quad x, y \in \mathbb{R}^n.$$

Then we formulate the following lemma.

Lemma 5.1.9. Let $m(\xi)$ is a positive continuous bounded function, $0 < s < k$, $k' < 1$, $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \cap \Lambda^m_{k,k'}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\varphi_s(\lambda) = (1+|\lambda|^2)^{s/2}$, $\lambda \in \mathbb{R}^n$. Suppose $\hat{g}_1^1(\cdot, \xi) = \hat{a}^1(\cdot,\xi)\varphi_s(\cdot)$ and $\hat{g}_2^2(\cdot,\xi) = \hat{a}^2(\cdot,\xi)\varphi_s(\cdot)$, $x, \xi \in \mathbb{R}^n$. Then we have

$$|g_i(x,\xi)| \leq C_{n,m,s}|m(\xi)|||a||, \ x, \xi \in \mathbb{R}^n, \tag{5.1.4}$$

where $i = 1, 2$, $||a|| = ||a||_{m,k,k'}$ and $C_{n,m,s}$ depends only on k or k'.

Proof. We shall prove the case $i = 2$ only, since the proof of the case $i = 1$ is similar. Without loss of generality, we may assume that $\varphi_s(\lambda) = \varphi_{s,\varepsilon}(\lambda) = (1 + |\lambda|^2)^{s/2}e^{-\varepsilon|\lambda|^2}$, $\lambda \in \mathbb{R}^n$ and $0 < \varepsilon < 1$. Then we have

$$\begin{aligned} g_2(x,\xi) &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{i\xi y}\hat{a}^2(x,y)\varphi_{s,\varepsilon}(y)dy \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \hat{\varphi}_{s,\varepsilon}(\eta)a(x,\eta+\xi)d\eta \\ &= I_1(x,\xi) + I_2(x,\xi), \quad x, \xi \in \mathbb{R}^n, \end{aligned}$$

where

$$I_1(x,\xi) = (\frac{1}{2\pi})^n \int_{|\eta|\leq 1} \hat{\varphi}_{s,\varepsilon}a(x,\eta+\xi)d\eta,$$

and

$$I_2(x,\xi) = (\frac{1}{2\pi})^n \int_{|\eta|\geq 1} \hat{\varphi}_{s,\varepsilon}a(x,\eta+\xi)d\eta,$$

By (1.2) and (5.1.2), we obtain

$$|I_2(x,\xi)| \leq C_{n,s,t}||a|| \int_{|\eta|\geq 1} |\eta|^{-1-t}|m(\xi)|d\eta, \quad t \in \mathbb{N}.$$

We get

$$|I_2(x,\xi)| \leq C_{n,m,s}|m(\xi)|||a||.$$

We now estimate I_1. First, we write I_1 in the form

$$I_1(x,\xi) = I_{1,1}(x,\xi) + I_{1,2}(x,\xi), \quad x, \xi \in \mathbb{R}^n,$$

where

$$I_{1,1}(x,\xi) = \left(\frac{1}{2\pi}\right)^n \int_{|\eta|\leq 1} \hat{\varphi}_{s,\varepsilon}(\eta)(a(x,\eta+\xi) - a(x,\xi))d\eta,$$

and

$$I_{1,2}(x,\xi) = \left(\frac{1}{2\pi}\right)^n \int_{|\eta|\leq 1} \hat{\varphi}_{s,\varepsilon}(\eta)d\eta \cdot a(x,\xi).$$

By (1.2) and (5.1.3), we get

$$|I_{1,1}(x,\xi)| \leq C_{n,s}|m(\xi)|||a|| \int_{|\eta|\leq 1} \frac{1}{|\eta|^{n+s-k'}}d\eta \leq C_{n,s}|m(\xi)|||a||,$$

where $C_{n,s}$ depends on k'.

Since $\varphi_{s,\varepsilon}(0) = 1$ and $\int_{|\eta|>1} |\hat{\varphi}_{\eta,\varepsilon}(\eta)| d\eta \leq C_{n,s}$, we obtain

$$|I_{1,2}(x,\xi)| \leq C_{n,s} |m(\xi)| ||a||,$$

where $C_{n,s}$ depends on k'.

Corollary 5.1.10. Let $m(\xi)$ is bounded continuous function, $0 < s < k < 1$, $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \cap \Lambda_k^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $\varphi_s(\lambda) = (1 + |\lambda|^2)^{s/2}$, $\lambda \in \mathbb{R}^n$. Suppose $\hat{g}^1(\cdot, \xi) = \hat{a}^1(\cdot, \xi)\varphi_s(\cdot)$, $\xi \in \mathbb{R}^n$. Then

$$|g(x,\xi)| \leq C_{n,m,s} |m(\xi)| ||a||, \quad x, \xi \in \mathbb{R}^n,$$

where $||a|| = ||a||_{k,k'}$ and $C_{n,m,s}$ depends only on k.

Proof. This is an immediate consequence of Lemma (5.1.9).

Corollary 5.1.11. Let $m(\xi)$ is bounded continuous function, $0 < s < k < 1$, $0 < s' < k' < 1$, $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \cap \Lambda_{k,k'}^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $\varphi_{\tilde{s}}(\lambda) = (1 + |\lambda|^2)^{\tilde{s}/2}$, $\lambda \in \mathbb{R}^n$ and $\tilde{s} = s, s'$. Suppose $\hat{g}(\lambda, y) = \hat{a}(\lambda, y)\varphi_s(\lambda)\varphi_{s'}(y)$, $y, \lambda \in \mathbb{R}^n$. Then we have

$$|g(x,\xi)| \leq C_{n,m,s} |m(\xi)| ||a||, \quad x, \xi \in \mathbb{R}^n,$$

where $||a|| = ||a||_{m,k,k'}$ and $C_{n,m,s}$ depends only on k, k'.

Proof. Without loss of generality, we may assume that $\varphi_{\tilde{s}}(\lambda) = \varphi_{\tilde{s},\varepsilon}(\lambda) = (1+|\lambda|^2)^{\tilde{s}/2} e^{-\varepsilon|\lambda|^2}$, $\lambda \in \mathbb{R}^n$, $\tilde{s} = s$, s' and $0 < \varepsilon < 1$. First, we have

$$g(x,\xi) = \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\varphi}_{s,\varepsilon}(z)\hat{\varphi}_{s',\varepsilon}(\eta) a(z + x, \eta + \xi) dz d\eta$$

$$= \left(\frac{1}{2\pi}\right)^{2n} \sum_{j=1}^{4} I_j(x,\xi), \quad x, \xi \in \mathbb{R}^n,$$

where

$$I_1(x,\xi) = \int_{|z|\leq 1} \int_{|\eta|\leq 1} \hat{\varphi}_{s,\varepsilon}(z)\hat{\varphi}_{s',\varepsilon}(\eta) a(z + x, \eta + \xi) dz d\eta,$$

$$I_2(x,\xi) = \int_{|z|\leq 1} \int_{|\eta|>1} \hat{\varphi}_{s,\varepsilon}(z)\hat{\varphi}_{s',\varepsilon}(\eta) a(z + x, \eta + \xi) dz d\eta,$$

$$I_3(x,\xi) = \int_{|z|>1} \int_{|\eta|\leq 1} \hat{\varphi}_{s,\varepsilon}(z)\hat{\varphi}_{s',\varepsilon}(\eta) a(z + x, \eta + \xi) dz d\eta,$$

and

$$I_4(x,\xi) = \int_{|z|>1} \int_{|\eta|>1} \hat{\varphi}_{s,\varepsilon}(z)\hat{\varphi}_{s',\varepsilon}(\eta) a(z + x, \eta + \xi) dz d\eta,$$

By (1.2), (5.1.2) and Peetre's inequality, we get

$$|I_4(x,\xi)| \leq C_{n,m,s} |m(\xi)| ||a||,$$

where $C_{n,m,s}$ depends on k, k'.

We now estimate I_1 only and leave the estimates of I_2 and I_3 to the reader. We write estimate I_1 in the form

$$I_1(x,\xi) = \sum_{t=1}^{4} I_{1,t}(x,\xi), \quad x,\xi \in \mathbb{R}^n,$$

where

$$I_{1,1}(x,\xi) = \int_{|z|\le 1}\int_{|\eta|\le 1} \hat{\varphi}_{s,\varepsilon}(z)\hat{\varphi}_{s',\varepsilon}(\eta)(a(z+x,\eta+\xi) - a(z+x,\xi) - a(x,\eta+\xi) + a(x,\xi))dzd\eta,$$

$$I_{1,2}(x,\xi) = \int_{|z|\le 1}\int_{|\eta|\le 1} \hat{\varphi}_{s,\varepsilon}(z)\hat{\varphi}_{s',\varepsilon}(\eta)(a(z+x,\xi) - a(x,\xi))dzd\eta,$$

$$I_{1,3}(x,\xi) = \int_{|z|\le 1}\int_{|\eta|\le 1} \varphi_{s,\varepsilon}\hat{(}z)\varphi_{s',\varepsilon}(\eta)(a(x,\eta+\xi) - a(x,\xi))dzd\eta,$$

and

$$I_{1,4}(x,\xi) = \int_{|z|\le 1}\int_{|\eta|\le 1} \hat{\varphi}_{s,\varepsilon}(z)\hat{\varphi}_{s',\varepsilon}(\eta)dzd\eta \cdot a(x,\xi).$$

By (1.2) and (5.1.3), we get

$$|I_{1,t}(x,\xi)| \le C_{n,m,s}|m(\xi)|||a||,$$

where $t = 1,2,3,4$ and $C_{n,m,s}$ depends on k, k'.

We give the following lemma of Hwang [56] which is related to the Winger function.

Lemma 5.1.12. For $u,\varphi \in C_0^\infty(\mathbb{R}^n)$, we define

$$g(x,\xi) = \int_{\mathbb{R}^n} e^{-iy\xi}\varphi(x-y)u(y)dy,$$

and

$$h(x,\xi) = \int_{\mathbb{R}^n} e^{ix\lambda}\varphi_i(\xi+\lambda)u(\lambda)d\lambda,\ x,\xi \in \mathbb{R}^n.$$

Then we have

$$||g||_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = ||h||_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = (2\pi)^{n/2}||\varphi||_{L^2(\mathbb{R}^n)}||u||_{L^2(\mathbb{R}^n)}.$$

To prove the L^p- boundedness of pseudodifferential operators, we also need the following lemma which is related to the Hausdorff-Young inequality and Paley's inequality [57]. It can be found in [56] and [55].

Lemma 5.1.13. If $1 < p \le 2$, $1/p + 1/q = 1$ and $p \le r \le q$, then

$$\left(\int_{\mathbf{R}^n} |\xi|^{-n(1-r/q)}|\hat{f}(\xi)|^r d\xi\right)^{1/r} \le C_p||f||_{L^p(\mathbb{R}^n)}.$$

Remark 5.1.14. In this work, Lemma 5.1.13 is applied in the case of $r = 2$.

Now we use the following partition of unity.

Let $r > 0$ and $s = 1,...,n$. We define

$$\Gamma_{s,r} = \{\xi \in \mathbb{R}^n | \xi = (\xi_1,...,\xi_n),\ |\xi_t| \le r|\xi_s|\ if\ t \ne s\}.$$

Then we can find $W_0 \in C_0^\infty(\mathbb{R}^n)$ and $W_s \in C^\infty(\mathbb{R}^n)$, $s = 1, ..., n$, such that the following conditions hold:

1) $0 \leq W_s \leq 1, \quad s = 0, 1, ..., n,$

2) *supp* $W_0 \subseteq \{|\xi| \leq 1\}$, *supp* $W_s \subseteq \Gamma_{s,3/2} \cap \{|\xi| \geq 1/2\}$, $W_s(\xi) = W_s(\frac{\xi}{|\xi|})$ for $\xi \geq 1$, and $W_s(\xi) = 1$ for $\xi \in \Gamma_{s,1/2}$ and $|\xi| \geq 1$, $s = 1, ..., n$,

3) $\sum_{s=0}^n W_s \equiv 1$,

4) for $\alpha \in \mathbb{N}^n$, there exists a constant $C_\alpha > 0$ such that

$$|\partial_\xi^\alpha W_s(\xi)| \leq C_\alpha (1 + |\xi|)^{-|\alpha|}, \ \xi \in \mathbb{R}^n \ \ and \ s = 1, ..., n.$$

To prove the L^p- boundedness, $2 < p < \infty$, of pseudodiferential operators, we need to study the Fourier transform of the following functions:

$$\psi_s(\xi) = W_s(\xi) \frac{1}{1 + i\xi_s^{[n/p]}} \frac{1}{(1 + |\xi|^2)^{\frac{1}{2}(n/p - [n/p] + \varepsilon/2)}}, \tag{5.1.5}$$

where $\xi \in \mathbb{R}^n$, $s = 1, ..., n$, W_s are defined as above and ε, $\varepsilon > 0$, is so small that $n/p + \varepsilon/2 \notin \mathbb{N}$, $n/p - [n/p] + \varepsilon < 1$, $[n/p + \varepsilon] = [n/p]$ and $n/q - [n/q] \neq \varepsilon/2$ with $1/p + 1/q = 1$.

It is clear that $\psi_s \in L^p(\mathbb{R}^n)$, and to proof that that $\hat{\psi}_s \in L^q(\mathbb{R}^n)$, we give following lemma of Hwang [57] without proof.

Lemma 5.1.15. Let ψ_s be defined as in (5.1.5). Then we have

$$|\hat{\psi}_s(x)| \leq C_{n,\varepsilon} |x|^{-n/q + \varepsilon/2} \ for \ 0 < |x| \leq 1, \tag{5.1.6}$$

$$|\hat{\psi}_s(x)| \leq C_{n,t} |x|^{-t} \ for \ |x| > 1, \quad and \ t \in \mathbb{N} \qquad (5.1.7),$$

where $x \in \mathbb{R}$ and $1/p + 1/g = 1$ with $2 < p < \infty$.

Corollary 5.1.16. For $\xi \in \mathbb{R}^n$ and $2 < p < \infty$, we define

$$\psi(\xi) = \frac{1}{(1 + |\xi|^2)^{\frac{1}{2}(n/p + \varepsilon/2)}},$$

where $\varepsilon, \varepsilon > 0$, is so small that $n/p + \varepsilon/2 \notin \mathbb{N}$, $n/p - [n/p] + \varepsilon < 1$, $[n/p + \varepsilon] = [n/p]$ and $n/p - [n/q] \neq \varepsilon/2$ with $1/p + 1/q = 1$.

Then we have

$$|\hat{\psi}(x)| \leq C_{n,\varepsilon} |x|^{-n/q + \varepsilon/2} \ for \ 0 < |x| \leq 1,$$

and

$$|\hat{\psi}(x)| \leq C_{n,t} |x|^{-t} \ \ for \ |x| > 1 \ \ and \ t \in \mathbb{N},$$

where $x \in \mathbb{R}^n$.

Proof. By an argument similar to the proof of Lemma 5.1.15, Corollary 6.1.16 is obtained.

Main results

First, set $m(\xi)$ bounded continuous function, $||a|| = ||a||_{m,k}$ if $a \in \Lambda_k^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $||a|| = ||a||_{m,k,k'}$ if $a \in \Lambda_{k,k'}^m(\mathbb{R}^n \times \mathbb{R}^n)$. Then we have the following theorems.

Theorem 5.1.17. Let $1 < p \leq 2$, $k > n/2$, $k \notin \mathbb{N}$, E a compact subset of $\mathbb{R}^n$ and $\Omega_1 = \{x \in \mathbb{R}^n | d(x, E) \leq 1\}$. If $a \in \Lambda_k^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $supp\ a \subseteq E \times \mathbb{R}^n$, then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(E)$ with its norm bounded by $C_{E,n,p,k}|\Omega_1|^{1/p}|m(\xi)|||a||$, where $||$ denotes the Lebesgue measure.

Theorem 5.1.18. Let $2 < p < \infty$, $k > n/p$, $k \notin \mathbb{N}$ and E a compact subset of $\mathbb{R}^n$. If $a \in \Lambda_k^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $supp\ a \subseteq E \times \mathbb{R}^n$, then $a(x, D)$ is continuous from $L^p_{loc}(\mathbb{R}^n)$ to $L^p(E)$.

Theorem 5.1.19. let $1 < p \leq 2$, $k > n/2$, $k' > n/p$ and $k, k' \notin \mathbb{N}$. If $a \in \Lambda_{k,k'}^m(\mathbb{R}^n \times \mathbb{R}^n)$, then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with its norm bounded by $C_{n,p,k,k'}|m(\xi)|||a||$.

Theorem 5.1.20. Let $2 < p < \infty$, $k > n/p$, $k' > n/2$ and $k, k' \notin \mathbb{N}$. If $a \in \Lambda_{k,k'}^n(R^n \times \mathbb{R}^n)$, then $a(x, D)$ is continuous from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with its norm bounded by $C_{n,p,k,k'}|m(\xi)|||a||$.

Proofs of the main results

Proof of Theorem 5.1.17. Without loss of generality, we may assume that

$$a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \cap \Lambda_k^m(\mathbb{R} \times \mathbb{R}^n).$$

Let $k = n/2 + \varepsilon$ and $\varphi_2(\lambda) = (1 + |\lambda|^2)^{\frac{1}{2}(n/2 - [n/2] + \varepsilon/2)}$, where $\lambda \in \mathbb{R}^n$ and $\varepsilon, \varepsilon > 0$, is so small that $n/2 - [n/2] - \varepsilon < 1$ and $[n/2 + \varepsilon] = [n/2]$. Then for $u, v \in C_0^\infty(\mathbb{R}^n)$, we have

$$(a(x, D)u, v) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi)\hat{u}(\xi)\bar{v}(x) d\xi dx. \tag{5.1.8}$$

We write (5.1.8) in the form

$$(a(x, D)u, v) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{a}^1(\lambda, \xi)\hat{u}(\xi)\bar{\hat{v}}(\lambda + \xi) d\lambda d\xi$$

$$= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{a}^1(\lambda, \xi)\varphi_2(\lambda)\varphi_2^{-1}(\lambda)\hat{u}(\xi)\bar{\hat{v}}(\lambda + \xi) d\lambda d\xi$$

$$= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\lambda} b(x, \xi)\hat{u}(\xi)\varphi_2^{-1}(\lambda)\bar{\hat{v}}(\lambda + \xi) d\lambda d\xi dx, \tag{5.1.9}$$

where

$$\hat{b}^1(\cdot, \xi) = \hat{a}^1(\cdot, \xi)\varphi_2(\cdot), \quad \xi \in \mathbb{R}^n.$$

Making use of the partition of unity W_s, $s = 0, 1, ..., n$, we write (5.1.9) in the form

$$(a(x, D)u, v) = \left(\frac{1}{2\pi}\right)^{2n} \sum_{s=0}^{n} I_s,$$

where

$$I_s = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\lambda} b(x, \xi)\hat{u}(\xi) W_s(\lambda)\varphi_2^{-1}(\lambda)\bar{\hat{v}}(\lambda + \xi) d\lambda d\xi dx, \quad s = 0, 1, ..., n.$$

We estimate I_1 only. Integrating the above integral with respect to x first and making the use of the identity

$$\frac{1}{1 + i\lambda_1^{[n/2]}}(1 - (-i)^{1-[n/2]}\partial_{x_1}^{[n/2]})(e^{-ix\lambda}) = e^{ix\lambda},$$

we write I_1 in the form

$$I_1 = I_{1,1} + (i)^{1-[n/2]} I_{1,2},$$

where

$$I_{1,1} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} b(x,\xi)\hat{u}(\xi)h(x,\xi)d\xi dx,$$

and

$$I_{1,2} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{b}(x,\xi)\hat{u}(\xi)h(x,\xi)d\xi dx, \tag{5.1.10}$$

with

$$\tilde{b}(x,\xi) = \partial_{x_1}^{[n/2]}(b(x,\xi)),$$

$$h(x,\xi) = \int_{\mathbb{R}^n} e^{-ix\lambda}\psi(\lambda)\bar{\hat{v}}(\lambda+\xi)d\lambda, \quad x,\xi \in \mathbb{R}^n, \tag{5.1.11}$$

and

$$\psi(\lambda) = W_1(\lambda)\varphi_2^{-1}(\lambda)\frac{1}{1+i\lambda_1^{[n/2]}}, \ \lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n.$$

We shall estimate $I_{1,2}$ only, since the estimate of $I_{1,1}$ is similar. First, we write (5.1.11) in the form

$$h(x,\xi) = \int_{\mathbb{R}^n} e^{i(x-z)\xi}\hat{\psi}(z)\bar{v}(x-z)dz. \tag{5.1.12}$$

Substituting (5.1.12) into (5.1.10), we write $I_{1,2}$ in the form

$$I_{1,2} = J_1 + J_2,$$

where

$$J_1 = \int_{\Omega_1} \int_{\mathbb{R}^n} \tilde{b}(x,\xi)\hat{u}(\xi)h(x,\xi)d\xi dx,$$

and

$$J_2 = \int_{R\backslash\Omega_1} \int_{\mathbb{R}^n} \tilde{b}(x,\xi)\hat{u}(\xi)h(x,\xi)d\xi dx,$$

We first estimate J_1. By Hölder's inequality, Corollary 5.1.10, Lemma 5.1.13 and Parseval's formula, we obtain

$$|J_1| \le C\int_{\Omega_1}\left(\int_{\mathbb{R}^n}|m(\xi)\hat{u}(\xi)|^2 d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n}|h(x,\xi)|^2 d\xi\right)^{1/2} dx$$

$$\le C|m(\xi)|||u||_{L^p(\mathbb{R}^n)}\int_{\Omega_1}\left(\int_{\mathbb{R}^n}|\hat{\psi}(z)\bar{v}(x-z)|^2 dz\right)^{1/2} dx,$$

where $C = C_{E,n,p,k}||a||$ and $1 < p \le 2$.

By Hölder's inequality and Minkowski's inequality, we get

$$|J_1| \le C|\Omega_1|^{1/p}|m(\xi)|||u||_{L^p(\mathbb{R}^n)}||\psi||_{L^2(\mathbb{R}^n)}||v||_{L^q(\mathbb{R}^n)}, \quad 1/p + 1/q = 1.$$

Now, we estimate J_2. We have

$$\tilde{b}(x,\xi) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \partial_{x_1}^{[n/2]}(a(x+y,\xi))\hat{\varphi}(y)dy$$

$$= \left(\frac{1}{2\pi}^{n}\right) \int_{R^n} (\partial_{x_1}^{[n/2]} a)(y,\xi)\hat{\varphi}(y-x)dy, \quad x, \xi \in \mathbb{R}^n,$$

and $|x-y| > 1$ for $x \in \mathbb{R}^n \setminus \Omega_1$ and $y \in E$. Hence, by Hölder's inequality, Corollary 5.1.10, Lemma 5.1.13, and Parseval's formula, we have

$$|J_2| \le C||u||_{L^p(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus \Omega_1} A(x) \left(\int_{\mathbb{R}^n} |\hat{\psi}(z)\bar{v}(x-z)|^2 dz \right)^{1/2} dx,$$

where $C = C_{E,n,p,k,}||a||$, $1 < p \le 2$, and

$$A(x) = \int_{\mathbb{R}^n} \frac{\chi_E(y)}{|x-y|^{n+1}} dy, \quad x \in \mathbb{R}^n,$$

with

$$\chi_E = \begin{cases} 1 & if\ y \in E \\ 0 & if\ y \notin E. \end{cases}$$

By Hölder's inequality and Minkowski's inequality, we obtain

$$|J_2| \le C|E|^{1/p}||u||_{L^p(\mathbb{R}^n)}||\psi||_{L^2(\mathbb{R}^n)}||v||_{L^q(\mathbb{R}^n)}, \quad 1/p + 1/q = 1.$$

Proof of Theorem 5.1.18. Without loss of generality, we may assume that

$$a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \cap \Lambda_k^m(\mathbb{R}^n \times \mathbb{R}^n).$$

Let $k = n/p + \varepsilon$ and $\varphi_p(\lambda) = (1+|\lambda|^2)^{\frac{1}{2}(n/p-[n/p]+\varepsilon/2)}$, where $\lambda \in \mathbb{R}^n$ and $\varepsilon, \varepsilon > 0$, is so small that $n/p + \varepsilon/2 \notin \mathbb{N}$, $n/p - [n/p] + \varepsilon < 1$, $[n/p+\varepsilon] = [n/p]$ and $n/q - [n/q] \neq \varepsilon/2$ with $1/p + 1/q = 1$. It is enough to show that the conclusion holds in every open ball. So fix a ball, say B. Then for $u, v \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and *supp* $u \subseteq B$, we have

$$(a(x,D)u, v) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix\xi} a(x,\xi)\hat{u}(\xi)\bar{v}(x) d\xi dx.$$

Since the arguments are similar to the proof of Theorem 5.1.17, we only study the following lemma.

Lemma 5.1.20'. For $u, v \in C_0^\infty(\mathbb{R}^n)$ and *supp* $u \subseteq B$, we define

$$J = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{b}(x,\xi)\hat{u}(\xi)h(x,\xi) d\xi dx, \tag{5.1.13}$$

where

$$\tilde{b}(x,\xi) = \partial_{x_1}^{[n/p]}(b(x,\xi)), \quad x = (x_1, ..., x_n),\ \xi \in \mathbb{R}^n,\ 2 < p < \infty,$$

with

$$\hat{b}^1(\cdot,\xi) = \hat{a}^1(\cdot,\xi)\varphi_p(\cdot),$$

$$h(x,\xi) = \int_{\mathbb{R}^n} e^{-ix\lambda}\psi_p(\lambda)\bar{\hat{v}}(\lambda+\xi)d\lambda, \quad x, \xi \in \mathbb{R}^n, \tag{5.1.14}$$

with

$$\psi_p(\lambda) = W_1(\lambda)\varphi_p^{-1}(\lambda)\frac{1}{1+i\lambda_1^{[n/p]}}, \quad \lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n,$$

and W_1 is defined as in the proof of Theorem 5.1.17. Then we have

$$|J| \le C_{E,n,p,k}||a|||B|^{\frac{p-2}{2p}}|\Omega_1|^{1/p}||u||_{L^p(B)}||v||_{L^q(\mathbb{R}^n)},$$

where $|B|$, Ω_1 denote the Lebesque measure of B, Ω_1, respectively, with $\Omega_1 = \{x \in \mathbb{R}^n | d(x,E) \le 1\}$, and $1/p + 1/q = 1$ with $2 < p < \infty$.

Proof. First, we write (5.1.14) in the form

$$h(x,\xi) = \int_{\mathbb{R}^n} e^{i(x-z)\xi}\hat{\psi}(z)\bar{v}(x-z)dz. \tag{5.1.15}$$

Substituting (5.1.15) into (5.1.13), we write J in the form $J = J_1 + J_2$, where

$$J_1 = \int_{\Omega_1}\int_{\mathbb{R}^n} \tilde{b}(x,\xi)\hat{u}(\xi)h(x,\xi)d\xi dx,$$

and

$$J_2 = \int_{\mathbb{R}^n\setminus\Omega_1}\int_{\mathbb{R}^n} \tilde{b}(x,\xi)\hat{u}(\xi)h(x,\xi)d\xi dx,$$

By an argument similar to the proof of Theorem 5.1.17, we have

$$|J_1| \le C\int_{\Omega_1}\left(\int_{\mathbb{R}^n}|\hat{u}(\xi)|^2 d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n}|m(\xi)h(x,\xi)|^2 d\xi\right)^{1/2} dx$$

$$\le C|m(\xi)|||u||_{L^2(B)}\int_{\Omega_1}\left(\int_{\mathbb{R}^n}|\hat{\psi}(z)\bar{v}(x-z)|^q dz\right)^{1/q} dx,$$

$$|J_1| \le C|m(\xi)|||u||_{L^2(B)}\int_{\mathbb{R}^n\setminus\Omega_1} A(x)\left(\int_{\mathbb{R}^n}|\hat{\psi}(z)\bar{v}(x-z)|^q dz\right)^{1/q} dx,$$

where $C_{n,p,k}||a||$, A is defined as in the proof of Theorem 5.1.17 and $2 < p < \infty$.

By duality and Fubini's theorem, we obtain

$$|J_1| \le C|B|^{\frac{p-2}{2p}}|m(\xi)|||u||_{L^p(B)}|E|^{1/p}||\hat{\psi}(z)||_{L^q(\mathbb{R}^n)}||v||_{L^q(\mathbb{R}^n)},$$

$$|J| \le C|B|^{\frac{p-2}{2p}}|m(\xi)|||u||_{L^p(B)}|E|^{1/p}||\hat{\psi}(z)||_{L^q(\mathbb{R}^n)}||v||_{L^q(\mathbb{R}^n)}.$$

Then Theorem 5.1.18 follows by applying Lemma 5.1.14

Proof of Theorem 5.1.19. Without loss of generality, we may assume that $a \in C_0^\infty(\mathbb{R}\times\mathbb{R}) \cap \Lambda_{k,k'}^m(\mathbb{R}^n\times\mathbb{R}^n)$. Let $k = n/2+\varepsilon$, $k' = n/p+\varepsilon$ and $\varphi_p(\lambda) = (1+|\lambda|^2)^{\frac{1}{2}(n/p-[n/p]+\varepsilon/2)}$, where $\lambda \in \mathbb{R}^n$ and $\varepsilon, \varepsilon > 0$, is so small that $n/p - [n/p] + \varepsilon < 1$ and $[n/p+\varepsilon] = [n/p]$. Then for $u, v \in C_0^\infty(\mathbb{R}^n)$, we have

$$(a(x,D)u, v) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\lambda}a(x,\xi)\hat{u}(\xi)\bar{v}(x)d\xi dx. \tag{5.1.16}$$

We write (5.1.16)in the form

$$(a(x,D)u,v) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{a}(\lambda,y)f(\lambda,y)dyd\lambda$$

$$= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{a}(\lambda,y)\varphi_p(y)\varphi_2(\lambda)\varphi_p^{-1}(y)\varphi_1^{-1}(\lambda)f(\lambda,y)dyd\lambda$$

$$= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(x,\xi)e^{-i\xi y}e^{-i\xi y}e^{-ix\lambda}\varphi_p^{-1}\varphi_2^{-1}(\lambda)f(\lambda,y)dyd\lambda d\xi, \tag{5.1.17}$$

where

$$\hat{b}(\lambda,y) = \hat{a}(\lambda,y)\varphi_p(y)\varphi_2(\lambda), \quad 1<p\le 2, \tag{5.1.18}$$

and

$$f(\lambda,y) = \int_{\mathbb{R}^n} e^{iw\lambda}u(w+y)\bar{v}(w)dw, \quad \lambda,y\in\mathbb{R}^n. \tag{5.1.19}$$

Making use of the partition of unity W_s, $s=0,1,...,n$, we write (5.1.17) in the form

$$(a(x,D)u,v) = (\frac{1}{2\pi})^{2n}\sum_{s,t=0}^{n} I_{s,t},$$

where

$$I_{s,t} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(x,\xi)e^{-i\xi y}e^{-ix\lambda}W_s(y)\varphi_p^{-1}(y)W_t(\lambda)\varphi_2^{-1}(\lambda)f(\lambda,y)dyd\lambda d\xi dx,$$

$$s,t=0,1,...,n.$$

We shall estimate $I_{1,1}$ only, since the estimates of the cases are similar. By an argument similar to the proof of Theorem 5.1.17, we use the following method:

1) We integrate the above integral with respect to ξ first and make use of the identity

$$\frac{1}{1+iy_1}^{[n/p]}(1-(-i)^{1-[n/p]}\partial_{\xi_1}^{[n/p]})(e^{-i\xi y}) = e^{-i\xi y}.$$

2) We integrate the result of (1) with respect to x first and make use of identity

$$\frac{1}{1+iy_1}^{[n/p]}(1-(-i)^{1-[n/2]}\partial_{x_1}^{[n/2]})(e^{-ix\lambda}) = e^{-ix\lambda}.$$

Then we obtain

$$I_{1,1} = J_1 + (i)^{1-[n/2]}J_2 + (i)^{1-[n/p]}J_3 + (i)^{1-[n/2]}(i)^{1-[n/p]}J_4,$$

where

$$J_k = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b_k(x,\xi)\Delta(x,\xi)d\xi dx, \quad k=1,2,3,4,$$

with

$$b_1(x,\xi) = b(x,\xi),$$

$$b_2(x,\xi) = \partial_{x_1}^{[n/2]}(b(x,\xi)),$$

$$b_3(x,\xi) = \partial_{\xi_1}^{[n/p]}(b(x,\xi)),$$

$$b_4(x,\xi) = \partial_{x_1}^{[n/2]}\partial_{\xi_1}^{[n/p]}(b(x,\xi)),$$

$$\Delta(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\xi y}e^{-ix\lambda}\psi_p(y)\psi_2(\lambda)f(\lambda,y)dyd\lambda,\ x,\xi\in\mathbb{R}^n, \tag{5.1.20}$$

and

$$\psi_p(y) = W_1(y)\frac{1}{1+iy_1^{[n/p]}}\varphi_p^{-1}(y),\quad y=(y_1,...,y_n)\in\mathbb{R}^n,\quad 1<p\le 2.$$

We shall estimate J_4 only, since the other cases are similar. First, we estimate the following integral:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |m(\xi)\Delta(x,\xi)|d\xi dx.$$

By Lemma 5.1.3 and $\psi_p\in L^2(\mathbb{R}^n)$, we see that the integral in (5.1.19) is in $L^1(\mathbb{R}^n\times\mathbb{R}^n)$. Therefore, without loss of generality, we consider the following integral:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}} |m(\xi)\Delta_\delta(x,\xi)|d\xi dx, \tag{5.1.21}$$

where

$$\Delta_\delta(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\xi y}e^{-ix\lambda}\psi_{p,\delta}\psi_{2,\delta}(\lambda)f(\lambda,y)dyd\lambda,\ x,\xi\in\mathbb{R}^n, \tag{5.1.22}$$

with

$$\psi_{p,\delta}(y) = \psi_p(y)e^{-\delta|y|^2},\ y\in\mathbb{R}^n,\ 1<p\le 2 \textit{ and } 0<\delta<1. \tag{5.1.23}$$

We now give a proposition that will help us to study (5.1.20).

Proposition 5.1.21. For $u,v\in C_0^\infty$, $1<p\le 2$ and $0<\delta<1$, let Δ_δ be defined as in (5.1.21). Then we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |m(\xi)\Delta_\delta(x,\xi)|d\xi dx \le C_{n,p,k,k'}||u||_{L^p(\mathbb{R}^n)}||v||_{L^q(\mathbb{R}^n)},\ 1/p+1/q=1.$$

Proof. Substituting (5.1.18) into (5.1.21), writing $\bar{v}(w)$ in the form

$$\bar{v}(w) = \left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^n} e^{-iw\eta}\bar{\hat{v}}(\eta)d\eta,\ w\in\mathbb{R}^n,$$

and making the change of variables $w+y\to w$, we write Δ_δ in the form

$$\Delta_\delta(x,\xi) = \left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi-\eta)}e^{-iw\xi}$$

$$\times\left(\int_{\mathbb{R}^n} e^{i(w-x)\lambda}\hat{\psi}_{p,\delta}\psi_{2,\delta}(\lambda+\eta-\xi)d\lambda\right)u(w)\bar{\hat{v}}(\eta)dwd\eta,\ x,\xi\in\mathbb{R}^n.$$

By Taylor's expansion formula, we write $\psi_{2,\delta}$ in the form

$$\psi_{2,\delta}(\lambda+\eta-\xi) = \sum_{|\alpha|\le 4n}\frac{\lambda^\alpha}{\alpha!}\psi_{2,\delta}^{(\alpha)}(\eta-\xi)$$

$$+(4n+1)\sum_{|\alpha|=4n+1}\frac{\lambda^\alpha}{\alpha!}\int_0^1(1-\theta)^{4n+1}\psi^\alpha_{2,\delta}(\eta-\xi+\theta\lambda)d\theta,\ \lambda,\eta,\xi\in\mathbb{R}^n.$$

Substituting (5.1.23) into (5.1.22), we have

$$\Delta_\delta(x,\xi)=\left(\frac{1}{2\pi}\right)^n\sum_{|\alpha|\le 4n}\frac{1}{\alpha!}g_{\alpha,p,\delta}(x,\xi)h_{\alpha,2,\delta}(x,\xi)+\left(\frac{1}{2\pi}\right)^n(4n+1)\Box_\delta(x,\xi),\quad s,\xi\in\mathbb{R}^n,$$

where

$$g_{\alpha,p,\delta}(x,\xi)=(i)^{-|\alpha|}\int_{\mathbb{R}^n}e^{-iw\xi}\psi^{(\alpha)}_{p,\delta}(w)u(x+w)dw, \tag{5.1.24}$$

$$h_{\alpha,2,\delta}(x,\xi)=\int_{\mathbb{R}^n}e^{-ix\eta}\psi^\alpha_{2,\delta}(\eta-\xi)\bar{\hat{v}}(\eta)d\eta, \tag{5.1.25}$$

and

$$\Box_\delta(x,\xi)=\sum_{|\alpha|=4n+1}\frac{1}{\alpha!}\int_0^1(1-\theta)^{4n+1}$$

$$\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}e^{i(w-x)\lambda}\psi^\alpha_{2,\delta}(\eta-\xi+\theta\lambda)\lambda^\alpha\hat{\psi}_{p,\delta}(\lambda)d\lambda\right)e^{ix(\xi-\eta)}e^{-iw\xi}u(w)\bar{\hat{v}}(\eta)dwd\eta d\theta. \tag{5.1.26}$$

We now give a lemma to help us study the following integral:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m(\xi)|^m g_{\alpha,p,\delta}(x,\xi)h_{\alpha,2,\delta}(x,\xi)|d\xi dx, \tag{5.1.27}$$

for $|\alpha|\le 4n$, $1<p\le 2$ and $0<\delta<1$.

Lemma 5.1.22. For $u,v\in C_0^\infty(\mathbb{R}^n)$, $|\alpha|\le 4n$, $1<p\le 2$ and $0<\delta<1$, let $g_{\alpha,p,\delta}$ and $h_{\alpha,2,\delta}$ be defined as in (5.1.24) and (5.1.25), respectively. Then we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m(\xi)g_{\alpha,p,\delta}(x,\xi)h_{\alpha,2,\delta}(x,\xi)|d\xi dx\le C_{n,p,k,k'}|m(\xi)|||u||_{L^p(\mathbb{R})}||v||_{L^q(\mathbb{R}^n)},\ 1/p+1/q=1.$$

Proof. First, we write (5.1.25) in the form

$$h_{\alpha,2,\delta}(x,\xi)=\int_{\mathbb{R}^n}e^{-i\xi z}\hat{\psi}^{(\alpha)}_{2,\delta}(z)\bar{v}(x-z)dz. \tag{5.1.28}$$

Substituting (5.1.28) into (5.1.27), by Holder's inequality, Lemma 5.1.13 and Parseval's formula, we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m(\xi)g_{\alpha,p,\delta}(x,\xi)h_{\alpha,2,\delta}(x,\xi)|d\xi dx$$

$$\le\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|m(\xi)g_{\alpha,p,\delta}(x,\xi)|^2d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n}|h_{\alpha,2,\delta}(x,\xi)|^2d\xi\right)^{1/2}dx$$

$$\le C\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\psi^\alpha_{p,\delta}(w)u(x+w)|^pdw\right)^{1/p}\left(\int_{\mathbb{R}^n}|\hat{\psi}^\alpha_{2,\delta}(z)\bar{v}(x-z)|^2dz\right)^{1/2}dx,$$

where $C=C_{n,p,k,k'}$.

By Hölder's inequality, Fubini's theorem and Minkowski's inequality, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m(\xi)g_{(\alpha,p,\delta)}(x,\xi)h_{\alpha,2,\delta}(x,\xi)|d\xi dx$$

$$\leq C_{n,p,k,k'}||\psi_{p,\delta}^{(\alpha)}||_{L^p(\mathbb{R})}||u||_{L^p(\mathbb{R}^n)}||\psi_{2,\delta}^{(\alpha)}||_{L^2(\mathbb{R}^n)}||v||_{L^q(\mathbb{R}^n)},\ 1/p+1/q=1.$$

We now give a lemma to help us study $\Box_\delta$.

Lemma 5.1.23. For $u, v \in C_0^\infty$, $1 < p \leq 2$ and $0 < \delta < 1$, let $\Box_\delta$ be defined as in (5.1.26). Then we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m(\xi)\Box_\delta(x,\xi)|d\xi dx \leq C_{n,p,k,k'}|m(\xi)|||u||_{L^p(\mathbb{R}^n)}||v||_{L^q(\mathbb{R}^n)},\ 1/p+1/q=1.$$

Proof. First, we study the following integral:

$$\int_{\mathbb{R}^n} e^{i(w-x)\lambda}\psi_{2,\delta}^{(\alpha)}(\eta-\xi+\theta\lambda)(\lambda^\alpha\hat{\psi}_{p,\delta}(\lambda))d\lambda, \tag{5.1.29}$$

where $w, x, \eta, \xi \in \mathbb{R}^n$, $0 \leq \theta \leq 1$ and $|\alpha| = 4n+1$.

Making use of the following identity:

$$(\prod_{s=1}^n \frac{1}{1+i(w_s-x_s)})(\prod_{s=1}^n(1+\partial_{\lambda_s}))(e^{i(w-x)\lambda}) = e^{i(w-x)\lambda},$$

we write (5.1.29) in the form

$$\int_{\mathbb{R}^n} e^{i(w-x)\lambda}\psi_{2,\delta}^{\alpha}(\eta-\xi+\theta\lambda)(\lambda^\alpha\hat{\psi}_{p,\delta}(\lambda))d\lambda$$

$$= (\prod_{s=1}^n \frac{1}{1+i(w_s-x_s)})\sum_{\beta\in T}(-1)^{|\beta|}\sum_{\gamma\leq\beta}\int_{\mathbb{R}^n} e^{i(w-x)\lambda}\partial_\lambda^\gamma(\psi_{2,\delta}^\alpha(\eta-\xi+\theta\lambda))\partial_\lambda^{\beta-\gamma}(\lambda^\alpha\hat{\psi}_{p,\delta}(\lambda))d\lambda, \tag{5.1.30}$$

with

$$T = \{(\beta_1,\beta_2,...,\beta_n)\in\mathbb{N}^n|\beta_t = 0\ \ or\ 1, t=1,...,n\}.$$

Substituting (5.1.30) into (5.1.26), we get

$$\Box_\delta(x,\xi) = \sum_{|\alpha|=4n+1}\frac{1}{\alpha!}\sum_{\beta\in T}(-1)^{|\beta|}\sum_{\gamma\leq\beta}\int_0^1\int_{\mathbb{R}^n}(1-\theta)^{4n+1}\partial_\lambda^{\beta-\gamma}(\lambda^\alpha\hat{\psi}_{p,\delta}(\lambda))$$

$$\times e^{-ix(\lambda-\xi)}\tilde{g}(x,\xi,\lambda)\tilde{h}_{\alpha,\gamma}(x,\xi,\theta\lambda)d\lambda d\theta,\ x,\xi\in\mathbb{R}^n, \tag{5.1.31}$$

where

$$\tilde{g}(x,\xi,\lambda) = \int_{\mathbb{R}^n} e^{iw(\lambda-\xi)}(\prod_{s=1}^n\frac{1}{1+i(w_s-x_s)})u(w)dw,$$

and

$$\tilde{h}_{\alpha,\gamma}(x,\xi,\theta\lambda) = \int_{\mathbb{R}^n} e^{-ix\eta}\partial_\lambda^\gamma(\psi_{2,\delta}^\alpha(\eta-\xi+\theta\lambda))\bar{\hat{v}}(\eta)d\eta,\ \lambda\in\mathbb{R}^n. \tag{5.1.32}$$

By an argument similar to the proof of Lemma 5.1.13, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m(\xi)\tilde{g}(x,\xi,\lambda)\tilde{h}_{\alpha,\gamma}(x,\xi,\theta\lambda)|d\xi dx$$

$$\leq C_{n,p}||l||_{L^p(\mathbb{R}^n)}||u||_{L^p(\mathbb{R}^n)}\theta^{|r|}||\psi_{2,\delta}^{(\alpha+\gamma)}||_{L^2(\mathbb{R}^n)}||v||_{L^q(\mathbb{R}^n)}, \tag{6.1.33}$$

where

$$l(x) = \prod_{s=1}^{n} \frac{1}{1+ix_s} \; and \; 1/p + 1/q = 1.$$

Also, we have

$$\int_{\mathbb{R}^n} |\partial_\lambda^\beta(\lambda^\alpha \hat{\psi}_{p,\delta}(\lambda))| d\lambda \leq C_n, \tag{5.1.34}$$

with $|\alpha| = 4n+1$ and $\beta \in T$. Therefore, (5.1.29)-(5.1.31) imply

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |m(\xi)\Box_\delta(x,\xi)| d\xi dx \leq \sum_{|\alpha|=4n+1} \frac{1}{\alpha!} \sum_{\beta\in T}\sum_{\gamma\leq\beta} \int_0^1 \int_{\mathbb{R}^n} (1-\theta)^{4n+1} |\partial_\lambda^{\beta-\gamma}(\lambda^\alpha \hat{\psi}_{p,\delta}(\lambda))|$$

$$\times \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |m(\xi)\tilde{g}(x,\xi,\lambda)\tilde{h}_{\alpha,\gamma}(x,\xi,\theta\lambda)| d\xi dx \right) d\lambda d\theta$$

$$\leq C_{n,p,k,k'} |m(\xi)| ||u||_{L^p(\mathbb{R}^n)} ||v||_{L^q(\mathbb{R}^n)}, \; 1/p+1/q=1, \; 1<p\leq 2.$$

Thus, Proposition 5.1.21 gives

$$|J_4| \leq C_{n,p,k,k'} |m(\xi)| ||a|| ||u||_{L^p(\mathbb{R}^n)} ||v||_{L^q(\mathbb{R}^n)}, \; 1/p+1/q=1, \; 1<p\leq 2.$$

Proof of Theorem 5.1.4. Without loss of generality, we may assume that $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \cap \Lambda_{k,k'}^m(\mathbb{R}^n \times \mathbb{R}^n)$. Let $k = n/p + \varepsilon$, $k' = n/2 + \varepsilon$ and $\varphi_{p'}(\lambda) = (1+|\lambda|^2)^{\frac{1}{2}(n/p' - [n/p'] + \varepsilon/2)}$, where $\lambda \in \mathbb{R}^n$, $2 \leq p' < \infty$ and $\varepsilon, \varepsilon > 0$, is so small that $n/p' + \varepsilon/2 \notin \mathbb{N}$, $n/p' - [n/p']\varepsilon < 1$, $[n/p' + \varepsilon] = [n/p]$ and $n/q - [n/q] \neq \varepsilon/2$ with $1/p + 1/q = 1$. Then for $u, v \in C_0^\infty$, we have

$$(a(x,D)u, v) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix\xi} a(x,\xi)\hat{u}(\xi)\bar{v}(x) d\xi dx.$$

Sice the following arguments are similar to the proof of Theorem 5.1.19, we shall only study the following lemma, which is similar to Lemma 5.1.13

Lemma 5.1.24. For $u, v \in C_0^\infty$, $|\alpha| \leq 4n$, $2 < p < \infty$ and $0 < \delta < 1$, we define $g_{\alpha,2,\delta}$ and $h_{\alpha,p,\delta}$ as follows:

$$g_{\alpha,2,\delta}(x,\xi) = (i)^{-|\alpha|} \int_{\mathbb{R}^n} e^{-iw\xi} \psi_{2,\delta}^{(\alpha)}(w) u(x+w) dw,$$

$$h_{\alpha,p,\delta}(x,\xi) = \int_{\mathbb{R}^n} e^{-ix\eta} \psi_{p,\delta}^{(\alpha)}(\eta-\xi)\bar{\hat{v}}(\eta) d\eta, \quad x,\xi \in \mathbb{R}^n,$$

with

$$\psi_{p',\delta}(y) = \psi_{p'}(y) e^{-\delta|y|^2}, \quad \psi_{p'}(y) = W_1(y) \frac{1}{1+iy_1^{[n/p']}} \varphi_{p'}^{-1}(y),$$

$$y = (y_1, ..., y_n) \in \mathbb{R}^n, \quad 2 \leq p' < \infty,$$

and W_1 is defined as in the proof of Theorem 5.1.18. Then we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |m(\xi)| g_{\alpha,2,\delta}(x,\xi) h_{\alpha,p,\delta}(x,\xi)| d\xi dx$$

$$\leq C_{n,p,k,k'} |m(\xi)| ||u||_{L^p(\mathbb{R}^n)} ||v||_{L^q(\mathbb{R}^n)}, \; 1/p+1/q=1.$$

Proof. First, we write (5.1.33) in the form

$$h_{\alpha,p,\delta}(x,\xi) = \int_{\mathbb{R}^n} e^{-i\xi z}\hat{\psi}^{(\alpha)}_{p,\delta}(z)\bar{v}(x-z)dz.$$

By an argument similar to that in the proof of Lemma 5.1.13, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |m(\xi)g_{\alpha,2,\delta}(x,\xi)h_{\alpha,p,\delta}(x,\xi)|d\xi dx$$

$$\leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|g_{\alpha,2,\delta}(x,\xi)|^2 d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n}|m(\xi)||h_{\alpha,p,\delta}(x,\xi)|^2 d\xi\right)^{1/2} dx$$

$$\leq C\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\psi^{(\alpha)}_{2,\delta}(w)u(x+w)|^2 dw\right)^{1/2}\left(\int_{\mathbb{R}^n}|\hat{\psi}^{(\alpha)}_{p,\delta}(z)\bar{v}(x-z)|^q dz\right)^{1/q} dx$$

$$\leq C||\psi^{\alpha}_{2,\delta}||_{L^p(\mathbb{R}^n)}||u||_{L^p(\mathbb{R}^n)}||\hat{\psi}^{\alpha}_{p,\delta}||_{L^q(\mathbb{R}^n)}, \quad 1/p+1/q=1,$$

where $C = C_{n,p,k,k'}$.

Thus, by an argument similar to the proof of Theorem 5.1.19, Lemma 5.1.12, and Lemma 5.1.13, Theorem 5.1.20 is obtained.

Part 6

Appendix

Appendix A

A Few Basic Definitions and Theorems of Functional Analysis

The aim of this chapter is to recall some basics definitions and theorems, which are useful for our generally calculus.

6.1 Closed and closable operators

Throughout this section, E denotes a Banach space (over $\mathbb{K}$ or $\mathbb{C}$) with norm $||\cdot||$. By $\mathcal{L}(E)$, we denote the space of all bounded linear operators on E.

Definition 1.1. An operator $B : D(B) \subseteq E \to E$ is called a closed operators if the graph

$$G(B) := \{(u; Bu), u \in D(B)\}$$

is closed in $E \times E$.

This definition can be rephrased as follows:

If $(x_n)_n \in D(B)$ is such that $x_n \to x$ and $Bx_n \to y$ in E (as $n \to \infty$), then $x \in D(B)$ and $y = Bx$.

Note also that B is a closed operator if and only if $D(B)$ endowed with the graph norm $||\cdot|| + ||B||$ is a complete space.

Definition 1.2. Let $B : D(B) \subseteq E \to E$ be an operator on E. A scalar $\lambda \in \mathbb{K}$ is the resolvent set of B if $\lambda I - B$ is invertible (from D(B) into E) and its inverse $(\lambda I - B)^{-1}$ is a bounded operator on E. For such λ, the operator $(\lambda I - B)^{-1}$ is the resolvent of B at λ.

The set

$$\rho(B) := \{\lambda \in \mathbb{K}, \lambda I - B \text{ is invertible and } (\lambda I - B)^{-1} \in \mathcal{L}(E)\}$$

is called the resolvent set of B.

The complement of $\rho(B)$ in K

$$\sigma(B) := \mathbb{K} \setminus \rho(B)$$

is called the spectrum of B.

Proposition 1.3. 1) Assume that B is a closed operator on a Banach space E. Then a scalar λ is in $\rho(B)$ if and only if $\lambda I - B$ is invertible (from $D(B)$ into E).

2) If the resolvent set $\rho(B)$ is not empty, then B is a closed operator.

Definition 1.4. An operator B on a Banach space E is closable if there exists a closed operator $C : D(C) \subseteq E \to E$ such that $D(B) \subseteq D(C)$ and $Bu = Cu$ for all $u \in D(B)$. In other words, B has a closed extension C.

Assume that B is a closable operator on a Banach space E. One can define the smallest closed extension $\bar{B}$ of B as follows:

$$D(\bar{B}) = \{u \in E \ s.t. \ \exists u_n \in D(B) : lim_n u_n = u, \quad lim_{n,m}[Bu_n - Bu_m] = 0\} \qquad (6.1),$$

and if u and $(u_n)_n$ are as in (6.1) we set

$$\bar{B}u := lim_n Bu_n, \qquad (6.2)$$

where the limits are taken with respect to the norm of E.

One shows easily that $\bar{B}$ is closed operator and every closed extension of B is also an extension of $\bar{B}$.

If B is an operator such that $\bar{B}$, defined by (6.1) and (6.2), is well defined (i.e., $\bar{B} = lim_n Bu_n$ does not depend on the choice of the sequence (u_n)), then $\bar{B}$ is a closed extension of B. Consequently, B is closable if and only if $\bar{B}$ is a well defined operator.

Let now $u \in D(\bar{B})$ and let $u_n \in D(B)$, $v_n \in D(B)$ be two sequences which converge to u and such that $Bu_n - Bu_m \to 0$ and $Bv_n - Bv_m \to 0$ as $n, m \to \infty$. Thus, Bu_n and Bv_n converge to some w and w' in E. Now, $barB$ is well defined if and only if $w = w'$. Thus, we have proved the following characterization of closable operators.

Proposition 1.5. A linear operator B on E is closable if and only if it satisfies the following property:

if $(u_n) \in D(B)$ is any sequence such that $u_n \to 0$ and $Bu_n \to v$ (in E), then $v = 0$.

Definition 1.6. Let B be an operator with domain $D(B)$ on a Banach space E. A linear subspace of $D(B)$ is called a core of B if it is dense in $D(B)$, endowed with the graph norm $|| \cdot || + ||B||$.

Let B act on a Banach space E and D a linear subspace of $D(B)$. The restriction of B to D is the operator

$$B|_D u := Bu \ for \ \ u \in D = D(B|_D).$$

The next result follows easily from the previous definitions.

Proposition 1.7. Let B be a closed operator on a Banach space E and D a linear subspace of $D(B)$. Then, D is a core of B if and only if the closure of $B|_D$ is B, i.e., $\bar{B}|_D = B$.

6.2 Function spaces and Fourier transform

We first define spaces of smooth functions, $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$, and $E'(\Omega)$, $D'(\Omega)$ are spaces of distributions. Let Ω is a smooth paracompact manifold and open subset of R^n, when $\Omega = R^n$, we define the Schwartz space $\mathcal{S}$ of rapidly decreasing functions, and its dual $\mathcal{S}'$.

We define, on functions on R^n, the seminorms $q_{j,k}$ by

$$q_{j,k}(u) = \sup_{x\in R^n} (1+|x|^2)^{j/2}|D^\alpha u(x)| : |\alpha| \le k.$$

Definition 2.1. The space $\mathcal{S}(R^n)$ consists of smooth functions u on R^n for which each $q_{j,k}(u)$ is finite, with the Frechet space topology determined by these seminorms. It dual is denoted $\mathcal{S}'(R^n)$.

We have defined the differential operators D^α on functions on R^n by

$$D_j = i^{-1}\frac{\partial}{\partial x_j}, \quad D^\alpha = D_1^{\alpha_1}...D_n^{\alpha_n}.$$

If $P(\xi)$ is a polynomial,

$$P(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha,$$

denote by $P(D)$ the differential operator

$$P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha.$$

If

$$p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha, \quad a_\alpha \in C^\infty(\Omega),$$

let

$$p(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha.$$

The differential operators are continuously $p(x,D) : C^\infty(\Omega) \to C^\infty(\Omega)$ and $p(x,D) : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ on $\Omega \subset R^n$.

If the coefficients $a_\alpha \in C^\infty(R^n)$ satisfy the slow growth condition

$$|D^\beta a_\alpha(x)| \le C_{\alpha\beta}(1+|x|^2)^N, \quad N = N(\alpha,\beta),$$

then $p(x,D) : \mathcal{S}(R^n) \to \mathcal{S}(R^n)$ and similarly $p(x,D) : \mathcal{S}'(R^n) \to \mathcal{S}'(R^n)$.

The Fourier transform. If $u \in L^1(R^n)$, we define by

$$\hat{u}(\xi) = (2\pi)^{-n/2}\int u(x)e^{-ix\xi}dx \tag{6.3}$$

where $x\cdot\xi = x_1\xi_1 + ... + x_n\xi_n$. If $u \in L^1$ implies $\hat{u} \in L^\infty$ and $||\hat{u}||_{L^\infty} \le (2\pi)^{-n/2}||u||_{L^1}$. We denote $\hat{u}(\xi)$ by $\mathfrak{F}u(\xi)$. Changing the sign in the exponent in (6.3), we define

$$\mathfrak{F}u(\xi) = \tilde{u}(\xi) = (2\pi)^{-n/2}\int u(x)e^{-ix\xi}dx.$$

Proposition 2.2. If $u \in \mathcal{S}(R^n)$, then $\hat{u}, \tilde{u} \in \mathcal{S}(R^n)$.

Proof. If $u \in \mathcal{S}$, then we can differentiate (6.3) under the integral sign to obtain

$$D_\xi^\alpha \hat{u}(\xi) = (2\pi)^{-n/2} \int u(x)(-x)^\alpha e^{ix\xi} dx. \tag{6.4}$$

Now, realizing that $\xi^\beta e^{-ix\xi} = (-1)^{|\beta|} D_x^\beta e^{-ix\xi}$ and integrating (6.4) by parts we get

$$\xi^\beta D_\xi^\alpha \hat{u}(\xi) = (2\pi)^{-n/2}(-1)^{|\alpha|+|\beta|} \int D_x^\beta (x^\alpha u(x)) e^{-ix\xi} dx,$$

which implies that each $\xi^\beta D_\xi^\alpha \hat{u} \in L^\infty(R^n)$, provided $u \in \mathcal{S}(R^n)$. This yields $\hat{u} \in \mathcal{S}$, and similarly one has $\tilde{u} \in \mathcal{S}$.

Theorem 2.3. The Fourier transform $\mathfrak{F} : \mathcal{S}(R^n) \to \mathcal{S}(R^n)$ is an isomorphism; in fact $\mathfrak{F}\mathfrak{F}^* = \mathfrak{F}^*\mathfrak{F} = I$, so in particular, for $u \in \mathcal{S}$,

$$u(x) = (2\pi)^{-n/2} \int \hat{u}(\xi) e^{ix\xi} d\xi. \tag{6.5}$$

Proof. By (6.3) implies $\mathfrak{F}^*\mathfrak{F} = I$ and $\mathfrak{F}\mathfrak{F}^* = I$ is similar, so we can prove only (6.5)

$$\begin{aligned} (2\pi)^{-n/2} \int \hat{u}(\xi) e^{ix\xi} d\xi &= (2\pi)^{-n} \int \int u(y) e^{i(x-y)\xi} dy d\xi \\ &= \lim_{\varepsilon \to 0} (2\pi)^{-n} \int \int u(y) e^{i(x-y)\xi - \varepsilon|\xi|^2} dy d\xi \\ &= \lim_{\varepsilon \to 0} (2\pi)^{-n} \int \{ \int e^{i(x-y)\xi - \varepsilon|\xi|^2} d\xi \} u(y) dy \\ &= \lim_{\varepsilon \to 0} \int p(\varepsilon, x-y) u(y) dy \end{aligned} \tag{6.6}$$

where

$$p(\varepsilon, x) = (2\pi)^{-n} \int eix\xi - \varepsilon|\xi|^2 d\xi. \tag{6.7}$$

We evaluate (6.7)

$$p(\varepsilon, x) = (4\pi)^{-n/2} e^{-|x|^2/4\varepsilon} = \varepsilon^{-n/2} q(x/\sqrt{\varepsilon}) \tag{6.8}$$

where $q(x) = p(1, x) = (4\pi)^{/n/2} e^{-|x|^2/4}$. If $q \in \mathcal{S}(R^n)$, by (6.8)we have

$$\int_{R^n} q(x) dx = 1.$$

For any bounded continuous $u(x)$ to verify that,

$$= \lim_{\varepsilon \to 0} \int p(\varepsilon, x-y) u(y) dy = u(x). \tag{6.9}$$

(6.6) and (6.8) yield (6.5).

To establish (6.9), $p(\varepsilon, x)$ defined by (6.7) is an analytic function of $x \in \mathcal{C}^n$ to verify that

$$p(\varepsilon, ix) = (4\pi)^{-n/2} e^{|x|^2/4\varepsilon}, \quad x \in R^n \tag{6.10}$$

Now,

$$p(\varepsilon, ix) = (2\pi)^{-n} \int e^{-x\xi - \varepsilon|\xi|^2} d\xi$$

$$= (2\pi)^{-n} e^{|x|^2/4\varepsilon} \int e^{-|(x/2\sqrt{\varepsilon}) + \sqrt{\varepsilon}\xi|^2} d\xi$$

$$= (2\pi)^{-n} e^{|x|^2/4\varepsilon} \int e^{-\varepsilon\xi^2} d\xi$$

$$= (2\pi)^{-n} \varepsilon^{-n/2} e^{|x|^2/4\varepsilon} \int_{R^n} e^{-|\xi|^2} d\xi,$$

so to prove (6.10), we show that

$$\int_{R^n} e^{-|\xi|^2} d\xi = \pi^{n/2}.$$

If

$$A = \int_{-\infty}^{\infty} e^{-|\xi|^2 d\xi},$$

then

$$\int_{R^n} e^{-|\xi|^2} d\xi = A^n,$$

but

$$A^2 = \int_{R^2} e^{-|\xi|^2} d\xi = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = 2\pi \int_0^{\infty} e^{-r^2} r dr = \pi,$$

so we are done. The proof of Theorem 2.3 is complete.

We define

$$\mathfrak{F} : \mathcal{S}'(R^n) \to \mathcal{S}'(R^n)$$

by

$$(u, \mathfrak{F}\omega) = (\mathfrak{F}^* u, \omega). \tag{6.11}$$

The (6.11) holds for $\omega \in \mathcal{S}(R^n)$ and defines the unique continuous extension of $\mathfrak{F}$ from $\mathcal{S}$ to $\mathcal{S}'$, similarly $\mathfrak{F}^* : \mathcal{S}' \to \mathcal{S}'$. From Theorem 2.3 it follows $\mathfrak{F}\mathfrak{F}^* = \mathfrak{F}^*\mathfrak{F} = I$ on $\mathcal{S}'(R^n)$, $\mathfrak{F}$ and $\mathfrak{F}^*$ are isomorphism from $\mathcal{S}'$ to itself.

Integrating by parts that (6.5) implies

$$D^\alpha u(x) = (2\pi)^{-n/2} \int \xi^\alpha \hat{u}(\xi) e^{ix\xi} d\xi,$$

$$(-x)^\beta u(x) = (2\pi)^{-n/2} \int D_\xi^\beta \hat{u}(\xi) e^{ix\xi} d\xi \tag{6.12}$$

where $u \in \mathcal{S}(R^n)$.

Thus $D^\alpha = \mathfrak{F}^{-1}\xi^\alpha\mathfrak{F}$, $x^\beta = \mathfrak{F}^{-1}(-D)^\beta\mathfrak{F}$ on $\mathcal{S}(R^n)$, similarly on $\mathcal{S}'(R^n)$. From $\mathfrak{F}^{-1} = \mathfrak{F}^*$ on $\mathcal{S}$ it follows

$$||\hat{u}||_{L^2}^2 = (\mathfrak{F}u, \mathfrak{F}u) = (\mathfrak{F}^*\mathfrak{F}u, u) = (u, u) = ||u||_{L^2}^2, \quad u \in \mathcal{S}. \tag{6.14}$$

By (6.5) follows $\mathfrak{F}$ has a unique extension as an isometry $\mathfrak{F} : L^2(R^n) \to L^2(R^n)$, similarly also $\mathfrak{F}^*$ on L^2 and $\mathfrak{F}^*\mathfrak{F} = \mathfrak{F}\mathfrak{F}^* = I$ on L^2 in fact by using Plancherel's theorem both of them are unitary operators.

We define the convolution

$$u * v(x) = \int_{R^n} u(x)v(x-y)dy, \quad u, v \in \mathcal{S}(R^n),$$

then $u * v \in \mathcal{S}$, we say $\mathcal{S} * \mathcal{S} \subset \mathcal{S}$ and have $L^1 * L^1 \subset L^1, C_0^\infty * \mathcal{S} \subset \mathcal{S},\ C_0^\infty * C^\infty \subset C^\infty$.

By Fubini's theorem we have for $u, v \in \mathcal{S}(R^n)$

$$(u * v)^\wedge(\xi) = (2\pi)^{n/2}\hat{u}(\xi)\hat{v}(\xi).$$

The delta function $\delta \in \mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$ is defined by $< u, v >= u(0)$ and also $\hat{\delta}(\xi) = (2\pi)^{-n/2}$, $\delta * \omega = \omega$, $P(D)\omega = P(D)\delta * \omega$. By (6.12) for a polynomial $P(\xi)$, we have

$$(P(D)u)^\wedge(\xi) = P(\xi)\hat{u}(\xi)$$

or

$$P(D) = \mathfrak{F}^{-1}P(\xi)\mathfrak{F}. \tag{6.15}$$

For general function $p(\xi)$ has

$$p(D) = \mathfrak{F}^{-1}p(\xi)\mathfrak{F}. \tag{6.16}$$

If $p(\xi)$ satisfies the slow growth condition, $p(D)$ maps $\mathcal{S}(R^n)$ and $\mathcal{S}'(R^n)$ to themselves and by Plancherel's theorem $p(D) : L^2 \to L^2$ if $p \in L^\infty(R^n)$.

The Fourier series has defined on $T^n = R^n/2\pi Z^n$ by

$$u(x) = \sum_{m \in Z^n} \hat{u}(m)e^{imx}, \quad where \quad \hat{u}(m) = (2\pi)^{-n}\int_{T^n} u(x)e^{-imx}dx, \quad u \in \mathcal{D}'(T^n)$$

or

$$\hat{u}(m) = (2\pi)^{-n} < u, e^{-imx} >$$

By using Plancherel's theorem we have

$$\sum_m |\hat{u}(m)|^2 = (2\pi)^{-n}\int_{T^n} |u(x)|^2dx.$$

There is

$$D^\alpha u(x) = \sum_{m \in Z^n} m^\alpha \hat{u}(m)e^{imx}.$$

It follows that $u \in C^\infty(T^n)$ iff $\hat{u}(m)$ is a rapidly decreasing sequence in Z^n,

$$\sup_{m \in Z^n} (1 + |m|)^k|\hat{u}(m)| < \infty, \forall k.$$

By duality, $u \in \mathcal{D}'(T^n)$ iff $\hat{u}(m)$ is a polynomially bounded sequence for some ℓ, $|\hat{u}(m)| \le C(1 + |m|)^\ell$.

As in (6.15) for p on Z^n we have

$$p(D)u = \sum p(m)\hat{u}(m)e^{imx}.$$

Thus $p(D): C^\infty \to C^\infty$ and $p(D): \mathcal{D}'(T^n) \to \mathcal{D}'(T^n)$ provided $p(m)$ is polynomially bounded, while $p(D): L^2(T^n) \to L^2(T^n)$ iff $p(m)$ is bounded.

We define **the Sobolev space** $H^k(R^n) = \{u : u \in L^2(R^n), D^\alpha u \in L^2(R^n), |\alpha| \leq k\}$. By the Plancherel formula this is equivalent $H^k(R^n) = \{u : \xi^\alpha \hat{u}(\xi) \in L^2(R^n), (1+|\xi|)^k \hat{u} \in L^2(R^n), |\alpha| \leq k\}$.

Definition 2.4. For $s \in R$, $H^s(R^n)$ is the set of $u \in \mathcal{S}'(R^n)$ such that $\hat{u}$ is locally square integrable and $(1+|\xi|)^s |\hat{u}| \in L^2(R^n)$.

We defined norm by

$$||u||_{H^s}^2 = \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$$

which, when $s = k$ is an integer, is equivalent to

$$\sum_{|\alpha| \leq k} ||D^\alpha u||_{L^2}^2.$$

$H^s(R^n)$ is a Hilbert space, and the Fourier transform is an isometric isomorphism $\mathfrak{F} : H^s(R^n) \to L^2(R^n, (1+|\xi|^2)^s d\xi)$.

The Schwartz space $\mathcal{S}(R^n)$ is dense in $L^2(R^n, (1+|\xi|^2)^s d\xi)$, and $\mathfrak{F}^{-1}\mathcal{S} = \mathcal{S}$, it follows that $\mathcal{S}(R^n)$ is dense in $H^s(R^n)$.

It is easy to see that, $|p(\xi)| \leq C(1+|\xi|)^m$, then $p(D) = \mathfrak{F}^{-1} p(\xi) \mathfrak{F}$ maps $H^s(R^n) \to H^{s-m}(R^n)$. If $a(x) \in \mathcal{S}$, then $au = (2\pi)^{n/2} \mathfrak{F}^{-1}(\hat{a} * \hat{u}) \in H^s(R^n)$ provided $u \in H^s(R^n)$. If $s = k \geq 0$ is a integer, then $u \to au$ is map from $H^k(R^n)$ to itself provided

$$|D^\beta a(x)| \leq c_\beta, \quad \forall \beta \geq 0. \tag{6.16}$$

It follows by duality that $u \to au$ maps $H^{-k}(R^n)$ to itself, if $p(x,\xi) = \sum_{|\alpha| \geq m} a_\alpha \xi^\alpha$ and $a_\alpha(x)$ satisfies (6.16), then $p(x, D) : H^s(R^n) \to H^s(R^n)$.

Theorem 2.5. If $s > n/2$, then each $u \in H^s(R^n)$ is bounded and continuous.

Proof. We must prove $\hat{u}(\xi) \in L^1(R^n)$. Using Cauchy's inequality, we get

$$\int |\hat{u}(\xi)| d\xi \leq (\int |\hat{u}(\xi)|^2 (1+|\xi|^2)^s) d\xi)^{1/2} (\int (1+|\xi|^2)^{-s} d\xi)^{-1/2}$$

second factor is finite for $s > n/2$, we can conclude $u(x)$ vanishes at infinity.

Corollary. 2.6. If $s > (n/2) + k$, then $H^s(R^n) \subset C^k(R^n)$.

If $s = (n/2) + \alpha,\ 0 < \alpha < 1$, we can obtain Holder continuity. If $u \in C^\alpha(R^n),\ 0 < \alpha < 1$, u is bounded

$$|u(x+y) - u(x)| \leq c|y|^\alpha, \ |y| \leq 1.$$

Proposition 2.7. If $s = (n/2) + \alpha,\ 0 < \alpha < 1$, then $H^s(R^n) \subset C^\alpha$.

Proof. We use the Fourier inversion formula for $u \in H^s(R^n)$

$$|u(x+y) - u(x)| = (2\pi)^{-n/2} |\int \hat{u}(\xi) e^{ix\xi}(e^{iy\xi} - 1) d\xi|$$

$$\leq c(\int |\hat{u}|^2(1+|\xi|)^{n+2\alpha}d\xi)^{1/2} \times (\int |e^{iy\xi}-1|^2(1+|\xi|)^{-n-2\alpha}d\xi)^{1/2}.$$

if $|y| \leq 1$,

$$\int |e^{iy\xi}-1|^2(1+|\xi|)^{-n-2\alpha}d\xi \leq c\int_{|\xi|\leq|y|^{-1}} |y|^2|\xi|^2(1+|\xi|)^{-n-2\alpha}d\xi$$

$$+4\int_{|\xi|\geq|y|^{-1}}(1+|\xi|)^{-n-2\alpha}d\xi \leq c|y|^2\int_0^{|y|^{-1}}\frac{r^{n+1}}{(1+r)^{n+2\alpha}}dr + 4c\int_{|y|^{-1}}^{\infty}\frac{r^{n-1}}{(1+r)^{n+2\alpha}}dr$$

$$\leq c|y|^2 + c|y|^2\left\{\begin{array}{ll}\frac{|y|^{2\alpha-2}-1}{2\alpha-2} & (0<\alpha<1)\\ \log\frac{1}{|y|} & (\alpha=1)\end{array}\right\} + c|y|^{2\alpha}$$

if $|y| \leq 1/2$,

$$|u(x+y)-u(x)| \leq c|y|^{\alpha} \quad if \quad 0<\alpha<1 \quad or \quad \leq c|y|(\log\frac{1}{|y|})^{1/2} \quad if \quad \alpha=1.$$

Note the different modulus of continuity for $\alpha = 1$. Elements of $H^{n/2+1}(R^n)$ needn't be Lipshitz, and elements of $H^{n/2}(R^n)$ needn't be bounded. In fact, if $\hat{u}(\xi) = (1+|\xi|)^{-n}/\log(2+|\xi|)$, then $u \in H^s(R^n)$ and $u \notin L^\infty(R^n)$.($\forall \hat{u} \geq 0,\ if\ \hat{u} \notin L^1$, then $u \notin L^\infty$.) By duality, Theorem 2.2 implies that all finite measures on R^n belong to $u \in H^s(R^n)$ for $s < -(n/2)$, particularly $\delta \in H^{-n/2-\varepsilon}(R^n), \varepsilon > 0$ and $D^\alpha\delta \in H^{-n/2-|\alpha|-\varepsilon}(R^n)$.

Sobolev imbedding result is that

$$H^s(R^n) \subset L^q(R^n), \quad q = \frac{2n}{n-2s}, \ if\ 0 \leq s < \frac{1}{2}n.$$

We consider the behavior of the trace map $\tau : \mathcal{S}(R^n) \to \mathcal{S}(R^{n-1})$ with $x = (x_1, x')$, $\tau u(x') = u(0, x')$.

Theorem 2.8. The map τ extends uniquely to a continuous linear operator

$$\tau : H^s(R^n) \to H^{s-1/2}(R^{n-1}), \quad if\ s > 1/2.$$

Proof. Let $f = \tau u$, $u \in \mathcal{S}$, we have $\hat{f}(\xi') = \int \hat{u}(\xi)d\xi_1$ and

$$|\hat{f}(\xi')|^2 \leq (\int |\hat{u}(\xi)|^2(1+|\xi|)^{2s}d\xi_1)^{1/2}(\int(1+|\xi|)^{-2s}d\xi_1)^{1/2}$$

if $s > 1/2$, we estimate the last factor

$$\int(1+|\xi|)^{-2s}d\xi_1 \leq c\int(1+|\xi'|^2+\xi_1^2)^{-s}d\xi_1 = c'(1+|\xi'|^2)^{-s+1/2}. \tag{6.17}$$

Therefore,

$$(1+|\xi'|^2)^{s-1/2}|\hat{f}(\xi')|^2 \leq c\int |\hat{u}(\xi)|^2(1+|\xi|)^{2s}d\xi,$$

and integrating with respect to ξ' gives

$$||f||^2_{H^{s-1/2}(R^n)} \leq c||u||^2_{H^s(R^n)}.$$

Theorem 2.9. The restriction map $\tau : H^s(R^n) \to H^{s-1/2}(R^n)$, $\;s > 1/2$, is onto.

Proof. Take $g \in H^{s-1/2}(R^{n-1})$, $\hat{g}(\xi')$.

Let

$$\varphi(\xi) = \hat{g}(\xi')\frac{(1+|\xi'|^2)^{s-1/2}}{(1+|\xi|^2)^s}$$

and let $u = \tilde{\varphi}(x)$, $\hat{u} = \varphi$. We claim $u \in H^s(R^n)$, $u(0,x') = cg(x')$, $c = const > 0$ and

$$(1+|\xi|^2)^s|\varphi(\xi)|^2 = |\hat{g}(\xi')|^2(1+|\xi'|^2)^{s-1/2}\frac{(1+|\xi'|^2)^{s-1/2}}{(1+|\xi|^2)^s} \tag{6.18}$$

by (6.15) $\int(1+|\xi|^2)^{-s}d\xi_1 \le c(1+|\xi'|^2)^{-s+1/2}$, so left side of (6.16) has finite integral, giving $u \in H^s(R^n)$. Meanwhile

$$\int \varphi(\xi)d\xi_1 = \hat{g}(\xi')(1+|\xi'|^2)^{s-1/2}\int(1+|\xi|^2)^{-s}d\xi_1 = c\hat{g}(\xi_1), \quad c \ne 0,$$

so $u(0,x') = cg(x')$.

If the operator $(1-\Delta)^{s/2}$ is defined by $p(D) = \mathfrak{F}^{-1}p(\xi)\mathfrak{F}$ with $p(\xi) = (1+|\xi|^2)^{s/2}$, then

$$H^s(R^n) = (1-\Delta)^{-s/2}L^2(R^n). \tag{6.19}$$

For $s > 0$, $H^s(R^n) = \mathcal{D}((1-\Delta)^{s/2})$, the domain of self-adjoint operator $(1-\Delta)^{s/2}$ and also $H^{-s}(R^n)$ is the dual to $H^s(R^n)$.

We can replace L^2 by L^p and study such spaces. We consider **Sobolev spaces** L^p style $W_p^k(R^n) = \{u : u \in L^p(R^n), D^\alpha u \in L^p(R^n), |\alpha| \le k\}$. In analogy to (6.19) of $H^s(R^n)$ define $\mathcal{S}_p^s(R^n)$ by $\mathcal{S}_p^s(R^n) = (1-\Delta)^{-s/2}L^p(R^n)$, $\;s \in R$.

Theorem 2.10. Let $p(\xi)$ be a smooth function on R^n such that

$$|\xi|^{|\alpha|}|p^\alpha(\xi)| \le C_\alpha, \quad |\alpha| \le [\frac{n}{2}] + 1.$$

Then, for $1 < p < \infty$,

$$p(D) : L^p(R^n) \to L^p(R^n),$$

with operator norm bounded by $C(p)\sum_\alpha C_\alpha$.

Where $C(p) = Cp$ for $p \ge 2$, $C(p) = C/p - 1$ for $1 < p \le 2$. We establish

$$W_p^k(R^n) = \mathcal{S}_p^k(R^n), \quad 1 < p < \infty.$$

In fact, $|\alpha| \le k$, $D^\alpha(1-\Delta)^{-k/2} : L^p \to L^p$ by Theorem 2.5 and $\mathcal{S}_p^k \subset W_p^k$. Conversely $u \in W_p^k$ Theorem 2.5 implies for $|\alpha| \le k$, $D^\alpha(1-\Delta)^{-k/2}D^\alpha u \in L^p$, so $(1-\Delta)^{-k/2}(1-\Delta)^k u \in L^p$, or $(1-\Delta)^{k/2}u \in L^p$ and $u \in \mathcal{S}_p^k$.

We claim

$$[L^p(R^n), \mathcal{S}_p^s(R^n)]_\theta = \mathcal{S}_p^{s\theta}(R^n), \quad 1 < p < \infty.$$

The Theorem 2.5 implies that

$$||(1-\Delta)^{-iy}||_{\mathcal{S}(L^p)} \le C_p(1+|y|)^n. \tag{6.20}$$

If $v \in \mathcal{S}_p^{s,\theta}$, let

$$u(z) = e^{z^2}(1-\Delta)^{(-z+\theta)s/2}v.$$

Then $u(\theta) = e^{\theta^2}v$, $u(iy) = e^{-y^2}(1-\Delta)^{-iys/2}((1-\Delta)^{s\theta/2}v)$ is bounded in $L^p(R^n)$ and $u(1+iy) = e^{(1+iy)^2}(1-\Delta)^{-s/2}(1-\Delta)^{-is/2y}((1-\Delta)^{s\theta/2}v)$ is bounded in $\mathcal{S}_p^s(R^n)$, so $u \in \mathcal{H}_{L^p(R^n),\mathcal{S}_p^s(R^n)}(\Omega)$ which implies $\mathcal{S}_p^{s\theta} \subset [L^p(R^n), \mathcal{S}_p^s(R^n)]_\theta$. Generalizing (6.18), we establish for $1 \le p \le \infty$,

$$[\mathcal{S}_p^\sigma(R^n), \mathcal{S}_p^s(R^n)]_\theta = \mathcal{S}_p^{\sigma(1-\theta)+s\theta}(R^n).$$

Generalizing Rellich's theorem for $s \ge 0$, $1 < p < \infty$,

$$I : \mathcal{S}_p^{s+\sigma}(\Omega) \to \mathcal{S}_p^s(\Omega) \;\; compact, \;\sigma > 0.$$

By using the Marcinkiewicz multiplier theorem for the torus, we have $p(D) : \mathcal{D}'(T^n) \to \mathcal{D}'(T^n)$ and $p(D) : L^p(T^n) \to L^p(T^n)$, $1 < p < \infty$. For multipliers on the torus that Marcinkiewicz proved his theorem. We take $\varphi \in C_o^\infty(R^n)$, $\varphi(\xi) = 1$ for $|\xi| \le 1$, let $p_\varepsilon = (1+|\xi|^2)^{-\sigma/2}\varphi(\varepsilon\xi)$. Then each $p_\varepsilon(D)$ has finite rank, so is compact. Marcinkiewicz multiplier theorem implies

$$||(1-\Delta)^{-\sigma/2} - p_\varepsilon(D)||_{\mathcal{S}(L^p)} \to 0 \;\; as \;\; \varepsilon \to 0, \;\;\; 1 < p < \infty,$$

so $(1-\Delta)^{-\sigma/2}$ is a norm limit of compact operators.

The Sobolev imbedding theorem, Theorem 6.2 has the following generalization:

$$\mathcal{S}_p^s(R^n) \subset C(R^n) \cap L^\infty(R^n) \;\; if \; s > \frac{n}{p}.$$

To prove this it suffices to show that $(1-\Delta)^{-s/2}\delta \in L^{p'}(R^n)$ if $s > (n/p)$, since $u = (1-\Delta)^{-s/2}\delta(1-\Delta)^{s/2}u$, and

$$\psi(x) = (1-\Delta)^{-s/2}\delta = (2\pi)^{/n}\int e^{ix\xi}(1+|\xi|^2)^{-s/2}d\xi.$$

We can show $\psi(x) = (1-\Delta)^{-s/2}\delta$ is smooth for $x \ne 0$, rapidly decreasing as $|x| \to \infty$, and satisfies the estimate

$$|\psi(x)| \le C|x|^{-n+s}, \;\;\; |x| \le 1, \;\;\; s < n.$$

Bergh and Lofstrom take the generalization result is

$$\mathcal{S}_p^s(R^n) \subset L^q(R^n), \;\;\; q = \frac{np}{n-ps}, \;\;\; 0 \le s < n/p.$$

The trace theorems for restrictions to $\partial\Omega$ of $u \in \mathcal{S}_p^s(\Omega)$ are more subtle for $p \ne 2$ than for $p = 2$. τu loses $1/p$ derivatives, generally, but doesn't belong to $\mathcal{S}^{s-1/p}(\partial\Omega)$, necessarily, but rather to a Besov space:

$$\tau : \mathcal{S}_p^s(\Omega) \to B_p^{s-1/p}(\partial\Omega), \;\;\; s > \frac{1}{p}.$$

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