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On preservation of ageing under minimum for dependent random lifetimes

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Abstract

In this note we consider a vector (X_1, X_2) of lifetimes whose dependence is described by means of an Archimedean copula, and we provide conditions useful to describe the relationships between ageing properties of X_1 and X_2 and ageing properties of their minimum. Examples are also provided.

Keywords: Ageing notions, IFR, NBU, Minimum, Archimedean copulas, Frailty models.

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1 Introduction

Nonparametric classes of distributions are commonly considered in reliability theory to describe positive or negative ageing of items or components having random lifetimes. Examples of these classes, also referred as *ageing notions* or *ageing properties*, are the IFR (*Increasing in Failure Rate*) and NBU (*New Better than Used*), which describe positive ageing, and the DFR (*Decreasing in Failure Rate*) and NWU (*New Worst than Used*), which describe negative ageing (definition of these notions is provided next). Preservation properties of ageing notions under operations of interest in reliability like addition, mixture or construction of coherent systems, have been extensively studied; see, e.g., Barlow and Proschan (1981) for a monograph on this topic, and related applications.

It is a well-known fact that, dealing with independent lifetimes, some of these ageing notions are closed with respect to the reliability operations listed above, and to minimum in particular, i.e., it is known that if the lifetimes of two (or more) components satisfy one specific ageing property, then the same holds, for example, for their minimum. However, it is also a known fact that this assertion is no more true when dependence exists between the lifetimes of the components, as shown, e.g., in Example 2.1 described later.

In many cases of applied interest, the idealized conditions of independence between components' lifetimes must be abandoned, being not realistic at all. For this reason, the analysis of relationships between dependence and ageing have become issues of central interest in the fields of reliability theory (see, e.g., Bassan and Spizzichino, 2005, Navarro and Spizzichino, 2010, Navarro et al., 2006, or Navarro et al., 2013, and references therein). In particular, conditions such that series and parallel systems of dependent components are comparable according to various stochastic orders have been studied in recent literature. Analysis in this direction have been performed for example in Joo and Mi (2010), where the case of dependence between components described by Gumbel bivariate distributions is studied in details, in Yilmaz (2011), where comparisons in failure rate order of lifetimes' systems are considered, or in Ucer and Gurler (2012), where the mean residual life function of a parallel system of non independent components is studied. See also Navarro and Shaked (2010) and references therein, where interesting conditions for IFR and DFR property of series and parallel systems are provided. The aim of this note is to provide a contribution on this topic by means of an analysis on the conditions on the structure of dependence such that some ageing properties are preserved under minimum, i.e., such that ageing properties satisfied by the components of a series system, having dependent lifetimes, are preserved by the lifetime of the system. In particular, we will analyze here the case that the dependence between the random lifetimes is described by an Archimedean copula.

The paper is structured as follows. In Section 2 we provide the definitions of the aging notions considered through the paper and the notion of copula, and of Archimedean copulas in particular. Section 3 is devoted to the analysis of conditions on the survival copula of the vector (X_1, X_2) such that the aging properties IFR and DFR are preserved by $T = \min\{X_1, X_2\}$ (and viceversa), while Section 4 is devoted to the same analysis for the NBU and NWU aging notions. Throughout this note the terms increasing and decreasing should be read in non-strict sense.

2 Utility notions

The definitions of the ageing notions and of the dependence structures considered through the paper are provided in this section, together with some useful properties.

Let X be a random variable, and for each real $t \in \{t : P\{X > t\} > 0\}$ let $X_t = [X - t \big| X > t]$ denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. When X is a lifetime of a device then X_t can be interpreted as the residual lifetime of the device at time t, given that the device is alive at time t. Most of the more useful characterizations of aging are based on stochastic comparisons between the residual lifetimes X_0, X_t and X_{t+s}, with $t, t+s \in \{t : P\{X > t\} > 0\}$. Among others, the following well-known aging notions can be defined by comparisons among residual lifetimes. For their definitions, recall that two random variables X and Y are said to be ordered in the *usual stochastic order*, denoted $X \leq_{st} Y$, if $P(X > t) \leq P(Y > t)$ for all $t \in \mathbb{R}$, or, equivalently, if $E[\phi(X)] \leq E[\phi(Y)]$ for any increasing function ϕ such that the expectations exist.

Definition 2.1. *given a non–negative random lifetime X defined on $[0, +\infty)$ we say that*

$$X \in \textit{NBU [NWU]} \iff X_t \leq_{st} [\geq_{st}]\ X \ \textit{whenever}\ t \geq 0,$$

and that

$$X \in \textit{IFR [DFR]} \iff X_{t+s} \leq_{st} [\geq_{st}]\ X_t \ \textit{whenever}\ t, s \geq 0.$$

Let the lifetime X have cumulative distribution F_X and survival function $\overline{F}_X = 1 - F_X$ (i.e., $\overline{F}_X(t) = P(X > t), t \in \mathbb{R}^+$). Then the aging notions above can be restated in terms of the survival function $\overline{F}_X$, as stated in the following preliminary result.

Lemma 2.1. *Let X be a non-negative random lifetime having survival function $\overline{F}_X$. Then*

(a) *$X \in$ NBU [NWU] $\iff \overline{F}_X(t+s) \leq\ [\geq]\ \overline{F}_X(t)\overline{F}_X(s) \ \ \forall t, s \geq 0$;*

(b) *$X \in$ IFR [DFR] $\iff \overline{F}_{X_t}(s) = \frac{\overline{F}_X(t+s)}{\overline{F}_X(t)}$ is decreasing [increasing] in t for all $s \geq 0$.*

Moreover, whenever X has absolutely continuous distribution, thus admits a density f_X and a failure rate r_X, then

(c) *$X \in$ IFR [DFR] $\iff r_X(t) = \frac{f_X(t)}{\overline{F}_X(t)}$ is increasing [decreasing] in $t \geq 0$.*

The following statement is well-known and easy to prove (see, e.g., Barlow and Proschan, 1981).

Proposition 2.1. *Let X_1 and X_2 be two independent lifetimes, and let $T = \min\{X_1, X_2\}$ denote the lifetime of the corresponding series system. Then the IFR [DFR, NBU, NWU] property of X_1 and X_2 is preserved by T. Also, the IFR [DFR, NBU, NWU] property of T is preserved by X_1 and X_2 whenever they are identically distributed.*

Let us consider now a pair $\mathbf{X} = (X_1, X_2)$ of non-negative random variables. Let

$$\overline{F}(x, y) = P(X_1 > x, X_2 > y), \quad x, y \in \mathbb{R}^+,$$

be the corresponding joint survival function, and let

$$\overline{F}_1(x) = \overline{F}(x, 0) = P(X_1 > x) \text{ and } \overline{F}_2(x) = \overline{F}(0, x) = P(X_2 > x)$$

be the marginal univariate survival functions of X_1 and X_2, respectively. Assume that $\overline{F}$ is a continuous survival function which is strictly decreasing on each argument on $(0, \infty)$, and that $\overline{F}_1(0) = \overline{F}_2(0) = 1$.

As pointed out in recent literature (see, e.g., Nelsen, 2006), the dependence structure of a bivariate vector $\mathbf{X}$ can be usefully described by its survival copula K, defined as

$$K(u, v) = \overline{F}(\overline{F}_1^{-1}(u), \overline{F}_2^{-1}(v)),$$

where $(u, v) \in [0, 1] \times [0, 1]$. This function (which is unique under the assumption of continuity of the marginal survivals $\overline{F}_i$), together with the marginal survival functions $\overline{F}_1$ and $\overline{F}_2$, allows for a different representation of $\overline{F}$ in terms of the triplet $(\overline{F}_1, \overline{F}_2, K)$ useful to analyze dependence properties between X_1 and X_2, since the survival copula entirely describes it. For example, it is a well-known fact that concordance indexes like Kendall's τ or Spearman's ρ can be calculated by means of the survival copula. In particular, K describes positive [negative] dependence between the components X_1 and X_2 whenever $K(u, v) \geq [\leq]\ uv$ for all $(u, v) \in [0, 1]^2$; in this case we say that the vector (X_1, X_2) satisfies the *Positive Quadrant Dependence* [*Negative Quadrant Dependence*] property (shortly PQD [NQD]). See Nelsen (2006) for details.

Survival copulas, instead of ordinary copulas, are in particular considered in reliability and actuarial sciences, where survival functions instead of cumulative distributions are commonly studied. Among survival copulas, particularly interesting is the class of Archimedean survival copulas: a survival copula is said to be *Archimedean* if it can be written as

$$K(u, v) = \overline{W}(\overline{W}^{-1}(u) + \overline{W}^{-1}(v)) \ \forall u, v \in [0, 1] \tag{2.1}$$

for a suitable one-dimensional, continuous, strictly positive and strictly decreasing and convex survival function $\overline{W} : \mathbb{R}^+ \to [0, 1]$ such that $\overline{W}(0) = 1$. The function $\overline{W}$ is usually called the *generator* of the Archimedean survival copula K. As pointed out in Nelsen (2006), many standard survival copulas (such as the ones in Gumbel, Frank, Clayton and Ali-Mikhail-Haq families) are special cases of this class. Vectors of lifetimes having Archimedean survival copulas are of great interest in reliability, but also in many other applied contexts, being of this kind the dependence structure of frailty models (see Oakes, 1989). We refer the reader to Müller and Scarsini (2005), or Bassan and Spizzichino (2005), and references therein, for details, properties and applications of Archimedean survival copulas.

It is important to observe that when the vector $\mathbf{X} = (X_1, X_2)$ has an Archimedean survival copula, then its joint survival function $\overline{F}$ can be written as

$$\overline{F}(x, y) = \overline{W}(R_1(x) + R_2(y)), \quad x, y \in \mathbb{R}^+, \tag{2.2}$$

for two suitable continuous and strictly increasing functions $R_1, R_2 : [0,+\infty) \to [0,+\infty)$ such that $R_1(0) = R_2(0) = 0$ and $\lim_{x\to\infty} R_1(x) = \lim_{y\to\infty} R_2(y) = \infty$, where $\overline{W}$ is the survival function appearing in (2.1) (see Bassan and Spizzichino, 2005, for details). For example, in the frailty approach it is assumed that X_1 and X_2 are independent conditionally on some random environmental factor Θ, having conditional survival marginals $\overline{F}_{i,\theta}(t) = P[X_i > t|\Theta = \theta] = \overline{H}_i^{\theta}(t),\ i = 1, 2$, for some survival functions $\overline{H}_1$ and $\overline{H}_2$. Thus, for this model,

$$\begin{aligned}\overline{F}(t,s) &= E[\overline{H}_1^{\Theta}(t)\overline{H}_2^{\Theta}(s)] = E[\exp(\Theta(\ln \overline{H}_1(t))) \exp(\Theta(\ln \overline{H}_2(s)))] \\ &= \overline{W}(-\ln \overline{H}_1(t) - \ln \overline{H}_2(s)) = \overline{W}(R_1(t) + R_2(s)),\quad t, s \geq 0,\end{aligned}$$

where $\overline{W}(x) = E[\exp(-x\Theta)]$, and $R_i(t) = -\ln \overline{H}_i(t),\ i = 1, 2$. In this context, the survival copula is of Clayton type when the random parameter Θ has distribution in the Gamma family.

Note that when $\overline{F}$ is defined as in (2.2) then $\overline{F}_1(x) = \overline{F}(x, 0) = \overline{W}(R_1(x))$, $\overline{F}_2(y) = \overline{F}(0, y) = \overline{W}(R_2(y))$ and $\overline{W}^{-1}(x) = R_1(\overline{F}_1^{-1}(x)) = R_2(\overline{F}_2^{-1}(x))$.

Consider now a random vector $\mathbf{X} = (X_1, X_2)$ describing the lifetimes of two components, and denote with $T = \min\{X_1, X_2\}$ the lifetime of the series system composed by the two components. As shown in the following example, the preservation properties described in Proposition 2.1 are not necessarily satisfied whenever the lifetimes X_1 and X_2 are dependent.

Example 2.1. *Let X_1 and X_2 be two random variables having dependence described by a Frank survival copula, i.e., let their joint survival function given by*

$$\overline{F}(x, y) = \overline{F}_1(x) + \overline{F}_2(y) - 1 + \frac{1}{\ln \gamma} \ln\left(1 + \frac{(\gamma^{1-\overline{F}_1(x)} - 1)(\gamma^{1-\overline{F}_2(x)} - 1)}{\gamma - 1}\right),$$

where $\overline{F}_i$ denotes the survival function of X_i, $i = 1, 2$, and $\gamma \in (0, 1) \cup (1,\infty)$ (see Nelsen, 2006, for details on this family of copulas). Assume X_1 and X_2 to have exponential distributions with equal means 1. Let $\overline{F}_T(t) = \overline{F}(t, t)$ denote the survival function of the corresponding series system, i.e., of $T = min\{X_1, X_2\}$, and let $\overline{F}_{T_s}(t) = \frac{\overline{F}_T(s+t)}{\overline{F}_T(s)}$ denote the survival function of the residual lifetime $T_s = [T - s \mid T > s]$ at time $s \geq 0$. The survival functions $\overline{F}_T$ and $\overline{F}_{T_s}$ do intersect being, for example for $s = 1$ and $\gamma = 2$,

$$\overline{F}_T(0.5) = 0.348 > \overline{F}_{T_s}(0.5) = 0.341 \quad \textit{and} \quad \overline{F}_T(1) = 0.117 < \overline{F}_{T_s}(1) = 0.119.$$

Thus T can not be NBU or NWU (thus neither IFR or DFR), even if the margins are exponentially distributed (i.e., even if they satisfy all these ageing properties).

Here will study some relationships between the aging notions satisfied by lifetimes X_1, X_2 and T assuming that the dependence between X_1 and X_2 is described by an Archimedean survival copula. We will assume from now on that the lifetimes X_i have the same marginal distribution, thus that $R_1(t) = R_2(t) = R(t)$ for all $t \geq 0$, and that the joint survival distribution of (X_1, X_2) can be written as

$$\overline{F}(x, y) = \overline{W}(R(x) + R(y)),\quad x, y \geq 0. \tag{2.3}$$

Note that in this case the survival function of T can be written as $\overline{F}_T(t) = \overline{W}(2R(t)),\ t \geq 0$. Also, we will assume differentiability of the generator $\overline{W}$ of the Archimedean copula, as it always happens for example for frailty models.

3 IFR and DFR ageing notions

First, we consider the IFR and DFR cases. For simplicity of notations, let us denote with h the failure rate corresponding to the survival function $\overline{W}$, i.e., let $h(t) = w(t)/\overline{W}(t),\ t \geq 0$, where w denotes the derivative of $-\overline{W}$, i.e., $w(t) = -\overline{W}'(t)$.

Proposition 3.1. *Let (X_1, X_2) have joint survival function as in (2.3). Then the IFR [DFR] property of X_1 and X_2 is preserved by $T = \min\{X_1, X_2\}$ if the generator of the copula satisfies*

$$\frac{2h'(2s)}{h(2s)} \geq\ [\leq]\ \frac{h'(s)}{h(s)} \tag{3.1}$$

for all $s \geq 0$.

Proof. By (2.3) and Lemma 2.1(c), the IFR [DFR] property of the components $X_i, i = 1, 2$, and of the minimum T are equivalent to

$$\frac{w(R(t))R'(t)}{\overline{W}(R(t))} \text{ is increasing [decreasing] in } t > 0, \tag{3.2}$$

and

$$\frac{2w(2R(t))R'(t)}{\overline{W}(2R(t))} \text{ is increasing [decreasing] in } t > 0, \tag{3.3}$$

respectively.

Let $s = R(t)$, and observe that, being R an increasing function from 0 to ∞, the assertion holds if, and only if,

$$\frac{w(s)R'(R^{-1}(s))}{\overline{W}(s)} \text{ increasing in } s > 0 \quad \Rightarrow \quad \frac{w(2s)R'(R^{-1}(s))}{\overline{W}(2s)} \text{ increasing in } s > 0 \tag{3.4}$$

(and similarly for the decreasing case).

Let $g(s) = R'(R^{-1}(s))$. Thus (3.4) is equivalent to

$$h(s)g(s) \text{ increasing [decreasing] in } s > 0 \quad \Rightarrow \quad h(2s)g(s) \text{ increasing [decreasing] in } s > 0,$$

or,

$$h'(s)g(s) + h(s)g'(s) \geq\ [\leq]\ 0 \ \Rightarrow\ 2h'(2s)g(s) + h(2s)g'(s) \geq\ [\leq]\ 0,$$

i.e.

$$\frac{h'(s)}{h(s)} \geq\ [\leq]\ \frac{-g'(s)}{g(s)} \ \Rightarrow\ \frac{2h'(2s)}{h(2s)} \geq\ [\leq]\ \frac{-g'(s)}{g(s)}$$

for all $s \geq 0$. The latter is clearly satisfied under assumption (3.1), thus the assertion follows. $\square$

Using the same arguments as in the previous proof (but reverting the inequalities and the monotonicity properties) one can prove that the positive [negative] ageing of the minimum implies the positive [negative] ageing of the components.

Proposition 3.2. *Let (X_1, X_2) have joint survival function as in (2.3). Then the DFR [IFR] property of $T = \min\{X_1, X_2\}$ implies that also X_1 and X_2 are DFR [IFR] if the generator of the copula satisfies inequality (3.1).*

Examples of Archimedean copulas that satisfy the assumptions of the statements above are given here. Some counterexamples, showing that violations of the assumptions may undermine the stated preservations properties, are also provided.

Example 3.1. *Let (X_1, X_2) have a Clayton survival copula, whose generator is $\overline{W}(t) = (\theta t + 1)^{-\frac{1}{\theta}}, t \in \mathbb{R}^+$, with $\theta > 0$. It is easy to verify that*

$$h(s) = \frac{1}{\theta s + 1},$$

so that

$$\frac{h'(s)}{h(s)} = \frac{-2\theta}{2\theta s + 2} \quad \text{and} \quad \frac{2h'(2s)}{h(2s)} = \frac{-2\theta}{2\theta s + 1}.$$

Thus, we have

$$\frac{2h'(2s)}{h(2s)} \leq \frac{h'(s)}{h(s)}$$

and assumption (3.1) with inequality $\leq$ is clearly satisfied. Thus from Propositions 3.1 and 3.2 we can assert that the minimum among X_1 and X_2 is DFR whenever they are DFR, or that they are IFR whenever the minimum is IFR. It should be pointed out that this property of the Clayton copula has been already noticed (but using different arguments) in Yilmaz (2011), Subsection 4.1.

Example 3.2. *A second case for which assumption (3.1) is satisfied, but with inequality in the opposite direction, is when (X_1, X_2) has a Gumbel-Barnett survival copula, whose generator is $\overline{W}(t) = \exp\left[\frac{1-\exp(t)}{\theta}\right], t \in \mathbb{R}^+$, with $\theta \in (0, 1] \subseteq \mathbb{R}$. It is easy to verify that now*

$$h(s) = \frac{1}{\theta}\exp(s),$$

so that

$$\frac{2h'(2s)}{h(2s)} = 2 > 1 = \frac{h'(s)}{h(s)}$$

that is, assumption (3.1) is satisfied, with inequality $\geq$ for any $s \geq 0$. Thus in this case from Propositions 3.1 and 3.2 we can assert that the minimum among X_1 and X_2 is IFR whenever they are IFR, or that they are DFR whenever the minimum is DFR.

Example 3.3. *Another case that satisfies assumption (3.1), with equality, is the Gumbel-Hougaard survival copula, whose generator is $\overline{W}(t) = \exp(-t^{\frac{1}{\theta}}), t \in \mathbb{R}^+$, with $\theta \geq 1$. In fact, in this case it is easy to verify that*

$$h(s) = \frac{1}{\theta}s^{\frac{1}{\theta}-1},$$

so that

$$\frac{h'(s)}{h(s)} = \left(\frac{1}{\theta} - 1\right)\frac{1}{s} \quad \text{and} \quad \frac{2h'(2s)}{h(2s)} = \left(\frac{1}{\theta} - 1\right)\frac{1}{s}.$$

Thus, in this particular case one can assert that the minimum among X_1 and X_2 is IFR [DFR] whenever they are IFR [DFR], or that they are IFR [DFR] whenever the minimum is IFR [DFR].

It is interesting to observe that the Archimedean copula considered in Example 3.1 describes positive dependence, i.e., it is PQD, while the copula considered in Example 3.2 describes a case of negative dependence between the lifetimes X_1 and X_2, i.e., it is NQD. Thus the impression one can have is that under positive dependence the negative ageing notion DFR of margins is preserved by the minimum, while for negative dependence is the positive ageing IFR that is preserved by the minimum. Unfortunately, in general this assertion is not always verified, as shown in Example 2.1, where the vector is PQD for $\gamma \in (1, \infty)$ and NQD for $\gamma \in (0, 1)$, but the preservation property is not satisfied in both cases. This is shown also in the following example, where the copula is PQD for $\theta \in (0, 1)$ and NQD for $\theta \in [-1, 0)$.

Example 3.4. *Let (X_1, X_2) have an Ali-Mikhail-Haq survival copula, whose generator is $\overline{W}(t) = \frac{1-\theta}{\exp(t)-\theta}, t \in \mathbb{R}^+$, with $\theta \in [-1, 1)$. It is easy to verify that here*

$$h(s) = \frac{\exp(s)}{\theta - \exp(s)},$$

so that

$$\frac{h'(s)}{h(s)} = \frac{\theta}{\theta - \exp(s)} \quad \text{and} \quad \frac{2h'(2s)}{h(2s)} = \frac{2\theta}{\theta - \exp(2s)}.$$

The inequalities in (3.1) do not hold (for $\leq$ or $\geq$) for any $s \geq 0$ and for any fixed $\theta \in [-1, 1)$, since the two functions above cross at $s = \ln(1 + \sqrt{1-\theta})$. Now, assume that (X_1, X_2) has margins exponentially distributed with failure rate $\lambda = 1$. In this case we have

$$\exp(-t) = \overline{F}_1(t) = \overline{F}_2(t) = \overline{W}(R(t)) = \frac{\theta - 1}{\theta - \exp(R(t))},$$

so that

$$R(t) = \ln[\theta + e^t(1-\theta)].$$

Recall that $T = \min\{X_1, X_2\}$ is IFR [DFR] if, and only if, $\overline{F}_{T_s}(t)$ is decreasing [increasing] in s for all t, and NBU [NWU] if, and only if, $\overline{F}_{T_0}(t) \geq [\leq]\ \overline{F}_{T_s}(t)$ for all $t, s \geq 0$. But, for $\theta = -0.5$,

$$\overline{F}_{T_0}(0.1) = 0.815, \quad \overline{F}_{T_{0.1}}(0.1) = 0.809, \quad \text{and} \quad \overline{F}_{T_3}(0.1) = 0.816,$$

so that T is not IFR or NBU (and neither DFR or NWU).

It is also interesting to observe that inequality (3.1) is not a necessary condition for the preservation of aging properties by the minimum (or viceversa). This is shown in the following example.

Example 3.5. *Let the vector* (X_1, X_2) *have survival function defined as in (2.3) where*

$$\overline{W}(t) = \exp(-(t^2 + 2t)) \quad \text{and} \quad R(t) = \frac{1}{2}\ln(t+1),$$

for $t \geq 0$*. Straightforward calculations show that in this case* $h(s) = 2s + 2$ *and* $g(s) = R'(R^{-1}(s)) = \frac{1}{2}\exp(-2s),\ s \geq 0$*, so that both the inequalities*

$$\frac{h'(s)}{h(s)} = \frac{2}{2s+2} \leq \frac{-g'(s)}{g(s)} = 2 \quad \text{and} \quad \frac{2h'(2s)}{h(2s)} = \frac{4}{4s+2} \leq \frac{-g'(s)}{g(s)} = 2$$

are satisfied for all $s \geq 0$*, i.e., both the margins* X_1, X_2 *and the minimum* T *satisfy the DFR property. However, it does not hold the condition* $h'(s)/h(s) \geq 2h'(2s)/h(2s)$ *stated in Proposition 3.1 for the preservation of DFR under minimum.*

The last example of this section shows that positive (or negative) dependence is not a necessary condition for inequality (3.1).

Example 3.6. *Let* (X_1, X_2) *have as survival copula the Archimedean copula numbered as 2 in Nelsen (2006), page 116, i.e., with generator function* $\overline{W}(t) = 1 - t^{\frac{1}{\theta}}$, $\theta \in [1, \infty)$, $t \in [0, 1]$*. It is not difficult to verify that in this case*

$$h(s) = \frac{s^{\frac{1}{\theta}-1}}{\theta(1 - s^{\frac{1}{\theta}})},$$

so that

$$\frac{h'(s)}{h(s)} = \frac{(1-\theta)(1 - s^{\frac{1}{\theta}}) + s^{\frac{1}{\theta}}}{\theta s(1 - s^{\frac{1}{\theta}})},$$

$$\frac{2h'(2s)}{h(2s)} = \frac{(1-\theta)(1 - (2s)^{\frac{1}{\theta}}) + (2s)^{\frac{1}{\theta}}}{\theta s(1 - (2s)^{\frac{1}{\theta}})}$$

and that $\frac{2h'(2s)}{h(2s)} > \frac{h'(s)}{h(s)}$ *is satisfied for any* $s \in (0, 0.5)$*. Thus Propositions 3.1 and 3.2 can be applied, and one can assert that the minimum between* X_1 *and* X_2 *is IFR whenever they are IFR (and viceversa for DFR). However, this vector is neither PQD or NQD, being* $K(u, v) < uv$ *for* $u = v = 0,2$ *and* $k(u, v) > uv$ *for* $u = v = 0.5$*.*

4 NBU and NWU ageing notions

In this section we describe specific cases where the weaker NBU and NWU aging notions (with respect to IFR and DFR, respectively) pass on from components to the series system (or viceversa). The first case deals with a copula describing positive dependence.

Proposition 4.1. *Let the vector* (X_1, X_2) *have joint survival function as in (2.3), and let it have a Clayton survival copula. Then the NWU property of* X_1 *and* X_2 *is preserved by* $T = \min\{X_1, X_2\}$*.*

Proof. Recall that the generator of the Clayton copula is the function $\overline{W}(t) = (\theta t + 1)^{-\frac{1}{\theta}}$, where $\theta \geq 0$. Thus, X_1 and X_2 satisfy the NWU property if

$$(1 + \theta R(t+s))^{-\frac{1}{\theta}} \geq (1 + \theta R(s))^{-\frac{1}{\theta}} \cdot (1 + \theta R(t))^{-\frac{1}{\theta}}$$

for all $s, t \geq 0$, i.e., if and only if

$$R(t) + R(s) + \theta R(t)R(s) \geq R(t+s) \text{ for all } s, t \geq 0. \tag{4.1}$$

Since $\overline{F}_T(t) = \overline{W}(2R(t))$, the NWU property of T is equivalent to

$$(1 + 2\theta R(t+s))^{-\frac{1}{\theta}} \geq (1 + 2\theta R(s))^{-\frac{1}{\theta}} \cdot (1 + 2\theta R(t))^{-\frac{1}{\theta}}$$

for all $s, t \geq 0$, i.e.,

$$R(t) + R(s) + 2\theta R(t)R(s) \geq R(t+s) \text{ for all } s, t \geq 0. \tag{4.2}$$

Obviously, (4.2) is satisfied whenever (4.1) holds, thus the assertion follows. □

Using the same arguments as in the previous proof, but reverting the inequalities, one can prove that the positive ageing of the minimum implies the positive ageing of the components.

Proposition 4.2. *Let the vector (X_1, X_2) have joint survival function as in (2.3), and let it have a Clayton survival copula. Then the NBU property of $T = \min\{X_1, X_2\}$ implies that also X_1 and X_2 satisfy the NBU property.*

The next statement refers to the Archimedean copula numbered as 12 in Nelsen (2006), page 116. This is again a case of positive dependence between the components of the vector (i.e., the copula is PQD).

Proposition 4.3. *Let the vector (X_1, X_2) have joint survival function as in (2.3), and let it have a survival copula having generator $\overline{W}(t) = (1 + t^{\frac{1}{\theta}})^{-1}$, with $\theta \geq 1$. Then $T = \min\{X_1, X_2\}$ is NWU if X_1 and X_2 satisfy the NWU property.*

Proof. In this case the joint survival function is given by

$$\overline{F}(t, s) = (1 + (R(t) + R(s))^{\frac{1}{\theta}})^{-1}$$

so that the marginal survival functions are

$$\overline{F}_i(t) = (1 + (R(t))^{\frac{1}{\theta}})^{-1}, \ i = 1, 2,$$

while the survival function of the minimum is given by

$$\overline{F}_T(t) = (1 + (2R(t))^{\frac{1}{\theta}})^{-1}.$$

Thus, the margins are NWU if, and only if,

$$(1 + (R(t+s))^{\frac{1}{\theta}})^{-1} \geq (1 + (R(t))^{\frac{1}{\theta}})^{-1}(1 + (R(s))^{\frac{1}{\theta}})^{-1}$$

i.e., if and only if

$$(R(t+s))^{\frac{1}{\theta}} \leq (R(t))^{\frac{1}{\theta}} + (R(s))^{\frac{1}{\theta}} + (R(t)R(s))^{\frac{1}{\theta}} \tag{4.3}$$

for all $t, s \geq 0$. On the other hand, the minimum T is NWU if, and only if,

$$(1 + (2R(t+s))^{\frac{1}{\theta}})^{-1} \geq (1 + (2R(t))^{\frac{1}{\theta}})^{-1}(1 + (2R(s))^{\frac{1}{\theta}})^{-1}$$

i.e., if and only if

$$(R(t+s))^{\frac{1}{\theta}} \leq (R(t))^{\frac{1}{\theta}} + (R(s))^{\frac{1}{\theta}} + (2R(t)R(s))^{\frac{1}{\theta}}.$$

The latter is clearly satisfied under validity of (4.3), thus the assertion. □

As for other cases, by using the same argument and reverting the inequalities, one can verify the following statement.

Proposition 4.4. *Let the vector (X_1, X_2) have joint survival function as in (2.3), and let it have a survival copula having generator $\overline{W}(t) = (1 + t^{\frac{1}{\theta}})^{-1}$, with $\theta \geq 1$. Then X_1 and X_2 satisfy the NBU property if $T = \min\{X_1, X_2\}$ is NBU.*

The next statement deals with the Gumbel-Barnett Archimedean copula, which describes negative dependence.

Proposition 4.5. *Let the vector (X_1, X_2) have joint survival function as in (2.3), and let it have a Gumbel-Barnett survival copula. Then $T = \min\{X_1, X_2\}$ is NBU if, and only if, X_1 and X_2 satisfy the NBU property, and if*

$$R(t) + R(s) \geq R(t+s) \text{ for all } s, t \geq 0, \tag{4.4}$$

i.e if R is subadditive.

Proof. The joint survival function of this family is given by

$$\overline{F}(t,s) = \exp\{\frac{1 - \exp(R(t) + R(s))}{\theta}\}$$

while the survival functions of the $X_i, i = 1, 2$, and of their minimum T are, respectively,

$$\overline{F}_i(t) = \exp\{\frac{1 - \exp(R(t))}{\theta}\},$$

and

$$\overline{F}_T(t) = \exp\{\frac{1 - \exp(2R(t))}{\theta}\}.$$

Thus, the margins are NBU if, and only if,

$$\exp\{\frac{1 - \exp(R(t+s))}{\theta}\} \leq \exp\{\frac{1 - \exp(R(t))}{\theta}\} \exp\{\frac{1 - \exp(R(s))}{\theta}\}$$

i.e., if and only if

$$\exp\{R(t)\} + \exp\{R(s)\} \leq 1 + \exp\{R(t+s)\} \tag{4.5}$$

for all $t, s \geq 0$. On the other hand, the minimum T is NBU if, and only if,

$$\exp\{\frac{1-\exp(2R(t+s))}{\theta}\} \leq \exp\{\frac{1-\exp(2R(t))}{\theta}\}\exp\{\frac{1-\exp(2R(s))}{\theta}\}$$

i.e., if and only if

$$\exp\{2R(t)\} + \exp\{2R(s)\} \leq 1 + \exp\{2R(t+s)\}. \tag{4.6}$$

But inequality (4.4) implies

$$\exp\{R(t) + R(s)\} \geq \exp\{R(t+s)\} \text{ for all } s, t \geq 0,$$

or

$$-2\exp\{R(t) + R(s)\} \leq -2\exp\{R(t+s)\} \text{ for all } s, t \geq 0, \tag{4.7}$$

and by (4.5) we obtain

$$(\exp\{R(t)\} + \exp\{R(s)\})^2 \leq (1 + \exp\{R(t+s)\})^2,$$

i.e

$$\exp\{2R(t)\} + \exp\{2R(s)\} + 2\exp\{R(t) + R(s)\} \leq 1 + \exp\{2R(t+s)\} + 2\exp(R(t+s)). \tag{4.8}$$

Now, adding both sides of inequalities (4.7) and (4.8) one gets inequality (4.6). The thesis follows. □

For what it concerns the relationships between positive (or negative) dependence and preservation of ageing, the impression for the case of NBU and NWU notions is the same as for the IFR and DFR case, i.e., that under positive dependence the negative ageing notion NWU of margins is preserved by the minimum, while for negative dependence is the positive ageing NBU that is preserved by the minimum. But, again, in general this assertion is not always verified, as shown in Example 2.1.

In particular, there also exists one case where both positive and negative ageing are preserved by the minimum of dependent lifetimes. This is the case of the Gumbel-Hougaard survival copula, whose generator is $\overline{W}(t) = \exp(-t^{\frac{1}{\theta}}), \theta \geq 1$.

Proposition 4.6. *Let the vector (X_1, X_2) have joint survival function as in (2.3), and let it have a Gumbel-Hougaard survival copula. Then $T = \min\{X_1, X_2\}$ is NBU [NWU] if, and only if, X_1 and X_2 satisfy the NBU [NWU] property.*

Proof. In this case we have that the joint survival function of (X_1, X_2) is

$$\overline{F}(t, s) = \exp[-(R(t) + R(s))^{\frac{1}{\theta}}], \quad t, s \geq 0,$$

while the survival functions of the $X_i, i = 1, 2$, and of their minimum T are, respectively,

$$\overline{F}_i(t) = \exp[-(R(t))^{\frac{1}{\theta}}]$$

and

$$\overline{F}_T(t) = \exp[-(2R(t))^{\frac{1}{\theta}}].$$

By Lemma 2.1(c) the lifetimes X_i are NBU [NWU] if, and only if,

$$\exp[-(R(t+s))^{\frac{1}{\theta}}] \leq \ [\geq] \ \exp[-(R(t))^{\frac{1}{\theta}}] \cdot \exp[-(R(s))^{\frac{1}{\theta}}]$$

i.e., if and only if

$$(R(t+s))^{\frac{1}{\theta}} \geq \ [\leq] \ (R(t))^{\frac{1}{\theta}} + (R(s))^{\frac{1}{\theta}} \tag{4.9}$$

for all $t, s \geq 0$.

On the other hand, the minimum is NBU [NWU] if, and only if,

$$\exp[-(2R(t+s))^{\frac{1}{\theta}}] \leq \ [\geq] \ \exp[-(2R(t))^{\frac{1}{\theta}}] \exp[-(2R(s))^{\frac{1}{\theta}}]$$

i.e., if and only if (4.9) holds. The assertion follows. $\square$

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