On the second solution to a critical growth Robin problem.

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Abstract

We investigate the existence of the second mountain-pass solution to a Robin problem, where the equation is at critical growth and depends on a positive parameter λ. More precisely, we determine existence and nonexistence regions for this type of solutions, depending both on λ and on the parameter in the boundary conditions.

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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a smooth and bounded domain and let $2^* = \frac{2n}{n-2}$ be the critical Sobolev exponent. We consider the Robin problem

$$\begin{cases} -\Delta u = \lambda(1+u)^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u_\nu + cu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where c, $\lambda > 0$ and u_ν denotes the outer normal derivative of u on $\partial\Omega$.

As pointed out in the seminal paper [9], the interest in problems like (1) is due to their similarity to some geometrical and physical variational problems where a lack of compactness also occurs (recall that the embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$ is not compact).

A solution u_λ to (1) is called *minimal* if $u_\lambda \leq u$ a.e. in Ω, for any other solution u to (1). Furthermore, we say that a solution u is *regular* if $u \in L^\infty(\Omega)$. From [5] we know

Proposition 1. *For every $c > 0$, there exists $\lambda^* = \lambda^*(c) > 0$ such that:*

(i) for $0 < \lambda < \lambda^*$ problem (1) admits a minimal regular solution u_λ;

(ii) for $\lambda = \lambda^*$ problem (1) admits a unique regular solution u^*;

(iii) for $\lambda > \lambda^*$ problem (1) admits no solution.

Furthermore, the map $c \mapsto \lambda^*(c)$ is strictly increasing and $\lambda^*(c) \to 0$, as $c \to 0$.

When $c = 0$, (1) reduces to the Neumann problem (for which no positive solutions exist), whereas the limit case $c \to +\infty$ may be seen as the Dirichlet problem. Indeed, Proposition 1 includes well-known results for the Dirichlet problem, see [9, 13, 16, 19].

Under Dirichlet boundary conditions, due to [9], we know that the equation in (1) admits, besides the minimal solution u_λ, a larger *mountain-pass* solution U_λ (see Section 2 for the definition) for every

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$\lambda \in (0, \lambda^*_{Dir})$, where λ^*_{Dir} is the extremal parameter for the Dirichlet problem. One of the purposes of the present paper is to investigate, for any $c > 0$ and $\lambda \in (0, \lambda^*(c))$, the existence of a larger mountain-pass solution U_λ to problem (1). This represents a further step towards a complete description of the set of solutions to (1).
Let $H(x)$ be the mean curvature of $\partial\Omega$ at x and let

$$H_{\max} := \max_{x \in \partial\Omega} H(x). \tag{2}$$

We show

Theorem 1. *Let $\lambda^*(c)$ be as in Proposition 1. For every $c > 0$, there exists $0 \le \Lambda(c) < \lambda^*(c)$ such that problem (1) admits, besides the minimal solution u_λ, a mountain-pass solution U_λ for any $\Lambda(c) < \lambda < \lambda^*(c)$. Furthermore, the map $(0, +\infty) \ni c \mapsto \Lambda(c)$ is nondecreasing and the following statements hold*

- *(i) if $n = 3$ and $c > 0$ or $n \ge 4$ and $0 < c < \frac{n-2}{2} H_{\max}$, then $\Lambda(c) = 0$. Moreover, if $n = 4, 5$, then $\Lambda(\frac{n-2}{2} H_{\max}) = 0$.*
- *(ii) if $n \ge 4$, there exists $K = K(\Omega) \ge \frac{n-2}{2} H_{\max}$ such that if $c > K$, then $\Lambda(c) > 0$, U_λ exists up to $\lambda = \Lambda(c)$ and does not exist if $0 < \lambda < \Lambda(c)$.*

Note that, arguing as in [6], any mountain-pass solution to (1) is regular. Hence, by elliptic regularity, it solves (1) in a classical sense.
When $\Lambda(c) > 0$, one may wonder if different kinds of solutions exist for $\lambda \in (0, \Lambda(c))$. If $\Omega = B$, the unit ball, in [5] explicit radial solutions to (1) have been determined for every $\lambda \in (0, \lambda^*(c))$. We briefly recall their construction. For $c > 0$ and $\eta > \eta_0(c)$, where

$$\eta_0(c) := \max\{0, \frac{n-2}{c} - 1\}\,, \tag{3}$$

consider the function

$$\varphi(\eta) := \frac{[n(n-2)]^{n-2}}{c^4} \, \frac{[c(1+\eta) - n + 2]^4 \eta^{n-2}}{(1+\eta)^{2n}}. \tag{4}$$

It is readily seen that $\varphi(\eta_0) = 0 = \lim_{\eta \to +\infty} \varphi(\eta)$, that φ attains a global maximum at

$$\overline{\eta} := \frac{n + 2 + \sqrt{(n+2)^2 - 4c(n-2-c)}}{2c},$$

that φ increases on $(\eta_0, \overline{\eta})$ and decreases on $(\overline{\eta}, +\infty)$. Hence, for any $\lambda \in (0, \lambda_n(c))$, where $\lambda_n(c) := (\varphi(\overline{\eta}))^{1/(n-2)}$,

$$\text{there exist} \quad \eta_i = \eta_i(\lambda, c) \quad (i = 1, 2) \quad \text{such that } \varphi(\eta_i) = \lambda^{n-2}. \tag{5}$$

If $\lambda = \lambda_n(c)$, then $\eta_1 = \eta_2 = \overline{\eta}$. Finally, we recall by [5]

Proposition 2. *Let $\Omega = B \subset \mathbb{R}^n$ $(n \ge 3)$. Then, if $\lambda_n(c) > 0$ and $\eta_0 < \eta_2 \le \overline{\eta} \le \eta_1$ are defined as in (5), we have*

- *(i) for every $\lambda \in (0, \lambda_n(c))$, there exist two radial solutions of problem (1), the minimal solution u_{η_1} and a larger solution u_{η_2}, given by*

$$u_{\eta_i}(x) = \left(\frac{n(n-2)\eta_i}{\lambda}\right)^{(n-2)/4} (\eta_i + |x|^2)^{-(n-2)/2} - 1, \qquad i = 1, 2\,;$$

(ii) *the extremal parameter satisfies* $\lambda^*(c) = \lambda_n(c)$ *and the extremal solution* u^* *of* (1) *is given by* $u^*(x) := u_{\overline{\eta}}(x)$.

Letting $c \to +\infty$ in Proposition 2, one recovers known results for the corresponding Dirichlet problem, see [16, Section 5]. In particular, $\lambda_n(c) \nearrow \lambda^*_{Dir}$, see also [19, Section VI].
In Section 4 we show that the larger solution u_{η_2} in Proposition 2 has high energy when $c > \frac{n-2}{2}$ and λ is sufficiently small. Combining this with the fact that u_{η_1} and u_{η_2} are the only radial solutions to (1), we prove

Theorem 2. *Let* $\Omega = B \subset \mathbb{R}^n$ $(n \geq 3)$ *and* $\lambda_n(c)$ *be as in Proposition* 2. *Then*

(i) *if* $0 < c \leq \frac{n-2}{2}$, *problem* (1) *admits, besides the minimal solution, a radial mountain-pass solution* U_λ *for every* $0 < \lambda < \lambda_n(c)$;

(ii) *if* $c > \frac{n-2}{2}$, *there exists* $\Lambda_{rad}(c) > 0$ *such that problem* (1) *admits, besides the minimal solution, a radial mountain-pass solution* U_λ *if and only if* $\Lambda_{rad}(c) \leq \lambda < \lambda_n(c)$. *Furthermore, the map* $(\frac{n-2}{2}, +\infty) \ni c \mapsto \Lambda_{rad}(c)$ *is increasing and* $\lim_{c \to (\frac{n-2}{2})^+} \Lambda_{rad}(c) = 0$.

In both cases (i) *and* (ii), $U_\lambda = u_{\eta_2}$ *as given in Proposition* 2.

Let $\Lambda(c)$ be as in Theorem 1. When $\Omega = B$, from Theorem 2, we infer $\Lambda(c) \leq \Lambda_{rad}(c)$. Hence, $\Lambda(\frac{n-2}{2}) = 0$ for every $n \geq 3$. On the other hand, we do not know if, as in the Dirichlet case [17], any (smooth) solution to (1) in the ball is radially symmetric. Namely, if $\Lambda(c) = \Lambda_{rad}(c)$ for every $c > 0$. When $n = 3$, this is false. Indeed, by combining the statements of Theorems 1 and 2, we deduce the following

Corollary 1. *Let* $\Omega = B \subset \mathbb{R}^3$, $c > \frac{1}{2}$ *and* $\Lambda_{rad}(c) > 0$ *be as in Theorem* 2. *Then, for every* $0 < \lambda < \Lambda_{rad}(c)$, *problem* (1) *admits, besides the minimal solution, a mountain-pass solution which is not radial.*

A couple of remarks are in order. The proof of Theorem 1 is obtained by studying a suitable Robin problem at critical growth, see Section 2. The lower order perturbations considered include nonlinearities of the form: $\lambda(a(x)u + u^q)$, where $\lambda > 0$, a is a positive measurable function in $L^\infty(\Omega)$ and $1 < q < 2^* - 1$. A critical threshold for the exponent q turns out to be

$$2_T := \frac{2(n-1)}{n-2}, \tag{6}$$

the so-called *trace exponent*. If $2_T - 1 < q < 2^* - 1$, existence of mountain-pass solutions to the corresponding Robin problem is known from [27]. When $1 < q \leq 2_T - 1$, λ is sufficiently small and c is sufficiently large, we show nonexistence of mountain-pass solutions, see Theorem 4 in Section 2. We should mention that the role of the trace exponent in existence and nonexistence results is well-known for the corresponding Neumann problem (with $\lambda < 0$), see for instance the survey article [15]. In this case, one has existence if $1 < q < 2_T - 1$ and nonexistence if $2_T - 1 \leq q < 2^* - 1$, see [10, 14] and references therein. The "inversion", between the existence and nonexistence regions, is basically due to the sign of λ. Roughly speaking, in the Robin case ($c > 0$ and $\lambda > 0$) the subcritical term lowers the functional, while in the Neumann case ($c = 0$ and $\lambda < 0$) it increases the energy of solutions, see Section 2.
As a by-product of the above mentioned nonexistence results, we derive a Sobolev type inequality. First, from [22] (see also [1]), we recall that there exists $C = C(\Omega) \geq \frac{n-2}{2} H_{\max}$ such that, for every $c \geq C(\Omega)$, there holds

$$\int_\Omega |\nabla u|^2 \, dx + c \int_{\partial\Omega} u^2 \, d\sigma \geq \frac{S}{2^{2/n}} |u|^2_{2^*} \quad \text{for every } u \in H^1(\Omega). \tag{7}$$

Here and in the following, $|.|_p$ denotes the usual norm in $L^p(\Omega)$ and S is the best Sobolev constant, namely

$$S = \inf\{|\nabla u|_2^2;\ u \in \mathcal{D}^{1,2}(\mathbb{R}^n),\ |u|_{2^*} = 1\}. \tag{8}$$

If $\Omega = B$, then $C(\Omega) = \frac{n-2}{2}$, see [8]. We also refer to [30] and references therein for some variants to (7) involving L^2 interior and L^{2_T} boundary norms.
Let $a(x)$ be a positive measurable function in $L^\infty(\Omega)$. For every $c > 0$, we set

$$\lambda_1^a(c) := \inf_{u \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\, dx + c \int_{\partial\Omega} u^2\, d\sigma}{\int_\Omega a(x) u^2\, dx}\,. \tag{9}$$

Namely, $\lambda_1^a(c)$ is the first eigenvalue (with weight a) of $-\Delta$ under Robin boundary conditions.
Finally, for every $0 < \lambda < \lambda_1^a(c)$, we define the norm

$$\|u\|^2_{\lambda,c} := \int_\Omega |\nabla u|^2\, dx + c \int_{\partial\Omega} u^2\, d\sigma - \lambda \int_\Omega a(x)\, u^2\, dx \tag{10}$$

and we state

Theorem 3. *Let $n \geq 3$, 2_T as in (6), $C(\Omega)$ as in (7), $\lambda_1^a(c)$ as in (9) and $\|.\|_{\lambda,c}$ as in (10). There exists $K = K(\Omega) \geq C(\Omega)$ such that for every $c > K$ there exists $0 < \Lambda = \Lambda(c) < \lambda_a^1(c)$ such that*

$$\|u\|^2_{\Lambda,c} \geq \frac{S}{2^{2/n}}\, |u|^2_{2^*} \left(1 + \frac{4}{n \cdot 2_T}\, \frac{\Lambda |u|^{2_T}_{2_T}}{\sqrt{\Lambda^2 |u|^{2\cdot 2_T}_{2_T} + 4\|u\|^2_{\Lambda,c} |u|^{2^*}_{2^*}} + \Lambda |u|^{2_T}_{2_T}}\right) \tag{11}$$

for every $u \in H^1(\Omega) \setminus \{0\}$.

As can be checked, if $u \in H^1(\Omega) \setminus \{0\}$,

$$1 + \frac{n-2}{n(n-1)} > \left(1 + \frac{4}{n \cdot 2_T}\, \frac{\Lambda |u|^{2_T}_{2_T}}{\sqrt{\Lambda^2 |u|^{2\cdot 2_T}_{2_T} + 4\|u\|^2_{\Lambda,c} |u|^{2^*}_{2^*}} + \Lambda |u|^{2_T}_{2_T}}\right) > 1\,.$$

The paper is organized as follows. In Section 2 we give some existence and nonexistence results for a suitable model problem at critical growth. This allows to prove Theorem 1 in Section 3. In Section 4 we prove Theorem 2, while in Section 5 we derive inequality (11).

2 The model problem

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a smooth and bounded domain and let $H^1(\Omega)$ be the usual Sobolev space endowed with the norm

$$\|u\|^2 := \int_\Omega |\nabla u|^2\, dx + \int_\Omega u^2\, dx.$$

For any $c > 0$ fixed, the following norm

$$\|u\|^2_c := \int_\Omega |\nabla u|^2\, dx + c \int_{\partial\Omega} u^2\, d\sigma$$

is equivalent to $\|.\|$, see for instance [25, A.9 Theorem]. As in Section 1, we will denote with $|.|_p$ the usual $L^p(\Omega)$ norm and with 2^* and 2_T the critical Sobolev and trace exponents.

Motivated by problem (1), in the spirit of [9] (see our Section 3), we consider the following model problem

$$(P_\lambda)\begin{cases} -\Delta u = u^{2^*-1} + f_\lambda(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u_\nu + cu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda, c > 0$ and f_λ is a lower order perturbation. More precisely, we assume that

(f1) $f_\lambda(x,s) \geq 0$ is measurable with respect to x, continuous with respect to $s \geq 0$ and $\sup\{f_\lambda(x,s) : x \in \Omega, 0 \leq s \leq C\} < +\infty$, for every $C > 0$. Furthermore, the map $\lambda \mapsto f_\lambda(x,s)$ is increasing for a.e. $x \in \Omega$ and for every $s > 0$, and $f_0(x,s) \equiv 0$;

(f2) f_λ can be written as $f_\lambda(x,s) = \lambda a(x)s + g_\lambda(x,s)$, where a is a positive bounded measurable function and

$$\begin{aligned} &g_\lambda(x,s) = o(s) \text{ as } s \to 0^+\,, \text{ uniformly with respect to a.e. } x \in \Omega\,; \\ &g_\lambda(x,s) = o(s^{2^*-1}) \text{ as } s \to +\infty\,, \text{ uniformly with respect to a.e. } x \in \Omega\,; \\ &g_\lambda(x,s) + s^{2^*-1} > 0, \text{ for every } s > 0 \text{ and a.e. } x \in \Omega\,. \end{aligned} \tag{12}$$

The same equation was studied in [9] but under Dirichlet boundary conditions. When $c = 0$ and $f_\lambda(x,u) = -a(x)u - \lambda u^q$, problem (P_λ) was studied in several papers. Existence of least energy solutions (see (16) for the definition) was proved in [27], for $1 < q < 2_T - 1$ and $n \geq 3$. Existence and nonexistence of least energy solutions were proved in [10] for $q = 2_T - 1$ and $n \geq 5$, and in [14] for $1 < q < 2^* - 1$ and $n \geq 3$. See also [3, 12, 28] and the survey article [11].
Less is known under Robin boundary conditions. When $f_\lambda(x,u) \equiv 0$ and $0 < c < \frac{n-2}{2} H_{\max}$, with $H_{\max}$ as defined in (2), existence of least energy solutions is known from [2]. If $f_\lambda(x,u) = f(x,u) = a(x)u + b(x)u^q$, where b is a bounded and positive function and $2_T - 1 < q < 2^* - 1$, existence of mountain-pass solutions was shown in [27, Corollary 4.1]. Also we mention that the case $f_\lambda(x,u) = \lambda a(x)u$ was studied in [23] and [24] by means of a suitable transformation sending the Robin problem into a Neumann problem. Finally, we refer to [20] and [21] where the case $\Omega = B$, $f_\lambda(x,u) = \lambda a(x)u$ and u radial is dealt.
We consider weak solutions $u \in H^1(\Omega)$ to (P_λ), namely such that

$$\int_\Omega \nabla u \cdot \nabla v\, dx + c\int_{\partial\Omega} uv\, d\sigma = \int_\Omega \Big(u^{2^*-1} + \lambda a(x)u + g_\lambda(x,u)\Big)\, v\, dx \quad \text{for every } v \in H^1(\Omega)\,. \tag{13}$$

Let $\lambda_1^a(c)$ be as defined in (9). Standard calculus arguments show that $\lambda_1^a(c)$ is achieved by a unique positive function φ_1^a. Testing (13) with $v = \varphi_1^a$, by the third assumption in (12), we readily deduce that (P_λ) admits solutions if and only if $\lambda < \lambda_1^a(c)$.
On the other hand, for any $\lambda \in (0, \lambda_1^a(c))$, we set

$$\mu_1^a(\lambda, c) := \inf_{u \in H^1(\Omega)\setminus\{0\}} \frac{\|u\|_c^2 - \lambda \int_\Omega a(x)\, u^2\, dx}{|u|_2^2}\,, \tag{14}$$

the first eigenvalue of the operator $-\Delta - \lambda a(x)$ under Robin boundary conditions. It turns out that $\mu_1^a(\lambda, c) > 0$ and the minimum is achieved by a unique (up to a multiplicative constant) function ϕ_1^a strictly of one sign in Ω, see [5, Lemma 12]. By (14) it follows that, for any $c > 0$ and for any $\lambda \in (0, \lambda_1^a(c))$, the norm $\|.\|_{\lambda,c}$ in (10) is equivalent to $\|.\|_c$ and, in turn, to $\|.\|$.
Weak solutions to (P_λ) are the nonzero critical points of the functional

$$J_{\lambda,c}(u) := \frac{1}{2}\|u\|_{\lambda,c}^2 - \frac{1}{2^*}|u|_{2^*}^{2^*} - \int_\Omega G_\lambda(x,u)\, dx, \tag{15}$$

where $G_\lambda(x,u)=\int_0^u g_\lambda(x,s)\,ds$. In order to deal with nonnegative solutions, one has to consider the modified functional where $|u|_{2^*}^{2^*}$ is replaced by $|u^+|_{2^*}^{2^*}$ and $g_\lambda(x,u)=0$ for $u<0$. These substitutions do not affect the analysis below.

Exploiting either the fact that $\|.\|_{\lambda,c}$ is a norm equivalent to $\|.\|$ and the growth conditions assumed on g_λ, it is readily seen that $J_{\lambda,c}$ has a mountain-pass structure for any $c>0$ and $0<\lambda<\lambda_1^a(c)$, see [9, 27]. We set

$$M(\lambda,c):=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{\lambda,c}(\gamma(t)),$$

where $\Gamma:=\{\gamma\in C^0\left([0,1],H^1(\Omega)\right):\gamma(0)=0,\ J_{\lambda,c}(\gamma(1))<0\}$. We also recall that a natural constraint for $J_{\lambda,c}$ is the so-called Nehari manifold:

$$\mathcal{N}_{\lambda,c}:=\{u\in H^1(\Omega)\setminus\{0\}:J'_{\lambda,c}(u)[u]=0\}.$$

Arguing as in [29, Chapter 4], one may check that, for any $u\in H^1(\Omega)\setminus\{0\}$, there exists a unique $t_{\lambda,c}=t_{\lambda,c}(u)>0$ such that $t_{\lambda,c}(u)u\in\mathcal{N}_{\lambda,c}$ and the maximum of $J_{\lambda,c}(tu)$ is achieved at $t=t_{\lambda,c}(u)$. The map $H^1(\Omega)\setminus\{0\}\ni u\mapsto t_{\lambda,c}(u)\in(0,+\infty)$ is continuous, while the map $u\mapsto t_{\lambda,c}(u)u$ defines an homeomorphism between the unit ball of $H^1(\Omega)$ and $\mathcal{N}_{\lambda,c}$. Furthermore, there holds

$$\inf_{u\in\mathcal{N}_{\lambda,c}} J_{\lambda,c}(u)=\inf_{u\in H^1(\Omega)\setminus\{0\}}\sup_{t\geq 0} J_{\lambda,c}(tu)=\inf_{u\in H^1(\Omega)\setminus\{0\}} J_{\lambda,c}(t_{\lambda,c}(u)u)=M(\lambda,c). \tag{16}$$

Minimizers to $J_{\lambda,c}(u)$ in $\mathcal{N}_{\lambda,c}$ are usually called least energy solutions to (P_λ). Hence, we shall equivalently refer to *least energy* or *mountain-pass* solutions to (P_λ).

Some computations show that

$$t_{\lambda,c}(u)\|u\|_{\lambda,c}^2-t_{\lambda,c}^{2^*-1}(u)|u|_{2^*}^{2^*}-\int_\Omega g_\lambda(x,t_{\lambda,c}(u)u)u\,dx=0\,, \tag{17}$$

for every $u\in H^1(\Omega)$. Then, since by assumption $\lambda a(x)s^2+g_\lambda(x,s)s=f_\lambda(x,s)s\geq 0$ for every $s\geq 0$ and a.e. $x\in\Omega$, we get

$$t_{\lambda,c}(u)\leq\left(\frac{\|u\|_c^2}{|u|_{2^*}^{2^*}}\right)^{(n-2)/4}. \tag{18}$$

Next we state a compactness result which is obtained by slightly modifying [9, Theorem 2.2] and [27, Theorem 2.1].

Lemma 1. *For $c>0$ and $\lambda\in(0,\lambda_1^a(c))$, the functional $J_{\lambda,c}$ admits a Palais Smale sequence at level $M=M(\lambda,c)$, namely there exists a sequence $\{u_m\}_{m\geq 0}\subset H^1(\Omega)$ such that*

$$J_{\lambda,c}(u_m)\to M\,,\quad J'_{\lambda,c}(u_m)\to 0\quad in\quad (H^1(\Omega))'.$$

If furthermore

$$M(\lambda,c)<\frac{S^{n/2}}{2n},$$

then there is a solution $u\in H^1(\Omega)$ of (P_λ) such that $u_m\to u$ in $H^1(\Omega)$ (up to a subsequence) and $J_{\lambda,c}(u)=M(\lambda,c)$.

Proof. The existence of a Palais Smale sequence $\{u_m\}_{m\geq 0}$ follows by the mountain-pass structure of the functional $J_{\lambda,c}$, see [9, Theorem 2.2]. We prove the compactness issue.

By assumption, we have that

$$\frac{1}{2}\|u_m\|_{\lambda,c}^2-\frac{1}{2^*}|u_m|_{2^*}^{2^*}-\int_\Omega G_\lambda(x,u_m)\,dx=M+o(1) \tag{19}$$

and

$$\langle u_m, \varphi\rangle_{\lambda,c} - \int_\Omega |u_m|^{2^*-2} u_m \varphi\, dx - \int_\Omega g_\lambda(x, u_m)\varphi\, dx = o(\|\varphi\|) \quad \text{for every } \varphi \in H^1(\Omega) \tag{20}$$

as $m \to +\infty$, where $\langle ., .\rangle_{\lambda,c}$ denotes the scalar product associated to the norm $\|.\|_{\lambda,c}$.
Writing (20) with $\varphi = u_m$ and inserting this into (19), we get

$$\frac{1}{n}\, |u_m|_{2^*}^{2^*} = \int_\Omega \left(G_\lambda(x, u_m) - \frac{1}{2}\, g_\lambda(x, u_m)\, u_m \right) dx + M + o(1)\,. \tag{21}$$

By (12), for every $\varepsilon > 0$ there exists $C_1 > 0$ such that

$$|g_\lambda(x, s)| \le \varepsilon s^{2^*-1} + C_1 \quad \text{for } s > 0\,.$$

Exploiting the arbitrariness of ε and recalling that $g_\lambda(x, s) = 0$ for $s < 0$, (21) yields

$$|u_m|_{2^*}^{2^*} \le C_2\|u_m\|_{\lambda,c} + M + o(1)\,,$$

for some $C_2 > 0$. Comparing with (19) and exploiting (12), we conclude that

$$\|u_m\|_{\lambda,c}^2 \le C_3\|u_m\|_{\lambda,c} + C_4 + o(1)\,,$$

for some $C_3, C_4 > 0$. Hence, $\{u_m\}_{m\ge0}$ is bounded in $H^1(\Omega)$. Then, (up to a subsequence) there exists $u \in H^1(\Omega)$ such that

$$u_m \rightharpoonup u \quad \text{in } H^1(\Omega) \quad \text{and} \quad u_m \to u \quad \text{a.e. in } \Omega\,,$$

$$u_m\, |_{\partial\Omega} \to u\, |_{\partial\Omega} \quad \text{in } L^2(\partial\Omega) \quad \text{and} \quad u_m \to u \quad \text{in } L^q(\Omega)\,, \text{ for every } 1 \le q < 2^*.$$

Assume by contradiction that $u = 0$. As in [9, (2.26) and (2.27)], we deduce that

$$\int_\Omega g_\lambda(x, u_m)\, u_m\, dx \to 0 \quad \text{and} \quad \int_\Omega G_\lambda(x, u_m)\, dx \to 0.$$

Then, (20), with $\varphi = u_m$, gives

$$|\nabla u_m|_2^2 = |u_m|_{2^*}^{2^*} + o(1)$$

and, in turn, by (21) we get

$$|u_m|_{2^*}^{2^*} = nM + o(1) \quad \text{and} \quad |\nabla u_m|_2^2 = nM + o(1)\,. \tag{22}$$

This, combined with (7), implies

$$o(1) + nM = o(1) + |\nabla u_m|_2^2 \ge \frac{S}{2^{2/n}}\, |u_m|_{2^*}^2 = \frac{S}{2^{2/n}}\, (nM)^{2/2^*} + o(1)\,.$$

Namely,

$$M \ge \frac{S^{n/2}}{2n}\,,$$

a contradiction.
Let $u \ne 0$, (20) with $\varphi = u_m - u$ yields

$$|\nabla(u_m - u)|_2^2 = |u_m - u|_{2^*}^{2^*} + = o(1)\,,$$

where we have also exploited the Brezis-Lieb Lemma [7]. Then, since $\int_\Omega G_\lambda(x, u_m)\, dx \to \int_\Omega G_\lambda(x, u)\, dx$, by (19) and the Brezis-Lieb Lemma, we deduce

$$J_{\lambda,c}(u) + \frac{1}{n}|\nabla(u_m - u)|_2^2 = M + o(1)\,. \tag{23}$$

Writing (20) with $\varphi = u$ and passing to the limit, we get

$$\|u\|^2_{\lambda,c} = |u|^{2^*}_{2^*} + \int_\Omega g_\lambda(x,u)u \, + o(1)\,,$$

so that $u \in \mathcal{N}_{\lambda,c}$ (the Nehari manifold associated to $J_{\lambda,c}$). Then, by (16), we deduce that $J_{\lambda,c}(u) \geq M$. This, inserted into (23), implies that

$$\frac{1}{n}\,|\nabla(u_m - u)|_2^2 \leq o(1)\,,$$

from which the statement follows. □

Recall that the functions

$$U_\varepsilon(x) = \left(\frac{\varepsilon n(n-2)}{\varepsilon^2 n(n-2) + |x|^2}\right)^{\frac{n-2}{2}} \qquad (\varepsilon > 0) \tag{24}$$

achieve the best Sobolev constant (8) and solve the equation

$$-\Delta u = u^{2^*-1} \quad \mathbb{R}^n\,.$$

By exploiting the functions in (24), we prove

Lemma 2. *For every $c > 0$, the following statements hold:*

- *(i)* $M(\lambda, c) \leq \frac{S^{n/2}}{2n}$ *for every* $\lambda \in (0, \lambda_1^a(c))$*;*
- *(ii) the map* $(0, \lambda_1^a(c)) \ni \lambda \mapsto M(\lambda, c)$ *is nonincreasing (decreasing when* $M(\lambda, c) < \frac{S^{n/2}}{2n}$*) and continuous;*
- *(iii)* $\displaystyle\lim_{\lambda\to 0^+} M(\lambda, c) = \frac{S^{n/2}}{2n}$ *for every* $c \geq C(\Omega)$*, with* $C(\Omega)$ *as in (7), and* $\displaystyle\lim_{\lambda\to(\lambda_1^a(c))^-} M(\lambda, c) = 0$*, for every* $c > 0$*.*

Let $\lambda_\infty^a := \displaystyle\lim_{c\to+\infty} \lambda_1^a(c)$ *(which exists since* $\lambda_1^a(c)$ *is increasing). For every* $\lambda \in (0, \lambda_\infty^a)$*, there exists* $c_0 > 0$ *such that* $\lambda = \lambda_1^a(c_0)$ *and*

- *(iv) the map* $(c_0, +\infty) \ni c \mapsto M(\lambda, c)$ *is nondecreasing (decreasing when* $M(\lambda, c) < \frac{S^{n/2}}{2n}$*) and continuous.*

Arguing as in [23, Lemma 3.3], it is not difficult to check that λ_∞^a corresponds to $\lambda_{1,Dir}^a$, the first eigenvalue (with weight a) of $-\Delta$ under Dirichlet boundary conditions.

Proof. For $\varepsilon > 0$, let $U_\varepsilon(x)$ be as in (24). Put

$$\overline{U}_\varepsilon(x) := \frac{U_\varepsilon(x)}{|U_\varepsilon(x)|_{2^*}}$$

so that, by applying arguments similar to those in [9, Lemma 2.1] (see also (29) below), one has that

$$\sup_{t\geq 0} J_{\lambda,c}(t\overline{U}_\varepsilon) \leq \frac{1}{n}\,\|\overline{U}_\varepsilon\|_c^n\,,$$

for every $c > 0$ and $\lambda \in (0, \lambda_1^a(c))$. By the estimates performed in [1] and [2], we have that

$$\|\overline{U}_\varepsilon\|_c^2 = \frac{S}{2^{2/n}} + \alpha_n(\varepsilon)\,,$$

where $\alpha_n(\varepsilon) = \varepsilon + o(\varepsilon)$, if $n \geq 4$, while $\alpha_3(\varepsilon) = \varepsilon|\log(\varepsilon)| + O(\varepsilon)$, see also (30) below. Then, letting $\varepsilon \to 0$, statement (i) follows from (16).
Since the proof of statements (ii) and (iv) is the same as [10, Lemma 3.2], we omit it. The key point is the exploitation of the characterization (16). This has to be suitably combined with compactness arguments similar to those applied in the proof of Lemma 1, see also [14, Lemma 11].
Let us consider (iii). Set

$$I_c(u) := \frac{1}{2}\|u\|_c^2 - \frac{1}{2^*}|u|_{2^*}^{2^*} \quad \text{and} \quad s_c := \inf_{u\in\mathcal{N}_c} I_c(u) = \frac{1}{n}\inf_{u\in H^1(\Omega)\setminus\{0\}}\left(\frac{\|u\|_c^2}{\|u\|_{2^*}^2}\right)^{\frac{n}{2}}, \tag{25}$$

where $\mathcal{N}_c := \{u \in H^1(\Omega)\setminus\{0\} : I_c'(u)[u] = 0\}$, see (16). The estimates given above and (7) yield $s_c = \frac{S^{n/2}}{2n}$, for every $c \geq C(\Omega)$.
Let $\lambda_m \to 0^+$ as $m \to +\infty$. By (ii), there exists $\lim_{m\to+\infty} M(\lambda_m, c) = M_c$ and, by (16), $M_c \leq s_c$, for every $c > 0$. If $M_c = \frac{S^{n/2}}{2n}$, there is nothing to prove. Assume, by contradiction, that $M_c < \frac{S^{n/2}}{2n}$ for $c \geq C(\Omega)$. Then, $M(\lambda_m, c)$ is achieved by $u_m \in \mathcal{N}_{\lambda_m,c}$ and the sequence $\{u_m\}_{m\geq 0}$ turns out to be bounded in $H^1(\Omega)$, see the proof of Lemma 1. Thanks to $(f1)$ and $(f2)$, we may repeat the proof of Lemma 1 (with minor changes) to conclude that $u_m \to u \neq 0$ in $H^1(\Omega)$, where $u \in \mathcal{N}_c$. In particular, we get that $s_c \leq I_c(u) = \lim_{n\to+\infty} J_{\lambda_m,c}(u_m) = M_c < \frac{S^{n/2}}{2n}$, which is impossible for $c \geq C(\Omega)$.
Now we turn to the second part of (iii). Let ϕ_1^a be the first positive eigenfunction associated to $\mu_1^a(\lambda, c)$ as defined in (14). By the third assumption in (12), we get

$$J_{\lambda,c}(t_{\lambda,c}(\phi_1^a)\phi_1^a) \leq \frac{t_{\lambda,c}^2(\phi_1^a)}{2}\|\phi_1^a\|_{\lambda,c}^2 = \mu_1^a(\lambda, c)\frac{t_{\lambda,c}^2(\phi_1^a)}{2}\|\phi_1^a\|_2^2 =: \mu_1^a(\lambda, c)F(\phi_1^a).$$

The last term in the above equation goes to zero as $\lambda \to (\lambda_1^a(c))^-$. Indeed, $F(\phi_1^a)$ is bounded by (18) and, for every $c > 0$, the map $(0, \lambda_1^a(c)) \ni \lambda \mapsto \mu_1^a(\lambda, c)$ is continuous, decreasing and $\mu_1^a(\lambda, c) \searrow 0$ as $\lambda \to (\lambda_1^a(c))^-$. Then, recalling (16), we conclude. □

By Lemma 2, the following infimum is well-defined

$$\Lambda(c) := \inf\left\{0 < \lambda < \lambda_1^a(c) : \quad M(\lambda, c) < \frac{S^{n/2}}{2n}\right\}, \tag{26}$$

for any $c > 0$. Moreover, we have

Lemma 3. *Let $\Lambda(c)$ be as in (26), then the map $(0, +\infty) \ni c \mapsto \Lambda(c)$ is nondecreasing.*

Proof. Let $0 < c_1 < c_2$. If $\Lambda(c_2) = 0$, by Lemma 2, we readily get that $\Lambda(c_1) = \Lambda(c_2)$. Assume now $\Lambda(c_2) > 0$. Since the map $c \mapsto \lambda_1^a(c)$ is increasing, there exists $c_0 < c_2$ such that $\Lambda(c_2) = \lambda_1^a(c_0) < \lambda_1^a(c)$, for every $c > c_0$. Then, by Lemma 2-(iv), $M(\lambda, c_1) < M(\lambda, c_2) < \frac{S^{n/2}}{2n}$, for every $\lambda \in (\Lambda(c_2), \lambda_1^a(c_1))$ and for every $c_0 < c_1 < c_2$. Hence, $\Lambda(c_1) \leq \Lambda(c_2)$, for every $c_0 < c_1 < c_2$. The above argument, suitably iterated, proves the statement. □

Finally, we prove

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$, $\lambda_1^a(c)$ be as in (9) and $\Lambda(c)$ as in (26). Furthermore, we denote with b_λ a suitable positive bounded measurable function. Assume that f_λ satisfies $(f1)$ and $(f2)$, then problem (P_λ) admits a mountain-pass solution for every $\Lambda(c) < \lambda < \lambda_1^a(c)$, where* (*i*) *if* $0 < c < \frac{n-2}{2} H_{\max}$, *then* $\Lambda(c) = 0$;

(*ii*) *if* $c \geq \frac{n-2}{2} H_{\max}$ *and*

$$g_\lambda(x,s) \geq b_\lambda(x) s^q \quad \text{with} \quad 2_T - 1 < q < 2^* - 1 \quad \text{for every } s \geq 0 \text{ and a.e. } x \in \Omega\,, \tag{27}$$

then $\Lambda(c) = 0$. *If* $n \geq 4$ *and* (27) *holds with* $1 < q \leq 2_T - 1$, *then* $\Lambda(\frac{n-2}{2} H_{\max}) = 0$.

(*iii*) *If*

$$g_\lambda(x,s) \leq b_\lambda(x) s^q \quad \text{with} \quad 1 < q \leq 2_T - 1 \quad \text{for every } s \geq 0 \text{ and a.e. } x \in \Omega\,,$$

then there exists $K(\Omega) \geq \frac{n-2}{2} H_{\max}$ *such that, for every* $c > K$, $\Lambda(c) > 0$ *and* (P_λ) *admits a mountain-pass solution if and only if* $\Lambda(c) \leq \lambda < \lambda_1^a(c)$.

The first part of Theorem 4 is an immediate consequence of (26) and Lemma 1. A large part of statements (*i*) and (*ii*) is known from [2] and [27]. For completeness, we put the whole proofs in Section 2.1 below.

Concerning assertion (*iii*), we note that it includes the cases $g_\lambda(x,s) \equiv 0$ and $g_\lambda(x,s) \leq 0$. To get its proof, we apply a blow-up argument as $\lambda \to (\Lambda(c))^+$, in the spirit of the one developed for the Neumann problem (as $\lambda \to -\infty$) in [3, 12, 28]. See also [1], where a similar approach was adopted for problem (P_0) as $c \to +\infty$.

2.1 Proof of Theorem 4-(*i*) and (*ii*)

We only need to verify that there exists $w_0 \in H^1(\Omega)$, $w_0 \geq 0$ in Ω such that

$$\sup_{t \geq 0} J_{\lambda,c}(t w_0) < \frac{S^{n/2}}{2n}\,,$$

for every $0 < \lambda < \lambda_1^a(c)$ and for c in a suitable interval. Once this proved, Lemma 1 gives the conclusion.

For $\varepsilon > 0$, let $U_\varepsilon(x)$ be as in (24). For $\alpha > 0$, by [4] we recall the following estimates:

$$\int_\Omega |U_\varepsilon(x)|^\alpha\, dx \leq \begin{cases} C_1 \varepsilon^{n - \alpha\frac{n-2}{2}} + C_2 \varepsilon^{\alpha\frac{n-2}{2}} & \text{for} \quad \alpha \neq \frac{n}{n-2} \\ \varepsilon^{n/2}(C_1 + C_2|\ln \varepsilon|) & \text{for} \quad \alpha = \frac{n}{n-2}. \end{cases} \tag{28}$$

As in the proof of Lemma 2, let

$$\overline{U}_\varepsilon(x) = \frac{U_\varepsilon(x)}{|U_\varepsilon(x)|_{2^*}}\,.$$

By applying arguments similar to those of [9, Lemma 2.1], we get that

$$J_{\lambda,c}(t\overline{U}_\varepsilon) \leq \frac{1}{n} \|\overline{U}_\varepsilon\|_c^n - \int_\Omega \int_0^{t_\varepsilon \overline{U}_\varepsilon} f_\lambda(x,s)\, ds\, dx \quad \text{for every } t \geq 0\,, \tag{29}$$

where $t_\varepsilon = t_{\lambda,c}(\overline{U}_\varepsilon)$ is as in (17). Furthermore, following the proof of [9, Lemma 2.1], we have that $t_\varepsilon \to \frac{S^{(n-2)/4}}{2^{2^*}}$ as $\varepsilon \to 0$. By the estimates in [2], we know that

$$\|\overline{U}_\varepsilon\|_c^2 = \frac{\|U_\varepsilon\|_c^2}{|U_\varepsilon(x)|_{2^*}^2} = \frac{S}{2^{2/n}} + B_n \left(c - \frac{n-2}{2} H_{max}\right) \varepsilon + O(\varepsilon^2 |\log(\varepsilon)|)\,, \tag{30}$$

for some $B_n > 0$ and for $n \geq 4$. If $n = 3$, the same estimate holds but with $\varepsilon|\log(\varepsilon)|$ in place of ε and with $O(\varepsilon)$ in place of $O(\varepsilon^2|\log(\varepsilon)|)$. Then, since $f_\lambda(x,s) \geq 0$, statement (*i*) readily follows by combining (29) with (30).

Let us now turn to statement (ii). By assumption, since $a(x)$ is positive, we have that $f_\lambda(x,s) \geq b_\lambda(x)s^q$, where $2_T - 1 < q < 2^* - 1$. Hence, by (28),

$$-\int_\Omega \int_0^{t_\varepsilon \overline{U}_\varepsilon} f_\lambda(x,s)\,ds\,dx \leq -\frac{t_\varepsilon^{q+1}}{q+1}\int_\Omega b_\lambda(x)\,\overline{U}_\varepsilon^{q+1} = -C_{n,\lambda}\varepsilon^{n-(q+1)\frac{(n-2)}{2}},$$

with $C_{n,\lambda} > 0$ for $\lambda > 0$. By noting that $0 < n-(q+1)\frac{(n-2)}{2} < 1$, the conclusion follows by combining this with (29) and (30).
To get the proof of the second part of statement (ii), we simply note that, when $n \geq 4$, the above estimate still holds for $1 < q \leq 2_T - 1$. The only difference is that, here, $1 \leq n-(q+1)\frac{(n-2)}{2} < 2$. When $c = \frac{n-2}{2}\,H_{max}$, by (29) and (30), this suffices to lower the functional under the compactness threshold. If $n = 3$ and $c = \frac{1}{2}\,H_{max}$, the term to be lowered is $O(\varepsilon)$ and the growth condition (27) cannot be weakened.

2.2 Proof of Theorem 4-(iii)

For $n \geq 3$, we show that that there exists $C'(\Omega) \geq \frac{n-2}{2}\,H_{\max}$, such that $M(\Lambda(c),c)$ is attained for every $c > C'$. Were $\Lambda(c) = 0$, there would exist a sequence of functions $\{u_m\}_{m\geq 0}$ in $H^1(\Omega)$ which achieve $M(\lambda_m,c)$ as $\lambda_m \to 0^+$. Arguing as in the proof of Lemma 2-(iii), we deduce that $u_m \to u$ in $H^1(\Omega)$ and $u \neq 0$ (if $u = 0$, one gets a contradiction by repeating the proof below). Moreover, u achieves s_c as defined in (25). But s_c is constant, hence, it cannot be attained for $c > C(\Omega)$, with $C(\Omega)$ as in (7) (s_c is strictly increasing when achieved). We conclude that $\Lambda(c) > 0$, for every $c > K(\Omega) := \max\{C(\Omega), C'(\Omega)\}$. Similarly, when $\lambda \in (0,\Lambda(c))$, $M(\Lambda(c),c)$ is constant and cannot be achieved.

Let $c > C'(\Omega)$, with $C'(\Omega) \geq \frac{n-2}{2}\,H_{\max}$ to be fixed later, and $\lambda_m \to (\Lambda(c))^+$. Then, by Lemma 1, $M(\lambda_m,c)$ is achieved by a function $u_m \in H^1(\Omega)$. The sequence $\{u_m\}_{m\geq 0}$ is bounded in $H^1(\Omega)$ (see the proof of Lemma 1). Hence, up to a subsequence, $u_m \rightharpoonup u$ in $H^1(\Omega)$ as $m \to +\infty$. We assume that $u = 0$. Otherwise, by arguing as in the proof of Lemma 1, u is a mountain-pass solution to (P_Λ) and we conclude. It follows that

$$\lim_{m\to\infty} |\nabla u_m|_2^2 = \lim_{m\to\infty} |u_m|_{2^*}^{2^*} = \lim_{m\to\infty} nM(\lambda_m,c) = \frac{S^{n/2}}{2},$$

see (22).
By this, invoking [3, Lemma 3.7], we obtain that there exist $\delta_m > 0$ and $P_m \in \partial\Omega$ such that

$$\lim_{m\to\infty} \delta_m = 0\,, \quad \lim_{m\to\infty} \frac{dist(P_m,\partial\Omega)}{\delta_m} = 0 \quad \text{and} \quad \lim_{m\to\infty} |\nabla(u_m - U_{\delta_m,P_m})|_2^2 = 0\,, \tag{31}$$

where, recalling (24), we denote with $U_{\varepsilon,y}(x) := U_\varepsilon(x-y)$ for $\varepsilon > 0$ and $y \in \mathbb{R}^n$. Therefore, up to a subsequence, $P_m \to P \in \partial\Omega$.
Then, putting

$$\mathcal{M} := \{CU_{\varepsilon,y} : \quad C \in \mathbb{R},\, \varepsilon > 0,\, y \in \partial\Omega\}$$

and

$$d(\varphi,\mathcal{M}) := \inf\{|\nabla(\varphi-\psi)|_2^2 \quad \psi \in M\},$$

[3, Lemma 3.1] implies that $d(u_m,\mathcal{M})$ is achieved by some $C_mU_{\varepsilon_m,y_m}$. More precisely, there exist $m_0 > 0$, $\varepsilon_m > 0$, $C_m \in \mathbb{R}$, $y_m \in \partial\Omega$, $\omega_m \in H^1(\Omega)$ such that

$$u_m = C_mU_{\varepsilon_m,y_m} + \omega_m \quad \text{for every } m \geq m_0.$$

Furthermore, by [3, Lemma 2.3], up to a subsequence, $\varepsilon_m/\delta_m \to 1$, $C_m \to 1$, $y_m \to P$ and $\omega_m \to 0$ in $H^1(\Omega)$. Moreover, we have

$$\int_\Omega \nabla\omega_m \cdot \nabla U_{\varepsilon_m,y_m}\, dx = 0 \quad \text{for every } m \ge m_0.$$

Now we recall some estimates. By [2], we know that

$$|U_{\varepsilon_m,y_m}|_{2^*}^{2^*} = \frac{S^{n/2}}{2} - A_n\varepsilon_m + o(\varepsilon_m)\,, \tag{32}$$

for some $A_n > 0$.
By [3, Lemma 3.5] (with $L = 2$),

$$|u_m|_{2^*}^{2^*} = C_m^{2^*}|U_{\varepsilon_m,y_m}|_{2^*}^{2^*} + 2^*C_m^{2^*-1}\int_\Omega U_{\varepsilon_m,y_m}^{2^*-1}\omega_m + \frac{2^*(2^*-1)}{2}\,C_m^{2^*-2}\int_\Omega U_{\varepsilon_m,y_m}^{2^*-2}\omega_m^2 + o(\|\omega_m\|^2)\,. \tag{33}$$

By [14, (7.33)],

$$\int_\Omega U_{\varepsilon_m,y_m}^{2^*-1}\omega_m\, dx = O(\beta_n(\varepsilon_m)\|\omega_m\|)\,,$$

where

$$\beta_n(\varepsilon_m) = \begin{cases} \varepsilon_m & \text{if } n \ge 5 \\ \varepsilon_m|\log(\varepsilon_m)|^{2/3} & \text{if } n = 4 \\ \varepsilon_m^{1/2} & \text{if } n = 3 \end{cases}$$

and, by [14, (7.34)],

$$\int_\Omega U_{\varepsilon_m,y_m}^{2^*-2}\omega_m^2\, dx = O(\|\omega_m\|^2)\,. \tag{34}$$

Hence,

$$|u_m|_{2^*}^{2^*} = C_m^{2^*}|U_{\varepsilon_m,y_m}|_{2^*}^{2^*} + O(\beta_n(\varepsilon_m)\|\omega_m\| + \|\omega_m\|^2)\,. \tag{35}$$

Furthermore, by [1, (3.25)],

$$\int_{\partial\Omega} U_{\varepsilon_m,y_m}\,\omega_m\, dx = O(\beta_n(\varepsilon_m)\|\omega_m\|).$$

Finally, by [14, (7.28)],

$$\int_\Omega U_{\varepsilon_m,y_m}\,\omega_m\, dx = O(\gamma_n(\varepsilon_m))\|\omega_m\|),$$

where

$$\gamma_n(\varepsilon_m) = \begin{cases} \varepsilon_m^2 & \text{if } n \ge 7 \\ \varepsilon_m^2|\log(\varepsilon_m)|^{2/3} & \text{if } n = 6 \\ \varepsilon_m^{(n-2)/2} & \text{if } n = 3,4,5\,. \end{cases}$$

Hence, $\gamma_n(\varepsilon_m) = o(\beta_n(\varepsilon_m))$, for every $n \ge 4$, and $\gamma_3(\varepsilon_m) = \beta_3(\varepsilon_m)$.
Next we get a lower bound for $\|\omega_m\|_{\lambda_m,c}$.

Lemma 4. *Let $n \ge 3$, there exist $\delta > 0$ and $m_0 > 0$ such that, for all $m \ge m_0$,*

$$\|\omega\|_{\lambda_m,c}^2 \ge (2^*-1+\delta)\int_\Omega U_{\varepsilon_m,y_m}^{2^*-2}\omega^2\, dx + O(\beta_n^2(\varepsilon_m)\|\omega\|^2),$$

for every $c > 0$ and for all ω orthogonal to the tangent space to the manifold $\mathcal{M}$ at $(1, \varepsilon_m, y_m)$.

Proof. The proof follows the lines of [3, Lemmas 3.3 and 3.4]. The main difference is that the eigenvalue problem considered there has to be replaced by

$$\begin{cases} -\Delta u - \lambda_m a(x) u = \mu\, U_{\varepsilon_m,y_m}^{2^*-2} u & \text{in } \Omega \\ u_\nu + cu = 0 & \text{on } \partial\Omega. \end{cases} \tag{36}$$

Let $\{u_{j,\varepsilon_m}\}_{j\geq 1}$ be a complete set of orthonormal eigenfunctions to (36), that is

$$\int_\Omega U_{\varepsilon_m,y_m}^{2^*-2} u_{i,\varepsilon_m} u_{j,\varepsilon_m}\, dx = \delta_{ij}\,,$$

with corresponding eigenvalues μ_{i,ε_m}.
Now, putting $\Omega_m := \frac{\Omega - y_m}{\varepsilon_m}$, for every $u \in H^1(\Omega)$, we define

$$\widetilde{u}(x) = \varepsilon_m^{(n-2)/2} u(\varepsilon_m x + y_m) \quad x \in \Omega_m\,.$$

There holds

$$\lim_{m\to+\infty} \mu_{i,\varepsilon_m} = \mu_i \quad \text{and} \quad \lim_{m\to+\infty} \int_{\Omega_m} U_1^{2^*-2} (\widetilde{u}_{j,\varepsilon_m} - \widetilde{u}_j)^2\, dx = 0\,, \tag{37}$$

where the μ_i and $\widetilde{u}_j$ are the eigenvalues and eigenfunctions of

$$\begin{cases} -\Delta u = \mu\, U_1^{2^*-2} u & \text{in } \mathbb{R}^n_+ \\ u_\nu = 0 & \text{on } \partial\mathbb{R}^n_+ \\ \int_{\mathbb{R}^n_+} U_1^{2^*-2} u^2\, dx = 1\,. \end{cases} \tag{38}$$

We refer to [3, Lemma 3.3] for the details of the proof of (37). We simply note that, to get (37), one first writes (36) in terms of $\widetilde{u}$. Then, the "convergence" to (38) is ensured by the fact that $\lim_{m\to+\infty} \Omega_m = \mathbb{R}^n_+$, by (31), $\varepsilon_m \lambda_m \to 0$ (since λ_m is bounded) and $c\varepsilon_m \to 0$.
Once (37) is proved, the very same arguments of the proof of [3, Lemma 3.4] (see also [14, Lemma 16]) give the statement. □

Next we estimate

$$M(\lambda_m, c) = J_{\lambda_m,c}(t_{\lambda_m,c}(u_m) u_m) \geq \sup_{t\geq 0} \left[\frac{t^2}{2} \|u\|^2_{\lambda_m,c} - \frac{t^{2^*}}{2^*} |u_m|_{2^*}^{2^*} - \frac{t^{q+1}}{q+1} B(\lambda_m)\, |u_m|_{q+1}^{q+1} \right],$$

where

$$B(\lambda) = \begin{cases} 0 & \text{if } g_\lambda(x,s) \leq 0 \text{ or } g_\lambda(x,s) \equiv 0\,, \\ |b_\lambda(x)|_\infty & \text{if } g_\lambda(x,s) \geq 0\,. \end{cases}$$

Then, putting

$$Q_{\lambda,c}(u) := \frac{\|u\|^2_{\lambda,c}}{|u|^2_{2^*}}\,,$$

if $g_\lambda(x,s) \leq 0$ or $g_\lambda(x,s) \equiv 0$, we get

$$M(\lambda_m, c) \geq \frac{1}{n} \left(Q_{\lambda_m,c}(u_m)\right)^{\frac{n}{2}}\,. \tag{39}$$

If $g_\lambda(x,s) \geq 0$, we get

$$\begin{aligned} M(\lambda_m, c) &\geq J_{\lambda_m,c}\left(\left(\frac{\|u_m\|^2_{\lambda_m,c}}{|u_m|^{2^*}_{2^*}} \right)^{\frac{n-2}{4}} u_m \right) \\ &\geq \tfrac{1}{n} \left(Q_{\lambda_m,c}(u_m)\right)^{\frac{n}{2}} - \tfrac{B(\lambda_m)}{q+1} \left(\frac{(Q_{0,c}(u_m))^{\frac{n-2}{4}}}{|u_m|_{2^*}} \right)^{q+1} |u_m|_{q+1}^{q+1}\,, \end{aligned} \tag{40}$$

where we have exploited the fact that $\left(\frac{\|u_m\|^2_{\lambda_m,c}}{|u_m|^{2^*}_{2^*}}\right)^{\frac{n-2}{4}} \leq \frac{(Q_{0,c}(u_m))^{\frac{n-2}{4}}}{|u_m|_{2^*}}$.
From the estimates recalled before stating Lemma 4, we have that

$$\begin{aligned} Q_{\lambda_m,c}(u_m) = &\left(Q_{\lambda_m,c}(U_{\varepsilon_m,y_m}) + \frac{\|\omega_m\|^2_{\lambda_m,c}}{C_m^2 |U_{\varepsilon_m,y_m}|^2_{2^*}} + O(c\,\beta_n(\varepsilon_m)\|\omega_m\|) + O(\lambda_m\,\gamma_n(\varepsilon_m)\|\omega_m\|) \right) \\ &\left(1 - \frac{(2^*-1)}{C_m^2|U_{\varepsilon_m,y_m}|^{2^*}_{2^*}} \int_\Omega U^{2^*-2}_{\varepsilon_m,y_m}\omega_m^2 + O(\beta_n(\varepsilon_m)\|\omega_m\|) + o(\|\omega_m\|^2) \right). \end{aligned} \tag{41}$$

For $n \geq 4$, by (28), $|U_{\varepsilon_m,y_m}|_2^2 = o(\varepsilon_m)$ and, by (30), we deduce

$$Q_{\lambda_m,c}(U_{\varepsilon_m,y_m}) = \frac{S}{2^{2/n}} + B_n \left(c - \frac{n-2}{2}\, H_{max} \right) \varepsilon_m + o(\varepsilon_m)\,. \tag{42}$$

If $n = 3$ (recall that $|U_{\varepsilon_m,y_m}|_2^2 = O(\varepsilon_m)$), the same estimate holds but with $\varepsilon_m|\log(\varepsilon_m)|$ instead of ε_m and with $O(\varepsilon_m)$ instead of $o(\varepsilon_m)$.
In what follows we consider separately the case $g_\lambda(x,s) \leq 0$ and $g_\lambda(x,s) \geq 0$.

Case $g_\lambda(x,s) \leq 0$ **or** $g_\lambda(x,s) \equiv 0$. Inspired by [14, (7.37)], we use the inequality

$$c\,\beta_n(\varepsilon_m)\|\omega_m\| \leq \frac{\gamma}{2}\|\omega_m\|^2 + \frac{c^2\beta_n^2(\varepsilon_m)}{2\gamma} \quad \text{for all } \gamma > 0\,.$$

This and (42), inserted into (41), give

$$\begin{aligned} Q_{\lambda_m,c}(u_m) &= Q_{\lambda_m,c}(U_{\varepsilon_m,y_m}) - \frac{(2^*-1)}{C_m^2(S^{n/2}/2)^{(n-2)/n}} \int_\Omega U^{2^*-2}_{\varepsilon_m,y_m}\omega_m^2 + \frac{\|\omega_m\|^2_{\lambda_m,c}}{C_m^2(S^{n/2}/2)^{(n-2)/n}} \\ &+ O(c\beta_n(\varepsilon_m)\|\omega_m\|) + o(\|\omega_m\|^2) \\ &\geq \frac{S}{2^{2/n}} + B_n\left(c - \frac{n-2}{2}\,H_{max}\right)\varepsilon_m + o(\varepsilon_m) \\ &+ \frac{1}{C_m^2(S^{n/2}/2)^{(n-2)/n}}\left[(1-\gamma_1-\gamma_2)\|\omega_m\|^2_{\lambda_m,c} - (2^*-1)\int_\Omega U^{2^*-2}_{\varepsilon_m,y_m}\omega_m^2\right] - \frac{c^2\beta_n^2(\varepsilon_m)}{2\gamma_3}\,, \end{aligned} \tag{43}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ can be arbitrarily small (recall that the norms $\|.\|$ and $\|.\|_{\lambda,c}$ are equivalent, for every $\lambda < \lambda_1^a$) and $\gamma_3 > 0$. More precisely, we choose γ_1 and γ_2 so small that, by Lemma 4, the quantity in the square parentheses is greater than or equal to $o(\beta_n^2(\varepsilon_m))$. We conclude that, for every $n \geq 4$,

$$Q_{\lambda_m,c}(u_m) \geq \frac{S}{2^{2/n}} + B_n\left(c - \frac{n-2}{2}\,H_{max}\right)\varepsilon_m + o(\varepsilon_m) - \frac{c^2\beta_n^2(\varepsilon_m)}{2\gamma_3}\,.$$

Since $\beta_n^2(\varepsilon_m) = o(\,\varepsilon_m)$, for $c > C'(\Omega) = \frac{n-2}{2}\,H_{\max}$, the above inequality with (39) contradicts the definition of $\Lambda(c)$. When $n = 3$, the same estimate holds with $\varepsilon_m|\log(\varepsilon_m)|$ instead of ε_m and with $O(\varepsilon_m)$ instead of $o(\varepsilon_m)$. Then, since $\beta_3^2(\varepsilon_m) = o(\,\varepsilon_m|\log(\varepsilon_m)|)$, we conclude as for $n \geq 4$.

Case $g_\lambda(x,s) \geq 0$**.** The proof works similarly, except that now one has to take into account the extra term $|u_m|^{q+1}_{q+1}$, where $1 < q \leq 2_T - 1 = \frac{n}{n-2}$.
By [3, Lemma 3.5] (with $L = 2$) we have that

$$\begin{aligned} |u_m|^{q+1}_{q+1} &= C_m^{q+1}\int_\Omega U^{q+1}_{\varepsilon_m,y_m}\,dx + (q+1)C_m^q\int_\Omega U^q_{\varepsilon_m,y_m}\omega_m\,dx \\ &+ \frac{q(q+1)}{2}\,C_m^{q-1}\int_\Omega U^{q-1}_{\varepsilon_m,y_m}\omega_m^2\,dx + O(\textstyle\int_\Omega |\omega_m|^{q+1}\,dx)\,. \end{aligned} \tag{44}$$

By Holder inequality, Sobolev embedding and the estimates (28), we deduce

$$\int_\Omega U^q_{\varepsilon_m,y_m}\omega_m\,dx \le |U_{\varepsilon_m,y_m}|^q_{2nq/(n+2)}\,|\omega_m|_{2^*} \le O(\theta_{n,q}(\varepsilon_m)\,\|\omega_m\|)\,,$$

where

$$\theta^q_n(\varepsilon_m) = \begin{cases} \varepsilon_m^{n-(q+1)\frac{n-2}{2}} & \text{if} \quad \begin{array}{l} n\ge 6 \text{ and } 1<q\le 2_T-1 \text{ or} \\ n=3,4,5 \text{ and } \frac{n+2}{2(n-2)}<q\le 2_T-1 \end{array} \\ \varepsilon_m^{\frac{n+2}{4}}\,|\log(\varepsilon_m)|^{\frac{n+2}{2n}} & \text{if } n=3,4,5 \text{ and } q=\frac{n+2}{2(n-2)} \\ \varepsilon_m^{q\frac{n-2}{2}} & \text{if } n=3,4,5 \text{ and } 1<q<\frac{n+2}{2(n-2)}\,. \end{cases}$$

A further application of Holder inequality and Sobolev embedding, together with (28), give

$$\int_\Omega U^{q-1}_{\varepsilon_m,y_m}\omega_m^2\,dx \le |U_{\varepsilon_m,y_m}|^{q-1}_{(q-1)(n/2)}\,|\omega_m|^2_{2^*} \le \begin{cases} O\left(\varepsilon_m^{n-(q+1)\frac{n-2}{2}}\,\|\omega_m\|^2\right) & \text{if } 1<q<2_T-1 \\ O\left(\varepsilon_m|\log(\varepsilon_m)|^{2/3}\,\|\omega_m\|^2\right) & \text{if } q=2_T-1\,. \end{cases}$$

By inserting the above estimates into (44), we get

$$|u_m|^{q+1}_{q+1} \le O(\varepsilon_m^{n-(q+1)\frac{n-2}{2}}) + O\left(\theta^q_n(\varepsilon_m)\,\|\omega_m\|\right) + o(\|\omega_m\|^2)\,, \tag{45}$$

where $1 < n-(q+1)\frac{n-2}{2} < 2$, if $1<q<2_T-1$, while $n-(q+1)\frac{n-2}{2}=1$, if $q=2_T-1$. Furthermore, if $n\ge 4$, $\theta^q_n(\varepsilon_m)=o(\varepsilon_m)$, for every $1<q<2_T-1$, and $\theta_{n,q}(\varepsilon_m)=O(\varepsilon_m)$, if $q=2_T-1$. In $n=3$, $\theta^q_n(\varepsilon_m)=o(\beta_3(\varepsilon_m))$, for every $1<q\le 2_T-1$.
By (34) and (43), we have that

$$Q_{0,c}(u_m) = \frac{S}{2^{2/n}} + B_n\left(c-\frac{n-2}{2}H_{max}\right)\varepsilon_m + O(c\,\beta_n(\varepsilon_m)\|\omega_m\|) + O(\|\omega_m\|^2) + o(\varepsilon_m)$$

and subsequently, by (32) and (35), that

$$\frac{(Q_{0,c}(u_m))^{(n-2)/4}}{|u_m|_{2^*}} = \frac{D_n}{C_m}\left(1+E_n\left(c-\tfrac{n-2}{2}\,H_{max}\right)\varepsilon_m + o(\varepsilon_m) + O(c\,\beta_n(\varepsilon_m)\|\omega_m\|) + O(\|\omega_m\|^2)\right)$$

$$\left(1+O(\varepsilon_m)+O(\beta_n(\varepsilon_m)\|\omega_m\|+\|\omega_m\|^2)\right)$$

$$= \frac{D_n}{C_m}\left(1+E_n c\varepsilon_m + O(\varepsilon_m) + O(c\beta_n(\varepsilon_m)\|\omega_m\|) + O(\|\omega_m\|^2)\right)\,,$$

for some $D_n, E_n>0$. Note that, if $n=3$, one has to replace $c\varepsilon_m$ with $c\varepsilon_m|\log(\varepsilon_m)|$. Finally, by (45), we conclude that

$$\left(\frac{(Q_{0,c}(u_m))^{(n-2)/4}}{|u_m|_{2^*}}\right)^{q+1} |u_m|^{q+1}_{q+1} \le O(\varepsilon_m^{n-(q+1)\frac{n-2}{2}}) + o(\varepsilon_m) + o(c\varepsilon_m) + o(\|\omega_m\|^2)\,,$$

with, if $n=3$, $o(c\varepsilon_m|\log(\varepsilon_m)|)$ instead of $o(c\varepsilon_m)$ and adding the term $o(\beta_3(\varepsilon_m)\|\omega_m\|)$ from (45).
By repeating the proof of the case $g_\lambda(x,s)\le 0$ and exploiting Lemma 4 (whose proof does not depend on q), by (40), we get that

$$M(\lambda_m,c) \ge \frac{S^{n/2}}{2n} + B'_n\left(c-\frac{n-2}{2}\,H_{max}\right)\varepsilon_m + O(\varepsilon_m^{n-(q+1)\frac{n-2}{2}}) + o(\varepsilon_m) + o(c\varepsilon_m)$$
$$+ D'_n\left[(1-\gamma_3)\|\omega_m\|^2_{\lambda_m,c} - (2^*-1)\int_\Omega U^{2^*-2}_{\varepsilon_m,y_m}\omega_m^2\right]\,,$$

where $\gamma_3 > 0$ can be chosen so small that the term in the square parentheses is greater than or equal to $o(\beta_n^2(\varepsilon_m))$.
Summarizing, for $n \geq 4$,

$$M(\lambda_m, c) \geq \frac{S^{n/2}}{2n} + B_n' \left(c - \frac{n-2}{2} H_{max} \right) \varepsilon_m + O(\varepsilon_m^{n-(q+1)\frac{n-2}{2}}) + o(\varepsilon_m) + o(c\varepsilon_m) \,.$$

If $n = 3$, replace ε_m with $\varepsilon_m |\log(\varepsilon_m)|$, $o(c\varepsilon_m)$ with $o(c\varepsilon_m |\log(\varepsilon_m)|)$ and $o(\varepsilon_m)$ with $O(\varepsilon_m)$.
Hence, in both cases, there exists $C'(\Omega) \geq \frac{n-2}{2} H_{max}$ such that, for any $c > C'(\Omega)$, the above estimate contradicts the definition of $\Lambda(c)$.
We note that, when $n = 3$ or $n \geq 4$ and $q < 2_T - 1$, one can choose $C' = \frac{n-2}{2} H_{max}$.

Remark 1. Even if this is beyond the scope of the present work, we make a couple of remarks concerning the limit case $c \to +\infty$. As already noticed, $\lambda_1^a(c)$ converges to $\lambda_{1,Dir}^a$, the first eigenvalue (with weight a) of $-\Delta$ under Dirichlet boundary conditions. On the other hand, by Lemma 3, there exists $\lim_{c\to+\infty} \Lambda(c) = \Lambda_\infty$ and $\Lambda_\infty > 0$, if case (iii) of Theorem 4 occurs. For every $\lambda \in (\Lambda_\infty, \lambda_{1,Dir}^a)$, as in [23, Theorem 3.6], it can be proved that any least energy solution to problem (P_λ) converges in $H^1(\Omega)$, as $c \to +\infty$, to a least energy solution of the corresponding Dirichlet problem.

3 Proof of Theorem 1

Let $\lambda^*(c)$ be as in Proposition 1. For any $\lambda \in (0, \lambda^*(c))$, as in [13], we look for a second solution to problem (1) of the form $U_\lambda = u_\lambda + \lambda^{-(n-2)/4} u$, where u_λ is the minimal solution and $u > 0$ in Ω. Then, u solves problem (P_λ) of Section 2 with

$$f_\lambda(x, u) := (\lambda^{(n-2)/4}(1 + u_\lambda) + u)^{2^*-1} - \lambda^{(n+2)/4}(1 + u_\lambda)^{2^*-1} - u^{2^*-1} \geq 0 \,. \tag{46}$$

Since the map $(0, \lambda^*(c)) \ni \lambda \mapsto u_\lambda(x)$ is increasing for a.e. $x \in \Omega$ (see [5]), a direct inspection shows that also the map $(0, \lambda^*(c)) \ni \lambda \mapsto f_\lambda(x, s)$ is increasing, for a.e. $x \in \Omega$ and for every $s > 0$, and $f_0(x, s) \equiv 0$. Namely, assumption $(f1)$ holds. On the other hand, write $f_\lambda(x, s) = \lambda \, \overline{a}(x) s + g_\lambda(x, s)$, where $\overline{a}(x) := (2^* - 1)\,(1 + u_\lambda(x))^{2^*-2}$. Clearly, $\overline{a}$ is a measurable positive and bounded function (recall that u_λ is bounded). Since some computations show that g_λ satisfies (12), then $(f2)$ holds.
For our purposes, we notice that

$$\begin{cases} g_\lambda(x, s) < 0 & \text{if } n \geq 7 \\ g_\lambda(x, s) = 0 & \text{if } n = 6 \\ 0 < g_\lambda(x, s) \leq \eta \lambda^{3/4}(1 + u_\lambda) s^{4/3} \ \text{ for some } \eta & \text{if } n = 5 \\ g_\lambda(x, s) = 3\lambda^{1/2}(1 + u_\lambda) s^2 & \text{if } n = 4 \\ g_\lambda(x, s) > 5\lambda^{1/4}(1 + u_\lambda) s^4 & \text{if } n = 3 \end{cases} \tag{47}$$

for every $s > 0$. Namely, $f_\lambda(x, s)$ is linear, up to a bounded weight, only when $n = 6$ (sub-linear if $n \geq 7$ and super-linear for $n = 3, 4, 5$).
The role of $\lambda_1^a(c)$ in Section 2 is assumed here by $\lambda^*(c)$ (recall that the map $c \mapsto \lambda^*(c)$ is increasing by Proposition 1). In particular, if we define $\mu_1^{\overline{a}}(\lambda, c)$ as in (14), the same arguments of [13, Proposition 2.15] yield that $\mu_1^{\overline{a}}(\lambda, c) \to 0$ as $\lambda \to \lambda^*(c)$, for every $c > 0$. Then, all the analysis performed in the previous section applies and we may set $\Lambda(c)$ as in (26) (with $\lambda^*(c)$ instead of $\lambda_1^a(c)$).
To conclude, we note that, if u is a mountain-pass solution to (P_λ), with f_λ as in (46), and $U_\lambda = u_\lambda + \lambda^{-(n-2)/4} u$, then

$$J_{\lambda,c}(u) = \lambda^{(n-2)/4} \left(\frac{1}{2} \|U_\lambda\|_c^2 - \frac{\lambda}{2^*} \int_\Omega (1 + U_\lambda)^{2^*} \right) + C_\lambda := \lambda^{(n-2)/4} I_{\lambda,c}(U_\lambda) + C_\lambda \,.$$

Here, $J_{\lambda,c}$ is as in (15), $I_{\lambda,c}$ is the functional associated to (1) and $C_\lambda = \lambda^{n/2}(\frac{1}{2^*}\int_\Omega (1+u_\lambda)^{2^*}\, dx - \frac{1}{2}\int_\Omega (1+u_\lambda)^{2^*-1} u_\lambda \, dx)$. Namely, u and U_λ have the same variational characterization.
Finally, the proof of Theorem 1 follows by combining the statement of Theorem 4 with (47).

4 Proof of Theorem 2

For $\eta > 0$, we denote by

$$V_\eta(x) = \frac{(\eta n(n-2))^{(n-2)/4}}{(\eta + |x|^2)^{\frac{n-2}{2}}} \quad x \in B\,.$$

Let $\lambda_n(c)$, u_{η_1} and u_{η_2} be as in Proposition 2, where $\eta_1 = \eta_1(\lambda, c)$ and $\eta_2 = \eta_2(\lambda, c)$ are as in (5). For $c > 0$ and $0 < \lambda < \lambda_n(c)$, we set

$$W_\lambda(x) := \lambda^{\frac{n-2}{4}}(u_{\eta_2}(x) - u_{\eta_1}(x)) = V_{\eta_2}(x) - V_{\eta_1}(x) \quad x \in B\,.$$

Recall that u_{η_1} and u_{η_2} solve problem (1) and that $u_{\eta_1} = u_\lambda$, the minimal solution to (1). Then, W_λ solves (P_λ), as defined in Section 2, with $\Omega = B$ and f_λ as in (46). Furthermore, this is the only *radial* solution to (P_λ), see [5, proof of Theorem 5].
By (5), for every $c > 0$, $\eta_1(\lambda, c) \nearrow +\infty$ and $\eta_2(\lambda, c) \searrow \eta_0(c)$ as $\lambda \to 0^+$. Hence, if $c \in (0, n-2)$, since $\eta_0(c) > 0$, we get

$$\lim_{\lambda \to 0^+} W_\lambda(x) = V_{\eta_0}(x) \quad \text{for a.e. } x \in \Omega\,,$$

where $V_{\eta_0}(x)$ is known to be the only radial solution to (P_0). More precisely, by [27, Theorem 4.2], (P_0) admits a positive radial solution if and only if $c \in (0, n-2)$ and the solution is explicitly given by $V_{\eta_0}(x)$.
Let $J_{\lambda,c}$ be as in (15) with $\Omega = B$ and let f_λ be as in (46). We have that

$$J_{\lambda,c}(W_\lambda) = \frac{1}{2}\|W_\lambda\|_c^2 - \frac{1}{2^*}\left(\int_B (\lambda^{(n-2)/4}(1+u_{\eta_1}(x)) + W_\lambda(x))^{2^*} - \lambda^{n/2}(1+u_{\eta_1}(x))^{2^*}\, dx\right)$$
$$+\lambda^{(n+2)/4}\int_B (1+u_{\eta_1}(x))^{2^*-1} W_\lambda(x)\, dx.$$

On the other hand, since u_{η_1} and u_{η_2} solve problem (1), we deduce that

$$\|u_{\eta_1}\|_c^2 = \lambda \int_B (1+u_{\eta_1}(x))^{2^*-1} u_{\eta_1}(x)\, dx\,, \qquad \|u_{\eta_2}\|_c^2 = \lambda \int_B (1+u_{\eta_2}(x))^{2^*-1} u_{\eta_2}(x)\, dx$$

and

$$\int_B \nabla u_{\eta_1}(x) \cdot \nabla u_{\eta_1}(x)\, dx + c\int_{\partial B} u_{\eta_1}(x) u_{\eta_2}(x)\, d\sigma = \lambda \int_B (1+u_{\eta_1}(x))^{2^*-1} u_{\eta_2}(x)\, dx.$$

Exploiting the above identities, recalling the definition of W_λ and that $u_{\eta_i} = \lambda^{-(n-2)/4} V_{\eta_i} - 1$, we conclude that

$$J_{\lambda,c}(W_\lambda) = \frac{1}{n}\int_B (V_{\eta_2}^{2^*}(x) - V_{\eta_1}^{2^*}(x))\, dx - \frac{\lambda^{(n-2)/4}}{2}\int_B (V_{\eta_2}^{2^*-1}(x) - V_{\eta_1}^{2^*-1}(x))\, dx. \tag{48}$$

Next we show

Proposition 3. *Let $J_{\lambda,c}(W_\lambda)$ be as in (48), there holds*

$$J_{\lambda,c}(W_\lambda) < \frac{S^{n/2}}{2n} \qquad \text{for all } \lambda \in (\gamma_n(c), \lambda_n(c))\,,$$

where

$$\gamma_n(c) := \begin{cases} 0 & \text{if } 0 < c \le \frac{n-2}{2} \\ \frac{n(n-2)}{4}\left(\frac{2c-(n-2)}{2c}\right)^{4/(n-2)} & \text{if } c > \frac{n-2}{2} \end{cases}$$

and $\lim_{c\to+\infty}\gamma_n(c) = \lim_{c\to+\infty}\lambda_n(c) = \lambda^*_{Dir}$.
Furthermore, we have that

$$\lim_{\lambda\to(\lambda_n(c))^-} J_{\lambda,c}(W_\lambda) = 0 \quad \text{and} \quad \lim_{\lambda\to 0^+} J_{\lambda,c}(W_\lambda) = \begin{cases} \alpha(c) \in (0, \frac{S^{n/2}}{n}) & \text{for } 0 < c < n-2 \\ \frac{S^{n/2}}{n} & \text{for } c \ge n-2\,, \end{cases}$$

where the map $(0, n-2) \ni c \mapsto \alpha(c)$ *is increasing,* $\lim\limits_{c\to 0^+}\alpha(c) = 0$, $\alpha\left(\frac{n-2}{2}\right) = \frac{S^{n/2}}{2n}$ *and* $\lim\limits_{c\to(n-2)^-}\alpha(c) = \frac{S^{n/2}}{n}$.

Proof. First we prove the second part of the statement. As $\lambda \to (\lambda_n(c))^-$, $\eta_1(\lambda,c) \searrow \overline{\eta}$ and $\eta_2(\lambda,c) \nearrow \overline{\eta}$, for every $c > 0$, with $\overline{\eta}$ as in (5). Hence, $W_\lambda \to 0$ and $J_{\lambda,c}(W_\lambda) \to 0$ for a.e. $x \in B$.
Let $\lambda \to 0^+$, by (5), for every $c > 0$, $\eta_1(\lambda,c) \nearrow +\infty$ and $\eta_2(\lambda,c) \searrow \eta_0(c)$, with η_0 as in (3). In turn,

$$\int_B V_{\eta_1}^{2^*}(x)\,dx \to 0\,, \quad \int_B V_{\eta_1}^{2^*-1}(x)\,dx \to 0 \quad \text{and} \quad \int_B V_{\eta_2}^{2^*-1}(x)\,dx \to \eta_0(c)^{(n-2)/4}C_n\,,$$

for some $C_n > 0$. Hence,

$$\lim_{\lambda\to 0^+} J_{\lambda,c}(W_\lambda) = \lim_{\eta_2\searrow\eta_0(c)} \frac{1}{n}\int_B V_{\eta_2}^{2^*}\,dx.$$

If $c \in (0, n-2)$, $\eta_0(c) > 0$ and we have

$$\int_B V_{\eta_0}^{2^*}(x)\,dx = (n(n-2))^{n/2}\eta_0^{n/2}\int_B \frac{1}{(\eta_0 + |x|^2)^n}\,dx = (n(n-2))^{n/2}\eta_0^{n/2}\int_0^1 \frac{\omega_n\, r^{n-1}}{(\eta_0 + r^2)^n}\,dr.$$

For every $\eta > 0$, set $h(\eta) := \eta^{n/2}\int_0^1 \frac{\omega_n\, r^{n-1}}{(\eta+r^2)^n}\,dr$. Then,

$$h'(\eta) = \frac{n\,\omega_n\,\eta^{(n-2)/2}}{2}\int_0^1 \frac{(r^2-\eta)r^{n-1}}{(\eta+r^2)^{n+1}}\,dr =: \frac{n\,\omega_n\,\eta^{(n-2)/2}}{2}\,g(\eta).$$

Clearly, $g(\eta) < 0$ for any $\eta \ge 1$.
Let now $\eta \in (0,1)$, then

$$\begin{aligned} g(\eta) &= \int_0^{\sqrt{\eta}} \frac{(r^2-\eta)r^{n-1}}{(\eta+r^2)^{n+1}}\,dr + \int_{\sqrt{\eta}}^1 \frac{(r^2-\eta)r^{n-1}}{(\eta+r^2)^{n+1}}\,dr \\ &= \frac{1}{\eta^n}\left(\int_0^1 \frac{(y^2-1)\,y^{n-1}}{(1+y^2)^{n+1}}\,dy + \int_1^{1/\sqrt{\eta}} \frac{(y^2-1)\,y^{n-1}}{(1+y^2)^{n+1}}\,dy\right) \\ &= \frac{1}{\eta^n}\left(\int_0^1 \frac{(y^2-1)\,y^{n-1}}{(1+y^2)^{n+1}}\,dy + \int_{\sqrt{\eta}}^1 \frac{(1-s^2)\,s^{n-1}}{(1+s^2)^{n+1}}\,ds\right) < 0. \end{aligned}$$

Hence, $h(\eta)$ is a decreasing function. Since the map $(0, n-2) \ni c \mapsto \eta_0(c) \in (0, +\infty)$ is decreasing, we conclude that $(0, n-2) \ni c \mapsto \alpha(c) := \frac{1}{n} \int_B V_{\eta_0}^{2^*}(x)\, dx$ is increasing.
Since $\eta_0(\frac{n-2}{2}) = 1$, we have

$$\alpha\left(\frac{n-2}{2}\right) = \frac{1}{n} \int_B V_1^{2^*}(x)\, dx = \frac{1}{n} \left(n(n-2)\right)^{n/2} \int_0^1 \frac{\omega_n\, r^{n-1}}{(1+r^2)^n}\, dr$$

$$= \frac{1}{2n} (n(n-2))^{n/2} \int_0^{+\infty} \frac{\omega_n\, r^{n-1}}{(1+r^2)^n}\, dr = \frac{1}{2n} (n(n-2))^{n/2} \omega_n \frac{\Gamma(n/2)^2}{2\,\Gamma(n)} = \frac{S^{n/2}}{2n} .$$

Recall that

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{and} \quad S = \pi n(n-2) \left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{2/n} ,$$

see [26].
When $c \to (n-2)^-$, then $\eta_0(c) \searrow 0$ and similarly one gets

$$\lim_{c \to (n-2)^-} \alpha(c) = \lim_{\eta_0(c) \searrow 0} \frac{1}{n} \left(n(n-2)\right)^{n/2} \int_0^{1/\sqrt{\eta_0(c)}} \frac{\omega_n\, r^{n-1}}{(1+r^2)^n}\, dr = \frac{S^{n/2}}{n}.$$

Since $\eta_0(c) = 0$ for any $c \geq n-2$, the same holds for any c in this range.
Let now $\lambda > 0$. Computations analogous to those done above give $\frac{d}{d\eta} \left(\int_B V_\eta^\alpha(x)\, dx \right) < 0$ for all $\alpha > 0$, if $\eta \geq 1$ (and also if $\eta \in (0,1)$, when $\alpha = 2^*$). Then, when $\eta_2 \geq 1$, we deduce

$$J_{\lambda,c}(W_\lambda) < \frac{1}{n} \int_B (V_{\eta_2}^{2^*}(x) - V_{\eta_1}^{2^*}(x))\, dx \leq \frac{1}{n} \int_B V_{\eta_2}^{2^*}(x)\, dx \leq \frac{1}{n} \int_B V_1^{2^*}(x)\, dx = \frac{S^{n/2}}{2n}.$$

If $c \in (0, \frac{n-2}{2}]$, $\eta_0(c) \geq 1$ and subsequently $\eta_2(\lambda, c) \geq 1$, for every $\lambda \in (0, \lambda_n(c))$. Namely, the above estimate holds for every $\lambda \in (0, \lambda_n(c))$. When $c > \frac{n-2}{2}$, by (5), $\eta_2(\lambda, c) \geq 1$ if $\lambda \in (\gamma_n(c), \lambda_n(c))$, where $\gamma_n(c) := \varphi(1)^{1/(n-2)}$, with φ as in (4). To conclude we note that $\overline{\eta} \searrow 1$ as $c \to +\infty$. □

Proof of Theorem 2 completed.

The proof of statement (i) is a straightforward consequence of Lemma 1 and Proposition 3.
Let us now turn to (ii). For $c > \frac{n-2}{2}$ we set

$$\Lambda_{rad}(c) := \inf \left\{ 0 < \lambda < \lambda_n(c) : \quad M_{rad}(\lambda, c) < \frac{S^{n/2}}{2n} \right\} ,$$

where

$$M_{rad}(\lambda, c) = \inf_{u \in \mathcal{N}_{rad}} J_{\lambda,c}(u)$$

and $\mathcal{N}_{rad} := \{u \in H^1(\Omega) \setminus \{0\} : u(x) = u(|x|) \text{ and } J'_{\lambda,c}(u)[u] = 0\}$. As in the nonradial case, the map $\lambda \mapsto M_{rad}(\lambda, c)$ is nonincreasing and continuous, see Section 2.
Since W_λ is a radial solution to (P_λ) (with f_λ as in (46)), we infer $W_\lambda \in \mathcal{N}_{rad}$. Then, by Proposition 3, Λ_{rad} is well-defined and $\Lambda_{rad}(c) \leq \gamma_n(c)$, for every $c > 0$. Hence, $\lim_{c \to (\frac{n-2}{2})^+} \Lambda_{rad}(c) = 0$. The fact that the map $c \mapsto \Lambda_{rad}(c)$ is nondecreasing (increasing if $M_{rad}(\Lambda_{rad}(c), c)$ is achieved) follows as in Lemma 3.
On the other hand, by Lemma 1, for every $\lambda > \Lambda_{rad}$, (P_λ) admits a mountain-pass solution U_λ which turns out to be radial. Furthermore, $U_\lambda = W_\lambda$ (since there are no other radial solutions). Were $\Lambda_{rad} = 0$, W_λ would be a mountain-pass solution to (P_λ), for every $\lambda \in (0, \lambda_n(c))$. Since the map $\lambda \mapsto J_{\lambda,c}(W_\lambda)$ is continuous (η_1 and η_2 depend continuously from λ), this contradicts Proposition 3.
Hence, when $c > \frac{n-2}{2}$, $J_{\lambda,c}(W_\lambda) \geq \frac{S^{n/2}}{2n}$ for every $0 < \lambda \leq \Lambda_{rad}(c)$ and $J_{\lambda,c}(W_\lambda) < \frac{S^{n/2}}{2n}$ for every $\lambda \in (\Lambda_{rad}(c), \lambda_n(c)$, with $\Lambda_{rad}(c) > 0$. By continuity, $J_{\Lambda_{rad}(c),c}(W_{\Lambda_{rad}(c)}) = \frac{S^{n/2}}{2n}$ and we conclude.

5 Proof of Theorem 3

We follow the same notations of Sections 1 and 2. In the spirit of the computations performed in [10, Appendix A] (see also [18]), we deduce (11) from the nonexistence result of Theorem 4-(iii).
Consider problem (P_λ) of Section 2, with $f_\lambda(x,u) = \lambda a(x)u + \lambda u^{2_T-1}$. Assumptions $(f1)$ and $(f2)$ are satisfied. For every $0 < \lambda < \lambda_1^a(c)$, let $t_{\lambda,c}(u)$ be as in (17). With this choice of f_λ, $t_{\lambda,c}(u)$ can be explicitly computed and we get

$$t_{\lambda,c}(u) = \left(\frac{-\lambda |u|_{2_T}^{2_T} + \sqrt{\lambda^2 |u|_{2_T}^{22_T} + 4\|u\|_{\lambda,c}^2 |u|_{2^*}^{2^*}}}{2|u|_{2^*}^{2^*}} \right)^{\frac{n-2}{2}},$$

see [10, Appendix A]. This allows us to determine explicitly the function $\Psi_{\lambda,c}(u) := J_{\lambda,c}(t_{\lambda,c}(u)u)$. More precisely, for every $c > 0$, $0 < \lambda < \lambda_1^a(c)$ and $u \in H^1(\Omega) \setminus \{0\}$, we set

$$\delta_{\lambda,c}(u) := \frac{\lambda}{2} \frac{|u|_{2_T}^{2_T}}{\|u\|_{\lambda,c} |u|_{2^*}^{2^*/2}}$$

and we get $\Psi_{\lambda,c}(u) = \frac{1}{n} \left(\Phi_{\lambda,c}(u)\right)^{n/2}$, where

$$\Phi_{\lambda,c}(u) := Q_{\lambda,c}(u) \left(\sqrt{\delta_{\lambda,c}^2(u) + 1} - \delta_{\lambda,c}(u) \right)^{4/2^*} \left[1 - \frac{2}{2_T} \delta_{\lambda,c}(u) \left(\sqrt{\delta_{\lambda,c}^2(u) + 1} - \delta_{\lambda,c}(u) \right) \right]^{2/n},$$

with $Q_{\lambda,c}(u)$ as in Section 2.2.
We note that

$$0 < \sqrt{\delta_{\lambda,c}^2(u) + 1} - \delta_{\lambda,c}(u) \le 1 \quad \text{and} \quad 0 \le \delta_{\lambda,c}(u) \left(\sqrt{\delta_{\lambda,c}^2(u) + 1} - \delta_{\lambda,c}(u) \right) < \frac{1}{2} < \frac{2_T}{2}.$$

Then, recalling that

$$\left(1 - \frac{2}{2_T} y\right)^{2/n} \left(1 + \frac{4}{n \cdot 2_T} y\right) \le 1 \quad \text{for every } 0 \le y \le \frac{2_T}{2},$$

we estimate

$$\Phi_{\lambda,c}(u) \le \frac{Q_{\lambda,c}(u)}{\left(1 + \frac{4}{n \cdot 2_T} \delta_{\lambda,c}(u) \left(\sqrt{\delta_{\lambda,c}^2(u) + 1} - \delta_{\lambda,c}(u) \right)\right)}.$$

Let $K = K(\Omega) \ge C(\Omega) \ge \frac{n-2}{2} H_{\max}$, with $C(\Omega)$ as in (7), be as given in Theorem 4-(iii). Then, $M(\lambda, c) = \frac{S^{n/2}}{2n}$, for every $c > K(\Omega)$ and for every $\lambda \in (0, \Lambda(c)]$, with $0 < \Lambda = \Lambda(c) < \lambda_a^1(c)$. This and (16) yield $\Psi_{\lambda,c}(u) \ge \frac{S^{n/2}}{2n}$ for any $u \in H^1(\Omega) \setminus \{0\}$, $c > K(\Omega)$ and $\lambda \in (0, \Lambda(c)]$. By noting that

$$\delta_{\lambda,c}(u) \left(\sqrt{\delta_{\lambda,c}^2(u) + 1} - \delta_{\lambda,c}(u) \right) = \frac{\delta_{\lambda,c}(u)}{\sqrt{\delta_{\lambda,c}^2(u) + 1} + \delta_{\lambda,c}(u)} = \frac{\lambda |u|_{2_T}^{2_T}}{\sqrt{\lambda^2 |u|_{2_T}^{2 \cdot 2_T} + 4\|u\|_{\lambda,c}^2 |u|_{2^*}^{2^*}} + \lambda |u|_{2_T}^{2_T}}$$

then the statement follows from the estimate of $\Phi_{\lambda,c}(u)$ (and, in turn, of $\Psi_{\lambda,c}$) just performed.

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